Compsci 166 Homework 2

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Question 1.

For the following states, if they are written in the standard basis, rewrite them in the Hadamard basis. If they are written in the Hadamard basis, rewrite them in the standard basis.

1) $|\psi_1\rangle = -|0\rangle$

$$|\psi_1\rangle = -(\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle)$$

2) $|\psi_2\rangle = \frac{i}{\sqrt{2}}|0\rangle + \frac{1-i}{2}|1\rangle$

$$\begin{aligned}
|\psi_2\rangle &= \frac{i}{\sqrt{2}}(\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)) + \frac{1-i}{2}(\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)) \\
&= \frac{i}{2}(|+\rangle + |-\rangle) + \frac{1-i}{2\sqrt{2}}(|+\rangle - |-\rangle) \\
&= \frac{i\sqrt{2}+1-i}{2\sqrt{2}}|+\rangle + \frac{i\sqrt{2}-1+i}{2\sqrt{2}}|-\rangle
\end{aligned}$$

3) $|\psi_3\rangle = \frac{1}{\sqrt{3}}|+\rangle - \frac{1+i}{\sqrt{3}}|-\rangle$

$$\begin{aligned}
|\psi_3\rangle &= \frac{1}{\sqrt{3}}(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)) - \frac{1+i}{\sqrt{3}}(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)) \\
&= \frac{1}{\sqrt{6}}(|0\rangle + |1\rangle) - (\frac{1+i}{\sqrt{6}}(|0\rangle - |1\rangle)) \\
&= \frac{-i}{\sqrt{6}}|0\rangle + \frac{2+i}{\sqrt{6}}|1\rangle
\end{aligned}$$

4) $|\psi_4\rangle = \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle$

$$|\psi_4\rangle = |1\rangle$$

Recall the $\{|i\rangle, |-i\rangle\}$ basis where
$|i\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$
$|-i\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}$
Write the following states in the $\{|i\rangle, |-i\rangle\}$ basis, that is as a weighted sum of $|i\rangle$ and $|-i\rangle$.

5) $|\psi_1\rangle = |0\rangle$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|i\rangle + |-i\rangle)$$

6) $|\psi_2\rangle = |1\rangle$

$$|\psi\rangle_1 = \frac{i}{\sqrt{2}}(-|i\rangle + |-i\rangle)$$

7) $|\psi_3\rangle = |+\rangle$

$$\begin{aligned}|\psi_3\rangle &= \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}}(|i\rangle + |-i\rangle) + \frac{i}{\sqrt{2}}(-|i\rangle + |-i\rangle))\\ &= \frac{1-i\sqrt{2}}{2}|i\rangle + \frac{1+i\sqrt{2}}{2}|-i\rangle\end{aligned}$$

8) $|\psi_4\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle$

$$|\psi_4\rangle = \frac{1+i\sqrt{2}}{2}|i\rangle + \frac{1-i\sqrt{2}}{2}|-i\rangle$$

9) $|\psi_5\rangle \frac{\sqrt{3}}{1}|0\rangle - \frac{1}{2}|1\rangle$

$$\begin{aligned}|\psi_5\rangle &= \frac{\sqrt{3}}{2}(\frac{1}{\sqrt{2}}|i\rangle + \frac{1}{\sqrt{2}}|-i\rangle) - \frac{1}{2}(\frac{-i}{\sqrt{2}}|i\rangle + \frac{i}{\sqrt{2}}|-i\rangle)\\ &= \frac{\sqrt{3}+i}{2\sqrt{2}}|i\rangle + \frac{\sqrt{3}-i}{2\sqrt{2}}|-i\rangle\end{aligned}$$

10) $|\psi_6\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|-\rangle$

$$\begin{aligned}|\psi_6\rangle &= \frac{\sqrt{3}}{2}(\frac{1-i\sqrt{2}}{2}|i\rangle + \frac{1+i\sqrt{2}}{2}|-i\rangle) + \frac{1}{2}(\frac{1+i\sqrt{2}}{2}|i\rangle + \frac{1-i\sqrt{2}}{2}|-i\rangle)\\ &= \frac{\sqrt{3}-i\sqrt{6}+1+i\sqrt{2}}{4}(|i\rangle + |-i\rangle)\end{aligned}$$

Question 2.

1) Find the adjoint of the matrices X, Y, Z, A, B, C

$$X^\dagger = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Y^\dagger = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad Z^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A^\dagger = \begin{bmatrix} -\frac{i}{\sqrt{2}} & \frac{1-i}{2} \\ \frac{1+i}{2} & -\frac{i}{\sqrt{2}} \end{bmatrix} \quad B^\dagger = \begin{bmatrix} 0 & e^{-i2\pi/3} \\ e^{-i\pi/3} & 0 \end{bmatrix} \quad C^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$$

2) For each of the following circuits, find the state of the quibit at the end of the circuit, and what the possible outcomes and probabilities are if we measure in the stated basis.

Measure in the Hadamard basis: $|0\rangle - \boxed{X} - \boxed{Z} -$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Measure $|+\rangle : 1/2, |-\rangle : 1/2$

Measure in the standard basis: $|+\rangle - \boxed{X} - \boxed{H} -$

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Measure $|0\rangle : 1, |1\rangle : 0$

Measure in the standard basis: $|-\rangle - \boxed{X} - \boxed{H} -$

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Measure $|0\rangle : 0, |1\rangle : 1$

Measure in the Hadamard basis: $|0\rangle - \boxed{H} - \boxed{C} - \boxed{Y} -$

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ i \end{bmatrix}$$

Probability $|+\rangle$ is measured:

$$|\tfrac{1}{\sqrt{2}} \langle +| \begin{bmatrix} -1 \\ i \end{bmatrix} |^2 = | -\tfrac{1}{2} + \tfrac{i}{2}| = 1/2$$

$|-\rangle$ measured: $1/2$

Measure in the Hadamard basis: $|1\rangle - \boxed{C} - \boxed{Z} - \boxed{H} - \boxed{X} - \boxed{X} - \boxed{H} - \boxed{C}$

$$\begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}0\\ i\end{bmatrix}$$

Probability measuring $|1\rangle : 1$, $|0\rangle : 0$

Question 3.

Suppose we repeat the Elitzur-Vaidman bomb experiment but adjust our circuit as follows:

$$|0\rangle - \boxed{R_{\pi/6}} - \boxed{B} - \boxed{R_{\pi/6}} - \boxed{B} - \boxed{R_{\pi/6}} - $$

3.1) What are the possible outcomes and probabilities of those outcomes if the bomb is faulty?

Answer:
If the bomb is faulty, the B gates act as identity gates. Therefore, the experiment is equivalent to applying a $\pi/6$ rotation to $|0\rangle$ 3 times, resulting in the quibit state becoming $|1\rangle$. Therefore, the only outcome if the bomb is faulty is to measure $|1\rangle$ with probability 1.

3.2) What are the possible outcomes and probabilites of those outcomes if the bomb is not faulty?

Answer:
If the bomb is not faulty, it may or may not explode after the B gate measures the quibit. After the first rotation gate, the state of the quibit is $\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$, meaning that the bomb explodes after the first measurement 1/4 of the time. 3/4ths of the time, the bomb measures $|0\rangle$ and the quibit collapses to $|0\rangle$. This process repeats one more time if the initial measurement did not explode the bomb. If the second measurement also did not explode the bomb, then the quibit is measured $|0\rangle$ 3/4ths the time and $|1\rangle$ 1/4ths. The probabilty we measure $|0\rangle$ if the bomb is in fact working is therefore:

$$\frac{3}{4} * \frac{3}{4} * \frac{3}{4} = \frac{27}{64}$$

The probability we measure $|1\rangle$ is:

$$\frac{3}{4} * \frac{3}{4} * \frac{1}{4} = \frac{9}{64}$$

And the probability the bomb explodes is therefore:

$$\frac{64}{64} - \frac{9+27}{64} = \frac{7}{16}$$

3.3) How does this version of the experiment compare to what we did in class? Which is more effective at detecting a bomb that is not faulty without having it explode? Which has a higher probability that the bomb explodes.

Answer:
In the case that the bomb is not faulty, the probability that the scheme covered in class detected it was 1/4. However, the probability this scheme detects that the bomb is not faulty if that is the

case is 27/64, meaning it has a higher chance of detecting non-faulty bombs. An issue, however, is that this scheme will explode 7/16ths of the time, which is nearly half. This scheme is therefore much more dangerous.