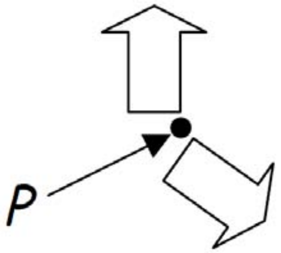
1. BASIC KNOWLEDGE

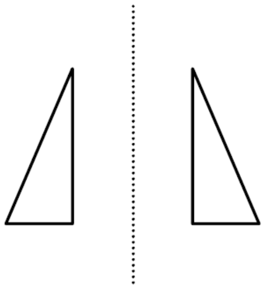
The Euclidean transformations are the most commonly used transformations. An Euclidean transformation is either a translation, a rotation, or a reflection. In an Euclidian transformations, lengths and angles are preserved.

Rotation means turning around a center. The distance from the center to any point on the shape stays the same. Every point makes a circle around the center.



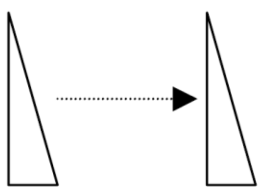
Reflection

A transformation that creates a mirror image of a figure; a flip. Reflection has the same size as the original image. The central line is called the mirror line. Mirror lines can be in any direction.



Translation

A transformation in which every point of a figure moves an equal distance in the same direction; a slide.



Translation simply means moving without rotating, resizing or anything else, just moving. To translate a figure, every point of the shape must move the same distance in the same direction.

When the transformations are done in a coordinate system, it is called the coordinate system transformation.

In this lecture, we will classify the transformations problems in AMC 8 or Mathcounts by two types: the coordinate system transformation, and the geometric objects transformation.

2. THE COORDINATE SYSTEM TRANSFORMATION

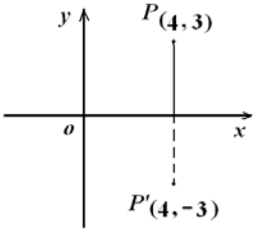
(1). $P(x_0, y_0)$ is a point. The image of P under reflections:

(*a*). In the *x*–axis	$(x_0, -y_0)$
(*b*). In the *y*–axis	$(-x_0, y_0)$
(*c*). In the line $x = a$	$(2a - x_0, y_0)$
(*d*). In the line $y = a$	$(x_0, 2a - y_0)$
(*e*). In the line $y = x$	(y_0, x_0)
(*f*). In the line $y = -x$	$(-y_0, -x_0)$
(*g*). In the line $y = x + m$	$(y_0 - m, x_0 + m)$
(*h*). In the line $y = -x + n$	$(n - y_0, n - x_0)$

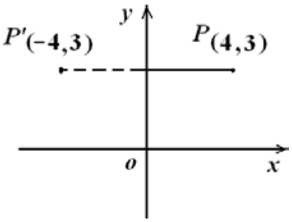
(*i*). In the point *A*(*a*, *b*) $(2a - x_0, 2b - y_0)$

(*j*). In the line *Ax* + *By* +*C* = 0 $(x_1, y_1): \begin{cases} A \cdot \dfrac{x_0 + x_1}{2} + B \cdot \dfrac{y_0 + y_1}{2} + C = 0 \\ A(y_0 - y_1) = B(x_0 - x_1) \end{cases}$

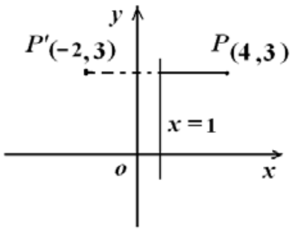
The figures for (*a*) to (*f*) are shown below using a sample point *P*(4,3).



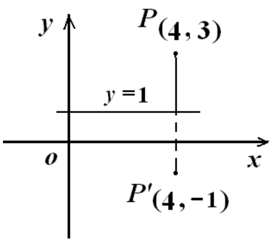
(*a*)



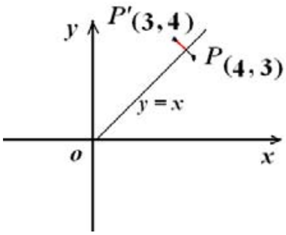
(*b*)



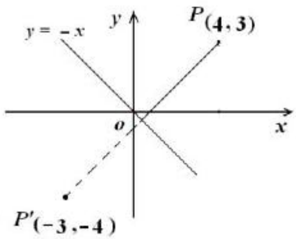
(*c*)



(*d*)



(*e*)



(*f*)

(2). The reflection of the line *Ax* + *By* +*C* = 0 in the point *P*(*a*, *b*):
Ax + *By* – (2*aA* + 2*bB* +*C*) = 0

Example 1. What are the coordinates of the point which is the reflection in the *y*-axis of the point whose coordinates are (5, – 3)?
(A) (– 5, – 3) (B) (5, – 3) (C) (– 3, 5) (D) (5, 3) (E) (3, – 5)

Example 2. If the point whose coordinates are (– 5, 3) is reflected about the line $y = -2$, what are the coordinates of its image?

(A) (– 5, – 7) (B) (5, – 7) (C) (– 7, 5) (D) (5, 7) (E) (7, – 5)

Example 3. The point (5, 3) is reflected about the line $x = 2$. The image point is then reflected about the line $y = 2$. The resulting point is (a, b). Compute $a + b$.

(A) 1 (B) 0 (C) 2 (D) 4 (E) 8

Example 4. What is the y-coordinate of the image when (5, 3) is reflected over the line $y = x$?

(A) 2 (B) 3 (C) 5 (D) 8 (E) 1

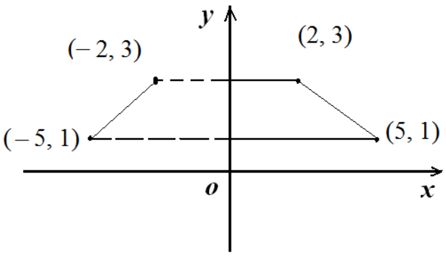
Example 5. When the point (– 3, – 4) is reflected about the line $y = -x$, what is the y-coordinate of its image?

(A) 4 (B) 3 (C) – 4 (D) – 3 (E) – 7

Example 6. The graph of the parabola $y = x^2 - 2$ is reflected with respect to the line $y = -x$. Write the equation of the resulting graph.

Example 7. The points (2, 3) and (5, 1) are reflected over the y-axis. Find the number of square units in the area of the quadrilateral whose vertices are the points and their images.

(A) 4 (B) 6 (C) 8 (D) 14 (E) 28



Example 8. The triangle with vertices at A(– 2, 2), B(– 8, 2), and C(– 8, – 1) is reflected about the line $y = 2x + 1$. Express the coordinates of the reflection of A as an ordered pair.

(A) (– 2, 0) (B) (0, – 2) (C) (– 2, 8) (D) (2, – 2) (E) (2, 0)

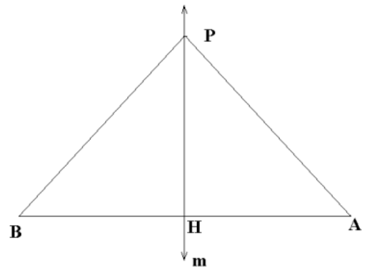
3. THE GEOMETRIC OBJECTS TRANSFORMATION

Figure reflection theorem

If a figure is determined by certain points, then its reflection image is the corresponding figure determined by the reflection images of those points.

The perpendicular bisector theorem

If a point is on the perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment.
Given: P is on the perpendicular bisector m of the segment AB. Prove: $PA = PB$.



Proof:
$\angle PBA = \angle PAB = 90°$, $BH = AH$, $PH = PH \Rightarrow \Delta PHA \cong \Delta PHB \Rightarrow PA = PB$

The reflection image of point A over the line m is the point B if and only if m is the perpendicular bisector of segment AB.

Segment symmetry theorem

A segment has exactly two symmetry lines:
(1) Its perpendicular bisector
(2) The line containing the segment.

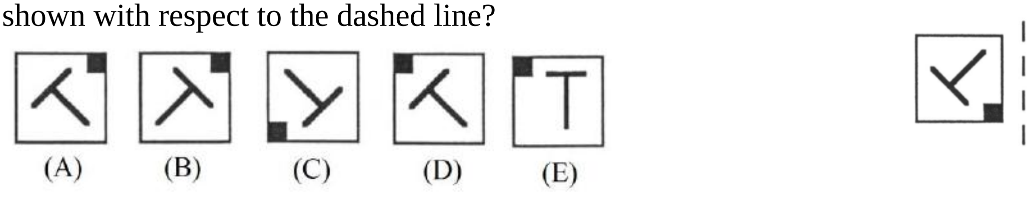
Angle symmetry theorem

The line containing the bisector of an angle is a symmetry line of the angle.

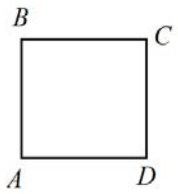
Summary of two important properties:

(1) The symmetric line of a shape separates the shape into two congruent parts.
(2) The symmetric line is the perpendicular bisector of the line segment connecting two symmetric points.

☆**Example 9.** Which of the five "T-like shapes" would be symmetric to the one shown with respect to the dashed line?



Example 10. Square *ABCD* is rotated 90° clockwise about its center, and reflected over a diagonal line determined by lower left and upper right vertices. The square is then reflected over a horizontal line through the center. What point now corresponds to the position originally occupied by *B*?



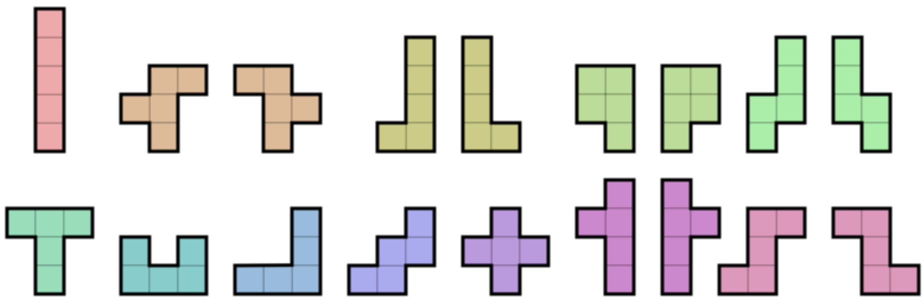
Example 11. A regular pentagon is rotated $d°$ clockwise around its center until it coincides with its original image. What is the smallest positive measure of degrees in d?

(A) 30° (B) 45° (C) 60° (D) 72° (E) 108°

Example 12. How many lines of symmetry does a square have?

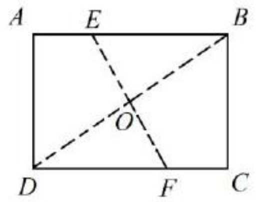
☆**Example 13.** How many of the eighteen pentominoes pictured below have at least one line of symmetry?

(A) 3 (B) 4 (C) 5 (D) 6 (E) 7



Example 14. Rectangle *ABCD* is folded along line *EF* so that point *B* falls on point *D*. If *AD* = 6 and *AB* = 8, find the length of the crease $\overline{EF}$.

(A) $\frac{15}{4}$ (B) $5\frac{5}{8}$ (C) $\frac{15}{2}$ (D) $\frac{5}{2}$ (E) $11\frac{1}{4}$



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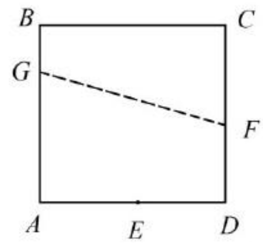
Example 15. In the figure shown, *ABCD* is a square piece of paper 6 cm on each side. Corner *C* is folded over so that it coincides with *E*, the midpoint of $\overline{AD}$. If $\overline{GF}$ represents the crease created by the fold, what is the length of $\overline{FD}$?



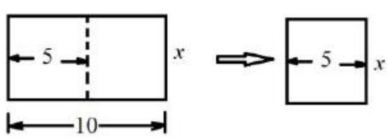
(A) 4/9 (B) 9/4 (C) 3/2 (D) 7/4 (E) $\frac{3\sqrt{5}}{2}$

Example 16. A rectangular sheet of paper measures 12" by 9". One corner is folded onto the diagonally opposite corner and the paper is creased. What is the length in inches of the crease?

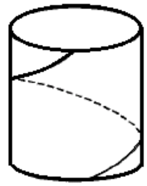
(A) $5\frac{5}{8}$ (B) 15/2 (C) 9/2 (D) 5/4 (E) $11\frac{1}{4}$

Example 17. A rectangular paper is folded along an axis of symmetry as shown. The shape of the resulting figure is similar to the shape of the original figure. Find x.

(A) 10 (B) 5 (C) 2 (D) $3\sqrt{2}$ (E) $5\sqrt{2}$



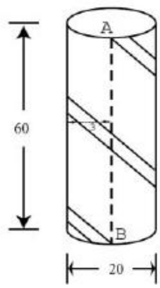
Example 18. A wire is wrapped around a cylinder forming a helix as in the picture. If the wire only goes around the cylinder once, and the height and diameter of the cylinder are both 10 cm, find the length of the wire in simplest radical form.



(A) $10\sqrt{1+\pi^2}$ (B) 10 (C) $20\sqrt{1+\pi^2}$

(D) $\pi\sqrt{10}$ (E) 10π

☆**Example 19.** A white cylindrical silo has a diameter of 20 feet and a height of 60 feet. A red stripe with a horizontal width of 3 feet is painted on the silo, as shown, making two complete revolutions around it. What is the area of the stripe in square feet?



(A) 120 (B) 180 (C) 240 (D) 160 (E) 480

Example 20. A 5 inch by 8 inch rectangular piece of paper can be rolled up to form either of two right circular cylinders, a cylinder with a height of 8 inches or a cylinder with a height of 5 inches. What is the ratio of the volume of the 8 inch tall cylinder to the volume of the 5 inch tall cylinder?

(A) 5/8 (B) 9/4 (C) 3/2 (D) 7/4 (E) $\frac{\sqrt{5}}{2}$

☆**Example 21.** Rectangle *PQRS* lies in a plane with *PQ* = *RS* = 3 and *QR* = *SP* = 4. The rectangle is rotated 90° clockwise about *R*, then rotated 90° clockwise about the point that *S* moved to after the first rotation. What is the length of the path traveled by point *P* ?

(A) $(\sqrt{2}+\sqrt{5})\pi$ (B) 6π (C) $\frac{9}{2}\pi$ (D) $(\sqrt{3}+2)\pi$ (E) $2\sqrt{10}\pi$