Compsci 166 Homework 3

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Question 1.

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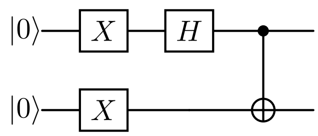
1) Design a two quibit circuit that starts in the state $|00\rangle$ and ends with $|\Psi^-\rangle$.



2) Write down $|0\rangle$ and $|1\rangle$ as a weighted sum of $|\pi/6\rangle$ and $|4\pi/6\rangle$ in the above basis.

$$|0\rangle = \frac{\sqrt{3}}{2}|\pi/6\rangle - \frac{1}{2}|4\pi/6\rangle$$
$$|1\rangle = \frac{1}{2}|\pi/6\rangle + \frac{\sqrt{3}}{2}|4\pi/6\rangle$$

3) If Alice measures her quibit in the standard basis, what are the probabilities of each outcome, and the state of the two quibits after the measurement?

$$|0\rangle : 1/2 \text{ Probability and state collapses to } |01\rangle$$
$$|1\rangle : 1/2 \text{ Probability and state collapses to } |10\rangle$$

4) If Alice instead chooses to measure in the $\{|\pi/6\rangle, |4\pi/6\rangle\}$ basis, what are the porbailities of each outcome, and the state of the two quibits after the measurement?

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}((\frac{\sqrt{3}}{2}|\pi/6\rangle - \frac{1}{2}|4\pi/6\rangle) \otimes |1\rangle - (\frac{1}{2}|\pi/6\rangle + \frac{\sqrt{3}}{2}|4\pi/6\rangle) \otimes |0\rangle)$$

Alice measures:

$$|\pi/6\rangle \text{ w/ probability } 1/2\text{, collapses to } \tfrac{\sqrt{3}}{2}|\pi/6\rangle|1\rangle - \tfrac{1}{2}|\pi/6\rangle|0\rangle$$
$$|4\pi/6\rangle \text{ w/ probability } 1/2\text{, collapses to } \tfrac{-1}{2}|4\pi/6\rangle|1\rangle - \tfrac{\sqrt{3}}{2}|4\pi/6\rangle|0\rangle$$

5) Verbally describe what happens to the second quibit when the first quibit of a $|\Psi\rangle$ state gets measures.

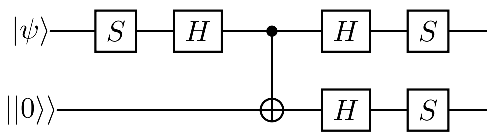
Answer:
The second quibit collapses into a state of $R_{-\theta}|0\rangle$ for $\theta =$ the state of the first quibit.

Question 2.

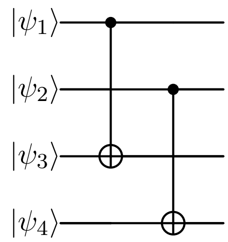
1) Determine whether a state $|\psi\rangle$ that we know to be one of the two states is clonable or not. Briefly justify your answer.

Answer:
Yes. Since $|i\rangle$ and $|-i\rangle$ form an orthonormal basis, there must exist a U that can clone a quibit that we know to be in one of the two above states.

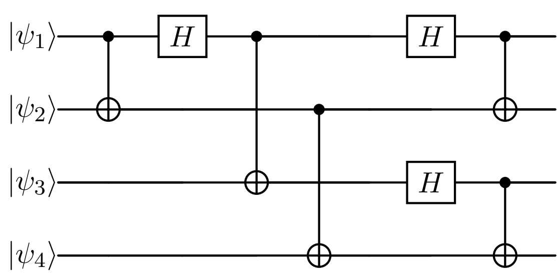
2) Design a quantum circuit that clones a state if we know it is in the above basis.



3) Design a 4 quibit quantum circuit that clones 2 quibit standard basis states.



4) Design a 4 quibit quantum circuit that clones the Bell basis states.



Question 3.

1) What does the serial code 010 get mapped to using this function?

Answer:

$$\begin{aligned} f(010) &= (2 \cdot 17 + 4)(\text{mod } 64) \\ &= 38 \end{aligned}$$

2) What does the serial code 110 get mapped to using this function?

Answer:

$$\begin{aligned} f(110) &= (6 \cdot 17 + 4)(\text{mod } 64) \\ &= 42 \end{aligned}$$

3) If a client handed you a bill with the serial number 010, what basis would you measure each quibit in to verify that it is in the correct state.

Answer:
The function with serial number 010 outputs 38 which is 100110 in binary. This bitstring translates to $|{+1+}\rangle$ which means that the first and third bits should be measured in the hadamard basis and the second bit in the standard basis.

4) If a client handed you a bill with the serial number 110, what basis would you measure each quibit in to verify that it is in the correct state.

Answer:
The function with serial number 110 outputs 42 which is 101010 in binary. This bitstring translates to $|+++\rangle$ which means that the first and third bits should be measured in the hadamard basis and the second bit in the standard basis.

5) The probability the verification is successful is the probability that $|0\rangle$ measures correctly for any of the given quibits. The probabilities for $|0\rangle, |1\rangle, |+\rangle, |-\rangle$ being measured correctly given $|\psi_1\rangle$ is $|0\rangle$ are $1, 0, 1/2, 1/2$ respectively, meaning that the probability that the verification succeeds is $1/2$.

6) Let $|\psi\rangle$ be the state received after the bank runs their verification process. If the verification was successful, what is the state of $|\psi\rangle$?

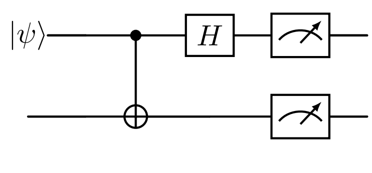
Answer:
The state of $|\psi\rangle$ must be the same for all $|\psi_i\rangle$ except for $i = 0$ in which case $|\psi_0\rangle = |\psi_0^\perp\rangle$ where $|\psi_0^\perp\rangle$ is the state orthogonal to the original first quibit's state.

7) Design a counterfeiting strategy to create a copy of $|\psi\rangle$ for all n quibits. How many times would we need to resubmit to the bank?

Answer:
In order to find a strategy to create a copy of $|\psi\rangle$ we will need to find out $|\psi_i\rangle$ for all i. We will generalize the strategy for finding each quibit. We first replace the quibit with $|0\rangle$ and try to verify it. If it passes, we know the correct bit cannot be a $|1\rangle$. We repeat this using $|0\rangle$ for the particular quibit until we are certain that $|\psi_i\rangle$ is $|0\rangle$ with a probability $> \epsilon$ for whatever value for ϵ we see fit. If we ever measure $|1\rangle$, we know that the correct quibit is either $|+\rangle$ or $|-\rangle$, so it takes one extra bill submission to determine which of the two it is. Therefore, to counterfit each $|\psi_i\rangle$, it will take $O(n)$ submissions.

Question 4.

1) Diagram of the circuit for the protocol



2)

$$\frac{1}{\sqrt{2}}(\alpha\,|001\rangle + \alpha\,|010\rangle + \beta\,|101\rangle + \beta\,|110\rangle)$$

$$\textbf{CNOT: } \frac{1}{\sqrt{2}}(\alpha\,|001\rangle + \alpha\,|010\rangle + \beta\,|111\rangle + \beta\,|100\rangle)$$

$$\textbf{H: } \frac{1}{2}(\alpha\,|001\rangle + \alpha\,|101\rangle + \alpha\,|010\rangle\,\alpha\,|110\rangle + \beta\,|011\rangle - \beta\,|111\rangle + \beta\,|000\rangle - \beta\,|100\rangle)$$

3) Alice measures:
$|00\rangle$: $\alpha\,|1\rangle + \beta\,|0\rangle$ Apply X gate
$|01\rangle$: $\alpha\,|0\rangle + \beta\,|1\rangle$ Apply I gate
$|10\rangle$: $\alpha\,|1\rangle - \beta\,|0\rangle$: Apply X then Z gate
$|11\rangle$: $\alpha\,|0\rangle - \beta 1$: Apply Z gate