

# What is Dolbeault cohomology

math center

## 1 Preliminaries on complex manifolds

A *complex structure*  $J$  on a real vector space  $V$  is a linear map  $J : V \rightarrow V$  such that  $J^2 = -\text{id}_V$ . It equivalently provides a  $\mathbb{C}$ -vector space structure on  $V$ , so after complexification, we have a natural splitting

$$V \otimes_{\mathbb{R}} \mathbb{C} \simeq V_J \oplus \overline{V_J}$$

into the two eigenspaces of  $J \otimes \mathbb{C}$  with eigenvalues  $\pm i$ .

An *almost complex manifold* is a smooth manifold  $M$  with a complex structure  $J$  on the real tangent bundle  $T$ . The above discussion yields a splitting of complex vector bundles

$$T_{1,0}^* := T^*, \quad T_{0,1}^* := \overline{T^*}, \quad T^* \otimes_{\mathbb{R}} \mathbb{C} \simeq T_{1,0}^* \oplus T_{0,1}^*.$$

Since the exterior algebra functor  $\Lambda^\bullet$  commutes with base change and colimits, we have

$$\Omega^i \otimes \mathbb{C} \simeq \Lambda^i(T^* \otimes \mathbb{C}) \simeq \bigoplus_{p+q=i} \Lambda^p(T_{1,0}^*) \otimes_{\mathbb{C}} \Lambda^q(T_{0,1}^*).$$

To ease notation, denote the sheaves of  $(p, q)$ -forms by  $\Omega^{p,q} \leftrightarrow \Lambda^p(T_{1,0}^*) \otimes \Lambda^q(T_{0,1}^*)$ . The (complexified) differential  $d$  then decomposes as

$$d = \partial + \overline{\partial},$$

where

$$\partial : \Omega^{p,q} \rightarrow \Omega^{p+1,q}, \quad \overline{\partial} : \Omega^{p,q} \rightarrow \Omega^{p,q+1}.$$

These clearly satisfy  $\partial^2 = 0$  and  $\overline{\partial}^2 = 0$ , and hence also  $\partial\overline{\partial} + \overline{\partial}\partial = 0$ .

In summary, the complexification of the de Rham complex  $(\Omega^\bullet, d)$  results in a bigraded complex  $(\Omega^{\bullet,\bullet}, \partial, \overline{\partial})$ .

**Definition 1.1.** The *Dolbeault complex* is the complex  $(\Omega^{p,\bullet}, \overline{\partial})$ . The cohomology of the complex of its global sections is called the *Dolbeault cohomology* and denoted by  $H^{p,q}(M)$ .

So for example, if  $p = 0$ , the Dolbeault complex is

$$0 \longrightarrow \Omega^{0,0} \xrightarrow{\overline{\partial}} \Omega^{0,1} \xrightarrow{\overline{\partial}} \Omega^{0,2} \longrightarrow \dots$$

and for higher  $p$ , the Dolbeault complex is obtained by tensoring the above with  $\Omega^{p,0}$ .

**Definition 1.2.** A *complex manifold*  $(M, \mathcal{O})$  is a smooth manifold  $(M, \mathcal{A})$  with a sheaf of holomorphic functions  $\mathcal{O} \subset \mathcal{A} \otimes \mathbb{C}$ , such that  $\mathcal{O}$  is locally isomorphic to the sheaf of holomorphic functions on open subsets of  $\mathbb{C}^n$ .

In this case,  $T_{1,0} \simeq T$ , as a  $\mathbb{C}$ -vector bundle, has a natural holomorphic structure. Hence it is sometimes called the *holomorphic tangent bundle*.

Finally, consider the following analogue of the de Rham complex

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{A} \otimes \mathbb{C} \xrightarrow{\overline{\partial}} \Omega^{0,1} \xrightarrow{\overline{\partial}} \Omega^{0,2} \longrightarrow \dots \quad (1)$$

All the sheaves  $\Omega^{\bullet,\bullet}$  are soft as  $\mathcal{A}$ -modules. To compute the cohomology, we have

**Theorem 1.3.** (1) is exact, so the Dolbeault complex is a soft resolution of  $\mathcal{O}$ .

We defer the proof of this to the next section.

**Corollary 1.4.** Given any holomorphic vector bundle  $E$ , the following sequence is a soft resolution for the corresponding locally free  $\mathcal{O}$ -module  $\mathcal{E}$ .

$$0 \longrightarrow \mathcal{E} \longrightarrow (\mathcal{A} \otimes \mathbb{C}) \otimes \mathcal{E} \xrightarrow{\bar{\partial}} \Omega^{0,1} \otimes \mathcal{E} \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \Omega^{0,n} \otimes \mathcal{E} \longrightarrow 0$$

*Proof.* Since  $\bar{\partial}$  is  $\mathcal{O}$ -linear, we apply  $-\otimes_{\mathcal{O}} \mathcal{E}$  to (1). □

Define  $\mathcal{E}' := (\mathcal{A} \otimes \mathbb{C}) \otimes_{\mathcal{O}} \mathcal{E}$  if we wish to consider the sheaf of smooth sections instead. Any toddler will know how to rewrite the above resolution as

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}' \xrightarrow{\bar{\partial}} \Omega^{0,1} \otimes \mathcal{E}' \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \Omega^{0,n} \otimes \mathcal{E}' \longrightarrow 0$$

where  $\otimes = \otimes_{\mathcal{A} \otimes \mathbb{C}}$ . In particular, we've explained previously that the vector bundles corresponding to  $\Omega^{p,0}$  are in fact holomorphic vector bundles. We quickly deduce that

$$0 \longrightarrow \mathcal{H}^p \longrightarrow \Omega^{p,0} \xrightarrow{\bar{\partial}} \Omega^{p,1} \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \Omega^{p,n} \longrightarrow 0$$

is a soft resolution for  $\mathcal{H}^p$ , the sheaf of holomorphic  $p$ -forms. Therefore

**Corollary 1.5.** The Dolbeault cohomology with coefficients in a holomorphic vector bundle  $E$  satisfies

$$H^{p,q}(M, E) \simeq H^q(M, \mathcal{E} \otimes \mathcal{H}^p).$$

In particular,  $H^{p,q}(M) \simeq H^q(M, \mathcal{H}^p)$ , and so  $H^{0,q}(M) \simeq H^q(M, \mathcal{O})$ .

## 2 Dolbeault-Grothendieck Lemma

In the results of this section, we will take smooth sections defined globally on some affine space  $\mathbb{C}^n$  and discuss their properties over a certain open subset  $V$ . However, it clearly suffices to define these functions on any neighborhood of  $\bar{V}$ .

**Lemma 2.1.** If  $V$  is a bounded open subset of  $\mathbb{C}$  and  $f \in \mathcal{A} \otimes \mathbb{C}(\mathbb{C})$  is compactly supported, then there exists  $g \in \mathcal{A} \otimes \mathbb{C}(\mathbb{C})$  such that  $\frac{\partial}{\partial \bar{z}} g = f$  on  $V$ . One such  $g$  is given by the formula

$$g(a) := \frac{1}{2\pi i} \int_V \frac{f(z)}{z - a} dz \wedge d\bar{z}, \quad \forall a \in \mathbb{C}.$$

*Proof.* First, let's check that this integral makes sense. For small  $\epsilon > 0$ , put  $W = B_\epsilon(a)$ . Since  $dz \wedge d\bar{z} = -2id\lambda = -2i\rho d\rho \wedge d\theta$ , where  $\lambda$  is the usual Lebesgue measure, we have

$$\int_W \left| \frac{f(z)}{z - a} \right| d\lambda = \int_{[0, 2\pi] \times [0, \epsilon]} \left| \frac{f(a + \rho e^{i\theta})}{\rho e^{i\theta}} \right| \rho d\rho d\theta = \int |f(a + \rho e^{i\theta})| d\rho d\theta < \infty,$$

where the first equality is the change of variable formula for non-negative functions, cf. *Folland, Theorem 2.47*. Away from  $W$ , the integrand is smooth. This shows that  $g$  is well defined.

If the singularity at  $a$  is in  $V$ , one way to deal with it is the following trick. Take a smooth function  $\mu$  such that  $\mu(W) = 1$  and  $\text{supp}(\mu) \subset V$ . Consider

$$\begin{aligned} g_1(a) &:= \int_V \frac{(\mu f)(z)}{z-a} dz \wedge d\bar{z} \\ &= \int_{\mathbb{C}} \frac{(\mu f)(z)}{z-a} dz \wedge d\bar{z} \\ &= \int_{\mathbb{C} \setminus \{0\}} \frac{(\mu f)(w+a)}{w} dw \wedge d\bar{w} \end{aligned}$$

where the substitution  $w = z - a$  is used. For the same reason as above, the function

$$\frac{\partial}{\partial \bar{a}} \frac{(\mu f)(w+a)}{w} = \frac{1}{w} \frac{\partial}{\partial \bar{a}} (\mu f)(w+a)$$

is Lebesgue integrable, so the [Leibniz integral rule](#) applies. Furthermore, notice that

$$\frac{1}{w} \frac{\partial}{\partial \bar{a}} (\mu f)(w+a) = \frac{1}{w} \frac{\partial}{\partial \bar{w}} (\mu f)(w+a) = \frac{\partial}{\partial \bar{w}} \frac{(\mu f)(w+a)}{w},$$

as  $1/w$  is holomorphic on the punctured plane. We thus have

$$\begin{aligned} \frac{\partial}{\partial \bar{a}} g_1(a) &= \int_{\mathbb{C} \setminus \{0\}} \frac{\partial}{\partial \bar{w}} \frac{(\mu f)(w+a)}{w} dw \wedge d\bar{w} \\ &= - \lim_{\varepsilon \rightarrow 0} \int_{B_{514}(0) \setminus B_\varepsilon(0)} d \left( \frac{(\mu f)(w+a)}{w} dw \right) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{|w|=\varepsilon} \frac{(\mu f)(w+a)}{w} dw \quad (\because \text{Stokes' formula}) \\ &= \lim_{\varepsilon \rightarrow 0} \int_0^{2\pi} \frac{(\mu f)(\varepsilon e^{i\theta} + a)}{\varepsilon e^{i\theta}} \frac{d(\varepsilon e^{i\theta})}{d\theta} d\theta \\ &= 2\pi i f(a). \end{aligned}$$

It remains to show that  $\frac{\partial}{\partial \bar{a}} g_2(a) = 0$ , where

$$\begin{aligned} g_2(a) &:= \int_V \frac{((1-\mu)f)(z)}{z-a} dz \wedge d\bar{z} \\ &= \int_{V \setminus W} \frac{((1-\mu)f)(z)}{z-a} dz \wedge d\bar{z}. \end{aligned}$$

This is obvious, since  $1/(z-a)$  is holomorphic away from  $W$ ; the Leibniz rule automatically applies. We have  $g = (g_1 + g_2)/(2\pi i)$ , and the proof is complete.  $\square$

Denote  $D_r := B_r(0)$ .

**Proposition 2.2** (Dolbeault-Grothendieck Lemma, local version). Fix  $n > 0$ ,  $r > 0$ , and let  $V = D_r^n$ . Given  $\omega \in \Omega^{p,q}(\mathbb{C}^n)$  with  $q > 0$ , if  $\bar{\partial}\omega = 0$ , then there exists  $\xi \in \Omega^{p,q-1}(\mathbb{C}^n)$  such that  $\omega = \bar{\partial}\xi$  on  $V$ .

*Proof.* WLOG assume  $p = 0$ . We will induct on  $n$ . But first, we need to strengthen the statement to let  $\omega$  and  $\xi$  depend holomorphically on a parameter  $t \in D^m$ , where  $m$  and the radii are arbitrary. When we use  $\bar{\partial}$ , we view  $t$  as fixed and never take its differential. We emphasize this by changing the notation to

$$\bar{\partial}_n \xi(t) \stackrel{!}{=} \omega(t).$$

The base case  $n = 1$  is handled by the above lemma.  $\xi$ , given by an explicit integral, depends holomorphically on  $t$ .

Let  $n \geq 2$  and assume the result for  $n - 1$ . Denote the holomorphic coordinates as  $z_1, \dots, z_n$ . We have a unique decomposition

$$\omega(t) =: \omega_1(t, z_n) \wedge d\bar{z}_n + \omega_2(t, z_n)$$

where  $\omega_1, \omega_2$  does not contain the form  $d\bar{z}_n$ . Hence they are forms on  $n - 1$  complex variables and  $m + 1$  parameters  $((t, z_n) \in D^{m+1})$ . We have

$$0 = \bar{\partial}_n \omega(t) = \left( \bar{\partial}_{n-1} \omega_1(t, z_n) + (-1)^q \frac{\partial}{\partial \bar{z}_n} \omega_2(t, z_n) \right) \wedge d\bar{z}_n + \bar{\partial}_{n-1} \omega_2(t, z_n).$$

Hence

$$\bar{\partial}_{n-1} \omega_2(t, z_n) = 0 \tag{2}$$

$$\bar{\partial}_{n-1} \omega_1(t, z_n) + (-1)^q \frac{\partial}{\partial \bar{z}_n} \omega_2(t, z_n) = 0 \tag{3}$$

Take  $r' > r$  and put  $V_1 = D_r^{n-1}$ ,  $V'_1 = D_{r'}^{n-1}$ . By using the induction hypothesis on (2), we obtain a form  $\alpha(t, z_n)$  such that  $\bar{\partial}_{n-1} \alpha = \omega_2$  on  $V'_1$ . Notice that

$$\frac{\partial}{\partial \bar{z}_n} \omega_2 = \frac{\partial}{\partial \bar{z}_n} \bar{\partial}_{n-1} \alpha = \bar{\partial}_{n-1} \frac{\partial}{\partial \bar{z}_n} \alpha.$$

Plugging this into (3), we get another form  $\beta(t, z_n)$ , satisfying on  $V_1$

$$\bar{\partial}_{n-1} \beta = \omega_1 + (-1)^q \frac{\partial}{\partial \bar{z}_n} \alpha.$$

We can now put

$$\xi_1 := \beta, \quad \xi_2 := \alpha, \quad \xi := \xi_1 \wedge d\bar{z}_n + \xi_2.$$

It is clear that  $\bar{\partial}_n \xi = \omega$  on  $V$ , and the proof is complete.  $\square$

*Proof of Theorem 1.3.* The exactness at the first three terms

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{A} \otimes \mathbb{C} \xrightarrow{\bar{\partial}} \Omega^{0,1}$$

is trivial. Let  $n = \dim_{\mathbb{C}}(M)$ . Fix  $q \in \mathbb{Z}_{>0}$ , a point in  $M$  and a neighborhood  $U$  biholomorphic to a polydisk  $D^n \subset \mathbb{C}^n$ . For all  $\omega \in \Omega^{0,q}(U)$  satisfying  $\bar{\partial}\omega = 0$ , the above proposition yields  $\xi \in \Omega^{0,q-1}(U)$  such that  $\bar{\partial}\xi = \omega$  holds locally.  $\square$

**Remark 2.3.** This proof is essentially the same as the de Rham theorem, where instead of the lemma, we simply use the Fundamental Theorem of Calculus there.

### 3 Čech cohomology

The Čech method is a way to directly compute sheaf cohomology.

Fix an open cover  $\mathcal{U} = \{U_i : i \in I\}$  of a space  $X$ . The associated *Čech nerve* is the simplicial set whose  $n$ -th simplices are set functions  $\sigma : [n] \rightarrow I$  such that

$$U_\sigma := U_{\sigma_0} \cap U_{\sigma_1} \cap \cdots \cap U_{\sigma_n} \neq \emptyset.$$

(This condition is not crucial here, but is devised for purposes such as the nerve theorem.) Given a presheaf  $\mathcal{F}$  of abelian groups, we define a *co-simplicial* abelian group  $A$ , with the group of  $n$ -simplices given by

$$A^n := \prod_{\sigma : [n] \rightarrow I} \mathcal{F}(U_\sigma).$$

In other words, an  $n$ -cochain is an assignment to each 'geometric  $n$ -simplex'  $\sigma$  of an element of  $\mathcal{F}(U_\sigma)$ . The co-degeneracy maps  $s^j : A^n \rightarrow A^{n-1}$  are evident, induced by the identity map of the group of sections over  $U_\sigma = U_{\sigma \circ s^j}$ . The co-face maps  $\delta^i : A^n \rightarrow A^{n+1}$  are defined component-wise by the composition

$$A^n \xrightarrow{\text{proj}} \mathcal{F}(U_{\sigma \circ \delta^i}) \longrightarrow \mathcal{F}(U_\sigma), \quad \sigma \in I^{[n+1]},$$

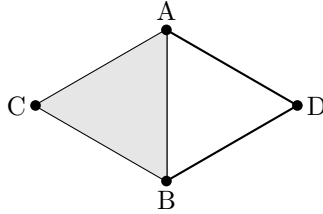
where the second morphism is the restriction.

Under the Dold-Kan correspondence, we then have the following notion of cohomology.

**Definition 3.1.** The Moore cochain complex given by  $A$  is called the *Čech complex* of  $\mathcal{F}$  with respect to the cover  $\mathcal{U}$ , denoted by  $\check{C}^\bullet(\mathcal{U}, \mathcal{F})$ . Its cohomology is denoted by  $\check{H}^\bullet(\mathcal{U}, \mathcal{F})$ .

By definition,  $\check{C}^n(\mathcal{U}, \mathcal{F}) = A^n$ , and  $d^n : A^n \rightarrow A^{n+1}$  is given by  $d^n = \sum_{i=0}^{n+1} (-1)^i \delta^i$ .

**Example 3.2.** Let's see a dumb example that may be helpful. Consider a geometrically realized finite simplicial complex, say this one below. Let  $\mathcal{F}$  be the constant sheaf  $\mathbb{Z}$ .



We will take an open cover indexed by the vertices, so  $I = \{A, B, C, D\}$  in this case. For each point  $P \in I$ , let  $U_P$  be the union of all simplices containing  $P$  minus the union of all simplices not containing  $P$ . Therefore, for any  $\sigma : [n] \rightarrow I$ ,  $U_\sigma \neq \emptyset$  if and only if  $\text{im}(\sigma)$  is a simplex. An  $n$ -cochain is an assignment  $f$  to each such  $\sigma$  of an integer. Let  $f$  be a closed 1-form. It equivalently satisfies the following relations:

$$\begin{aligned} df(\text{PPP}) &= 0 & f(\text{PP}) &= 0, & \forall P \in I, \\ df(\text{PQP}) &= 0 \Leftrightarrow f(\text{PQ}) = f(\text{QP}), & \forall \{P, Q\} &\neq \{C, D\}, \\ df(\text{ABC}) &= 0 & f(\text{BC}) - f(\text{AC}) + f(\text{AB}) &= 0. \end{aligned}$$

But clearly, for  $f$  to be a coboundary, we also need

$$f(\text{AB}) + f(\text{BD}) + f(\text{DA}) = 0.$$

This shows that  $\check{H}^1(\mathcal{U}, \mathbb{Z}) = \mathbb{Z}$ , agreeing with the singular cohomology and the sheaf cohomology of  $\mathbb{Z}$ . Soon, we shall see that this is because  $\mathcal{U}$  is a good cover.

Next, we decouple our definition of  $\check{\mathcal{C}}$  from the choice of the cover  $\mathcal{U}$ . This has an interesting side effect of effectively sheafifying  $\mathcal{F}$  as well.

**Definition 3.3.** Let  $\mathcal{U}, \mathcal{V}$  be open covers of  $X$ , indexed by  $I, J$ , respectively. A *refinement map* is a set function  $a : J \rightarrow I$  such that  $V_j \subset U_{a(j)}$  for all  $j \in J$ . If a refinement map exists, we say  $\mathcal{V}$  is a refinement of  $\mathcal{U}$ .

A refinement map as above induces a morphism  $\bar{a} : A_{\mathcal{U}} \rightarrow A_{\mathcal{V}}$  of cosimplicial objects by

$$\bar{a}(f)(\sigma) := f(a \circ \sigma), \quad \sigma : [n] \rightarrow J.$$

Here we omitted a restriction map of  $\mathcal{F}$  from  $U_{a \circ \sigma}$  to  $V_{\sigma}$ .

Before taking the limit the category of open covers, whose morphisms are the refinement maps, is not filtered. To fix this, we need the following lemma.

**Lemma 3.4.** If  $a, b$  are both refinement maps  $J \rightarrow I$ , then the induced  $\bar{a}, \bar{b}$  are homotopic.

*Proof.* The desired homotopy  $\bar{a} \Rightarrow \bar{b}$  is a family of maps  $h^{k,n} : A_{\mathcal{U}}^n \rightarrow A_{\mathcal{V}}^{n-1}$ ,  $0 \leq k \leq n-1$ . For  $\sigma : [n-1] \rightarrow J$  and  $0 \leq k \leq n-1$ , put

$$h^{k,n}(f)(\sigma) := f((b \circ \sigma_{\leq k}) \cup (a \circ \sigma_{\geq k})).$$

Again, the obvious restriction map is omitted. The notation should be self-explanatory. The conditions for  $\{h^{k,n}\}$  to give a homotopy are the following: (the superscripts indicating  $n$  is suppressed when appropriate)

- 1)  $h^{0,n}\delta^0 = \bar{a}^n, \quad h^{n-1,n}\delta^n = \bar{b}^n,$
- 2)  $h^k\delta^i = \delta^i h^{k+1}, \quad \text{if } i \leq k,$
- 3)  $h^k\delta^i = \delta^{i+1}h^k, \quad \text{if } i > k,$
- 4)  $h^k s^j = s^j h^{k-1}, \quad \text{if } j < k,$
- 5)  $h^k s^j = s^{j+1}h^k, \quad \text{if } j \geq k.$

All of these follows immediately from the definitions. □

**Corollary 3.5.** The cochain morphisms  $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \rightarrow \check{\mathcal{C}}^\bullet(\mathcal{V}, \mathcal{F})$  induced by  $a, b$  are homotopic. If  $\mathcal{V}$  is a refinement of  $\mathcal{U}$ , then we have a well-defined  $\check{H}^\bullet(\mathcal{U}, \mathcal{F}) \rightarrow \check{H}^\bullet(\mathcal{V}, \mathcal{F})$ .

*Proof.* The (cosimplicial) Dold-Kan functor carries homotopic maps to homotopic maps, cf. [stacks project](#). Thus they induce the same map at cohomology level. □

**Corollary 3.6.** For any presheaf  $\mathcal{F}$ ,  $\check{H}^\bullet(-, \mathcal{F})$  is a functor from the partial ordered set of open covers of  $X$  to the category of graded abelian groups.

*Proof.* Obvious. □

**Definition 3.7.** The filtered colimit of the above functor is called the *Čech cohomology*, denoted by

$$\check{H}^\bullet(X, \mathcal{F}) := \varinjlim_{\mathcal{U}} \check{H}^\bullet(\mathcal{U}, \mathcal{F}).$$

## 4 Example: Submanifold of $\mathbb{C}$

The first example of this theory is the Mittag-Leffler theorem in complex analysis.

Recall that a subset  $A \subset X$  is *relatively compact* if the closure of  $A$  in  $X$  is compact.

**Lemma 4.1.** Let  $U \subset \mathbb{C}$  be an open subset and let  $L$  be a compact subset of  $U$ . Let  $K$  be the union of  $L$  with all components of  $U \setminus L$  that are relatively compact in  $U$ . Then  $K$  is compact, and any holomorphic function on  $K$  can be uniformly approximated by holomorphic functions on  $U$ .

*Proof.* If we can show that every bounded component of  $\mathbb{C} \setminus K$  intersects  $\mathbb{C} \setminus U$ , then the result follows from Runge's approximation theorem, which we recall and prove below. To see this, suppose  $C$  is a bounded component of  $\mathbb{C} \setminus K$  fully contained in  $U$ . Then  $\partial C \subset K \subset U$ , so  $C$  is relatively compact in  $U$ . We then see that  $C$  is contained in a relatively compact component of  $\mathbb{C} \setminus L$ , a contradiction. □

**Theorem 4.2** (Runge). Let  $K \subset \mathbb{C}$  be compact, and let  $S \subset \mathbb{C}$  be that every bounded component of  $\mathbb{C} \setminus K$  intersects  $S$ . Let  $f$  be a holomorphic function on  $K$ . Then  $f$  can be uniformly approximated by rational functions whose poles all lie in  $S$ .

For simplicity, we put

$$R_S := \{g \in \Gamma(K, \mathcal{O}) : \exists \{R_n\} \longrightarrow g \text{ uniformly on } K\}$$

where  $R_n$  are rational functions with the only poles in  $S$ . It is easy to see that  $R_S$  is a  $\mathbb{C}$ -algebra. This will be used in the proof.

The theorem now states that  $R_S = \Gamma(K, \mathcal{O})$ .

*Proof.* Suppose  $f \in \mathcal{O}(V)$  where  $V$  is an open neighborhood of  $K$ . Take a compactly supported smooth  $\psi$  such that  $\psi \equiv 1$  on some neighborhood  $W$  of  $K$  and  $\text{supp}(\psi) \subset V$ . Applying Lemma 2.1 to  $\psi f$  on an expanding sequence of bounded open sets containing  $\text{supp}(\psi)$ , we obtain

$$\psi f(a) - \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial \psi}{\partial \bar{z}} \frac{f(z)}{z-a} dz \wedge d\bar{z} =: h(a) \in \mathcal{O}(\mathbb{C}).$$

By Cauchy's integral formula,  $h$  is given by a power series. Therefore it has an obvious polynomial approximation by the partial sums which is clearly uniformly convergent on  $K$ . The task is then to approximate the integral

$$g(a) := \int_{\mathbb{C}} \frac{\partial \psi}{\partial \bar{z}} \frac{f(z)}{z-a} dz \wedge d\bar{z} = \int_{V \setminus W} \frac{\partial \psi}{\partial \bar{z}} \frac{f(z)}{z-a} dz \wedge d\bar{z}.$$

for  $a \in W$ . Note that the integrand is smooth in both  $a$  and  $z$ .

Indeed,  $g(a)$  can be uniformly approximated by constructing the Riemann sum

$$\sum_j \frac{b_j}{z_j - a}, \quad \text{finite sum.}$$

where all  $b_j \in \mathbb{C}$  and  $z_j \notin K$ . The only issue is that  $z_j \notin S$  in general. Put

$$Z = \{z \in \mathbb{C} \setminus K : (a - z)^{-1} \in R_S\}.$$

It remains to prove that  $Z = \mathbb{C} \setminus K$ .

Suppose that  $z \in Z$  and put  $r = \text{dist}(z, K) > 0$ . If  $w \in B_r(z)$ , then the series  $\{Q_n\}$  given by

$$Q_n(a) := \sum_{k=0}^n \left( \frac{w - z}{a - z} \right)^k \in \mathbb{C}[a][(a - z)^{-1}] \subset R_S$$

is uniformly convergent on  $K$  by the Weierstrass M-test, and converges to

$$\left( 1 - \frac{w - z}{a - z} \right)^{-1} = \frac{a - z}{a - w}.$$

This shows that  $(a - w)^{-1} \in R_S$ , i.e.,  $w \in Z$ . Thus  $Z$  is open. Moreover,  $\partial Z \subset K$  is clear, so  $Z$  is also closed in  $\mathbb{C} \setminus K$ . It follows that every bounded component of  $\mathbb{C} \setminus K$  is in  $Z$ .

Finally, any  $z$  with  $|z| \geq 2 \sup_{a \in K} |a|$  is in  $Z$ , so the unbounded component is also in  $Z$ .  $\square$

From this, we can deduce a global version of Lemma 2.1.

**Proposition 4.3.** If  $U$  is an open subset of  $\mathbb{C}$  and  $f \in \mathcal{A} \otimes \mathbb{C}(U)$ , then there exists  $g \in \mathcal{A} \otimes \mathbb{C}(U)$  such that  $\frac{\partial}{\partial \bar{z}} g = f$ .

*Proof.* Pick a sequence  $\{K_n\}_n$  of compact sets such that  $\bigcup K_n = U$ ,  $K_n \subset \text{int}(K_{n+1})$  and no component of  $U \setminus K_n$  is relatively compact in  $U$ . Using Lemma 2.1, we get for each  $n$  a function  $g_n \in \mathcal{A} \otimes \mathbb{C}(U)$  such that  $\frac{\partial}{\partial \bar{z}} g_n = f$  on some open neighborhood of  $K_n$ . It follows that  $g_{n+1} - g_n$  is holomorphic on a neighborhood of  $K_n$ .

With Lemma 4.1, let  $u_n \in \mathcal{O}(U)$  such that  $|g_{n+1} - g_n - u_n| < 2^{-n}$  on  $K_n$ . Put

$$g := g_n + \sum_{m \geq n} (g_{m+1} - g_m - u_m) - u_1 - u_2 - \cdots - u_{n-1}, \quad \text{on } K_n.$$

$g$  is obviously a well defined function on  $U$ . Morera's theorem implies that a uniform limit of holomorphic functions is holomorphic, so  $g$  is the sum of  $g_n$  and a holomorphic function on  $K_n$ . This completes the proof.  $\square$

**Theorem 4.4** (Mittag-Leffler). If  $U$  is an open subset of  $\mathbb{C}$  and  $n \geq 1$ , then  $H^n(U, \mathcal{O}) = 0$ .

**Remark 4.5.** This result has a vast generalization known as Cartan's theorem B.

*Proof.* The above proposition shows that the complex of global sections of the Dolbeault complex is acyclic.  $\square$



**Corollary 4.6.** Let  $\mathcal{U} = \{U_i\}_{i \in I}$  be a family of open subsets in  $\mathbb{C}$  and  $f_{ij}$  be holomorphic functions on  $U_i \cap U_j$  for all  $i, j \in I$ . If they satisfy the cocycle condition

$$f_{ij} + f_{jk} + f_{ki} = 0$$

for all  $i, j, k \in I$ , then there are holomorphic functions  $f_i \in \mathcal{O}(U_i)$ , such that  $f_{ij} = f_i - f_j$ .

*Proof.* By Leray's theorem,  $\check{H}^1(\mathcal{U}, \mathcal{O}) = H^1(U, \mathcal{O}) = 0$ . □

**Definition 4.7.** A *meromorphic function* on a complex manifold  $X$  is a holomorphic map  $X \rightarrow \mathbb{P}^1$  that is not identically  $\infty$ . The sheaf of meromorphic functions will be denoted as  $\mathcal{M}$ .

We have an exact sequence of  $\mathcal{O}$ -modules:

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{M} \xrightarrow{\pi} \mathcal{M}/\mathcal{O} \longrightarrow 0$$

For each  $x \in X$ , the local map  $\pi_x$  takes the *principle part* of a meromorphic function at  $x$ .

**Theorem 4.8.** Let  $U$  be an open subset of  $\mathbb{C}$  and  $S$  be a discrete subset of  $U$ . Given, for each  $x \in S$ , a prescribed principle part

$$(\mathcal{M}/\mathcal{O})_x \ni \mathfrak{p}_x := \sum_{n < 0} b_n (z - x)^{-n}, \text{ finite sum.}$$

There exists a meromorphic function on  $U$  whose poles are precisely given by the  $\mathfrak{p}_x$ .

*Proof.* In the long exact sequence

$$0 \longrightarrow H^0(U, \mathcal{O}) \longrightarrow H^0(U, \mathcal{M}) \xrightarrow{\pi^*} H^0(U, \mathcal{M}/\mathcal{O}) \longrightarrow H^1(U, \mathcal{O})$$

By Mittag-Leffler's theorem,  $H^1(U, \mathcal{O}) = 0$ , so  $\pi^*$  is a surjection. □

**Remark 4.9.** We can obviously also prove this with the last corollary.

HHHHHHHHHH

## The End

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