

1. TERMS:

Fraction: A part of a whole or a quotient of two numbers, expressed as $\frac{a}{b}$. a and b are whole numbers. $b \neq 0$. $\frac{a}{b}$ is the same as $a \div b$ or a/b .

Proper fraction: A fraction in which the numerator is less than the denominator: $\frac{3}{5}$. Such a fraction has a value less than 1.

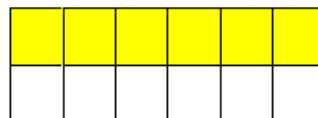
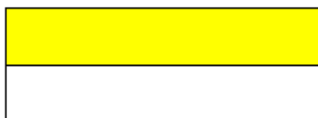
Improper fraction: A fraction in which the numerator is greater than or equal to the denominator: $\frac{5}{3}$. It has a value greater than or equal to 1.

Mixed number: A mixed number contains both a whole number part and a fraction part and can be written as an improper fraction: $2\frac{1}{3} = \frac{2 \times 3 + 1}{3} = \frac{7}{3}$.

2. PROPERTIES:

2.1. Equivalent Fraction (Cancellation Law): Two fractions are equal if they represent the same portion of a whole.

$$\frac{1}{2} = \frac{2}{4} = \frac{6}{12}$$



Examples: $\frac{38}{57} = \frac{2 \times \cancel{19}}{3 \times \cancel{19}} = \frac{2}{3}$; $\frac{38}{57} = \frac{38 \div 19}{57 \div 19} = \frac{2}{3}$

2.2. Fundamental Law of Fractions:

For any fraction $\frac{a}{b}$ and any number $c \neq 0$, $\frac{a}{b} = \frac{a \times c}{b \times c}$.

(The value of a fraction does not change if its numerator and denominator are multiplied by the same nonzero number).

Example. $\frac{2}{3} = \frac{2 \times 5}{3 \times 5} = \frac{10}{15}$.

2.3. Lowest (Reduced; Simplest) Term: A fraction in which the numerator and the denominator have no common terms except 1. The lowest terms are obtained by taking all the common factors out of the numerator and the denominator.

$\frac{10}{15}$ is not a fraction in the lowest term but $\frac{2}{3}$ is. ($\frac{10}{15} = \frac{2 \times 5}{3 \times 5} = \frac{2}{3}$).

2.4. Addition and Subtraction:

When working with fractions, only the numerators in fractions are added or subtracted.

(1). Two fractions having the same denominators:

We just add or subtract the numerators.

$$\begin{aligned} \frac{a}{b} + \frac{c}{b} &= \frac{a+c}{b} & \Rightarrow & \quad \frac{3}{5} + \frac{1}{5} = \frac{3+1}{5} = \frac{4}{5} \\ \frac{a}{b} - \frac{c}{b} &= \frac{a-c}{b} & \Rightarrow & \quad \frac{3}{5} - \frac{1}{5} = \frac{3-1}{5} = \frac{2}{5} \end{aligned}$$

(2). Two fractions having the different denominators:

We convert them to the same denominators first, and then add the numerators.

$$\begin{aligned} \frac{a}{b} + \frac{c}{d} &= \frac{a \times d}{b \times d} + \frac{c \times b}{b \times d} = \frac{a \times d + c \times b}{b \times d} \Rightarrow \\ \frac{1}{2} + \frac{2}{5} &= \frac{1 \times 5}{2 \times 5} + \frac{2 \times 2}{5 \times 2} = \frac{5+4}{10} = \frac{9}{10} \\ \frac{a}{b} - \frac{c}{d} &= \frac{a \times d}{b \times d} - \frac{c \times b}{b \times d} = \frac{a \times d - c \times b}{b \times d} \Rightarrow \\ \frac{1}{2} - \frac{2}{5} &= \frac{1 \times 5}{2 \times 5} - \frac{2 \times 2}{5 \times 2} = \frac{5-4}{10} = \frac{1}{10} \end{aligned}$$

2.5. Multiplication of Fractions

The numerator of the product is obtained by multiplying together the numerators. The denominator of the product is obtained by multiplying together the denominators.

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d} \quad \Rightarrow \quad \frac{2}{5} \times \frac{3}{7} = \frac{2 \times 3}{5 \times 7} = \frac{6}{35}$$

2.6. Division of Fractions

To divide by a fraction, we simply multiply by its reciprocal.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc} \quad \Rightarrow \quad \frac{2}{5} \div \frac{3}{7} = \frac{2}{5} \times \frac{7}{3} = \frac{2 \times 7}{5 \times 3} = \frac{14}{15}$$

The reciprocal of a number is obtained by switching the numerator and the denominator. For example, the reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$, and the reciprocal of 2 (note that 2 can be written as $\frac{2}{1}$) is $\frac{1}{2}$.

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}$$

3. PROBLEM SOLVING SKILLS

3.1. Comparing Fractions

(1). Same Denominator:

The fraction with a larger numerator is larger: $\frac{3}{5} > \frac{1}{5}$

(2). Same Numerator:

The fraction with a larger denominator is smaller: $\frac{3}{7} < \frac{3}{5}$

(3). Both the numerator and denominator are not the same

$$\frac{3 \times 11 = 33}{8} \quad ? \quad \frac{4 \times 8 = 32}{11} \quad \Rightarrow \quad 33 > 32 \quad \Rightarrow \quad \frac{3}{8} > \frac{4}{11}$$

Example 1. Mary made two pies that were exactly the same size. The first pie was a cherry pie, which she cut into 6 equal slices. The second was a pumpkin pie, which she cut into 12 equal pieces. Mary takes her pies to a party. People eat 3 slices of cherry pie and 6 slices of pumpkin pie. Did people eat more cherry pie or pumpkin pie?

Example 2. Peter has two cakes that are the same size. The first cake was chocolate, which he cut 12 equal parts. The second cake was marble, which he cut into 6 equal parts. His family eats 5 slices of chocolate cake and 3 slices of marble cake. Did they eat more chocolate cake or marble cake?

!

☆ **Example 3.** (AMC 8) What is the correct ordering of the three numbers $\frac{5}{19}$, $\frac{7}{21}$, and $\frac{9}{23}$, in increasing order?

(A) $\frac{9}{23} < \frac{7}{21} < \frac{5}{19}$

(B) $\frac{5}{19} < \frac{7}{21} < \frac{9}{23}$

(C) $\frac{9}{23} < \frac{5}{19} < \frac{7}{21}$

(D) $\frac{5}{19} < \frac{9}{23} < \frac{7}{21}$

(E) $\frac{7}{21} < \frac{5}{19} < \frac{9}{23}$

3.2. Sum of A Series of Fractions

Useful formulas:

$$\begin{aligned} \frac{1}{n(n+1)} &= \frac{1}{n} - \frac{1}{n+1} & \Rightarrow & \frac{1}{3(3+1)} = \frac{1}{3} - \frac{1}{3+1} = \frac{1}{3} - \frac{1}{4} \\ \frac{1}{n} &= \frac{1}{2n} + \frac{1}{2n} & \Rightarrow & \frac{1}{3} = \frac{1}{2 \times 3} + \frac{1}{2 \times 3} = \frac{1}{6} + \frac{1}{6} \\ \frac{1}{n(n+k)} &= \frac{1}{k} \left(\frac{1}{n} - \frac{1}{n+k} \right) & \Rightarrow & \frac{1}{3(3+2)} = \frac{1}{2} \left(\frac{1}{3} - \frac{1}{5} \right) \end{aligned}$$

Example 4. Find the sum: $\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \cdots + \frac{1}{11 \times 13}$.

Example 5. Calculate: $\frac{1}{2 \times 4} + \frac{1}{4 \times 6} + \cdots + \frac{1}{98 \times 100}$.

3.3. Continued Fractions

The simple continued fraction representation of a number is given by:

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \dots}}}}$$

where a_0 is an integer, any other a_i members are positive integers, and n is a non-negative integer.

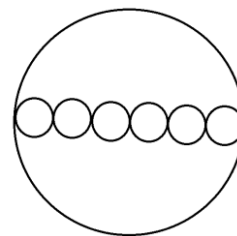
Example 6. Simplify: $1 + \frac{1}{1 + \frac{1}{1+1}}$. Express your answer as a common fraction.

Example 7. Simplify: $\frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}$. Express your answer as a common fraction.

3.4. Fraction Related to Geometry

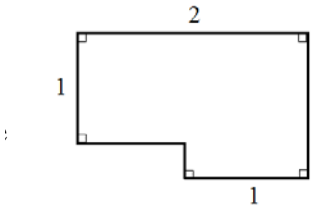
☆**Example 8.** Six pepperoni circles will exactly fit across the diameter of a 12-inch pizza when placed as shown. If a total of 20 circles of pepperoni are placed on this pizza without overlap, what fraction of the pizza is covered by pepperoni?

(A) $\frac{2}{3}$ (B) $\frac{5}{9}$ (C) $\frac{5}{7}$ (D) $\frac{5}{6}$ (E) $\frac{23}{36}$.



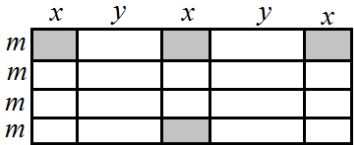
Example 9. The area of the figure shown is $\frac{11}{5}$. What is the perimeter of the figure?

ANSWER: 22/5 or 4.4



Example 10. In the figure shown, all angles are right angles and $y = 2x$. If m , x , and y are lengths of the segments indicated, what fraction of the figure is shaded?

- (A) $\frac{1}{7}$ (B) $\frac{1}{5}$ (C) $\frac{1}{14}$ (D) $\frac{3}{10}$ (E) $\frac{5}{14}$



3.5. Fraction Related to Numbers and Expressions

Example 11. What reduced common fraction is equivalent to $18\frac{1}{3}\%$?

Example 12. If $\frac{1}{2} + \frac{1}{5} + \frac{1}{8} > \frac{1}{x} + \frac{1}{6} + \frac{1}{8}$, then x could not be which of the following?

- (A) 1 (B) 2 (C) 3 (D) 4 (E) 8

Example 13. If a and b are integers such that $a + b > 160$ and $a/b = 0.15$, what is the smallest possible value of a ?

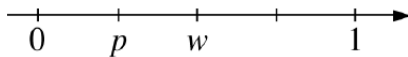
- (A) 140 (B) 21 (C) 24 (D) 3 (E) 15

Example 14. Which of the following numbers is between $\frac{1}{6}$ and $\frac{1}{5}$?

- (A) 0.14 (B) 0.15 (C) 0.16 (D) 0.17 (E) 0.26

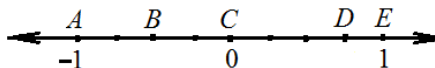
3.6. Fraction Related to Number Lines

Example 15. On the number line shown, the tick marks are equally spaced. What is the value of $w + p$?



- (A) $\frac{3}{4}$ (B) $\frac{2}{3}$ (C) $\frac{1}{2}$ (D) $\frac{1}{3}$ (E) $\frac{1}{4}$

Example 16. Dots are equally spaced on the number line shown. Which of the lettered points has a coordinate equal to $1 - (-\frac{1}{2})^2$?



(A) A

(B) B

(C) C

(D) D

(E) E

3.7. Fraction Applications

Example 17. An hour-long television program included 20 minutes of commercials. What fraction of the hour-long program was not commercials?

Example 18. A container is $\frac{3}{5}$ full of water. If 16 gallons of the water were removed from the container, it would be $\frac{1}{3}$ full. How many gallons of water does this container hold when it is completely full?

(A) 20

(B) 35

(C) 40

(D) 60

(E) 90.

Example 19. Roy planted corn on $\frac{2}{7}$ of his land. If he planted 60 acres of corn, how many acres of land does he have?

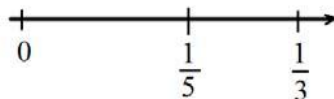
- (A) 90 (B) $112\frac{1}{2}$ (C) 135 (D) 210 (E) $337\frac{1}{2}$

☆**Example 20.** Peter's family ordered a 12-slice pizza for dinner. Peter ate two slices and shared another slice equally with his brother Tom. What fraction of the pizza did Peter eat?

- (A) $\frac{1}{8}$ (B) $\frac{1}{7}$ (C) $\frac{5}{24}$ (D) $\frac{1}{6}$ (E) $\frac{1}{24}$

MORE EXAMPLES

Example 21. The fraction halfway between $\frac{1}{5}$ and $\frac{1}{3}$ (on the number line) is



- (A) $\frac{1}{4}$ (B) $\frac{2}{15}$ (C) $\frac{4}{15}$ (D) $\frac{53}{200}$ (E) $\frac{8}{15}$

Example 22. How many more degrees of arc are there in $\frac{1}{5}$ of a circle than in $\frac{1}{6}$ of circle?

- (A) 9° (B) 12° (C) 24° (D) 30° (E) 36°

Example 23. Write the common fraction equivalent to $2\frac{1}{2}\%$.

Example 24. If $\frac{3}{8}$ of a number is $\frac{21}{2}$, what is $\frac{1}{7}$ of the number?

- (A) $3/2$ (B) $9/2$ (C) 4 (D) 6 (E) 28

Example 25. If $1/4 + 1/5 + 1/6 < 1/5 + 1/6 + 1/y$, then y could be which of the following?

- (A) 3 (B) 4 (C) 5 (D) 6 (E) 7

Example 26. Jenny had a pizza that was divided into 8 equal slices. She ate 3 of them. Alex has a pizza that is the same size, but his is divided into 4 equal slices. He ate 3 slices of his pizza. Who ate more pizza?

Answer: Alex

Example 27. Find the sum: $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{49 \times 50}$.

Answer: $\frac{49}{50}$

Example 28. Simplify: $\frac{1}{8 + \frac{1}{8 + \frac{1}{8}}}$.

Example 29. If $0 < a < b$, which of the following is greater than b/a ?
 (A) 1 (B) a/b (C) $1/(b/a)$ (D) $b/2a$ (E) $2b/a$

Answer: E