What is Dolbeault cohomology

math center

1 Preliminaries on complex manifolds

A *complex structure* J on a real vector space V is a linear map $J : V \to V$ such that $J^2 = -\mathrm{id}_V$. It equivalently provides a $\mathbb{C}$-vector space structure on V, so after complexification, we have a natural splitting

$$V \otimes_{\mathbb{R}} \mathbb{C} \simeq V_J \oplus \overline{V_J}$$

into the two eigenspaces of $J \otimes \mathbb{C}$ with eigenvalues $\pm i$.

An *almost complex manifold* is a smooth manifold M with a complex structure J on the real tangent bundle T. The above discussion yields a splitting of complex vector bundles

$$T^*_{1,0} := T^*, \quad T^*_{0,1} := \overline{T^*}, \quad T^* \otimes_{\mathbb{R}} \mathbb{C} \simeq T^*_{1,0} \oplus T^*_{0,1}.$$

Since the exterior algebra functor $\Lambda^\bullet$ commutes with base change and colimits, we have

$$\Omega^i \otimes \mathbb{C} \simeq \Lambda^i(T^* \otimes \mathbb{C}) \simeq \bigoplus_{p+q=i} \Lambda^p(T^*_{1,0}) \otimes_{\mathbb{C}} \Lambda^q(T^*_{0,1}).$$

To ease notation, denote the sheaves of (p,q)-forms by $\Omega^{p,q} \leftrightarrow \Lambda^p(T^*_{1,0}) \otimes \Lambda^q(T^*_{0,1})$. The (complexified) differential d then decomposes as

$$d = \partial + \overline{\partial},$$

where

$$\partial : \Omega^{p,q} \to \Omega^{p+1,q}, \quad \overline{\partial} : \Omega^{p,q} \to \Omega^{p,q+1}.$$

These clearly satisfy $\partial^2 = 0$ and $\overline{\partial}^2 = 0$, and hence also $\partial\overline{\partial} + \overline{\partial}\partial = 0$.

In summary, the complexification of the de Rham complex $(\Omega^\bullet, d)$ results in a bigraded complex $(\Omega^{\bullet,\bullet}, \partial, \overline{\partial})$.

Definition 1.1. The *Dolbeault complex* is the complex $(\Omega^{p,\bullet}, \overline{\partial})$. The cohomology of the complex of its global sections is called the *Dolbeault cohomology* and denoted by $H^{p,q}(M)$.

So for example, if $p = 0$, the Dolbeault complex is

$$0 \longrightarrow \Omega^{0,0} \xrightarrow{\overline{\partial}} \Omega^{0,1} \xrightarrow{\overline{\partial}} \Omega^{0,2} \longrightarrow \cdots$$

and for higher p, the Dolbeault complex is obtained by tensoring the above with $\Omega^{p,0}$.

Definition 1.2. A *complex manifold* $(M, \mathcal{O})$ is a smooth manifold $(M, \mathcal{A})$ with a sheaf of holomorphic functions $\mathcal{O} \subset \mathcal{A} \otimes \mathbb{C}$, such that $\mathcal{O}$ is locally isomorphic to the sheaf of holomorphic functions on open subsets of $\mathbb{C}^n$.

In this case, $T_{1,0} \simeq T$, as a $\mathbb{C}$-vector bundle, has a natural holomorphic structure. Hence it is sometimes called the *holomorphic tangent bundle*.

Finally, consider the following analogue of the de Rham complex

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{A} \otimes \mathbb{C} \xrightarrow{\overline{\partial}} \Omega^{0,1} \xrightarrow{\overline{\partial}} \Omega^{0,2} \longrightarrow \cdots \tag{1}$$

All the sheaves $\Omega^{\bullet,\bullet}$ are soft as $\mathcal{A}$-modules. To compute the cohomology, we have

Theorem 1.3. (1) is exact, so the Dolbeault complex is a soft resolution of $\mathcal{O}$.

We defer the proof of this to the next section.

Corollary 1.4. Given any holomorphic vector bundle E, the following sequence is a soft resolution for the corresponding locally free $\mathcal{O}$-module $\mathcal{E}$.

$$0 \longrightarrow \mathcal{E} \longrightarrow (\mathcal{A} \otimes \mathbb{C}) \otimes \mathcal{E} \xrightarrow{\overline{\partial}} \Omega^{0,1} \otimes \mathcal{E} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \Omega^{0,n} \otimes \mathcal{E} \longrightarrow 0$$

Proof. Since $\overline{\partial}$ is $\mathcal{O}$-linear, we apply $- \otimes_{\mathcal{O}} \mathcal{E}$ to (1). $\square$

Define $\mathcal{E}' := (\mathcal{A} \otimes \mathbb{C}) \otimes_{\mathcal{O}} \mathcal{E}$ if we wish to consider the sheaf of smooth sections instead. Any toddler will know how to rewrite the above resolution as

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}' \xrightarrow{\overline{\partial}} \Omega^{0,1} \otimes \mathcal{E}' \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \Omega^{0,n} \otimes \mathcal{E}' \longrightarrow 0$$

where $\otimes = \otimes_{\mathcal{A}\otimes\mathbb{C}}$. In particular, we've explained previously that the vector bundles corresponding to $\Omega^{p,0}$ are in fact holomorphic vector bundles. We quickly deduce that

$$0 \longrightarrow \mathcal{H}^p \longrightarrow \Omega^{p,0} \xrightarrow{\overline{\partial}} \Omega^{p,1} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \Omega^{p,n} \longrightarrow 0$$

is a soft resolution for $\mathcal{H}^p$, the sheaf of holomorphic p-forms. Therefore

Corollary 1.5. The Dolbeault cohomology with coefficients in a holomorphic vector bundle E satisfies

$$H^{p,q}(M, E) \simeq H^q(M, \mathcal{E} \otimes \mathcal{H}^p).$$

In particular, $H^{p,q}(M) \simeq H^q(M, \mathcal{H}^p)$, and so $H^{0,q}(M) \simeq H^q(M, \mathcal{O})$.

2 Dolbeault-Grothendieck Lemma

In the results of this section, we will take smooth sections defined globally on some affine space $\mathbb{C}^n$ and discuss their properties over a certain open subset V. However, it clearly suffices to define these functions on any neighborhood of $\overline{V}$.

Lemma 2.1. If V is a bounded open subset of $\mathbb{C}$ and $f \in \mathcal{A} \otimes \mathbb{C}(\mathbb{C})$ is compactly supported, then there exists $g \in \mathcal{A} \otimes \mathbb{C}(\mathbb{C})$ such that $\dfrac{\partial}{\partial \overline{z}} g = f$ on V. One such g is given by the formula

$$g(a) := \frac{1}{2\pi i} \int_V \frac{f(z)}{z-a} dz \wedge d\overline{z}, \quad \forall a \in \mathbb{C}.$$

Proof. First, let's check that this integral makes sense. For small $\epsilon > 0$, put $W = B_\epsilon(a)$. Since $dz \wedge d\overline{z} = -2i d\lambda = -2i\rho d\rho \wedge d\theta$, where λ is the usual Lebesgue measure, we have

$$\int_W \left| \frac{f(z)}{z-a} \right| d\lambda = \int_{[0,2\pi]\times[0,\varepsilon]} \left| \frac{f(a+\rho e^{i\theta})}{\rho e^{i\theta}} \right| \rho d\rho d\theta = \int \left| f(a+\rho e^{i\theta}) \right| d\rho d\theta < \infty,$$

where the first equality is the change of variable formula for non-negative functions, cf. *Folland, Theorem 2.47*. Away from W, the integrand is smooth. This shows that g is well defined.

If the singularity at a is in V, one way to deal with it is the following trick. Take a smooth function μ such that $\mu(W) = 1$ and $\operatorname{supp}(\mu) \subset V$. Consider

$$\begin{aligned} g_1(a) &:= \int_V \frac{(\mu f)(z)}{z-a} dz \wedge d\overline{z} \\ &= \int_{\mathbb{C}} \frac{(\mu f)(z)}{z-a} dz \wedge d\overline{z} \\ &= \int_{\mathbb{C}\setminus\{0\}} \frac{(\mu f)(w+a)}{w} dw \wedge d\overline{w} \end{aligned}$$

where the substitution $w = z - a$ is used. For the same reason as above, the function

$$\frac{\partial}{\partial \overline{a}} \frac{(\mu f)(w+a)}{w} = \frac{1}{w} \frac{\partial}{\partial \overline{a}} (\mu f)(w+a)$$

is Lebesgue integrable, so the Leibniz integral rule applies. Furthermore, notice that

$$\frac{1}{w} \frac{\partial}{\partial \overline{a}} (\mu f)(w+a) = \frac{1}{w} \frac{\partial}{\partial \overline{w}} (\mu f)(w+a) = \frac{\partial}{\partial \overline{w}} \frac{(\mu f)(w+a)}{w},$$

as $1/w$ is holomorphic on the punctured plane. We thus have

$$\begin{aligned} \frac{\partial}{\partial \overline{a}} g_1(a) &= \int_{\mathbb{C}\setminus\{0\}} \frac{\partial}{\partial \overline{w}} \frac{(\mu f)(w+a)}{w} dw \wedge d\overline{w} \\ &= -\lim_{\varepsilon \to 0} \int_{B_{514}(0)\setminus B_\varepsilon(0)} d\left(\frac{(\mu f)(w+a)}{w} dw \right) \\ &= \lim_{\varepsilon \to 0} \int_{|w|=\varepsilon} \frac{(\mu f)(w+a)}{w} dw \quad (\because \text{Stokes' formula}) \\ &= \lim_{\varepsilon \to 0} \int_0^{2\pi} \frac{(\mu f)(\varepsilon e^{i\theta} + a)}{\varepsilon e^{i\theta}} \frac{d\left(\varepsilon e^{i\theta}\right)}{d\theta} d\theta \\ &= 2\pi i f(a). \end{aligned}$$

It remains to show that $\dfrac{\partial}{\partial \overline{a}} g_2(a) = 0$, where

$$\begin{aligned} g_2(a) &:= \int_V \frac{\left((1-\mu)f\right)(z)}{z-a} dz \wedge d\overline{z} \\ &= \int_{V\setminus W} \frac{\left((1-\mu)f\right)(z)}{z-a} dz \wedge d\overline{z}. \end{aligned}$$

This is obvious, since $1/(z-a)$ is holomorphic away from W; the Leibniz rule automatically applies. We have $g = (g_1 + g_2)/(2\pi i)$, and the proof is complete. $\square$

Denote $D_r := B_r(0)$.

Proposition 2.2 (Dolbeault-Grothendieck Lemma, local version)**.** Fix $n > 0$, $r > 0$, and let $V = D_r^n$. Given $\omega \in \Omega^{p,q}(\mathbb{C}^n)$ with $q > 0$, if $\overline{\partial}\omega = 0$, then there exists $\xi \in \Omega^{p,q-1}(\mathbb{C}^n)$ such that $\omega = \overline{\partial}\xi$ on V.

Proof. WLOG assume $p = 0$. We will induct on n. But first, we need to strengthen the statement to let ω and ξ depend holomorphically on a parameter $t \in D^m$, where m and the radii are arbitrary. When we use $\overline{\partial}$, we view t as fixed and never take its differential. We emphasize this by changing the notation to

$$\overline{\partial}_n \xi(t) \overset{!}{=} \omega(t).$$

The base case $n = 1$ is handled by the above lemma. ξ, given by an explicit integral, depends holomorphically on t.

Let $n \geq 2$ and assume the result for $n - 1$. Denote the holomorphic coordinates as $z_1, \cdots, z_n$. We have a unique decomposition

$$\omega(t) =: \omega_1(t, z_n) \wedge d\overline{z_n} + \omega_2(t, z_n)$$

where ω_1, ω_2 does not contain the form $d\overline{z_n}$. Hence they are forms on $n - 1$ complex variables and $m + 1$ parameters ($(t, z_n) \in D^{m+1}$). We have

$$0 = \overline{\partial}_n \omega(t) = \left(\overline{\partial}_{n-1} \omega_1(t, z_n) + (-1)^q \frac{\partial}{\partial \overline{z_n}} \omega_2(t, z_n) \right) \wedge d\overline{z_n} + \overline{\partial}_{n-1} \omega_2(t, z_n).$$

Hence

$$\overline{\partial}_{n-1} \omega_2(t, z_n) = 0 \tag{2}$$

$$\overline{\partial}_{n-1} \omega_1(t, z_n) + (-1)^q \frac{\partial}{\partial \overline{z_n}} \omega_2(t, z_n) = 0 \tag{3}$$

Take $r' > r$ and put $V_1 = D_r^{n-1}$, $V_1' = D_{r'}^{n-1}$. By using the induction hypothesis on (2), we obtain a form $\alpha(t, z_n)$ such that $\overline{\partial}_{n-1} \alpha = \omega_2$ on V_1'. Notice that

$$\frac{\partial}{\partial \overline{z_n}} \omega_2 = \frac{\partial}{\partial \overline{z_n}} \overline{\partial}_{n-1} \alpha = \overline{\partial}_{n-1} \frac{\partial}{\partial \overline{z_n}} \alpha.$$

Plugging this into (3), we get another form $\beta(t, z_n)$, satisfying on V_1

$$\overline{\partial}_{n-1} \beta = \omega_1 + (-1)^q \frac{\partial}{\partial \overline{z_n}} \alpha.$$

We can now put

$$\xi_1 := \beta, \quad \xi_2 := \alpha, \quad \xi := \xi_1 \wedge d\overline{z_n} + \xi_2.$$

It is clear that $\overline{\partial}_n \xi = \omega$ on V, and the proof is complete. □

Proof of Theorem 1.3. The exactness at the first three terms

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{A} \otimes \mathbb{C} \xrightarrow{\overline{\partial}} \Omega^{0,1}$$

is trivial. Let $n = \dim_{\mathbb{C}}(M)$. Fix $q \in \mathbb{Z}_{>0}$, a point in M and a neighborhood U biholomorphic to a polydisk $D^n \subset \mathbb{C}^n$. For all $\omega \in \Omega^{0,q}(U)$ satisfying $\overline{\partial} \omega = 0$, the above proposition yields $\xi \in \Omega^{0,q-1}(U)$ such that $\overline{\partial} \xi = \omega$ holds locally. □

Remark 2.3. This proof is essentially the same as the de Rham theorem, where instead of the lemma, we simply use the Fundamental Theorem of Calculus there.

3 Čech cohomology

The Čech method is a way to directly compute sheaf cohomology.

Fix an open cover $\mathcal{U} = \{U_i : i \in I\}$ of a space X. The associated *Čech nerve* is the simplicial set whose n-th simplices are set functions $\sigma : [n] \to I$ such that

$$U_\sigma := U_{\sigma_0} \cap U_{\sigma_1} \cap \cdots \cap U_{\sigma_n} \neq \varnothing.$$

(This condition is not crucial here, but is devised for purposes such as the nerve theorem.) Given a presheaf $\mathcal{F}$ of abelian groups, we define a *co*-simplicial abelian group A, with the group of n-simplicies given by

$$A^n := \prod_{\sigma:[n]\to I} \mathcal{F}(U_\sigma).$$

In other words, an n-cochain is an assignment to each 'geometric n-simplex' σ of an element of $\mathcal{F}(U_\sigma)$. The co-degeneracy maps $s^j : A^n \to A^{n-1}$ are evident, induced by the identity map of the group of sections over $U_\sigma = U_{\sigma\circ s^j}$. The co-face maps $\delta^i : A^n \to A^{n+1}$ are defined component-wise by the composition

$$A^n \xrightarrow{\text{proj}} \mathcal{F}(U_{\sigma\circ\delta^i}) \longrightarrow \mathcal{F}(U_\sigma), \quad \sigma \in I^{[n+1]},$$

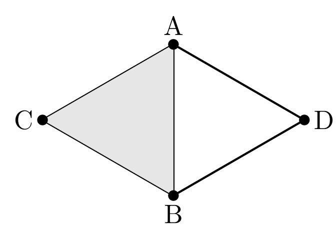
where the second morphism is the restriction.

Under the Dold-Kan correspondence, we then have the following notion of cohomology.

Definition 3.1. The Moore cochain complex given by A is called the *Čech complex* of $\mathcal{F}$ with respect to the cover $\mathcal{U}$, denoted by $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$. Its cohomology is denoted by $\check{\mathrm{H}}^\bullet(\mathcal{U}, \mathcal{F})$.

By definition, $\check{\mathcal{C}}^n(\mathcal{U}, \mathcal{F}) = A^n$, and $d^n : A^n \to A^{n+1}$ is given by $d^n = \sum_{i=0}^{n+1} (-1)^i \delta^i$.

Example 3.2. Let's see a dumb example that may be helpful. Consider a geometrically realized finite simplicial complex, say this one below. Let $\mathcal{F}$ be the constant sheaf $\mathbb{Z}$.



We will take an open cover indexed by the vertices, so $I = \{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\}$ in this case. For each point $\mathrm{P} \in I$, let U_P be the union of all simplices containing P minus the union of all simplices not containing P. Therefore, for any $\sigma : [n] \to I$, $U_\sigma \neq \varnothing$ if and only if $\mathrm{im}(\sigma)$ is a simplex. An n-cochain is an assignment f to each such σ of an integer. Let f be a closed 1-form. It equivalently satisfies the following relations:

$$\begin{array}{lcll} df(\mathrm{PPP}) = 0 & & f(\mathrm{PP}) = 0, & \forall \mathrm{P} \in I, \\ df(\mathrm{PQP}) = 0 & \Leftrightarrow & f(\mathrm{PQ}) = f(\mathrm{QP}), & \forall \{\mathrm{P},\mathrm{Q}\} \neq \{\mathrm{C},\mathrm{D}\}, \\ df(\mathrm{ABC}) = 0 & & f(\mathrm{BC}) - f(\mathrm{AC}) + f(\mathrm{AB}) = 0. & \end{array}$$

But clearly, for f to be a coboundary, we also need

$$f(\mathrm{AB}) + f(\mathrm{BD}) + f(\mathrm{DA}) = 0.$$

This shows that $\check{\mathrm{H}}^1(\mathcal{U}, \mathbb{Z}) = \mathbb{Z}$, agreeing with the singular cohomology and the sheaf cohomology of $\mathbb{Z}$. Soon, we shall see that this is because $\mathcal{U}$ is a good cover.

Next, we decouple our definition of $\check{\mathcal{C}}$ from the choice of the cover $\mathcal{U}$. This has an interesting side effect of effectively sheafifying $\mathcal{F}$ as well.

Definition 3.3. Let $\mathcal{U}, \mathcal{V}$ be open covers of X, indexed by I, J, respectively. A *refinement map* is a set function $a : J \to I$ such that $V_j \subset U_{a(j)}$ for all $j \in J$. If a refinement map exists, we say $\mathcal{V}$ is a refinement of $\mathcal{U}$.

A refinement map as above induces a morphism $\bar{a} : A_{\mathcal{U}} \to A_{\mathcal{V}}$ of cosimplicial objects by

$$\bar{a}(f)(\sigma) := f(a \circ \sigma), \quad \sigma : [n] \to J.$$

Here we omitted a restriction map of $\mathcal{F}$ from $U_{a\circ\sigma}$ to V_σ.

Before taking the limit the category of open covers, whose morphisms are the refinement maps, is not filtered. To fix this, we need the following lemma.

Lemma 3.4. If a, b are both refinement maps $J \to I$, then the induced $\bar{a}, \bar{b}$ are homotopic.

Proof. The desired homotopy $\bar{a} \Rightarrow \bar{b}$ is a family of maps $h^{k,n} : A^n_{\mathcal{U}} \to A^{n-1}_{\mathcal{V}}$, $0 \le k \le n-1$. For $\sigma : [n-1] \to J$ and $0 \le k \le n-1$, put

$$h^{k,n}(f)(\sigma) := f((b \circ \sigma_{\le k}) \cup (a \circ \sigma_{\ge k})).$$

Again, the obvious restriction map is omitted. The notation should be self-explanatory. The conditions for $\{h^{k,n}\}$ to give a homotopy are the following: (the superscripts indicating n is suppressed when appropriate)

1) $h^{0,n}\delta^0 = \overline{a}^n, \quad h^{n-1,n}\delta^n = \overline{b}^n,$
2) $h^k\delta^i = \delta^i h^{k+1}, \quad \text{if } i \le k,$
3) $h^k\delta^i = \delta^{i+1} h^k, \quad \text{if } i > k,$
4) $h^k s^j = s^j h^{k-1}, \quad \text{if } j < k,$
5) $h^k s^j = s^{j+1} h^k, \quad \text{if } j \ge k.$

All of these follows immediately from the definitions. □

Corollary 3.5. The cochain morphisms $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F})$ induced by a, b are homotopic. If $\mathcal{V}$ is a refinement of $\mathcal{U}$, then we have a well-defined $\check{\mathrm{H}}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathrm{H}}^\bullet(\mathcal{V}, \mathcal{F})$.

Proof. The (cosimplicial) Dold-Kan functor carries homotopic maps to homotopic maps, cf. stacks project. Thus they induce the same map at cohomology level. □

Corollary 3.6. For any presheaf $\mathcal{F}$, $\check{\mathrm{H}}^\bullet(-, \mathcal{F})$ is a functor from the partial ordered set of open covers of X to the category of graded abelian groups.

Proof. Obvious. ☐

Definition 3.7. The filtered colimit of the above functor is called the *Čech cohomology*, denoted by

$$\check{\mathrm{H}}^\bullet(X, \mathcal{F}) := \varinjlim_{\mathcal{U}} \check{\mathrm{H}}^\bullet(\mathcal{U}, \mathcal{F}).$$

4 Example: Submanifold of $\mathbb{C}$

The first example of this theory is the Mittag-Leffler theorem in complex analysis.

Recall that a subset $A \subset X$ is *relatively compact* if the closure of A in X is compact.

Lemma 4.1. Let $U \subset \mathbb{C}$ be an open subset and let L be a compact subset of U. Let K be the union of L with all components of $U \setminus L$ that are relatively compact in U. Then K is compact, and any holomorphic function on K can be uniformly approximated by holomorphic functions on U.

Proof. If we can show that every bounded component of $\mathbb{C} \setminus K$ intersects $\mathbb{C} \setminus U$, then the result follows from Runge's approximation theorem, which we recall and prove below. To see this, suppose C is a bounded component of $\mathbb{C} \setminus K$ fully contained in U. Then $\partial C \subset K \subset U$, so C is relatively compact in U. We then see that C is contained in a relatively compact component of $\mathbb{C} \setminus L$, a contradiction. ☐

Theorem 4.2 (Runge)**.** Let $K \subset \mathbb{C}$ be compact, and let $S \subset \mathbb{C}$ be that every bounded component of $\mathbb{C} \setminus K$ intersects S. Let f be a holomorphic function on K. Then f can be uniformly approximated by rational functions whose poles all lie in S.

For simplicity, we put

$$R_S := \{g \in \Gamma(K, \mathcal{O}) : \exists \{R_n\} \longrightarrow g \text{ uniformly on } K\}$$

where R_n are rational functions with the only poles in S. It is easy to see that R_S is a $\mathbb{C}$-algebra. This will be used in the proof.

The theorem now states that $R_S = \Gamma(K, \mathcal{O})$.

Proof. Suppose $f \in \mathcal{O}(V)$ where V is an open neighborhood of K. Take a compactly supported smooth ψ such that $\psi \equiv 1$ on some neighborhood W of K and $\mathrm{supp}(\psi) \subset V$. Applying Lemma 2.1 to ψf on an expanding sequence of bounded open sets containing $\mathrm{supp}(\psi)$, we obtain

$$\psi f(a) - \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial \psi}{\partial \bar{z}} \frac{f(z)}{z-a} dz \wedge d\bar{z} =: h(a) \ \in \mathcal{O}(\mathbb{C}).$$

By Cauchy's integral formula, h is given by a power series. Therefore it has an obvious polynomial approximation by the partial sums which is clearly uniformly convergent on K. The task is then to approximate the integral

$$g(a) := \int_{\mathbb{C}} \frac{\partial \psi}{\partial \bar{z}} \frac{f(z)}{z-a} dz \wedge d\bar{z} = \int_{V \setminus W} \frac{\partial \psi}{\partial \bar{z}} \frac{f(z)}{z-a} dz \wedge d\bar{z}.$$

for $a \in W$. Note that the integrand is smooth in both a and z.

Indeed, $g(a)$ can be uniformly approximated by constructing the Riemann sum

$$\sum_j \frac{b_j}{z_j - a}, \quad \text{finite sum.}$$

where all $b_j \in \mathbb{C}$ and $z_j \notin K$. The only issue is that $z_j \notin S$ in general. Put

$$Z = \left\{ z \in \mathbb{C} \setminus K : (a-z)^{-1} \in R_S \right\}.$$

It remains to prove that $Z = \mathbb{C} \setminus K$.

Suppose that $z \in Z$ and put $r = \operatorname{dist}(z, K) > 0$. If $w \in B_r(z)$, then the series $\{Q_n\}$ given by

$$Q_n(a) := \sum_{k=0}^{n} \left(\frac{w-z}{a-z} \right)^k \in \mathbb{C}[a][(a-z)^{-1}] \subset R_S$$

is uniformly convergent on K by the Weierstrass M-test, and converges to

$$\left(1 - \frac{w-z}{a-z} \right)^{-1} = \frac{a-z}{a-w}.$$

This shows that $(a-w)^{-1} \in R_S$, i.e., $w \in Z$. Thus Z is open. Moreover, $\partial Z \subset K$ is clear, so Z is also closed in $\mathbb{C} \setminus K$. It follows that every bounded component of $\mathbb{C} \setminus K$ is in Z.

Finally, any z with $|z| \geq 2 \sup_{a \in K} |a|$ is in Z, so the unbounded component is also in Z. $\square$

From this, we can deduce a global version of Lemma 2.1.

Proposition 4.3. If U is an open subset of $\mathbb{C}$ and $f \in \mathcal{A} \otimes \mathbb{C}(U)$, then there exists $g \in \mathcal{A} \otimes \mathbb{C}(U)$ such that $\dfrac{\partial}{\partial \bar{z}} g = f$.

Proof. Pick a sequence $\{K_n\}_n$ of compact sets such that $\bigcup K_n = U$, $K_n \subset \operatorname{int}(K_{n+1})$ and no component of $U \setminus K_n$ is relatively compact in U. Using Lemma 2.1, we get for each n a function $g_n \in \mathcal{A} \otimes \mathbb{C}(U)$ such that $\dfrac{\partial}{\partial \bar{z}} g_n = f$ on some open neighborhood of K_n. It follows that $g_{n+1} - g_n$ is holomorphic on a neighborhood of K_n.

With Lemma 4.1, let $u_n \in \mathcal{O}(U)$ such that $|g_{n+1} - g_n - u_n| < 2^{-n}$ on K_n. Put

$$g := g_n + \sum_{m \geq n} (g_{m+1} - g_m - u_m) - u_1 - u_2 - \cdots - u_{n-1}, \quad \text{on } K_n.$$

g is obviously a well defined function on U. Morera's theorem implies that a uniform limit of holomorphic functions is holomorphic, so g is the sum of g_n and a holomorphic function on K_n. This completes the proof. $\square$

Theorem 4.4 (Mittag-Leffler)**.** If U is an open subset of $\mathbb{C}$ and $n \geq 1$, then $H^n(U, \mathcal{O}) = 0$.

Remark 4.5. This result has a vast generalization known as Cartan's theorem B.

Proof. The above proposition shows that the complex of global sections of the Dolbeault complex is acyclic. $\square$

Corollary 4.6. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a family of open subsets in $\mathbb{C}$ and f_{ij} be holomorphic funtions on $U_i \cap U_j$ for all $i, j \in I$. If they satisy the cocycle condition

$$f_{ij} + f_{jk} + f_{ki} = 0$$

for all $i, j, k \in I$, then there are holomorphic functions $f_i \in \mathcal{O}(U_i)$, such that $f_{ij} = f_i - f_j$.

Proof. By Leray's theorem, $\check{\mathrm{H}}^1(\mathcal{U}, \mathcal{O}) = H^1(U, \mathcal{O}) = 0$. $\square$

Definition 4.7. A *meromorphic function* on a complex manifold X is a holomorphic map $X \to \mathbb{P}^1$ that is not identically ∞. The sheaf of meromorphic functions will be denoted as $\mathcal{M}$.

We have an exact sequence of $\mathcal{O}$-modules:

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{M} \xrightarrow{\pi} \mathcal{M}/\mathcal{O} \longrightarrow 0$$

For each $x \in X$, the local map π_x takes the *principle part* of a meromorphic function at x.

Theorem 4.8. Let U be an open subset of $\mathbb{C}$ and S be a discrete subset of U. Given, for each $x \in S$, a prescribed principle part

$$(\mathcal{M}/\mathcal{O})_x \ni \mathfrak{p}_x := \sum_{n<0} b_n (z - x)^{-n}, \text{ finite sum.}$$

There exists a meromorphic function on U whose poles are precisely given by the $\mathfrak{p}_x$.

Proof. In the long exact sequence

$$0 \longrightarrow H^0(U, \mathcal{O}) \longrightarrow H^0(U, \mathcal{M}) \xrightarrow{\pi^*} H^0(U, \mathcal{M}/\mathcal{O}) \longrightarrow H^1(U, \mathcal{O})$$

By Mittag-Leffler's theorem, $H^1(U, \mathcal{O}) = 0$, so π^* is a surjection. $\square$

Remark 4.9. We can obviously also prove this with the last corollary.

The End

Compiled on 2025/08/17.
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