Compsci 166 Homework 2

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Question 1.

For the following states, if they are written in the standard basis, rewrite them in the Hadamard basis. If they are written in the Hadamard basis, rewrite them in the standard basis.

1) $|\psi_1\rangle = -|0\rangle$

$$|\psi_1\rangle = -(\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle)$$

2) $|\psi_2\rangle = \frac{i}{\sqrt{2}}|0\rangle + \frac{1-i}{2}|1\rangle$

$$\begin{aligned}
|\psi_2\rangle &= \frac{i}{\sqrt{2}}(\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)) + \frac{1-i}{2}(\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)) \\
&= \frac{i}{2}(|+\rangle + |-\rangle) + \frac{1-i}{2\sqrt{2}}(|+\rangle - |-\rangle) \\
&= \frac{i\sqrt{2}+1-i}{2\sqrt{2}}|+\rangle + \frac{i\sqrt{2}-1+i}{2\sqrt{2}}|-\rangle
\end{aligned}$$

3) $|\psi_3\rangle = \frac{1}{\sqrt{3}}|+\rangle - \frac{1+i}{\sqrt{3}}|-\rangle$

$$\begin{aligned}
|\psi_3\rangle &= \frac{1}{\sqrt{3}}(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)) - \frac{1+i}{\sqrt{3}}(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)) \\
&= \frac{1}{\sqrt{6}}(|0\rangle + |1\rangle) - (\frac{1+i}{\sqrt{6}}(|0\rangle - |1\rangle)) \\
&= \frac{-i}{\sqrt{6}}|0\rangle + \frac{2+i}{\sqrt{6}}|1\rangle
\end{aligned}$$

4) $|\psi_4\rangle = \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle$

$$|\psi_4\rangle = |1\rangle$$

Recall the $\{|i\rangle, |-i\rangle\}$ basis where
$|i\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$
$|-i\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}$
Write the following states in the $\{|i\rangle, |-i\rangle\}$ basis, that is as a weighted sum of $|i\rangle$ and $|-i\rangle$.

5) $|\psi_1\rangle = |0\rangle$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|i\rangle + |-i\rangle)$$

6) $|\psi_2\rangle = |1\rangle$

$$|\psi\rangle_1 = \frac{i}{\sqrt{2}}(-|i\rangle + |-i\rangle)$$

7) $|\psi_3\rangle = |+\rangle$

$$\begin{aligned}|\psi_3\rangle &= \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}}(|i\rangle + |-i\rangle) + \frac{i}{\sqrt{2}}(-|i\rangle + |-i\rangle)) \\ &= \frac{1 - i\sqrt{2}}{2}|i\rangle + \frac{1 + i\sqrt{2}}{2}|-i\rangle\end{aligned}$$

8) $|\psi_4\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle$

$$|\psi_4\rangle = \frac{1 + i\sqrt{2}}{2}|i\rangle + \frac{1 - i\sqrt{2}}{2}|-i\rangle$$

9) $|\psi_5\rangle \frac{\sqrt{3}}{1}|0\rangle - \frac{1}{2}|1\rangle$

$$\begin{aligned}|\psi_5\rangle &= \frac{\sqrt{3}}{2}(\frac{1}{\sqrt{2}}|i\rangle + \frac{1}{\sqrt{2}}|-i\rangle) - \frac{1}{2}(\frac{-i}{\sqrt{2}}|i\rangle + \frac{i}{\sqrt{2}}|-i\rangle) \\ &= \frac{\sqrt{3} + i}{2\sqrt{2}}|i\rangle + \frac{\sqrt{3} - i}{2\sqrt{2}}|-i\rangle\end{aligned}$$

10) $|\psi_6\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|-\rangle$

$$\begin{aligned}|\psi_6\rangle &= \frac{\sqrt{3}}{2}(\frac{1 - i\sqrt{2}}{2}|i\rangle + \frac{1 + i\sqrt{2}}{2}|-i\rangle) + \frac{1}{2}(\frac{1 + i\sqrt{2}}{2}|i\rangle + \frac{1 - i\sqrt{2}}{2}|-i\rangle) \\ &= \frac{\sqrt{3} - i\sqrt{6} + 1 + i\sqrt{2}}{4}(|i\rangle + |-i\rangle)\end{aligned}$$

Question 2.

1) Find the adjoint of the matrices X, Y, Z, A, B, C

$$X^\dagger = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Y^\dagger = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad Z^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A^\dagger = \begin{bmatrix} -\frac{i}{\sqrt{2}} & \frac{1-i}{2} \\ \frac{1+i}{2} & -\frac{i}{\sqrt{2}} \end{bmatrix} \quad B^\dagger = \begin{bmatrix} 0 & e^{-i2\pi/3} \\ e^{-i\pi/3} & 0 \end{bmatrix} \quad C^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$$

2) For each of the following circuits, find the state of the quibit at the end of the circuit, and what the possible outcomes and probabilities are if we measure in the stated basis.

Measure in the Hadamard basis: $|0\rangle$ —$\boxed{X}$—$\boxed{Z}$—

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Measure $|+\rangle : 1/2, |-\rangle : 1/2$

Measure in the standard basis: $|+\rangle$ —$\boxed{X}$—$\boxed{H}$—

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Measure $|0\rangle : 1, |1\rangle : 0$

Measure in the standard basis: $|-\rangle$ —$\boxed{X}$—$\boxed{H}$—

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Measure $|0\rangle : 0, |1\rangle : 1$

Measure in the Hadamard basis: $|0\rangle$ —$\boxed{H}$—$\boxed{C}$—$\boxed{Y}$—

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ i \end{bmatrix}$$

Probability $|+\rangle$ is measured:

$$|\frac{1}{\sqrt{2}} \langle +| \begin{bmatrix} -1 \\ i \end{bmatrix} |^2 = | -\frac{1}{2} + \frac{i}{2}| = 1/2$$

$|-\rangle$ measured: $1/2$

Measure in the Hadamard basis: $|1\rangle - \boxed{C} - \boxed{Z} - \boxed{H} - \boxed{X} - \boxed{X} - \boxed{H} - \boxed{C}$

$$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ i \end{bmatrix}$$

Probability measuring $|1\rangle : 1$, $|0\rangle : 0$

Question 3.

Suppose we repeat the Elitzur-Vaidman bomb experiment but adjust our circuit as follows:

$|1\rangle - \boxed{C} - \boxed{Z} - \boxed{H} - \boxed{X} - \boxed{X} - \boxed{H} - \boxed{C}$