1 Lecture 2

Review:
Monotonicity: If $\alpha \models \beta$ then $\alpha \wedge \gamma \models \beta$.

Proof. Assume $\alpha \models \beta$, then $M(\alpha) \subseteq M(\beta)$. Then $M(\alpha \wedge \gamma) = M(\alpha) \cap M(\gamma) \subseteq M(\beta)$, so $\alpha \wedge \gamma \models \beta$. □

Literal: X, ¬X
Clause: Disjunction of literals
Conjunctive Normal Form (CNF): A conjunction of clauses. $(X \vee Y) \wedge (\neg Y \vee Z)$

1. Remove all logical connectives but $\neg, \wedge, \vee$ using equivalences. (i.e $\alpha \rightarrow \beta \equiv \neg\alpha \vee \beta$)
2. Push negations inward using De morgan's laws ($\neg(\alpha \wedge \beta) \equiv \neg\alpha \vee \neg\beta$ and $\neg(\alpha \vee \beta) \equiv \neg\alpha \wedge \neg\beta$) and double negation ($\neg(\neg\alpha) \equiv \alpha$).
3. Distribute $\vee$ over $\wedge$ using distributive law ($\alpha \vee (\beta \wedge \gamma) \equiv (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$).

Example: $\Delta = (A \vee B) \Rightarrow C$

$$
\begin{aligned}
&1.\Delta \equiv \neg(A \vee B) \vee C \\
&2. \equiv (\neg A \wedge \neg B) \vee C \\
&3. \equiv (\neg A \vee C) \wedge (\neg B \vee C)
\end{aligned}
$$

Clausal form is as follows, you represent $(\neg A \vee C) \wedge (\neg B \vee C)$ as the set of clauses $\{\{\neg A, C\}, \{\neg B, C\}\}$.

If α is true at world ω, then β is true at world ω. Is equivalent to $\omega \models \alpha \Rightarrow \omega \models \beta$.

1.1 Quantified Boolean Logic

1.1.1 Logical Operators

Conditioning (Restriction): |

Given a sentence Δ and variable P, condition $\Delta|P$ is the result of replacing every occurence of P in Δ with true and every occurrence of $\neg P$ in Δ with false.
Example: $\Delta = A \vee \neg B \vee C \vee \neg D$
$\Delta|B = A \vee C \vee \neg D$
$\Delta|B = A \vee B \vee \neg D$
$\Delta|\neg B = A \vee \text{true} \vee C \vee \neg D$
This is refered to Boole's expansion / Shannon's expansion: $\Delta = P \wedge (\Delta|P) \vee \neg P \wedge (\Delta|\neg P)$

$\Delta = (A \vee B \vee \neg C) \wedge (\neg A \vee D) \wedge (B \vee C \vee D)$
$\Delta|C = (A \vee B \vee \text{false}) \wedge (\neg A \vee D) \wedge (B \vee \text{true})$
$\Delta|C = \{\{A, B\}, \{\neg A, D\}\}$
$\Delta|A, B, C, D = \{\}$ Means True
$\Delta|A, \neg B, C, D = \{\{\}\}$ Means False

Empty Clause Means contradiction where empty set means True.

Existential Quantification: $\exists$

$\exists P \cdot \Delta = \Delta|P \vee \Delta|\neg P$ We have existentially quantified variable P out of Δ.
$\Delta = (A \Rightarrow B) \wedge (B \Rightarrow C)$
$\Delta = (\neg A \vee B) \wedge (\neg B \vee C)$
$\Delta|B = C$
$\Delta|\neg B = \neg A$
$\exists B \cdot \Delta = C \vee \neg A$

Forgetting operator / Existential Operator: You are getting rid of a variable while still containing all of the information about the other variables.
If α is a sentence that does not mention the variable P, then $\Delta \models \alpha$ iff $\exists P \cdot \Delta \models \alpha$.
Say we have a KB with 1000 variables and we only care about the output based on 10 of them. Then we can existentially quantify out the other 990 variables to get a smaller KB that only mentions the 10 variables we care about.

Universal Quantification: $\forall P \cdot \Delta = \Delta|P \wedge \Delta|\neg P$ We have universally quantified variable P out of Δ.
$\exists P \cdot \Delta \models \alpha = \neg(\Delta \models \forall P \cdot \alpha)$

2 Resolution

Resolution is referred to as an inference rule, or a rule for deduction. Modeus Ponens is a inference rule that says the following:

$$\Delta = \{..., \alpha, ..., \alpha \Rightarrow \beta\} \tag{1}$$

$$\frac{\alpha, \alpha \Rightarrow \beta}{\beta} \tag{2}$$

For resolution, input must be a CNF.

$$\frac{\alpha \vee X, \beta \vee \neg X}{\alpha \vee \beta}$$

Denominator is a resolvent and we say we resolved on variable X.

$$\Delta = (P \Rightarrow R) \wedge (Q \Rightarrow R) \wedge (\neg R) \wedge (P \vee Q)$$

1. $\{\neg P, R\}$
2. $\{\neg Q, R\}$
3. $\{\neg R\}$
4. $\{P, Q\}$
5. $\{\neg Q\}$ resolve 2 and 3. Unit resolution step
6. $\{P\}$ resolve 3 and 5. Unit resolution step
7. $\{R\}$ resolve 1 and 4. Unit resolution step
8. $\{\}$ resolve 6 and 7. Contradiction found.

If resolution is applied to a CNF then it is capable of discoverying a contradiction if one exists.

Refutation Theorem: $\Delta \models \alpha$ iff $\Delta \wedge \neg\alpha$ is inconsistent.
Resolution is refutation complete. (When applied to CNF)

$$\Delta = (A \vee B \Rightarrow C) \wedge (C \Rightarrow D)$$

$$\alpha = \neg D \Rightarrow \neg A$$

$$1.\{\neg A, C\}$$

$$2.\{\neg B, C\}$$

$$3.\{\neg C, D\}$$

$$4.\{D, A\}$$

$$5.\{\neg D\}$$

$$6.\{\neg A, C\} resolve 3 and 5.$$

$$7.\{\neg B, C\} resolve 3 and 5.$$

$$8.\{C\} resolve 6 and 4.$$

$$9.\{C\} resolve 7 and 4.$$

$$10.\{\} resolve 8 and 2.$$