Most continuous functions are nowhere differentiable

math center

The topic of this note is simply to show the Baire category theorem (BCT), which we state now.

Theorem 0.1 (Baire, Hausdorff)**.** Let (X, d) be a complete metric space, $\{U_n\}_{n\geq 1}$ be a sequence of dense open subsets of X. Then the intersection of all the U_n is dense.

Proof. Suppose that $V \neq \varnothing$ is an open set disjoint from $\bigcap U_n$. We inductively define two sequences $\{x_i\}$ and $\{r_i\}$, satisfying

- $B(x_1, r_1) \subset V$,
- $B(x_{i+1}, r_{i+1}) \subset B(x_i, \frac{1}{2}r_i) \cap U_i$,
- $r_i \leq 2^{-i}$ for all $i \geq 1$.

This makes $B(x_i, r_i)$ a strictly descending sequence of open balls.

To conclude the proof, we need to show that x_i is a Cauchy sequence, that its limit x lies in U_n, and that its limit also lies in V. This is clear, since we have

$$x_k \in B(x_i, r_i), \quad \forall k \geq i \geq 1,$$

$$\implies x \in \overline{B(x_i, r_i)} \subset \overline{B(x_{i-1}, \tfrac{1}{2}r_{i-1})} \subset D(x_{i-1}, \tfrac{1}{2}r_{i-1}) \subset B(x_{i-1}, r_{i-1})$$

for $i \geq 2$. Now, the last ball in the chain lies in V if $i = 2$, and in U_{i-2} otherwise. □

Remark 0.2. The result holds under a different assumption that X is locally compact Hausdorff.

Next, we give an application

The End

Compiled on 2025/10/29.
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