Computational Complexity 5

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Question 1.

For each of the following pairs of functions f, g determine whether $f = o(g)$, $g = o(f)$, or $f = \Theta(g)$ then find the first number n such that $f(n) < g(n)$:

a) $f(n) = n^2$, $g(n) = 2n^2 + 100\sqrt{n}$
b) $f(n) = n^{100}, g(n) = 2^{n/100}$
c) $f(n) = n^{100}, g(n) = 2^{n^{n/100}}$
d) $f(n)\sqrt{n}, g(n) = 2^{\sqrt{\log n}}$
e) $f(n) = n^{100}, g(n) = 2^{(\log n)^2}$
f) $f(n) = 1000n, g(n) = n\log n$

a) $f = \Theta(g)$

b) $f = o(g)$

$$\begin{aligned} n^{100} &< 2^{n/100} \\ 100\log n &< n/100 \\ 10000 &< n/\log n \end{aligned}$$

c) $f = o(g)$
d) $f = o(g)$
e) $f = o(g)$
f) $f = o(g)$

Question 2.

For each of the following recursively defined functions f, find a closed expression for a function g such that $f(n) = \Theta(g(n))$, and prove that this is the case.

(a) $f(n) = f(n-1) + 10$
(b) $f(n) = f(n-1) + n$
(c) $f(n) = 2f(n-1)$
(d) $f(n) = f(n/2) + 10$
(e) $f(n) = f(n/2) + n$
(f) $f(n) = 2f(n/2) + n$
(g) $f(n) = 3f(n/2)$
(h) $f(n) = 2f(n/2) + O(n^2)$

(a) For each $f(k)$, the total number of steps that needs to be done is $f(k-1)+O(1)$. Since it is a constant amount of work for n values, $f(n) = \Theta(n)$ in this case.
(b) For each $f(k)$, the total number of steps that needs to be done is $O(n)$. Since it is linear amount of work for n values, $f(n) = \Theta(n^2)$
(c) Expanded out, this function evaluates to $O(2^n)$, so $f = \Theta(2^n)$.
(d) Since there is $O(1)$ work being done $\log n$ times, $f = \log n$
(e) Since there is $O(k)$ work begin done $\log n$ times, where k is the input of f, $f = \Theta n$ since k is halving at every step
(f) Since there is $O(n)$ work being done $\log n$ times, $f = \Theta(n \log n)$
(g) Since $\log_2 3 > 0$, $f = \Theta(n^{log_2 3})$
(h) Since $n^2 > 1$, $f = n^2$

(0.3) The reason the machine does not break apart is due to the number of rotations needed from the first gear to move the final gear even a little. The number of turns to rotate the kth gear a single time is 50^k. Therefore the total number of rotations the last gear does in one minute is $50^{13}/212$ which is too large a number to notice any change in the last gears rotation.

Question 3.

Demonstrate that e is an irrational number.

Proof. For sake of contradiction, assume that e is a rational number, that is, that it can be written as $e = \frac{a}{b}$. Recall that e can be written as follows:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

Therefore, since $e = \frac{a}{b}$, we can write it as:

$$\frac{a}{b} = \sum_{n=0}^{\infty} \frac{1}{n!}$$

Since the summation iterates through all integers, we can split it into two separate summations, writing the ration a/b as:

$$\frac{a}{b} = \sum_{n=0}^{b} \frac{1}{n!} + \sum_{n=b+1}^{\infty} \frac{1}{n!}$$

Since all of the terms in the first summation are multiples of all future terms in the summation, every single term can be written as a fraction with $b!$ in the denominator, meaning that the first summation can be written as:

$$\frac{a}{b} = \frac{k}{b!} + \sum_{n=b+1}^{\infty} \frac{1}{n!}$$

For some integer k. $\square$