Solving Laplace Problems with Corner Singularities

Caelen Feller, Supervised by Professor Kirk M. Soodhalter

Abstract

In this project, I aim to further existing research in using rational function approximations to solve Laplace problems in two dimensions with corner singularities. I do this by:

- Contextualising the theory of Gopal and Trefethen's least-squares algorithm.
- Implementing their proposed algorithm.
- Extending it to handle discontinuous and non-Dirichlet boundary conditions.
- Exploring performance on elongated domains and non-convex domains.
- Comparing method against the standard finite and boundary element methods.

Introduction

Approximation theory studies the asymptotic behaviour as series of simple functions approach more complex ones. The speed of convergence is a common concern, measured in this project by the supremum norm of the error over the domain of interest. It was shown by Gopal & Trefethen that the approximation scheme in Equation 1

$$r(z) = \sum_{j=1}^{N_1} \frac{a_j}{z - z_j} + \sum_{j=0}^{N_2} b_j (z - z_*)^j \qquad (1)$$

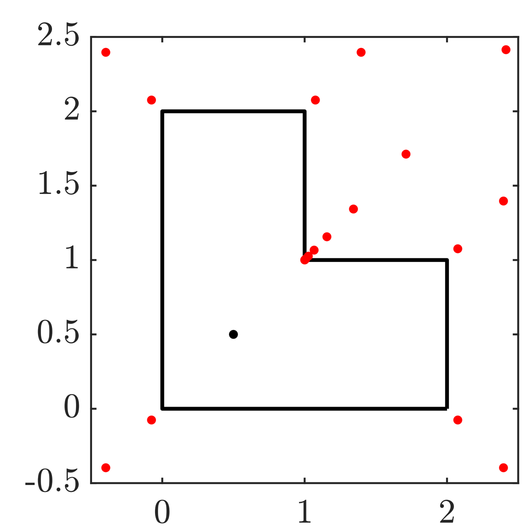
Approximation Scheme of [1]

can be used to find a holomorphic function on a convex polygon with root-exponential convergence given certain growth. Root-exponential convergence is defined by the order of the error in terms of the degree of the approximation is n,

$$\left\| f(x) - \hat{f}(x) \right\|_\infty = O\left(e^{-C\sqrt{n}}\right)$$

Theory

Gopal & Trefethen's result is inspired by two main theorems. In Runge's Theorem (1885), we have that polynomials can be used to approximate holomorphic functions with uniform convergence on simply-connected domains. The results of D.J. Newman (1964) show that rational functions approximate the absolute value function on $[a, b]$ with root-exponential convergence given exponentially clustered poles [2].



Poles(z_j) and center point (z_*) of Eqn.1 on a polygonal domain

Above are the poles of Gopal & Trefethen's approximation scheme, based on Newman's work, and the center point, based on Runge's Theorem. They show that given these fixed poles, on a convex, simply-connected, polygonal domain where the function being approximated obeys a growth condition, we have root-exponential convergence. This result can be applied to the solutions of Laplace equations as they are harmonic functions. Taking the real part of Eqn. 1 we can apply it to harmonic functions satisfying the same hypotheses using Hilbert transform to identify the harmonic conjugate, preserving the necessary properties.

Implementation

The algorithm below computes the coefficients of Eqn. 1 for a Laplace boundary value problem, specified in the form of either Dirichlet, Neumann, or Robin boundary data (specified values of solution, of its derivative, or a mixture).

Approximation Algorithm

For boundary Γ, with corners $\omega_1, \ldots, \omega_m$, boundary function h and error ϵ, with increasing values of n.

1. Fix $N_1 = O(mn)$ poles clustered outside corners exponentially.
2. Fix $N_2 + 1 = O(n)$ monomials, $1, \ldots, (z - z_*)^{N_2}$, and set $N = N_1 + N_2 + 1$
3. $M \approx 3N$ sample points on the boundary clustered near the corners.
4. Evaluate at sample points to form $M \times N$ matrix A and M-vector b
5. Solve the least-squares problem, $Ac \approx b$ for c.
6. Exit loop if $\|Ac - b\|_\infty < \epsilon$.

In step five, least-squares, rather than seek a true inverse to a problem without one, we instead seek to minimise the boundary error, here $\|Ac - b\|_\infty$, which guarantees the error within the domain by the maximum principle for harmonic functions. This is done in MATLAB using QR-factorisation, where A is decomposed into unitary and upper triangular factors for easier computations.

Extensions

Discontinuous boundary conditions are handled using weighting of error calculation based on the distance to the nearest corner. The solution can then still converge to zero in error despite corner discontinuities. Neumann boundary conditions are handled using a single-sided derivative estimate at each sampling point. The derivative is matched rather than the value of the solution itself. Curved boundaries use different parametrisation of boundaries, and cause issues when too unusual. Elongated domains have poor convergence behaviour, as shown in the report. This is addressed by adaptive boundary sampling point insertion, increasing the level of detail on long boundaries.

Recommendations

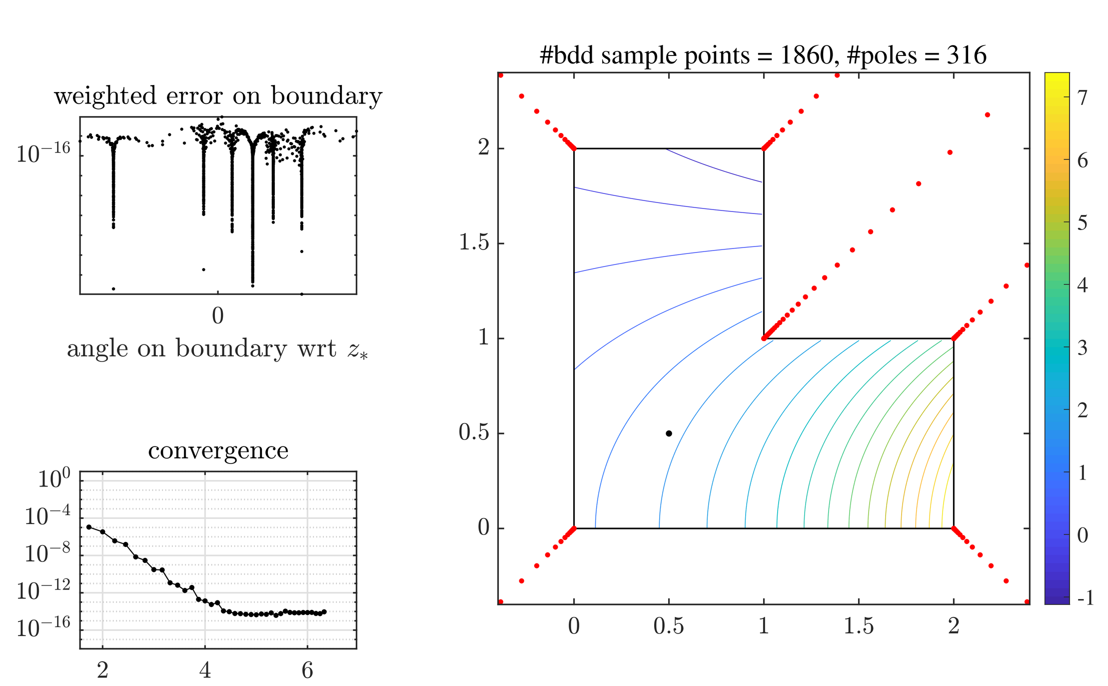
Work needs to be done to increase the accuracy cap on the method in cases. A fully rigorous proof or disproof of convergence for convex domains is necessary. This is a promising start to the research, though it needs to be extended to further dimensions and/or PDEs to see widespread use. Compared to the tests I ran using FEM/BEM in my report, this method is faster and more optimised due to its specialisation.

References

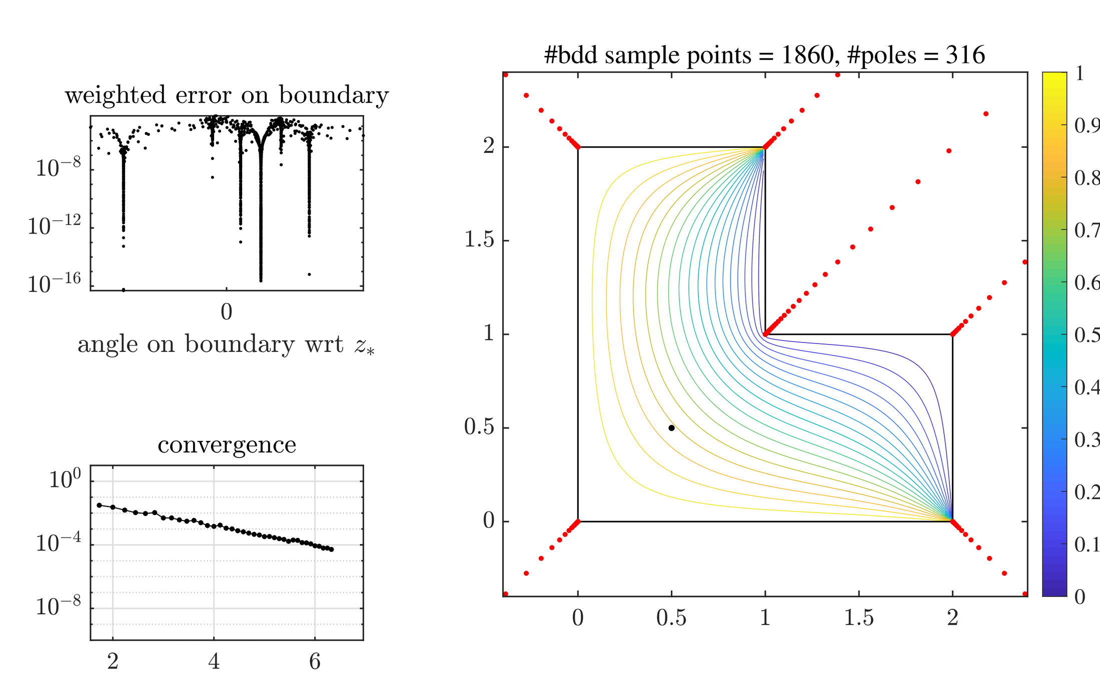
[1] Abinand Gopal and Lloyd N Trefethen. Solving laplace problems with corner singularities via rational functions. *SIAM Journal on Numerical Analysis*, 57(5):2074–2094, 2019. Implementation available at `https://people.maths.ox.ac.uk/trefethen/lightning.html`.

[2] Donald J Newman. Rational approximation to $|x|$. *The Michigan Mathematical Journal*, 11(1):11–14, 1964.

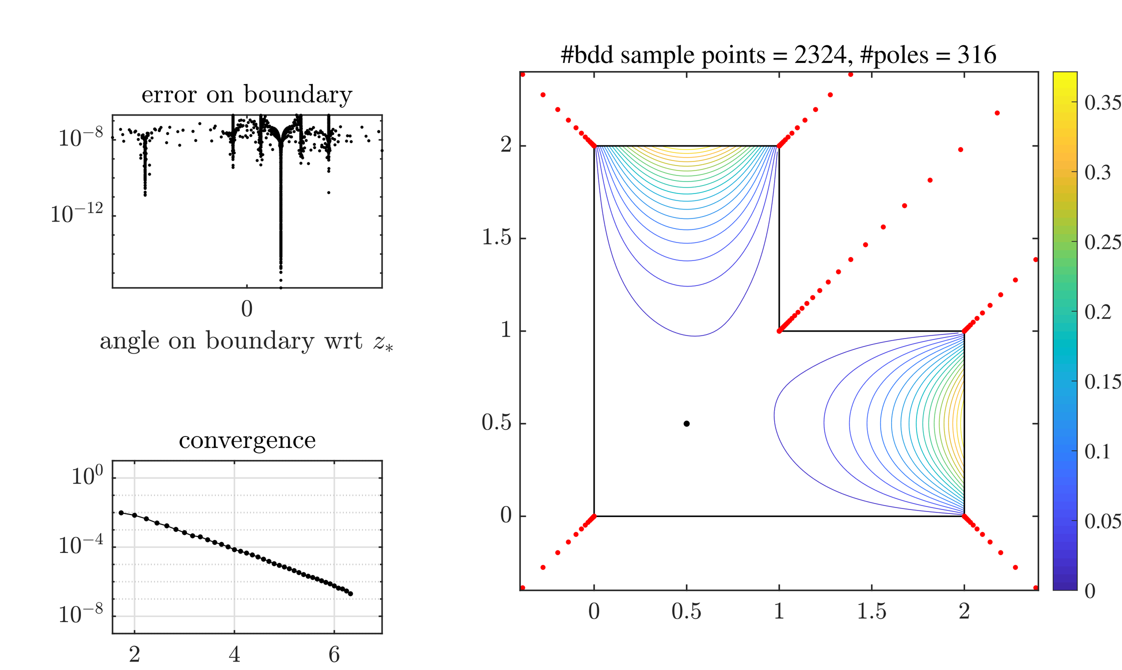
Results



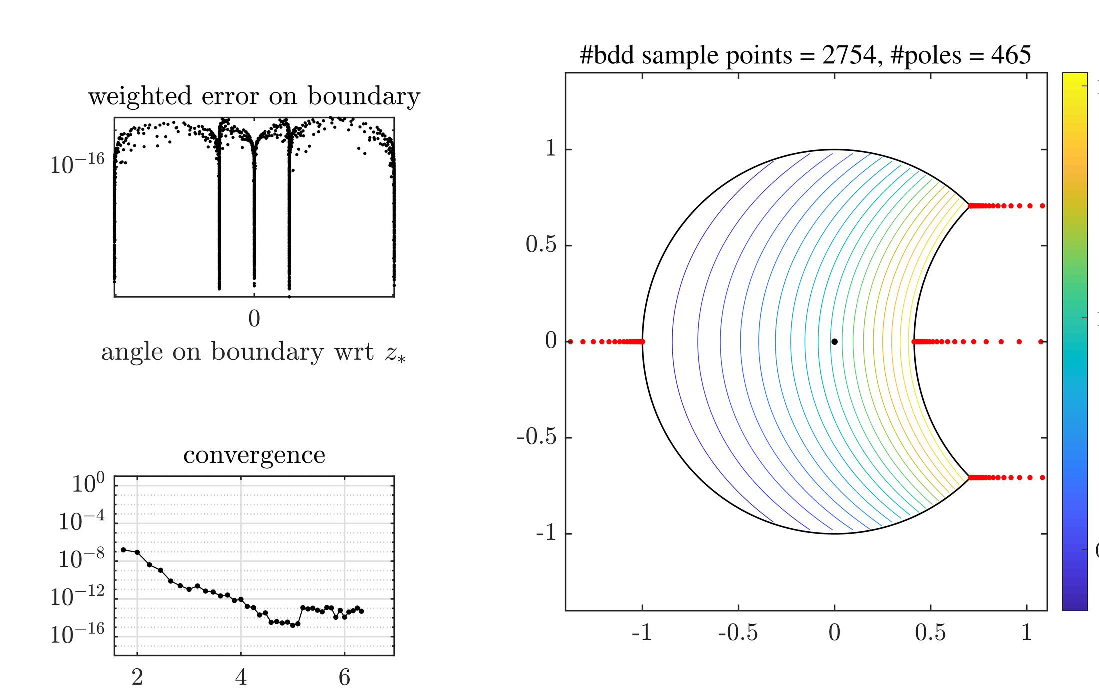
Result of program for $\Re(\exp(z))$



Result of program for discontinuous boundary with values of 0 or 1



Result of program for Robin boundary with Neumann values of 1 or Dirichlet values of 0



Result of program for curved boundary for $\Re(\exp(z))$