Central simple algebras

math center

This is a follow-up to the previous note.

1 Central simple algebras

Fix a field k.

Definition 1.1. An algebra A over k is called *central* if the center of A is the image of $k \hookrightarrow A$.

Example 1.2. The quaternions $\mathbb{H}$ form a central division algebra over $\mathbb{R}$.

Moreover, the only such algebras are $\mathbb{R}$ and $\mathbb{H}$. This is a consequence of the Frobenius theorem. Let's see why this is the case. Suppose A is a real division algebra with $\dim_{\mathbb{R}}(A) \geq 2$. Pick any element $x \in A \setminus \mathbb{R}$ and we identify $\mathbb{C} \simeq \mathbb{R}[x] \subset A$. In particular, A is a $\mathbb{C}$-vector space.

Let $\varphi : a \mapsto iai^{-1}$ where $i^2 = -1$. This is a $\mathbb{C}$-linear involution of A, i.e., $\varphi^2 = \mathrm{id}$. Thus the possible eigenvalues of φ are ± 1, and we can decompose A as a direct sum of eigenspaces:

$$A = U_1 \oplus U_{-1}.$$

Then U_1 is a finite dimensional $\mathbb{C}$-algebra, hence $U_1 = \mathbb{C}$ as $\mathbb{C}$ is algebraically closed. If $U_{-1} = 0$ then $A = \mathbb{C}$. Otherwise, pick $j \in U_{-1} \setminus \{0\}$. Left multiplication by j gives a $\mathbb{C}$-linear map $U_{-1} \hookrightarrow U_1$, so we must have $\dim_{\mathbb{C}}(U_{-1}) = \dim_{\mathbb{C}}(U_1) = 1$.

Now, $j^2 \in U_1 = \mathbb{C}$. But we also have $j^2 \in \mathbb{R} \oplus \mathbb{R}j$ due to the minimal polynomial of j having degree 2. Thus $j^2 \in \mathbb{R}$. It is clear that $j^2 < 0$. We see that $A = \mathbb{C} \oplus \mathbb{C} \cdot j/\sqrt{-j^2}$ is identified with $\mathbb{H}$.

2 Brauer group

The End

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