## 1. BASIC KNOWLEDGE

In this lecture, we learn how to solve the following types of problems using unconventional signs for the written notation of mathematical notions and reasoning: additions, subtractions, multiplications, divisions, exponents, and radicals.

All the rules of operations (addition, subtraction, multiplication, division, radicals, and exponents) we learnt from arithmetic and algebra are still valid with these symbols.

**Fundamental law of fractions:**

For any fraction $\frac{a}{b}$ and any number c ≠ 0, $\frac{a}{b}=\frac{a\times c}{b\times c}$.

**Division of fractions**

To divide by a fraction, we simply multiply by its reciprocal.

$$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}$$

**Power rules of exponents**

$a^m \times a^n = a^{m+n}$ ⇔ $a^{m+n} = a^m \times a^n$

$(a^m)^n = a^{mn}$ ⇔ $(ab)^n = a^n b^n$

$(\frac{a^m}{b^m}=\left(\frac{a}{b}\right)^m$ ⇔ $\left(\frac{a}{b}\right)^m=\frac{a^m}{b^m}$

**Properties of radicals**

$\sqrt{ab}=\sqrt{a}\times\sqrt{b}$ ($a$ > 0, and $b$ > 0)

$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$ ($a > 0$, and $b > 0$)

$\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m = a^{\frac{m}{n}}$

## Properties of absolute value

$|-x| = |x|$

$|x-y| = |y-x|$

$|xy| = |x| \cdot |y|$, and $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ $y \neq 0$.

## Square binomial

$(x+y)^2 = x^2 + 2xy + y^2$

$(x-y)^2 = x^2 - 2xy + y^2$

$x^2 - y^2 = (x-y)(x+y)$

## Number of divisors

For an integer $n$ greater than 1, let the prime factorization of $n$ be $n = p_1^a p_2^b p_3^c \dots p_k^m$, Where $a$, $b$, $c$,…, and $m$ are nonnegative integers, $p_1$, $p_2$, …, $p_k$ are prime numbers.

**The number of divisors** is: $d(n) = (a+1)(b+1)(c+1)\dots(m+1)$

## Sum of the positive divisors

The sum of divisors is:

$$\sigma(n) = (\frac{p_1^{a+1}-1}{p_1-1})(\frac{p_2^{b+1}-1}{p_2-1})\dots(\frac{p_k^{m+1}-1}{p_k-1})$$

Or $\sigma(n) = (p_1^{\ a} + p_1^{\ a-1} + ... + p_1^{\ 0})(p_2^{\ b} + p_2^{\ b-1} + ... + p_2^{\ 0})...(p_k^{\ m} + p_k^{\ m-1} + ... + p_k^{\ 0})$

## Patterns of the last digit of $a^n$

The last digits of $a^n$ have patterns shown in the table below.

| $n$ | 1 | 2 | 3 | 4 | Period |
|---|---|---|---|---|---|
| $2^n$ | **2** | **4** | **8** | **6** | 4 |
| $3^n$ | **3** | **9** | **7** | **1** | 4 |
| $4^n$ | **4** | **6** | | | 2 |
| $5^n$ | **5** | | | | 1 |
| $6^n$ | **6** | | | | 1 |
| $7^n$ | **7** | **9** | **3** | **1** | 4 |
| $8^n$ | **8** | **4** | **2** | **6** | 4 |
| $9^n$ | **9** | **1** | | | 2 |

For example, when $a = 2$,

$2^1 = 2$ $2^2 = 4$, $2^3 = 8$, $2^4 = 16$,

$2^5 = 32$, $2^6 = 64$, $2^7 = 128$, $2^8 = 256$,…

The last digits of $2^n$ demonstrate a pattern: 2, 4, 8, 6, 2, 4, 8, 6, etc…

## Pythagorean triples

A **Pythagorean triple** consists of three positive integers $a$, $b$, and $c$, such that $a^2 + b^2 = c^2$.

There are 16 primitive Pythagorean triples with $c < 100$:

| | | | |
|---|---|---|---|
| (3, 4, 5 ) | (5, 12, 13) | (8, 15, 17) | (7, 24, 25) |
| (20, 21, 29) | (12, 35, 37) | ( 9, 40, 41) | (28, 45, 53) |
| (11, 60, 61) | (16, 63, 65) | (33, 56, 65) | (48, 55, 73) |
| (13, 84, 85) | (36, 77, 85) | (39, 80, 89) | (65, 72, 97) |

## 2. PROBLEMS SOLVING

### 2. 1. Additions

☆**Example 1.** For any positive integer *n*, define (6)*n* to be the sum of the positive factors of *n*. For example, (6) = 1 + 2 + 3 + 6 = 12. Find ((18)) .

(A) 39 (B) 40 (C) 48 (D) 56 (E) 60

☆**Example 2.** For the positive integer *n*, let <*n*> denote the sum of all the positive divisors of *n* with the exception of *n* itself. For example, <4> = 1 + 2 = 3 and <12> = 1 + 2 + 3 + 4 + 6 = 16. What is <<<28>>>?

(A) 28 (B) 11 (C) 21 (D) 6 (E) 3

☆**Example 3.** Let ♣($x$) denote the sum of the digits of the positive integer $x$. For example, ♣ (8) = 8 and ♣ (123) = 1 + 2 + 3 = 6. For how many two-digit values of $x$ is ♣ ($x$) = 12?

(A) 3 (B) 4 (C) 6 (D) 9 (E) 7

**Example 4.** If $a$ ♦ $b = a^2 + ab - b^2$, then find (3 ♦ 2) ♦ 13.

(A) 11 (B) 95 (C) – 48 (D) 59 (E) 143

**Example 5.** If $a \triangle b = (a + b) + ab + b$, what is $(5 \triangle 7) \triangle 3$?

(A) 54 (B) 57 (C) 219 (D) 222 (E) 232

**Example 6.** For all real numbers $a$ and $b$, where $b \neq 0$, the operation ★is defined as $a \bigstar b = \dfrac{a^2 + b^2}{b^3}$.Compute the following, and express your answer as a common fraction: $(1\bigstar 2)^2/(2\bigstar 1)$.

(A) $\dfrac{1}{8}$ (B) $\dfrac{1}{5}$ (C) $\dfrac{125}{64}$ (D) $\dfrac{5}{64}$ (E) $\dfrac{25}{64}$

**Example 7.** If $a \bigtriangledown b = a^2 + 2ab + b^2$, what is the value of $(3 \bigtriangledown 2) \bigtriangledown 5$?
(A) 25 (B) 60 (C) 900 (D) 625 (E) 30

## 2. 2. Subtractions

☆**Example 8.** Define $x \otimes y = x^3 - y$. What is $h \otimes (h \otimes h)$?
(A) $-h$ (B) $-0$ (C) $h$ (D) $2h$ (E) $h^3$

**Example 9.** The operation $\oplus$ is defined as $m \oplus n = m^2 - mn - n^2$, and the operation $\bowtie$ is defined as $m \bowtie n = 2(m - n)$. Compute $(3\oplus4) \bowtie (4\oplus3)$.
(A) $-26$ (B) $-28$ (C) $-19$ (D) 28 (E) $-5$

**Example 10.** If for positive integers $a$ and $b$, $\langle ab\rangle = ab - a - b$, find the value of $a + b$ in the equation $\langle ab\rangle = 6$.

(A) 9 (B) 10 (C) 8 (D) 15 (E) 12

**Example 11.** Define ($x \triangle y$) to mean $2x - 3y$. Evaluate $((4 \triangle 3) \triangle (5 \triangle 3))$.

(A) –1 (B) 1 (C) 5 (D) –5 (E) 15

**Example 12.** Given that $a ✦ b = a^3 - b^2$, what is the value of $4 ✦ (2 ✦ 1)$?

(A) –7 (B) –15 (C) 8 (D) 15 (E) 7

**Example 13.** 10. Subfactorials, $!n$, are defined by the formula:

$$!n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$$

Express the following where for $x = 6$: $\dfrac{!x}{!(x-1)}$.

(A) $\dfrac{265}{44}$ (B) $\dfrac{11}{30}$ (C) $\dfrac{53}{24}$ (D) $\dfrac{53}{144}$ (E) 6.

**Example 14.** The operation ✿ is defined by $n$ ✿ $= n^2 - 1$. What is the value of the following: 10(3✿) – 5(4✿)?

(A) –5 (B) 99 (C) 8 (D) 15 (E) 5

## 2. 3. Multiplications

**Example 15.** If 4! means $4 \cdot 3 \cdot 2 \cdot 1$, what is the value of $\frac{5!}{3!}$?

(A) 20 (B) 40 (C) 30 (D) 60 (E) 80

**Example 16.** If $a \bigstar b$ is defined as $(a + 1)(b + 1)$, find $(2 \bigstar 3) \bigstar 4$.

(A) 120 (B) 65 (C) 56 (D) 11 (E) 20

**Example 17.** Two binary operations are defined by the rules $a \star b = a^3 - b^3$ and $a \triangledown b = (a + b)^3$. What is the value of $(2 \star 3) \triangledown 9$?

(A) 1000 (B) 729 (C) – 512 (D) 1 (E) –1000

**Example 18.** Given $a * b = ab + 1$, evaluate: $4 * [(6 * 8) + (3 * 5)]$ .

(A) 65 (B) 121 (C) 144 (D) 261 (E) 111

**Example 19.** If $a ☆ b = ab - 1$ and $a ★ b = a + b - 1$, what is the value of 4 ☆[(6 ★ 8)★(3 ☆ 5)]?

(A) 27 (B) 104 (C) 26 (D) 103 (E) 182

## 2. 4. Divisions

☆**Example 20.** For each pair of real numbers $a \neq b$, define the operation $*$ as $(a * b) = \dfrac{a+b}{a-b}$. What is the value of $((1*2)*4)$?

(A) –2/3 (B) – 1/7 (C) 0 (D) 1/2 (E) This value is not defined.

☆**Example 21.** Define $a@b = ab - b^2$ and $a\#b = a + b - ab^2$. What is $\frac{6@3}{6\#3}$?

(A) –1/5 (B) – 1/4 (C) 1/8 (D) 1/4 (E) 1/2

☆**Example 22.** For the nonzero numbers a, b, and c, define $(a,b,c) = \frac{abc}{a+b+c}$.

Find (2, 5, 8).

(A) 16/3 (B) 5 (C) 15/2 (D) 6 (E) 24

**Example 23.** Express 3 ∗ (4 ∗ 5) as a common fraction given $a * b = \frac{ab}{a+b}$.

(A) $\frac{20}{9}$ (B) 1 (C) $\frac{60}{47}$ (D) $\frac{20}{47}$ (E) $\frac{47}{60}$

**Example 24.** If $a \circledast b = \dfrac{(a+b)}{b}$, compute $(3 \circledast 1) \circledast 2$.

(A) 3 (B) 4 (C) 2 (D) 1 (E) 8

**Example 25.** If $a \star b = \dfrac{(\frac{1}{b}-\frac{1}{a})}{(a-b)}$, express $5 \star 7$ as a common fraction.

(A) $\dfrac{35}{3}$ (B) $\dfrac{1}{35}$ (C) $\dfrac{2}{35}$ (D) $\dfrac{4}{35}$ (E) $-\dfrac{1}{35}$

**Example 26.** Given $a * b = \dfrac{a+b}{a \cdot b}$, find $(5 * 6) * 1$.

(A) $\dfrac{30}{11}$ (B) $\dfrac{11}{30}$ (C) $\dfrac{41}{11}$ (D) $\dfrac{11}{41}$ (E) $\dfrac{985}{341}$

**Example 27.** If a ✪ b is defined as $\frac{a+b}{2}$, what is the value of 6 ✪(3✪5)?

(A) 1 (B) 4 (C) 2 (D) 10 (E) 5

## 2. 5. Exponents

☆**Example 28.** The operation ⊗ is defined for all nonzero numbers by $a \otimes b = a^2/b$. Determine $[(1 \otimes 2) \otimes 4] - [1 \otimes (2 \otimes 4)]$.

(A) –15/16 (B) – 14/15 (C) 0 (D) 15/16 (E) 17/16

**Example 29.** If $x$ ✪ $y = (x^y)^x$, what is the units digit of 4 ✪ 10?

(A) 6 (B) 8 (C) 2 (D) 4 (E) 0

**Example 30.** If $a \blacklozenge b$ means $3a - 2^b$, then what value is associated with $4 \blacklozenge (2 \blacklozenge 3)$?

(A) $\frac{49}{4}$ (B) 8 (C) 16 (D) $\frac{47}{4}$ (E) – 2

**Example 31.** If $a \bigstar b = a^b + b^a$, what is $(4 \bigstar 3) \div (3 \bigstar 4)$?

(A) 256 (B) 145 (C) 1 (D) 5 (E) 2

**Example 32.** For natural numbers $a$ and $b$, $a \triangle b = b^a + 2ab$. Find the value of $(2 \triangle 3) - (3 \triangle 2)$.

(A) 1 (B) 41 (C) 37 (D) 12 (E) 0

**Example 33.** If $x \otimes y = x^y - x + y^y$, find the value of $(4 \otimes 2) - (3 \otimes 1)$.

(A) 16 (B) 1 (C) 13 (D) 17 (E) 15

**Example 34.** If $x \oplus y = (x^y)^x$, what is the units digit of $7 \oplus 5$?
(A) 7 (B) 9 (C) 3 (D) 1 (E) 6

**Example 35.** Given a △ b = $\dfrac{a^b}{b^a}$, and $a \,\square\, b = \dfrac{3a-b}{ab}$, find the common fraction equivalent to (2△3)□(2△1).
(A) $\dfrac{3}{8}$ (B) $\dfrac{24}{9}$ (C) $\dfrac{8}{9}$ (D) $\dfrac{8}{3}$ (E) $\dfrac{16}{9}$

## 2.6. Radicals

☆**Example 36.** For real numbers *a* and *b*, define $a \Diamond b = \sqrt{a^2 + b^2}$ . What is the value of $(8 \Diamond 15) \Diamond ((-15) \Diamond (-8))$ ?

(A) 0 (B) 13/2 (C) 15 (D) $17\sqrt{2}$ (E) 26

**Example 37.** Let ▽ be defined as ▽($a$, $b$) = $\sqrt{a^2 + b^2}$ , for all real numbers *a* and *b*. Find ▽ (▽ (▽(12, 5), 84), 132).

(A) 97 (B) 117 (C) 137 (D) 157 (E) 187

**Example 38.** Let ☆ be defined as ☆($a$, $b$) = $\sqrt{a^2 + b^2}$ , for all real numbers *a* and *b*. Find ☆(☆(16, 63), ☆(33, 56)) and express in simplest radical form.

(A) 65 (B) $65\sqrt{2}$ (C) $63\sqrt{2}$ (D) 130 (E) $56\sqrt{2}$

**Example 39.** If $x \heartsuit y = \sqrt{xy} + \frac{2}{x} - \frac{3}{2}$, find the value of (4 ♡ 4) ♡ 3. Express your answer as a common fraction.

(A) $\frac{13}{6}$ (B) 3 (C) $\frac{23}{6}$ (D) $\sqrt{17}$ (E) 7