Nakayama's Lemma

math center

This note tries to cover some basic commutative algebra. So *all rings are commutative.*

Indeed, Nakayama's lemma is used to reduce finite modules to linear algebra. Let's start with the linear algebra.

Theorem 0.1 (Hamilton-Cayley)**.** Let R be a ring and let $A \in \mathrm{Mat}_n(R)$. Then $\chi_A(A) = 0$.

Proof. There are of course a ton of ways to prove this. The most standard proof is

$$\chi_A(t) := \det(t - A) \implies \chi_A(A) = \det(0) = 0.$$

In more details, let $Y = R[t]$ be a formal polynomial ring, and $V := R^{\oplus n}$ has a Y-module structure given by $t \rightsquigarrow A$. Consider the base change $V_Y = Y^{\oplus n}$ of V and $t - A \in \mathrm{End}(V_Y)$. Observe that the composition

$$V \hookrightarrow V_Y \longrightarrow \mathrm{coker}\,(t - A)$$

is an isomorphism of Y-modules. Thus $\chi_A(t) = (t - A)^\vee (t - A)$ annihilates V. □

Remark 0.2. Alternatively, observe that this is a purely formal proposition and it suffices to give a proof over the field $\mathbb{Q}(X_{11}, X_{12}, \cdots, X_{nn})$. Since a matrix A is non-diagonalizable if and only if the discriminant of the determinant of $t - A$ vanishes, the set these matrices is a Zariski closed set. Indeed, an infinite field, every nonempty Zariski open set is dense. Hence we may reduce to the case where A is diagonal, which is trivial.

1 Nakayama

Theorem 1.1 (Nakayama's lemma, 1st version)**.** Let R be a ring and I be an ideal. If M is a finitely generated module satisfying $IM = M$, then there exists $\alpha \in 1 + I$ which annihilates M.

Proof. Pick a set $\{e_i\}_{i=1}^n$ of generators of M. The condition implies elements $a_{ij} \in I$ satisfying

$$e_i = \sum_j a_{ij} e_j, \quad \forall i.$$

□

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