1. TERMS

Equation: An equation is a mathematical sentence in which the equal sign "=" connects two algebraic expressions.

The following are equations:

$2^{10} = 1024;$ $\frac{1}{7} = 0.\overline{142857};$ $3\pi = 5x - 6;$ $A = \pi r^2$

Open sentence: An equation that contains one or more variables is called an open sentence. For example: $2x + 2 = 8$. The sentence is neither true nor false.

Solution: A solution is a value of the variable that makes the equation true (Other names: root, and zero).

For the equation $x + 5 = 9$, $x = 4$ is the solution since $4 + 5 = 9$.

When the variable in the equation is replaced with a constant so that the equation becomes true, the equation has been solved.
We will learn the following skills:

(1) get rid of the denominators
(2) add parentheses
(3) remove the parentheses
(4) isolate the variable
(5) combine like terms.

We will learn how to solve the following equations:

(1) one-variable linear equations
(2) literal equations
(3) quadratic equations
(4) system of linear equations
(5) system of nonlinear equations

2. SOLVING ONE-VARIABLE LINEAR EQUATIONS

One-variable: one unknown in the equation, such as in $3x = 9$. x is the only unknown.

Linear: the variable has the power of 1, such as in $3x = 9$. x has the power of 1 ($x = x^1$).

The simplest form of this kind of equations: $ax = b$ ($a \neq 0$), where both a and b are constant. x is the variable (unknown).

The solution is $x = \frac{b}{a}$ (divide both sides of the equation by a).

Example 3. Solve for x: $\frac{1}{3}(x-1)+\frac{3}{4}(x+1)=\frac{1}{2}(x-1)+\frac{2}{3}(x+1)$

(A) 4 (B) 5 (C) 3 (D) 6 (E) 10

☆**Example 4.** In a far-off land five fish can be traded for three loaves of bread and a loaf of bread can be traded for three bags of rice. How many bags of rice is one fish worth?

(A) 1/5 (B) 1/2 (C) 3/4 (D) $1\frac{4}{5}$ (E) $\frac{5}{9}$

☆**Example 5.** Before district play, the Unicorns had won 45% of their basketball games. During district play, they won six more games and lost two, to finish the season having won half their games. How many games did the Unicorns play in all?

(A) 48 (B) 50 (C) 52 (D) 54 (E) 60

3. SOLVING LITERAL EQUATIONS

A literal equation is an equation that contains one or more letters. These letters are constants but are not fixed values.
General case: x is the variable in the equation.

Case I. When $a \neq 0$, the equation has a unique solution: $x = \frac{b}{a}$.

Case II. When $a = b = 0$, the equation has infinite many solutions.
Case III. When $a = 0$ and $b \neq 0$, the equation has no solutions.

Examples 6. In the metric system, temperature is measured in degree Celsius (°C) in stead of degree Fahrenheit (°F). The formula is as follows: $C = \frac{5(F-32)}{9}$.

Solve for F.

Example 7. Solve for x: $a^2 + ax = 1 - x$

Example 8. If $x = \frac{a}{b+c} = \frac{b}{c+a} = \frac{c}{a+b}$, find the value for x.

4. SOLVING QUADRATIC EQUATIONS

The following equation is called quadratic equation:
$ax^2 + bx + c = 0$
where a, b, and c are real numbers with $a \neq 0$.
Square root property: The solution to $x^2 = k$ is $x = \sqrt{k}$ and $x = -\sqrt{k}$.

Example 9. Solve the equation $x^2 = 11$.

Example 10. Solve the equation $(x-4)^2 = 12$.

QUADRATIC FORMULA:

We will derive the quadratic formula below. The method used is called "completing the square" method. The method works for all quadratic equations.

$ax^2 + bx + c = 0$

Since $a \neq 0$, we can divide both sides by a:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Now we complete the square:

$$x^2 + 2 \times x \times \frac{b}{2a} + (\frac{b}{2a})^2 - (\frac{b}{2a})^2 + \frac{c}{a} = 0$$

$$x^2 + 2 \quad x \quad y \quad + \quad y^2 = (x+y)^2.$$

$$x^2 + 2 \times x \times \frac{b}{2a} + (\frac{b}{2a})^2 = (\frac{b}{2a})^2 - \frac{c}{a}$$

We can write the left side as a perfect square, and the right side as a single fraction:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Take the square root of each side: $x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$

Solve for x: $x_{1,2} = \frac{-b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Simplify: $x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Example 11. Solve $x^2 + 6x = 40$

Example 12. Solve $5x^2 = 10x - 4$ using the quadratic formula.

5. SOLVING SYSTEM OF LINEAR EQUATIONS

A group (two or more) of equations is called a system of equations. The solutions of a system of equations should satisfy all the equations. The most commonly used methods are (1) substitution method, and (2) elimination method.

Example 13. If $x + y = 12$ and $x - y = 8$, what is the value of $2x - xy$?
(A) 4 (B) 0 (C) 2 (D) 5 (E) 6

Example 14. If $3a + b = 17$ and $a + 1 = b$ what is the value of $a \cdot b$?
(A) 40 (B) 50 (C) 20 (D) 18 (E) 16

Example 15. If $x + y = 10$ and $2x - y = 11$, find the value of $x^2 + y^2$.
(A) 49 (B) 9 (C) 20 (D) 58 (E) 16

Example 16. Solve the system of equations and find the value of $z - y$.

$$\left.\begin{array}{l} \dfrac{x}{2} = \dfrac{y}{3} \\ \dfrac{y}{2} = \dfrac{z}{3} \\ x + y + z = 38 \end{array}\right\}$$

(A) 10 (B) 20 (C) 24 (D) 18 (E) 19

Example 17. If $\dfrac{5}{x+1} = \dfrac{5}{2x-1}$, what is the value of x?
(A) -2 (B) -1 (C) 0 (D) 1 (E) 2

Example 18. If $a > 0$ in the equations $x = 4a$ and $y = 4a^2 + 1$, find y in terms of x.
(A) $\frac{1}{4}x^2 + 1$ (B) $\frac{1}{4}x^2 + 4$ (C) $\frac{1}{2}x^2 + 1$ (D) $x^2 + 1$ (E) $x^2 + 4$

Example 19. In the equation $6(x - 7)(x - 2) = k$, k is a constant. If the roots of the equation are 7 and 2, what is the value of k ?
(A) 0 (B) 2 (C) 3 (D) 7 (E) 14

Example 20. Find the value of $y + z$ if $5x + 2y + 2z = 21$ and $5x + y + z = 11$.

(A) 10 (B) 12 (C) 3 (D) 7 (E) 11

Example 21. Solve:

$$\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=1 & (1) \\ \dfrac{1}{x}+\dfrac{1}{z}=2 & (2) \\ \dfrac{1}{y}+\dfrac{1}{z}=\dfrac{3}{2} & (3) \end{cases}$$

☆**Example 22.** In a jar of red, green, and blue marbles, all but 12 are red marbles, all but 16 are green, and all but 8 are blue. How many marbles are in the jar?

(A) 12 (B) 16 (C) 18 (D) 20 (E) 36

Example 23. Find the sum of the x-coordinates of the points of intersection of the graphs of the equations $y = |2x| - 2$ and $y = -|2x| + 2$.

(A) 1 (B) 6 (C) 8 (D) 0 (E) 3

Example 24. In the following system of equations, x can be expressed as m/n, where m and n are positive integers relatively prime. What is the value of $m + n$?

$$\begin{cases} \dfrac{xy}{x+y}=3 & (1) \\ \dfrac{yz}{y+z}=4 & (2) \\ \dfrac{zx}{x+z}=5 & (3) \end{cases}$$

(A) 130 (B) 143 (C) 127 (D) 120 (E) 167

6. SOLVING SYSTEM OF NONLINEAR EQUATIONS

A system of equations is called nonlinear system of equations if at least one equation is nonlinear.

Example 25. Solve the system:

$$\left.\begin{array}{l} x+y=5 \\ xy=4 \end{array}\right\}$$

Example 26. Solve the system:

$$\left.\begin{array}{l} 2x-3y=11 \\ xy=-5 \end{array}\right\}$$

Example 27. Solve the system:

$$\begin{cases} x^2+y^2=25 & (1) \\ x+y=7 & (2) \end{cases}$$

☆**Example 28.** Find x^2+y^2 if (x, y) is a solution to the system $xy=6$ and $x^2y+y^2x+x+y=63$.

(A) 81 (B) 140 (C) 69 (D) 63 (E) 67

☆**Example 29.** How many distinct points common to the curve $3x^2+y^2=13$ and $x^2+3y^2=115$?

(A) 1 (B) 2 (C) 3 (D) 0 (E) 4

☆**Example 30.** Find x^2+y^2 if x and y are positive integers such that

$xy+x+y=19$ and $x^2y+xy^2=84$.

(A) 193 (B) 25 (C) 153 (D) 103 (E) 74

PROBLEMS

Problem 1. Find n, the root of the equation: $n+(n+1)+(n+2)=-75$.

(A) -3 (B) -25 (C) -75 (D) -26 (E) 26

Problem 2. Find the value of y which makes the following true: $\frac{3(6+8y)}{10}=9$.

(A) 3 (B) 5 (C) 7 (D) 6 (E) 2

Problem 3. For what value(s) of x is the equation $11x-4(2x-3)=24$ true?

(A) 3 (B) 4 (C) 5 (D) 6 (E) 11

Problem 4. What is the value of x? $\frac{x-1}{3}-1=\frac{2x-1}{5}+1$.

(A) -32 (B) -23 (C) -27 (D) -45 (E) 32

Problem 5. Solve for d: $\frac{3d-1}{4d-4}=\frac{2}{3}$.

(A) -3 (B) -2 (C) -7 (D) -5 (E) 3

Problem 6. Solve the equation $a=-3(x-5b)$ for x.

Problem 7. Solve for x: $\frac{1}{3}m(x-n)=\frac{1}{4}(x+2m)$.

Problem 8. Find the greatest positive integer n such that $n^2-26n+30$ is at most 30.

(A) 19 (B) 25 (C) 26 (D) 13 (E) 27

Problem 9. One root of the equation $5x^2 + kx = 4$ is 2. What is the other?

(A) 10 (B) $-\frac{4}{5}$ (C) $-\frac{2}{5}$ (D) $\frac{2}{5}$ (E) $-\frac{2}{10}$

Problem 10. Given that a and b are positive numbers such that $a^2 + b^2 = 52$ and $a^2 - b^2 = 20$, what is the value of b?

(A) -4 (B) 6 (C) -6 (D) 16 (E) 4

Problem 11. How many different points of intersection are there for $x + y = 7$ and $y = x^2 - 7$?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Problem 12. Four distinct integers a, b, c, and d have the property that when added in pairs the sums 16, 19, 20, 21, 22 and 25 are obtained. What is the sum of four integers?

(A) 36 (B) 39 (C) 41 (D) 43 (E) 47

Problem 13. Given that $7x + 3y = 54$ and $3x + 7y = 46$, what is the value of $x + y$?

(A) 14 (B) 16 (C) 27 (D) 23 (E) 10

Problem 14. The sum of two numbers is 22. Their product is 72. What is the greater of the two numbers?

(A) 20 (B) 18 (C) 16 (D) 14 (E) 12

Problem 15. The product of two whole numbers is 60. If the difference between the two numbers is 11, what is the greater of the two numbers?

(A) 4 (B) 18 (C) 15 (D) -4 (E) -15

Problem 16. The difference between two positive integers is 45, and their product is 196. What is their sum?

(A) 49 (B) 53 (C) 784 (D) −53 (E) −49

Problem 17. The difference between two numbers is 1, and the sum of their squares is 141. What is the product of the numbers?

(A) 4970 (B) 4899 (C) 4830 (D) 994 (E) 2485

Problem 18. The sum of two numbers is 24. Their difference is 16. What is the larger number?

(A) 16 (B) 18 (C) 20 (D) 24 (E) 22

Problem 19. If $4x + y = 24$ and $x - y = 1$, what is the value of $x + y$?

(A) 6 (B) 9 (C) 12 (D) 14 (E) 22

Problem 20. Four dogs and 3 puppies weigh 74 pounds while 3 dogs and 4 puppies weigh 66 pounds. How many pounds does a dog plus a puppy weigh?

(A) 33 (B) 37 (C) 20 (D) 24 (E) 22

Problem 21. If $x + y = 5$, $x + z = 8$, and $y + z = 11$, what is the value of $x + y + z$?

(A) 8 (B) 10 (C) 12 (D) 14 (E) 20

Problem 22. If $x + y = 5$ and $xy = 3$, what is the value of $x^2 + y^2$?

(A) 25 (B) 19 (C) 12 (D) 14 (E) 6

Problem 23. Find xy such that $x + y = 10$ and $x^2 + y^2 = 178$.

(A) 39 (B) 33 (C) −78 (D) −39 (E) −49

Problem 24. Given that $\frac{1}{a} + \frac{1}{b} = \frac{7}{24}$ and $a + b = 14$, what is the product of a and b?

(A) 24 (B) 31 (C) 48 (D) 39 (E) 49

Problem 25. Given 3 positive integers $a, b,$ and c such that $a \times b = 10, b \times c = 6,$ $c \times a = 15$. Find $a \times b \times c$.

(A) 25 (B) 16 (C) 21 (D) 30 (E) 49

Problem 26. Three pencils and two erasers cost \$0.60. Two pencils and three erasers cost \$0.55. How much will it cost to buy seven pencils and seven erasers?

(A) \$1.15 (B) \$1.61 (C) \$1.48 (D) \$1.39 (E) \$1.49

Problem 27. Suppose $a, b,$ and c are positive integers such that $ab = 18, bc = 24$ and $ac = 48$. Find $a+b+c$ to the nearest integer.

(A) 12 (B) 24 (C) 21 (D) 17 (E) 18

Problem 28. Three friends arrange to rent a summer cabin. Harry pays twice as much as Mary, and Mary pays twice as much as Larry. If the total rent is \$350, how many dollars does Harry pay?

(A) 350 (B) 300 (C) 250 (D) 200 (E) 175

Problem 29. Several boys bought a canoe, each paying an equal amount. If there had been two fewer boys, each would have paid \$3.00 more. If there had been one boy more, each would have paid \$1.00 less. How many boys were there?

(A) 12 (B) 14 (C) 10 (D) 8 (E) 11

☆**Problem 30.** If the line $y = mx + 1$ intersects the ellipse $x^2 + 4y^2 = 3$ exactly once, what is the value of m^2?

(A) 1/12 (B) 3/4 (C) 2/5 (D) 1/6 (E) 1/7

Problem 31. Find the real solutions of the equation:

$$\begin{cases} x+y=2 & (1) \\ xy - z^2 = 1 & (2) \end{cases}$$

(A) (1, 1, 0) (B) (1, 0, 1) (C) (2, 1, −1) (D) (0,1, 1) (E) (−1, −1, 0)