Semisimple modules and rings

math center

1 Semisimple Modules

In the first part of this note, we give a criterion for semisimple modules.

Definition 1.1. An R-module M, unless stated otherwise, refers to a right R-module. Recall that M is simple if its only submodules are 0 and M itself.

Clearly $D := \mathrm{End}_R(M)$ is a division ring. Also, $\mathrm{End}_R(M^n) \simeq \mathrm{Mat}_n(D)$.

Definition 1.2. We say M is semisimple if it is a direct sum of simple modules, and R is semisimple (as a ring) if all R-modules are semisimple.

By collecting the isomorphic summands, we have the *isotypic decomposition*

$$M \simeq \bigoplus_{i \in I} L_i^{d_i}, \tag{1}$$

where the invariants $\{L_i\}$: mutually non-isomorphic simple submodules, $D_i = \mathrm{End}_R(L_i)$, $d_i > 0$.

If I and all d_i's are finite, we see that $d_i = \dim_{D_i}(\mathrm{Hom}(L_i, M))$, and

$$\mathrm{End}_R(M) \simeq \prod_{i \in I} \mathrm{Mat}_{d_i}(D_i) \tag{2}$$

Lemma 1.3. If M is semisimple and $N \subset M$ is a submodule, then N is a direct summand of M. In other words, all short exact sequences

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

split.

Proof. Let $M = \bigoplus_{i \in I} L_i$ where L_i are simple. With Zorn's lemma, pick a maximal $J \subset I$ such that $N \cap \sum_{j \in J} L_j = \varnothing$. To see $N + \sum_{j \in J} L_j = M$, we just need to notice that it contains every L_i. $\square$

Corollary 1.4. Subquotients and sums of semisimple modules are semisimple.

Proof. First part is clear from the proof. For a sum of semisimple modules M_i, note the surjective map $\bigoplus_i M_i \twoheadrightarrow \sum_i M_i$. $\square$

Proposition 1.5. The converse of Lemma 1 is also true.

Proof. Define the socle of M as

$$\mathrm{soc}(M) = \sum \{N \subset M : \text{simple submodule}\}.$$

By Corollary 1.4, $\mathrm{soc}(M)$ is the maximal semisimple submodule in M. By assumption we have $M \simeq \mathrm{soc}(M) \oplus H$ for some $H \subset M$. The conclusion follows from the following two lemmas. $\square$

Lemma 1.6. Simple modules.

1. A proper submodule $N \subset M$ is maximal if and only if M/N is simple.
2. Every nonzero finitely generated module has a maximal submodule.
3. Every nonzero module has a simple subquotient.

Proof. 1 is trivial. 3 follows from 2 because the cyclic submodule aR is finitely generated.

For 2, we use Zorn's lemma. Pick $a \in M \setminus \{0\}$. Put

$$\mathcal{S} = \{N \subset M : \text{proper submodule}\},$$

ordered by inclusion. Then $0 \in \mathcal{S}$. Every chain

$$N_1 \subset N_2 \subset \cdots \subset N_n \subset \cdots$$

in S eventually stabilizes since M is finitely generated. Thus the union $N = \bigcup_i N_i$ is in $\mathcal{S}$. By Zorn's lemma, there is a maximal element $\mathcal{S}$. This completes the proof. $\square$

Lemma 1.7. If every submodule of M is a direct summand, then any subquotient of M also has this property.

Proof. Suppose

$$V \subset N \subset M, \quad M = N \oplus N' = V \oplus V'.$$

Then we have $N = V \oplus (V' \cap N)$, so submodules inherit this property.

If a module has this property, then a quotient is isomorphic to a submodule. Hence, every subquotient of M is isomorphic to a submodule. This completes the proof. $\square$

2 Semisimple Rings

In this section, we give the classification of semisimple rings (definition 1.2).

Lemma 2.1. Let R be a ring. Denote by $\mathsf{Mod}\text{–}R$ the category of right R-modules. Let $\{e_{ij}\}$ be the standard basis of $\mathrm{Mat}_n(R)$. Then the functors

$$M \longmapsto M^{\oplus n}, \quad Ne_{11} \longleftarrow\!\shortmid N$$

gives an equivalence of categories $\mathsf{Mod}\text{–}R \simeq \mathsf{Mod}\text{–}\mathrm{Mat}_n(R)$.

Proof. One way of composing the functors clearly gives (up to natural isomorphism) the identity functor $M \longmapsto M$. The other order gives

$$N \longmapsto (Ne_{11})^{\oplus n} \simeq \bigoplus_{i=1}^{n} Ne_{ii} \simeq N.$$

The last isomorphism follows from the fact that e_{ii} is a set of orthogonal idempotents in $\mathrm{Mat}_n(R)$ that sums to 1. $\square$

Corollary 2.2. Let D be a division ring. Then $\mathrm{Mat}_n(D)$ is a semisimple ring, and every simple right $\mathrm{Mat}_n(D)$-module is isomorphic to D^n.

Proof. We know $\mathsf{Mod}\text{–}D$ pretty well; it's called linear algebra. $\square$

Recall the following result.

Proposition 2.3. Let $R = \prod_i R_i$ be a finite product of rings. Let e_i be the corresponding idempotents and π_i be the projections onto R_i. Then the functors

$$M \longmapsto (Me_i)_i, \quad \bigoplus_i \pi_i^* N_i \longleftarrow (N_i)_i$$

give an equivalence of categories $\mathsf{Mod}\text{–}R \simeq \prod_i \mathsf{Mod}\text{–}R_i$. Here π_i^* denotes the pullback functor along π_i.

Proof. One direction of composing the functors is trivial. The other is easy once idempotents are understood. $\square$

Lemma 2.4. If $f : R \to S$ is a surjective ring homomorphism, then the pullback functor $f^* : \mathsf{Mod}\text{–}S \to \mathsf{Mod}\text{–}R$ sends simple S-modules to simple R-modules.

Proof. Any R-submodule has to be an S-submodule as well. $\square$

Theorem 2.5. Semisimple rings. TFAE:

1. R is a semisimple ring.
2. R is semisimple as a module over itself.
3. R has a factorization as in (2), i.e., a *finite* product of matrix algebras over division rings.

Proof. $1 \Rightarrow 2$: trivial.

$2 \Rightarrow 3$: $R^{\mathrm{op}} = \mathrm{End}_R(R)$. If we know that as a module R is only a *finite* direct sum of simple modules, then (2) follows. But this is clear, as R is finitely generated.

$3 \Rightarrow 1$: By Corollary 2.2, R is a finite product of semisimple rings. We wish to show that the second functor in Proposition 2.3 sends (a vector of) semisimple modules to semisimple R-modules.

Each π_i is surjective, so π_i^* preserves simple modules by the above lemma.

It's well-known that the pullback commutes with both limits and colimits because they have both left and right adjoints (the two 'base change' functors). In particular, π_i^* commutes with direct sums. This completes the proof. $\square$

3 Wedderburn's Theorem

Definition 3.1. A ring R is simple if it has not nontrivial two-sided ideals.

Remark 3.2. Obviously, a semisimple ring need not be simple, since we've shown that it can be a product of two rings.

However, a simple ring R need not be semisimple either. Two examples are provided below.

Definition 3.3. A ring R is left (resp. right) Artinian if it is Artinian as a left (resp. right) R-module.

Theorem 3.4 (Wedderburn)**.** TFAE:

1. R is simple and semisimple.
2. R is a matrix algebra over a division ring.

3. R is simple and right (or left) Artinian.
4. R is semisimple and all simple R-modules are isomorphic.

Proof. $1 \Rightarrow 2$ is clear. $2 \Rightarrow 4$ is Corollary 2.2.

$2 \Rightarrow 3$: $R = \mathrm{Mat}_n(D)$ is a finite-dimensional D-vector space, hence Artinian.

$3 \Rightarrow 1$: Since R is right (say) Artinian, it has a minimal right ideal $\mathfrak{a}$. Then $R\mathfrak{a}$ is a nonzero two-sided ideal, hence $R = R\mathfrak{a}$. Therefore, as an R-module, R is a quotient of $\bigoplus_{r \in R} \mathfrak{a}$, hence semisimple by Corollary 1.4 and Therorem 2.5.

$4 \Rightarrow 2$: The isotypic decomposition of R_R only has one term, so $R^{\mathrm{op}} = \mathrm{End}_R(R)$ is a matrix algebra over a division ring. □

Example 3.5. Any simple ring that's not Artinian is therefore not semisimple.

Let k be a field, E be a k-vector space of countably infinite dimension, and $R = \mathrm{End}_k(E)$. Notice that $m = \{f \in R : \mathrm{rk}(f) < \infty\}$ is a maximal *two-sided* ideal. Indeed, if $f \in R \setminus m$, then f induces an isomorphism between two subspaces, both of which are isomorphic to E. Hence $1 \in RfR$ and m is maximal. We see that R/m is a simple ring.

However, R/m is clearly not left Artinian. Let $\{e_1, e_2, \cdots\}$ be a basis of E. Then

$$I_k = m + \{f \in R : k! \nmid n \Longrightarrow f(e_n) = 0\}$$

is a descending chain of left ideals of R which does not stabilize.

Example 3.6. Another standard example is the Weyl algebra, defined by

$$W = \mathbb{C}\langle x, \partial_x \rangle := T(V)/(p \otimes q - q \otimes p - 1)$$

where T denotes the tensor algebra, and V is a $\mathbb{C}$-vector space with basis $\{p, q\}$. This can be viewed as the *canonical commutation relation* from quantum mechanics. We see that

- W is spanned by $x^i {\partial_x}^j$ for $i, j \in \mathbb{N}$;
- W is simple, seen by applying $[\partial_x, -]$ multiple times;
- W is not a division ring, but it also has no nonzero zero-divisors. Easily seen by inspecting the leading term in ∂_x.

By clause 2 of Wedderburn's theorem, we conclude that W cannot be semisimple.

The End

This note partially follows Noncommutative algebra.
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