1 Sensitivity Conjecture

Theorem 1.1 (Sensitivity Conjecture)**.** *Consider an n dimensional hypercube, H_n where each vertex is represented by a length n bitstring and any two vertices share an edge iff they have a different bit at only 1 index. For any set S of the vertices of H_n where $|S| \geq 2^{n-1} + 1$:*

$$\Delta(S) \geq \sqrt{n}$$

In other words, the highest degree of a vertex in any subset including more than half of the vertices in the hypercube is at least the square root of the dimension of the hypercube.

Furthermore, this root n is actually a tight bound.

Lemma 1.2. For any graph G, let A_G be the adjacency matrix for G. The following holds:

$$|\lambda_1(A_G)| \leq \Delta(G)$$

Or for any graph, the largest eigenvalue is at most the max degree of the graph.

This generalizes the statement if spectral graph theory that says that if a graph G is $D-$regular, thenn all eigenvalues are less than or equal to D. The intuition behind this is that the normalized version of a D-regular graph has eigenvalue at most 1 so dividing the matrix by D results in eigenvalue at most D for A_G.

Proof. Suppose v is an eigenvector with eigenvalue λ. We have that

$$A_G \cdot v = \lambda \cdot v$$

The proof follows from the proof in the previous lectures. Assume v_i is the largest entry in eigenvector v. We have

$$\begin{aligned} |\lambda| \cdot |v_i| &= \sum_j A_{ij} \cdot v_j \\ &\leq |v_i| \cdot \sum_j A_{ij} \\ &\leq |v_i| \cdot D \end{aligned}$$

□

We extend this proof to allow for an adjacency matrix to have both positive and negative entries denoting an edge between two nodes

Signed-Adjacency Matrix: A matrix B is a "signed-adjacency matrix" of an undirected graph G if the following holds:

- B is symmetric
- $B_{ij} = \begin{cases} 0, & (i,j) \notin E, \\ \pm 1, & (i,j) \in E. \end{cases}$ Where E denotes the edges of G,

Since the proof holds identically for signed-adjacency matrices, we can make the following claim:

Lemma 1.3. If A_G is a signed-adjacency matrix for graph G:

$$|\lambda_1(A_G)| \leq \Delta(G)$$

The proof is pretty much the same as above but is done below:

Proof. Consider eigenvector v with eigenvalue λ. We can rewrite matrix multiplication as follows:

$$A_G \cdot v = \lambda \cdot v$$

Now consider the entry of v with the largest absolute value v_i. We know that:

$$|\lambda \cdot v_i| = |\sum_j A_G[i,j] \cdot v_j|$$

We can argue the following:

$$\begin{aligned} |\lambda| \cdot |v_i| &= \sum_j |A_G[i,j]| \cdot |v_j| \\ &\leq \sum_j |A_G[i,j]| \cdot |v_i| \\ &\leq D \cdot |v_i| \\ |\lambda| &\leq D \end{aligned}$$

$\square$

We now have the necessary spectral graph theory lemma's to approach the problem.

1.1 Cauchy Interlacing and Principal Submatrices

We will consider the matrix H_n that represents the adjacency matrix of the hypercube of n dimensions. We define the **Principal Submatrix** of an $n \times n$ square matrix S as the submatrix that is obtained by deleting the i^{th} row and column from S. The main idea is to demonstrate that a principal submatrix of H_n that keeps $2^{n-1}+1$ or more rows and columns has largest eigenvalue at least $\sqrt{n}$. The plan is as follows:

1. Pick a "magical" signing B_n.
2. Compute the eigenvalues of B_n
3. Use linear algebra to say something about $\lambda(B_{S,S})$ (The principal submatrix of B_N)

1.1.1 Cauchy Interlacing

Cauchy Interlacing is a lemma that states we have a real symmetric matrix M, and generate a principal submatrix M_{-1} with the i^{th} row and column removed, the eigenvalues of M_{-1} interlace the eigenvalues of M. In other words:

$$\lambda_1(M) \geq \lambda_1(M_{-1}) \geq \lambda_2(M) \geq \lambda_2(M_{-1}) \geq ... \geq \lambda_{n-1}(M_{-1}) \geq \lambda_n(M)$$

We do not prove this here, but use the statement to prove the Sensitivity Conjecture.

Extending on this idea, we can make a claim about the eigenvalues of M_{-2}, the principal submatrix of M with two rows and columns removed. Relative to the eigenvalues of M_{-1}, they interlace them similarly to how M_{-1}'s eigenvalues interlace M's. We generalize it here:

Theorem 1.4 (Cauchy Interlacing Theorem). *B is an $N \times N$ symmetric matrix and C is a $M \times M$ principal submatrix of B:*

$$\lambda_N(B) \leq \lambda_{N-1}(B) \leq ... \leq \lambda_1(B)$$

$$\lambda_M(C) \leq \lambda_{M-1}(C) \leq ... \leq \lambda_1(C)$$

Then we have that, $\lambda_{N-M+i}(B) \leq \lambda_i(C) \leq \lambda_i(B)$

Suppose B is a signing of H_n, then it has dimensions $2^n \times 2^n$ for $N = 2^n$. We want a lower bound on $\lambda(B_{S,S})$ for all $|S| \geq 2^{n-1}+1$.

Lemma 1.5. If B is a signing of H_n then $\forall S, |S| = 2^{n-1}+1$,

$$\lambda_{2^{n-1}}(B) = \lambda_{2^n-(2^{n-1}+1)+1}(B) \leq \lambda_1(B_{S,S}) \leq \lambda_1(B)$$

In otherwords, the 2^{n-1}'th eigenvalue is the lower bound of the highest degree vertex in S.

The goal is therefore to find a matrix B_n of H_n such that:

$$\sqrt{n} \leq \lambda_{2^{n-1}}(B_n)$$

1.2 Constructing B_N

First, we will demonstrate what happens with a normal adjacency matrix for H_n. We define H_n as follows:

$$H_N[i \in \{0,1\}^N, j \in \{0,1\}^N] = \begin{cases} 1, & i,j \text{ Differ in one location.} \\ 0, & \text{Otherwise} \end{cases}$$

For a non-signed matrix for H_n, the eigenvalues are $-2, \_, \_, 2$. The matrix for $n = 2$ looks something along the lines of:

$$H_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

We can recursively build H_3 by observing that we are essentially connecting two copies of H_2 together, and then drawing N edges between the two copies if and only if the two vertices differ in a single bit. Since the two copies represent the last $N-1$ bits being the same, we can represent H_3 as follows:

$$H_3 = \begin{bmatrix} H_2 & I_4 \\ I_4 & H_2 \end{bmatrix}$$

Where I is the identity matrix of size $2^{n-1} \times 2^{n-1}$. Generalizing this, we have:

$$H_n = \begin{bmatrix} H_{n-1} & I_{2^{n-1}} \\ I_{2^{n-1}} & H_{n-1} \end{bmatrix}$$

The eigenvalues of these types of adjacency matrices have $\binom{n}{r}$ eigenvalues of magnitude $n(1-\frac{1}{n})^r$ which drop off exponentially.

1.3 Finalizing Proof

Lemma 1.6. $\forall n, \exists$ a signing B_n of H_n such that:

(a) $B_n^2 = n \cdot I_{2^n}$

(b) $\text{Trace}(B_n) = 0$

Furthermore, if we prove the above, we prove the Sensitivity Conjecture.

Claim: If B_n satisfies the above two properties, then $\lambda_{2^{n-1}}(B_n) = \sqrt{n}$.

Proof. Firstly, we know that squaring a matrix squares the eigenvalues of the matrix. Since $n \cdot I_{2^n}$ has n eigenvalues of n, the eigenvalues of B_n must be $\pm\sqrt{n}$. Since the trace of a matrix is the sum of its eigenvalues, and the trace of B_n is 0, we know that half the eigenvalues must be $\sqrt{n}$ and half must be $-\sqrt{n}$. Therefore, the 2^{n-1}'th largest eigenvalue must be $\sqrt{n}$. This completes the claim as it shows that H_n must have largest degree at least $\sqrt{n}$ in any subset of size $2^{n-1}+1$. $\square$

We construct the signing B_n inductively with a recursive structure. Firstly, we have for $N = 1$:

$$A_{H_1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Here we can clearly say that $A_{H_1}^2 = I_1$ and that the trace is 0, satisfying the inductive property. Furthermore, we know that any proper signing satisfies the Trace $= 0$ property as there are no self loops in the hypercube. For $N = 2$, we have:

$$B_2 = \begin{bmatrix} A_{H_1} & I_2 \\ I_2 & -A_{H_1} \end{bmatrix}$$

Firstly, we know this is a valid signing of H_2 as the non-zero entries correspond to edges in the hypercube as we saw earlier. We just show that $B_2^2 = 2 \cdot I_{2^1}$.

$$\begin{aligned} B_2^2 &= \begin{bmatrix} A_{H_1} & I_2 \\ I_2 & -A_{H_1} \end{bmatrix} \cdot \begin{bmatrix} A_{H_1} & I_2 \\ I_2 & -A_{H_1} \end{bmatrix} \\ &= \begin{bmatrix} A_{H_1}^2 + I_2 & A_{H_1} - A_{H_1} \\ -A_{H_1} + A_{H_1} & I_2 + A_{H_1}^2 \end{bmatrix} \\ &= \begin{bmatrix} 2 \cdot I_2 & 0 \\ 0 & 2 \cdot I_2 \end{bmatrix} = 2 \cdot I_4 \end{aligned}$$

This generlizes for any $k+1$ as follows:

$$B_{k+1} = \begin{bmatrix} B_k & I_{2^k} \\ I_{2^k} & -B_k \end{bmatrix}$$

We can show inductively that $B_{k+1}^2 = (k+1) \cdot I_{2^{k+1}}$ assuming $B_k^2 = k \cdot I_{2^k}$.

$$\begin{aligned} B_{k+1}^2 &= \begin{bmatrix} B_k & I_{2^k} \\ I_{2^k} & -B_k \end{bmatrix} \cdot \begin{bmatrix} B_k & I_{2^k} \\ I_{2^k} & -B_k \end{bmatrix} \\ &= \begin{bmatrix} B_k^2 + I_{2^k} & B_k - B_k \\ -B_k + B_k & I_{2^k} + B_k^2 \end{bmatrix} \\ &= \begin{bmatrix} (k+1) \cdot I_{2^k} & 0 \\ 0 & (k+1) \cdot I_{2^k} \end{bmatrix} = (k+1) \cdot I_{2^{k+1}} \end{aligned}$$

Proving the lemma and the proof!

1.4 Summary

The proof of the Sensitivity Conjecture is as follows:

1. Suffices to have a signing such that $\forall S, |S| = 2^{n-1}+1$, $\lambda_1(B_{S,S}) \geq \sqrt{n}$
2. Suffices by Cauchy Interlacing: Find a signing B_n such that $\lambda_{2^{n-1}}(B_n) \geq \sqrt{n}$.
3. Suffices to have a signing $B_n, B_n^2 = n \cdot I_{2^n}$ and $\mathrm{Trace}(B_n) = 0$
4. Use recursive construction to build B_n.

Remark: $\sqrt{n}$ is tight. $\exists S, |S| \geq 2^{n-1}+1$ st. $\Delta(H_n[S,S]) \leq \sqrt{n}$.

$$S = \{x : f(x) \cdot par(x) = 1\}$$

Where f is the following:

$$f(x) = 1 \Longleftrightarrow \exists \text{ a column that is all 1's } : \bigvee_{j=1}^{k} \left(\bigwedge_{i=1}^{k} x_{ij} \right)$$