Scripps Declassified Math Survival Guide: Algebra Refresher

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Aside: This is a brand-new set of notes. If you notice any mistakes, or have suggestions for improving the clarity of these notes, please let me know!

1 Introduction

This lecture is designed to provide incoming students with a quick refresher on basic algebraic problem-solving techniques. Topics to be covered are: order of operations, exponent and logarithm rules, the quadratic equation, and systems of equations.

2 Order of Operations

The order of operations helps us "unpack" mathematical expressions, so that we can simplify arithmetic problems and solve algebraic equations. It tells you exactly what sequence in which to perform the necessary operations. A helpful mnemonic is "PEMDAS":

P - Parentheses

E - Exponents

M - Multiplication

D - Division

A - Addition

S - Subtraction

To simplify an arithmetic expression using PEMDAS (see, Warm-up I), complete the mathematical operations in the order described above. To simplify an algebraic equation, apply PEMDAS in reverse order (see, Warm-up II-III). Note that the order in which the addition/subtraction operations are completed is often interchangeable. The same is true for multiplication/division. When in doubt, however, the rule is to work from left to right.

EX Warm-up I.

Let $a \in \mathbb{R}$ ("a is some real number"). Simplify the following (i.e. do not solve for any variable),

$$\frac{6a(3+2)}{3} - 5 = ?$$

Solution.

Apply PEMDAS in forward order, because we're simplifying a mathematical expression.

$$\begin{aligned} \frac{6a(3+2)}{3} - 5 &= \frac{6a \cdot 5}{3} - 5 && \text{Parentheses.} \\ &= \frac{30a}{3} - 5 && \text{Multiplication.} \\ &= 10a - 5. && \text{Division.} \end{aligned}$$

$\boxed{\textbf{EX}}$ Warm-up II.

Let $a \in \mathbb{R}$. Solve for unknown x.

$$\frac{6a(3+2)}{3} - 5 = 3x - 5.$$

Solution.

Apply PEMDAS in reverse order, because we're solving for an unknown.

$$\begin{aligned} \frac{6a(3+2)}{3} - 5 &= 3x - 5 \\ 10a - 5 &= 3x - 5 && \text{Simplifying LHS (see Warm-up I).} \\ 10a &= 3x && \text{Adding 5 to both sides.} \\ x &= \frac{10a}{3}. && \text{Dividing both sides by 3.} \end{aligned}$$

$\boxed{\textbf{EX}}$ Warm-up III–Trickier!

Let $a \in \mathbb{R}$. Solve for unknown x.

$$\frac{6a(3+2)}{3} - 5 = \ln(3x) - 5.$$

Solution.

Apply PEMDAS in reverse order, because we're solving for an unknown.

$$\begin{aligned} \frac{6a(3+2)}{3} - 5 &= \ln(3x) - 5 \\ 10a - 5 &= \ln(3x) - 5 && \text{Simplifying LHS (see Warm-up I).} \\ 10a &= \ln(3x) && \text{Adding 5 to both sides.} \\ 10a &= \log_e(3x) && \text{Def'n of natural log.} \\ e^{10a} &= 3x && \text{Def'n logarithm (see (1) below).} \\ x &= \frac{1}{3}e^{10a}. && \text{Dividing both sides by 3.} \end{aligned}$$

3 Logarithms & Exponents[3]

As a reminder, some of the key properties of logarithms and exponents are described below.

Exponents

Let $a, b \in \mathbb{R}$ ("a and b are real numbers"), and $m, n \in \mathbb{I}$ ("m and n are integers"). Then,

- $a^m a^n = a^{m+n}$
- $(a^m)^n = a^{mn}$
- $(ab)^m = a^m a^n$
- $\frac{a^m}{a^n} = a^{m-n}, a \neq 0$
- $(\frac{a}{b})^m = \frac{a^m}{b^m}, b \neq 0$
- $a^{-m} = \frac{1}{a^m}, a \neq 0$
- $\sqrt[m]{a} = a^{1/m}$
- $a^0 = 1, a \neq 0$

Logarithms

Now take $a, b, x \in \mathbb{R}_+$ (a, b, and x are positive, real numbers) and $y \in \mathbb{R}$. By definition,

$$y = \log_a x \leftrightarrow x = a^y. \tag{1}$$

Aside: Take just the LHS ("left hand side") of the bidirectional: $y = log_a x$. *Now take just the RHS ("right hand side") of this equation:* $log_a x$. *I think of this mathematical statement as being equivalent to the question, "a to the power of what equals x?". For example, consider* $ln_e 1 = ?$. *In words, "e to the power of what equals 1?". The answer is* 0*, by an identity listed below. So,* $log_e 1 = 0$.

Recall that a is the logarithmic base. Common bases include log base 10,

$$\log_{10} a = \log(a)$$

and the natural log,

$$\log_e a = \ln(a).$$

Furthermore,

- $\log_a xy = \log_a x + \log_a y$
- $\log_a \frac{x}{y} = \log_a x - \log_a y$
- $\log_a x^y = y \cdot \log_a x$
- $\log_a a^x = x$
- $a^{\log_a x} = x$
- $\log_a a = 1, a > 0$
- $\log_a 1 = 0, a > 0$
- $\log_b a = \frac{\ln a}{\ln b} = \frac{\log_{10} a}{\log_{10} b}$ "Change of base formula"

4 Quadratic Formula

The quadratic formula is a powerful tool for solving quadratic equations, which are polynomials of the following form:

$$ax^2 + bx + c = 0, \tag{2}$$

where x is some unknown and a, b, and c are constants.

Recall that polynomials always have the same number of solutions as the degree on the leading order term (i.e. the unknown variable with the largest exponent). This means that a quadratic equation has two solutions, called roots. These roots may be (1) real and distinct, (2) real and repeated, or (3) a pair of complex conjugates. We'll stick to real solutions, here.

The quadratic formula, then, is as follows:

$$x_+, x_- = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{3}$$

where x_+ and x_- are positive and negative roots, respectively.

To apply the quadratic formula, manipulate the given equation so that it looks like (2). Identify constants a, b, and c, and substitute into (3). Lastly, simplify as much as possible.

Aside: The quadratic formula will always solve a quadratic equation. However, there are sometimes more elegant approaches, like factoring or completing the square.

$\boxed{\textbf{EX}}$ Quadratic Formula.[1]

$$x^2 + 4x - 9 = 0.$$

Solution.

First, check that the quadratic equation to be solved matches the form of (2)–it does!. Next, identify the constants a, b, and c:

$$a = 1 \qquad b = 4 \qquad c = -9.$$

Then, substitute a, b, and c into the quadratic formula:

$$\begin{aligned} x_+, x_- &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-4 \pm \sqrt{4^2 - 4(1)(-9)}}{2(1)} \end{aligned}$$

and simplify,

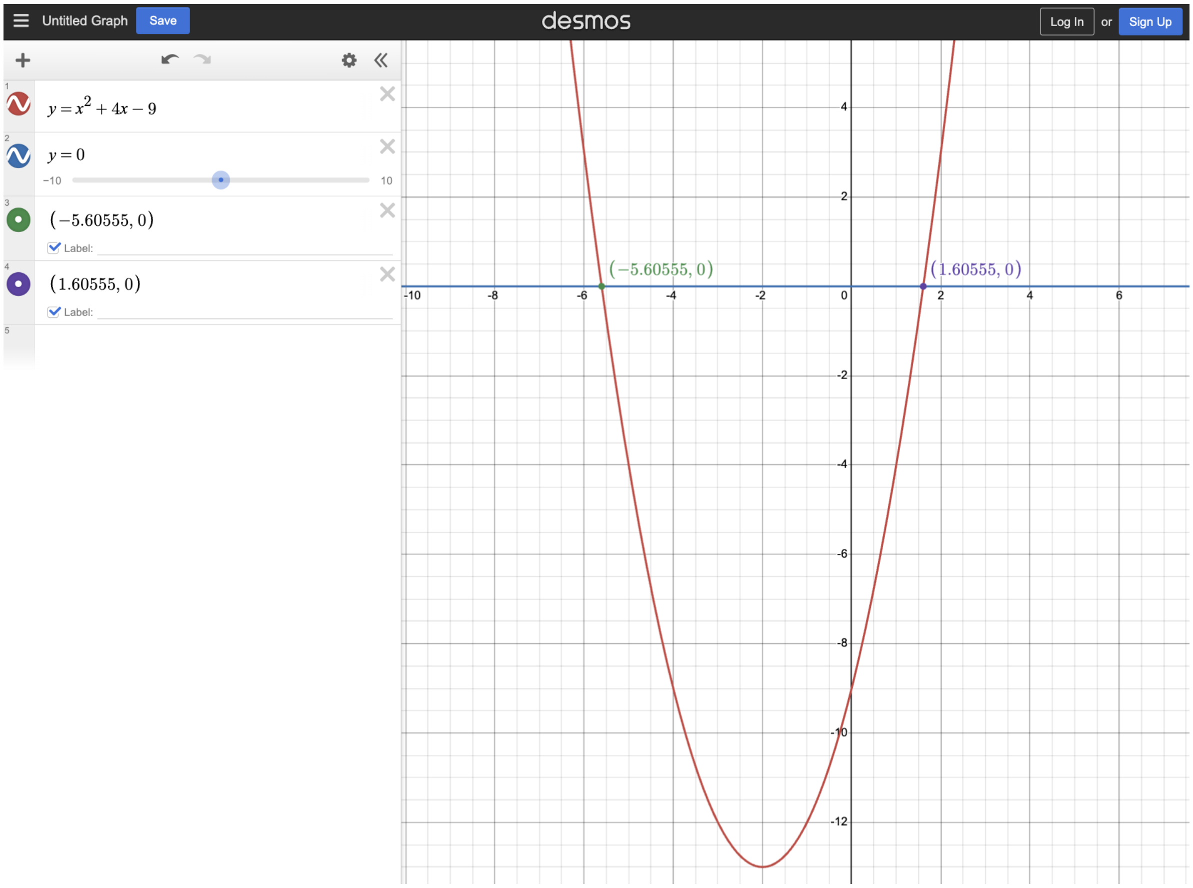
$$\begin{aligned} x_+, x_- &= \frac{-4 \pm \sqrt{4^2 - 4(1)(-9)}}{2(1)} \\ &= \frac{-4 \pm \sqrt{16 + 36}}{2} \\ &= \frac{-4 \pm \sqrt{52}}{2} \\ &= \frac{-4 \pm \sqrt{13 \cdot 4}}{2} \\ &= \frac{-4 \pm 2\sqrt{13}}{2} \\ &= -2 \pm \sqrt{13}. \end{aligned}$$

Our solutions, then, are the following:

$$x_+ = -2 + \sqrt{13} \approx 1.61 \qquad x_- = -2 - \sqrt{13} \approx -5.61.$$

We can also check our work graphically, by plugging the LHS ($x^2 + 4x - 9$) and RHS (0) of our original equation into graphing software, and determining where the two curves intersect.

Aside: I really like Desmos (https://www.desmos.com/calculator) for this, but MATLAB and Python both have utilities that will work.



5 Systems of Equations

A system of equations is a set of at least two equations, containing at least two unknowns, which are meant to be solved together.

It is possible for a system of equations to have either (1) exactly one solution, (2) infinitely many solutions, or (3) no solutions. Cases (1)-(2) describe a consistent system. Case (3) describes an inconsistent, meaning intractable, system. Graphical representations of all three cases are included below for a linear, 2x2 system:

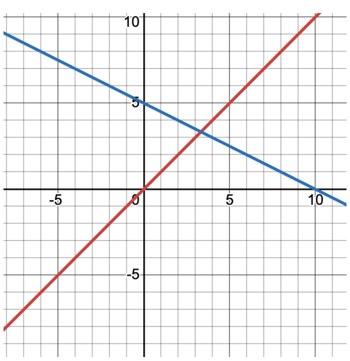


Figure 1: A *consistent* and *determinant* system has exactly one solution.

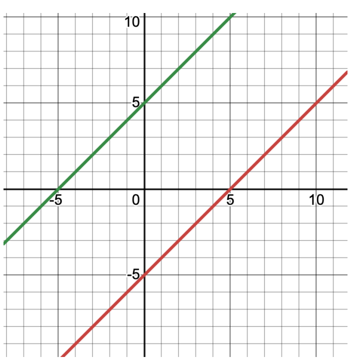


Figure 2: An *inconsistent* system has no solutions.

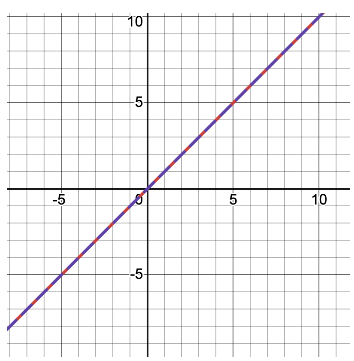


Figure 3: A *consistent* and *indeterminate* system has infinitely many solutions.

In order to be tractable, it is necessary (but not sufficient!) that a system of equations contain at least as many constituent equations as unknown variables. When presented with a new system of equations, it's often a good idea to check on this condition first–when it isn't met, then a unique solution to the system does not exist.

Aside: The formal way of deciding which of the above three cases a system falls into requires calculating matrix determinants–a linear algebra technique–and applying Cramer's Rule. We won't cover Cramer's Rule here, but it's helpful to know that it exists.

EX Mixing Water Masses.[2]

Imagine you're taking shipboard measurements along an equatorial transect, and collect a (very large!) water sample with a volume of 1 m^3, temperature of 2°C and salinity of 34.78 psu. You know from previous profiles in the region, and SIO210, that the sample is made up of two water masses: Antarctic Bottom Water (AABW) and North Atlantic Deep Water (NADW). You also recall that AABW typically has temperature around -0.5°C and salinity of approximately 34.67 psu. NADW usually has temperature of about 3°C and salinity near 34.9 psu.[4]

Calculate the volume of each of the two water mass constituents.

Aside: AABW and NADW are two water masses (special kinds of water with specific temperature, salinity, density, etc) which together make up most of the ocean below ∼4000 m. There's evidence that under a changing climate system, the amount and properties of AABW and NADW being formed could change–this is concerning, because both NADW and AABW help regulate circulation and overturning in the global ocean.[5,6]

Solution.

Conservation of tracer (i.e. salt) and energy (i.e. temperature) give the following equations:

$$\rho_A V_A C_p T_A + \rho_N V_N C_p T_N = \rho V C_p T \qquad \text{Conservation salt.} \tag{4}$$
$$\rho_A V_A S_A + \rho_N V_N S_N = \rho V S. \qquad \text{Conservation temperature.} \tag{5}$$

Dividing (4) and (5) through by ρ, and making a Boussinesq approximation ($\frac{\rho_A}{\rho} \approx \frac{\rho_N}{\rho} \approx 1$), yields a 2x2 system of equations:

$$V_A T_A + V_N T_N = VT \tag{6}$$
$$V_A S_A + V_N S_N = VS. \tag{7}$$

Notice that we have two equations and two unknowns. We proceed with two approaches to solving this problem: (1) by substitution, and (2) graphically. Alternative methods for solving systems of equations are additionally listed below.

Aside: Additional solution methods include row reduction, Cramer's Rule, and elimination of variables. The first two, row reduction and Cramer's Rule require some linear algebra background. The third, elimination of variables, can be very tidy, but is only applicable under very specific circumstances.

5.1 Substitution

Substitution entails solving one equation in the system for one unknown, than substituting this expression into the remaining equation(s). It doesn't matter which equation or variable you choose to solve for first, but some choices may result in tidier, easier math.

For example, we begin by solving (6) for V_B:

$$V_A T_A + V_N T_N = VT \tag{8}$$
$$V_N = \frac{VT - V_A T_A}{T_N}. \tag{9}$$

Now, substitute the new expression for V_N into (7). It turns out that the math is slightly nicer if we solve (7) for V_N first:

$$V_A S_A + V_N S_N = VS \tag{10}$$
$$V_N = \frac{VS - V_A S_A}{S_N}. \tag{11}$$

Then, setting the two expressions for V_N equal to each other,

$$\frac{VT - V_A T_A}{T_N} = \frac{VS - V_A S_A}{S_N} \tag{12}$$
$$V_A = \frac{S_N VT - T_N VS}{S_N T_A - T_N S_A}. \tag{13}$$

Next, we may substitute our solution for V_A into (9) or (10) to find V_N:

$$V_N = \frac{VS - V_A S_A}{S_N} \tag{14}$$
$$= \frac{VS - (\frac{S_N VT - T_N VS}{S_N T_A - T_N S_A}) S_A}{S_N}. \tag{15}$$

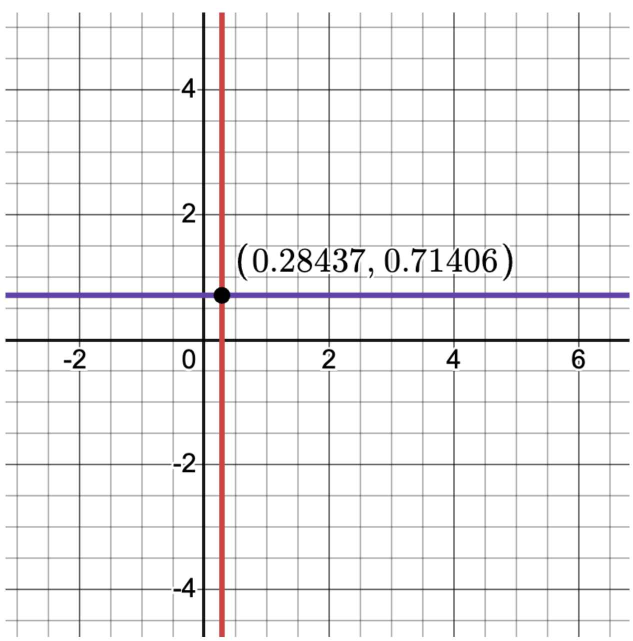
Finally, plugging in our known quantities yields solutions $V_A \approx 0.284$ m^3 and $V_N \approx 0.714$ m^3.

Aside: If you're paying close attention, you might have noticed that $V_N + V_A \neq 1$! This is because as the two water masses mix, they are subject to thermal expansion and haline contraction (i.e. mass, but not volume, is conserved for this system).

Aside: The substitution method is a kind of "brute force" approach for solving systems of equations–it's straightforward and always works, but isn't very clever and sometimes gets messy.

5.2 Graphically

We can quickly check our solution by plotting (8) and (9), and locating where they intersect.



Aside: The graphical method is efficient, but doesn't provide much mathematical intuition as to how the solution was obtained. In your coursework, your professors will often prefer a more explicit solution–solving by graphing is a great way to check your work, but typically shouldn't be the only method you attempt.

6 References

[1] https://docs.google.com/document/d/1XpBGxah9UauVig83NRROMrFhNlti0GIWGauEHxB_itI/edit?usp=sharing
[2] https://webspace.science.uu.nl/ dijks101/DO/solution_1.1.pdf
[3] https://wou.edu/mathcenter/files/2015/09/Exponents-and-Logarithms.pdf
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[5] Lago, V. & England, M. H. (2019). *Projected Slowdown of Antarctic Bottom Water Formation in Response to Amplified Meltwater Contributions.* American Meteorological Society.
[6] Muschitiello, F. et al. (2019). *Deep-water circulation changes lead North Atlantic climate during deglaciation.* Nature Communications.