Solving Laplace Problems with Corner Singularities

Caelen Feller

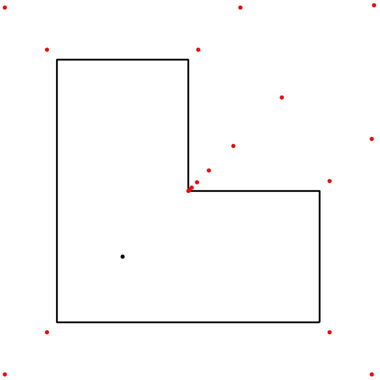
Supervised by: Prof. Kirk M. Soodhalter

January 24 2020

Theory Overview

Approximation Scheme

$$r(z) = \sum_{j=1}^{N_1} \underbrace{\frac{a_j}{z - z_j}}_{\text{Newman}} + \sum_{j=0}^{N_2} \underbrace{b_j (z - z_*)^j}_{\text{Runge}}$$



Background

A short timeline of results used

- 1885 - Carl Runge proves Runge's theorem, allowing uniform convergence using polynomials within the bulk.
- 1964 - D.J. Newman proves root-exponential convergence for $|x|$ using rational approximations.
- 2003 - Herbert Stahl proves root-exp for $|x|^{\alpha}$
- 2019 - Trefethen et. al show root-exp. convergence for this.

Definition (Root-Exponential Convergence)

Given a sequence $\{x_n\}$, converging to a true value x. Convergence of the sequence is root-exponential iff

$$\epsilon_n = |x_n - x| = O\left(\exp\left(-C\sqrt{N}\right)\right), \ N \to \infty, \ C > 0$$

Goals

- Investigation and implementation of Trefethen's results.
- Extension of theory and/or implementation completed for:
  - Non-continuous boundary conditions.
  - Non-Dirichlet boundary conditions.
  - Curved domains.
- Verification of Trefethen's results vs traditional methods

Root-Convergence Theorem

Theorem

Let Ω be a convex polygonal domain with corners $\omega_1 \ldots, \omega_m$, $f \in \mathcal{O}(\Omega)$, with holom. continuation to $\Delta_{\epsilon_k}(\omega_k) \backslash$ the exterior bisector of domain at ω_k.

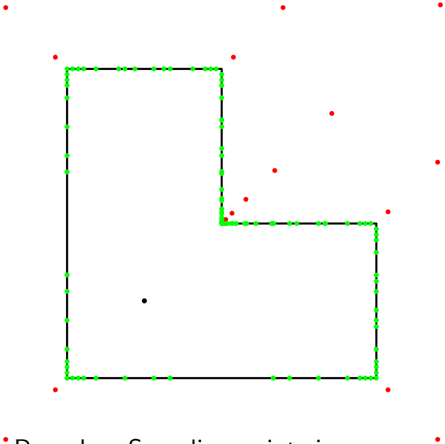
If $\exists\ \delta > 0$ s.t. $|f(z) - f(\omega_k)| = O(|z - \omega_k|^{\delta}),\ z \to \omega_k \implies$ $\exists$ degree n rational functions $\{r_i\},\ i > 0$ s.t. $|f - r|_{\Omega} = O(\exp(-C\sqrt{n})), C > 0,\ n \to \infty$, with the finite poles of each r_n clustered exponentially along the exterior bisectors of the domain such that the number of poles near a corner grows in proportion to n, as $n \to \infty$

Algorithm

For boundary Γ, with corners $\omega_1, \ldots, \omega_m$, boundary function h and error ϵ, with increasing values of n.

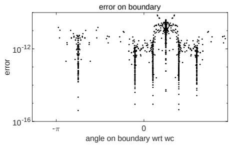
1. Fix $N_1 = O(mn)$ poles clustered outside corners.
2. Fix $N_2 + 1 = O(n)$ monomials, $1, \ldots, (z - z_*)^{N_2}$, and set $N = N_1 + N_2 + 1$
3. $M \approx 3N$ sample points on the boundary clustered near the corners.
4. Evaluate at sample points to form $M \times N$ matrix A and M-vector b
5. **Solve the least-squares problem**, $Ac \approx b$ for c.
6. Exit loop if $\|Ax - b\|_\infty < \epsilon$.

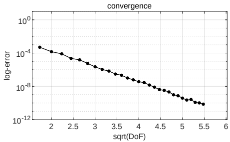
Boundary Sampling Method

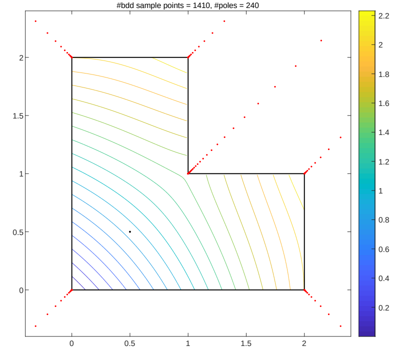


Boundary Sampling points in green

Full Solution

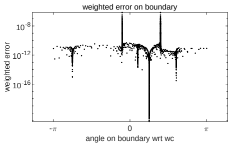


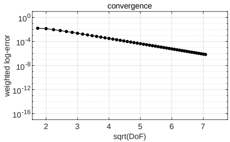


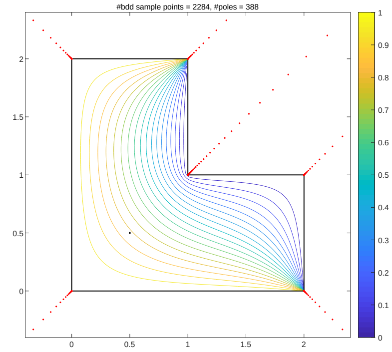


L-Shaped domain with solution

Demonstration

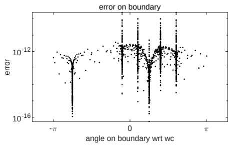


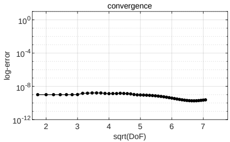


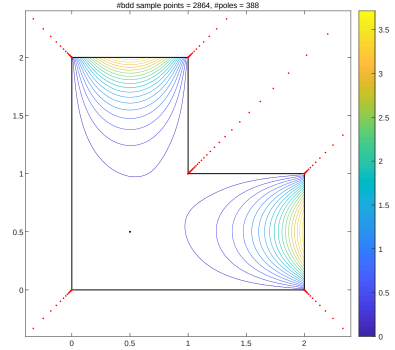


Discontinuous Boundary Conditions

Demonstration

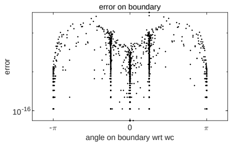


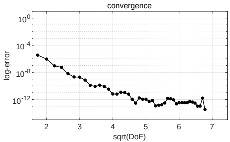


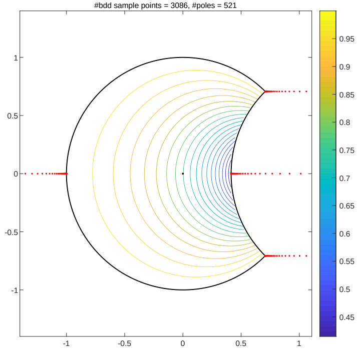


Robin Boundary Conditions

Demonstration

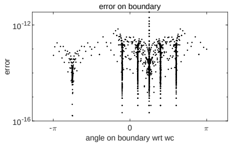


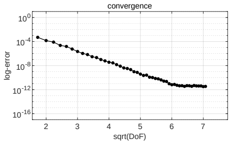


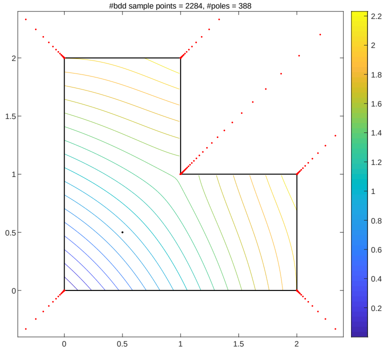


Curved boundaries

Limit to Accuracy







Upper bound to accuracy $1e^{-10}$

Further Goals

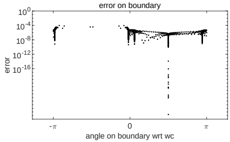
In the remainder of the time allocated for this project, I wish to extend this method to cover at least the following cases:

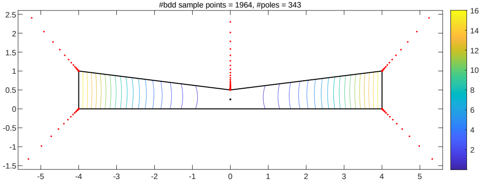
- Elongated domains
- Domains with slits & multiply connected domains
- Non-convex domains & more complex curved domains
- Transmission problems

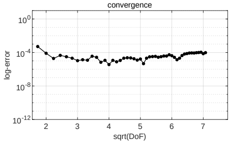
I wish to address the lack of stability in the solution for high precision, whether is a software or analytic issue, and understand where this cap in accuracy comes from.

Thank you!

Elongated Domain

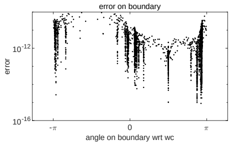


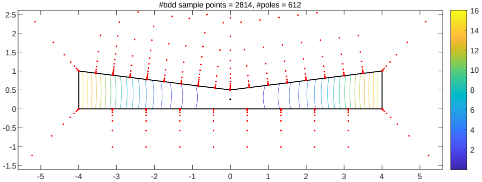


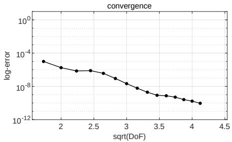


No convergence in case of elongated domain

Elongated Domain

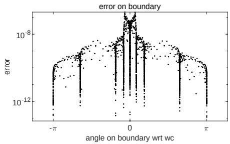


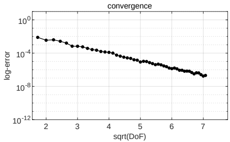


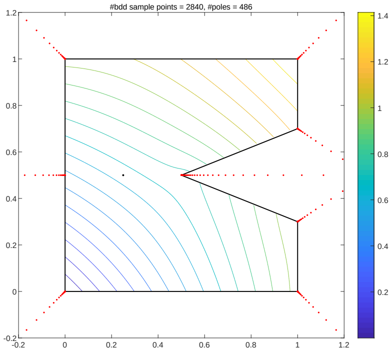


Potential fix to domain

Re-Entrant Spike

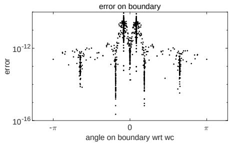


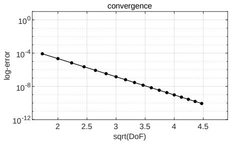


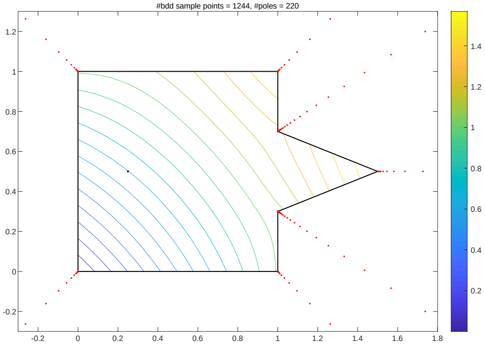


Re-Entrant Spike converges slowly

Salient Spike







Salient Spike converges faster

Harmonic Functions

Harmonic Functions

- Harmonic functions can be associated with holomorphic ones using Hilbert transform.
- Imaginary part of transformed function can also be extended across boundaries and about slits with correct behaviour.
- Previous theorem thus holds for harmonic functions
- Laplace solutions are harmonic