(7). Drawing Solid and Dash Lines

Example 7. Three friends – math teacher Mr. White, science teacher Mr. Black, and history teacher Mr. Redhead – met in a cafeteria. “It is interesting that one of us has white hair, another one has black hair, and the third has red hair, though no one’s name gives the color of their hair” said the black-haired person. “You are right,” answered White. What color is the history teacher’s hair?

(8). Back one step and forward two

Example 8. There's a box of three hats: one black and two white. Andy and Betsy (each very smart and very logical) each place a hat on his or her head, while blindfolded. One by one, each child removes his blindfold and (without using a mirror) gets one opportunity to guess the color of the hat on his own head. If any of the two guesses correctly, everyone gets to go to the park!

First, Betsy removes her blindfold. She sees the hats that Andy is wearing, but admits that she is unable to discern her own hat color.

Then Andy says: “I can answer with my blindfold on! I know what color hat I am wearing." What color is Andy's hat?

Example 9. There's a box of five hats: two black and three white. Andy, Betsy, and Charles (each very smart and very logical) each place a hat on his or her head, while blindfolded. One by one, each child removes his blindfold and (without using a mirror) gets one opportunity to guess the color of the hat on his own head. If any of the three guesses correctly, everyone gets to go to the park!

First, Charles removes his blindfold. He sees the hats that the others are wearing, but admits that he is unable to discern his own hat color.

Next, Betsy removes her blindfold, and sadly reveals that she too is not able to determine the color of her own hat.

Finally, Andy pipes up and says “I can answer with my blindfold on! I know what color hat I am wearing." What color is Andy's hat?

(9). Squeezing method

Example 10. (2014 Mathcounts National) Larry tells Mary and Jerry that he is thinking of two consecutive integers from 1 to 10. He tells Mary one of the numbers, and he tells Jerry the other number. Then the following conversation occurs between Mary and Jerry:
Mary: I don’t know your number.
Jerry: I don’t know your number, either.

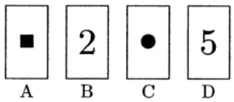
Mary: Ah, now I know your number.
Assuming both Mary and Jerry used correct logic, what is the sum of the possible numbers Mary could have?

MORE EXAMPLES

Example 11. Squares are faster than circles, hexagons are slower than triangles, and hexagons are faster than squares. Which of these shapes is the slowest?

(A) Squares (B) Circles (C) Hexagons (D) Triangles (E) None of them

Example 12. Four cards are constructed so that there is either a circle or a square on one side and an odd or even number on the other side. The cards are placed on a table as shown. Which cards must be turned to prove the following: Every square has an even number on the other side?



Example 13. Classroom window was broken. The principal had four students in his office. He knew that one of them did it, and he also knew that only one of the students told the truth, but not sure which one.
Alex said: Bob did;

Bob said: Dean did;
Cam said: not me;
Dean said: Bob lied.

Who broke the window?

Example 14. A sealed envelope contains a card with a single digit on it. Three of the following statements are true, and the other is false.
I. The digit is 1.
II. The digit is not 2.
III. The digit is 3.
IV. The digit is not 4.
Which one of the following must necessarily be correct?

(A) I is true. (B) I is false. (C) II is true. (D) III is true. (E) IV is false

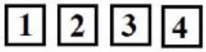
Example 15. A centipede climbs a 40-foot tree. Each day he climbs 5 feet, and each night he slides down 3 feet. In how many days will the centipede reach the top of the tree?

(A) 19 (B) 18 (C) 17 (D) 20 (E) 21

Example 16. Alex has 6 coins. Five of the 6 coins weigh the same and one coin is heavier. If Alex had a balance scale, what is the least number of times he could weigh coins to be sure he could determine which coin was heavier?

Example 17. In a horse race game on a computer, Secretariat, Man-Of-War, Affirmed and Citation finished in first through fourth places (not necessarily in that order), with no ties. Man-Of-War finished second or fourth. Affirmed did not win the race. Citation or Secretariat finished third. Man-Of-War beat Secretariat. What is the name of the horse that finished fourth?

Example 18. Below are the four labeled boxes. Each box is painted a different color. There is a red box, which is next to a blue box. There is a green box, which is next to the red box and a yellow box. Which box could be painted red?



(A) 1 only (B) 2 only (C) 3 only (D) 2 or 3 (E) 1 or 4

☆**Example 19.** (AMC 8) Amy, Bill and Celine are friends with different ages. Exactly one of the following statements is true.
I. Bill is the oldest.
II. Amy is not the oldest.
III. Celine is not the youngest.
Rank the friends from the oldest to the youngest.
(A) Bill, Amy, Celine (B) Amy, Bill, Celine (C) Celine, Amy, Bill
(D) Celine, Bill, Amy (E) Amy, Celine, Bill

☆**Example 20.** (AMC 8) Five friends compete in a dart-throwing contest. Each one has two darts to throw at the same circular target, and each individual's score is the sum of the scores in the target regions that are hit. The scores for the target regions are the whole numbers 1 through 10. Each throw hits the target in a region with a different value. The scores are: Alice 16 points, Ben 4 points, Cindy 7 points, Dave 11 points, and Ellen 17 points. Who hits the region worth 6 points?

(A) Alice (B) Ben (C) Cindy (D) Dave (E) Ellen

PROBLEMS

Problem 1. There are 9 apparently identical balls, except that one is heavier than the other 8. What is the smallest number of balance scale weighings required to ensure identification of the “odd” ball?
(A) 9 (B) 3 (C) 4 (D) 1 (E) 2

Problem 2. A kitchen pantry has five shelves, each containing a specific kind of food. The spices are on the shelf directly below the vegetables, the fruits are above the bread, and the vegetables are 3 shelves below the cereals. Which kind of food is on the third shelf?
(A) vegetables (B) fruits (C) bread (D) cereals (E) spices

Problem 3. At Hope Middle School, Mr. Eye, Mr. Love and Mr. Problems teach science, mathematics, and history—but not necessarily in that order. The history teacher, who was an only child, has the least experience. Mr. Problems, who married Mr. Eye’s sister, has more experience than the science teacher. Who teaches science?

Problem 4. Five coins look the same, but one is a counterfeit coin with a different weight than each of the four genuine coins. Using a balance scale, what is the least number of weighings needed to ensure that, in every case, the counterfeit coin is found and is shown to be heavier or lighter?
(A) 5 (B) 4 (C) 3 (D) 2 (E) 1

Problem 5. A centipede climbs a 40-foot tree. Each day he climbs 5 feet, and each night he slides down 2 feet. In how many days will the centipede reach the top of the tree?
(A) 14 (B) 13 (C) 12 (D) 8 (E) 20

Problem 6. Adam, Ben, Charles, David and Ed were waiting in line. Adam is between Ben and Chase. Ben is between David and Adam. Ed is also between David and Adam. Ben is between David and Ed. Who is in the middle of the line?

(A) Adam (B) Ben (C) Charles (D) David (E) Ed

Problem 7. Five cards are lying on a table as shown. Each card has a letter on one side and a whole number on the other side. Jane said, “If a vowel is on one side of any card, then an even number is on the other side.” Mary showed Jane was wrong by turning over one card. Which card did Mary turn over?

(A) 5 (B) 4 (C) 3 (D) 2 (E) 1

Problem 8. A centipede crawl a tree 75-inches high, starting from the ground. Each day it crawls 5 inches, and each night it slides down 4 inches. When will it first reach the top of the tree?
(A) 15 (B) 18 (C) 19 (D) 72 (E) 71.

Problem 9. There are 4 cards on the table with the symbols *a*, *b*, 4, and 5 written on their visible sides. What is the smallest number of cards we need to turn over to find out whether the following statement is true: “If an even number is written on one side of a card then a vowel is written on the other side?

Problem 10. Each of the cards shown below has a number on one side and a letter on the other. How many of the cards must be turned over to prove the correctness of the statement: Every card with a vowel on one side has a prime number on the other side.

(A) 7 (B) 6 (C) 5 (D) 4 (E) 3

Problem 11. Three kids are playing pitcher, catcher and infielder. Sam is not the catcher. The infielder lives next to Sam. The catcher and John go to the same school. What position does Alex play?

Problem 12. Cookies were missing, taken by either Alex, Bob, or Charles.
Each person said:

Alex: I did not take the cookies.
Bob: Charles took the cookies.
Charles: That is true

If at least one of them lied and at least one told the truth, who took the cookies?

PROBLEMS

Problem 1. There are 9 apparently identical balls, except that one is heavier than the other 8. What is the smallest number of balance scale weighings required to ensure identification of the "odd" ball?

(A) 9 (B) 3 (C) 4 (D) 1 (E) 2

Problem 2. A kitchen pantry has five shelves, each containing a specific kind of food. The spices are on the shelf directly below the vegetables, the fruits are above the bread, and the vegetables are 3 shelves below the cereals. Which kind of food is on the third shelf?

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Problem 10. Each of the cards shown below has a number on one side and a letter on the other. How many of the cards must be turned over to prove the correctness of the statement: Every card with a vowel on one side has a prime number on the other side.

(A) 7 (B) 6 (C) 5 (D) 4 (E) 3

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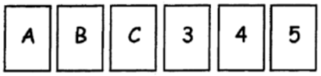
Problem 12. Cookies were missing, taken by either Alex, Bob, or Charles.
Each person said:

Alex: I did not take the cookies.
Bob: Charles took the cookies.
Charles: That is true

If at least one of them lied and at least one told the truth, who took the cookies?

Problem 13. Each of the cards shown has a number on one side and a letter on the other. How many of the cards must be turned over to prove the correctness of this statement for these cards: “If a card has a vowel on one side, then it has a prime number on the other side?”



(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Problem 14. If all alligators are ferocious creatures and some creepy crawlers are alligators, which statement(s) must be true?

I. All alligators are creepy crawlers.
II. Some ferocious creatures are creepy crawlers.
III. Some alligators are not creepy crawlers.

(A) I only (B) II only (C) III only (D) II and III only
(E) None must be true

Problem 15. A number of bacteria are placed in a container. One second later each bacterium divides into two, the next second each of the resulting bacteria divided in two again, et al. After one minute the container is full. When was the container half full?
(A) 58 (B) 59 (C) 60 (D) 120 (E) 119

Problem 16. If the two statements below are true, which of the following statements must also be true?
(1) Alex sometimes goes to adventure movies.

(2) Betsy never goes to comedy movies.

I. Alex never goes to comedy movies.
II. Betsy sometimes goes to adventure movies.
III. Alex and Betsy never go to comedy movies together.

(A) I only (B) II only (C) III only (D) I and III (E) II and III

Problem 17. The four children in the Jones family are Alex, Bob, Cathy, and Debra. Bob is neither the youngest nor the oldest. Debra is one of the two younger children. Cathy is the oldest child. Alex is often taking care of his younger brother and sister. Who is the youngest child?

(A) Bob (B) Debra (C) Alex (D) Cathy
(E) It cannot be determined from the information

Problem 18. Sam is not a member of the math club, then from which of the following statements can it be determined whether or not Sam is in the science club?

(A) Anyone in the math club is not in the science club.
(B) No one is in both the math club and the science club.
(C) Anyone who is not in the math club is not in the science club.
(D) Everyone in the math club is in the science club.
(E) Some people who are not in the math club are not in the science club.

Problem 19. If the statement "If a number is in list A, it is not in list B" is true, which of the following statements must also be true?
(A) If a number is not in list A, it is in list B.
(B) If a number is not in list B, it is in list A.
(C) If a number is in list B, it is not in list A.
(D) If a number is in list B, it is in list A.
(E) If a number is in list A, it is also in list B.

Problem 20. The Hope Middle School has three clubs: math, reading, and writing. Five students from a family each participated in one club only. The statements below are about what these five students participated. If *n* is the number of students who participated in the reading club, which of the following statements is true?

The first student participated in the math club.
The second student did not participate in the math club.
The third student participated in the reading club.
The fourth student participated in the same club as the first student.
The fifth student participated in the same club as the second student.

(A) *n* must be 1. (B) *n* must be 2. (C) *n* must be 3. (D) *n* must be 1 or 2.
(E) *n* must be 1 or 3.

Problem 21. If the statement "Some integers in set *X* are odd" is true, which of the following must also be true?

(A) If an integer is odd, it is in set *X*. (B) If an integer is even, it is in set *X*.
(C) All integers in set *X* are odd. (D) All integers in set *X* are even.
(E) Not all integers in set *X* are even.

Problem 22. If all boys in the math club are good at math. Which of the following statements must be true?

(A) No boy whose math is not good is a member of the math club.
(B) All boys whose math is good are members of the math club.
(C) All boys who are not members of the math are not good at math.
(D) Every member of the math club whose math is good is a boy.
(E) There is one boy in the math club whose math is not good.

Problem 23. At Hope High School, some members of the math club are on the science team and no members of the science team are 9th graders. Which of the following must also be true?

(A) No members of the math club are 9th graders.
(B) Some members of the math club are 9th graders.

(C) Some member of the math club are not 9th graders.
(D) More 9th graders are on the science team than are in the math club.
(E) More 9th graders are in the math club than are on the science team.

Problem 24. The teacher whispers positive integer A to Anna, B to Brett, and C to Chris. The students don't know one another's numbers but they do know that the sum of their numbers is 14. Anna says, "I know that Brett and Chris have different numbers". Then Brett says, "I already knew that all three of our numbers were different". Finally, Chris announces, "Now I know all three of our numbers". What is the product ABC? (Mathcounts)

1.TERMS:

Fraction: A part of a whole or a quotient of two numbers, expressed as $\frac{a}{b}$. a and b are whole numbers. $b \neq 0$. $\frac{a}{b}$ is the same as $a \div b$ or a/b.

Proper fraction: A fraction in which the numerator is less than the denominator: $\frac{3}{5}$. Such a fraction has a value less than 1.

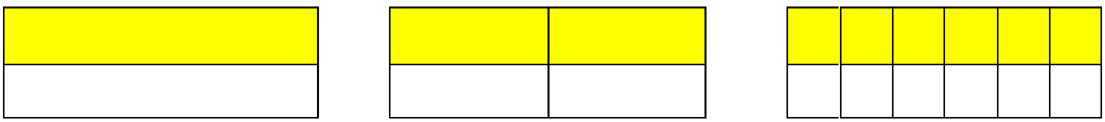
Improper fraction: A fraction in which the numerator is greater than or equal to the denominator: $\frac{5}{3}$. It has a value greater than or equal to 1.

Mixed number: A mixed number contains both a whole number part and a fraction part and can be written as an improper fraction: $2\frac{1}{3} = \frac{2\times 3+1}{3} = \frac{7}{3}$.

2. PROPERTIES:

2.1. Equivalent Fraction (Cancellation Law): Two fractions are equal if they represent the same portion of a whole.

$$\frac{1}{2} = \frac{2}{4} = \frac{6}{12}$$



Examples: $\frac{38}{57} = \frac{2\times \cancel{19}}{3\times \cancel{19}} = \frac{2}{3}$; $\frac{38}{57} = \frac{38\div 19}{57\div 19} = \frac{2}{3}$

2.2. Fundamental Law of Fractions:

For any fraction $\frac{a}{b}$ and any number $c \neq 0$, $\frac{a}{b} = \frac{a\times c}{b\times c}$.

(The value of a fraction does not change if its numerator and denominator are multiplied by the same nonzero number).

Example. $\frac{2}{3}=\frac{2\times 5}{3\times 5}=\frac{10}{15}$.

2.3. Lowest (Reduced; Simplest) Term: A fraction in which the numerator and the denominator have no common terms except 1. The lowest terms are obtained by taking all the common factors out of the numerator and the denominator.

$\frac{10}{15}$ is not a fraction in the lowest term but $\frac{2}{3}$ is. ($\frac{10}{15}=\frac{2\times 5}{3\times 5}=\frac{2}{3}$).

2.4. Additon and Subtraction:

When working with fractions, only the numerators in fractions are added or subtracted.

(1). Two fractions having the same denominators:

We just add or subtract the numerators.

$$\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b} \qquad \Rightarrow \qquad \frac{3}{5}+\frac{1}{5}=\frac{3+1}{5}=\frac{4}{5}$$

$$\frac{a}{b}-\frac{c}{b}=\frac{a-c}{b} \qquad \Rightarrow \qquad \frac{3}{5}-\frac{1}{5}=\frac{3-1}{5}=\frac{2}{5}$$

(2). **Two fractions having the different denominators**:

We convert them to the same denominators first, and then add the numerators.

$$\frac{a}{b}+\frac{c}{d}=\frac{a\times d}{b\times d}+\frac{c\times b}{b\times d}=\frac{a\times d+c\times b}{b\times d} \Rightarrow$$

$$\frac{1}{2}+\frac{2}{5}=\frac{1\times 5}{2\times 5}+\frac{2\times 2}{5\times 2}=\frac{5+4}{10}=\frac{9}{10}$$

$$\frac{a}{b}-\frac{c}{d}=\frac{a\times d}{b\times d}-\frac{c\times b}{b\times d}=\frac{a\times d-c\times b}{b\times d} \Rightarrow$$

$$\frac{1}{2}-\frac{2}{5}=\frac{1\times 5}{2\times 5}-\frac{2\times 2}{5\times 2}=\frac{5-4}{10}=\frac{1}{10}$$

2.5. Multiplication of Fractions

The numerator of the product is obtained by multiplying together the numerators. The denominator of the product is obtained by multiplying together the denominators.

$$\frac{a}{b}\times\frac{c}{d}=\frac{a\times c}{b\times d} \qquad \Rightarrow \qquad \frac{2}{5}\times\frac{3}{7}=\frac{2\times 3}{5\times 7}=\frac{6}{35}$$

2.6. Division of Fractions

To divide by a fraction, we simply multiply by its reciprocal.

$$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc} \qquad \Rightarrow \qquad \frac{2}{5}\div\frac{3}{7}=\frac{2}{5}\times\frac{7}{3}=\frac{2\times 7}{5\times 3}=\frac{14}{15}$$

The reciprocal of a number is obtained by switching the numerator and the denominator. For example, the reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$, and the reciprocal of 2 (note that 2 can be written as $\frac{2}{1}$) is $\frac{1}{2}$.

$$\frac{\;\frac{a}{b}\;}{\;\frac{c}{d}\;}=\frac{ad}{bc}$$

3. PROBLEM SOLVING SKILLS

3.1. Comparing Fractions

(1). Same Denominator:

The fraction with a larger numerator is larger: $\frac{3}{5}>\frac{1}{5}$

(2). Same Numerator:

The fraction with a larger denominator is smaller: $\frac{3}{7} < \frac{3}{5}$

(3). Both the numerator and denominator are not the same

$3\times 11=33 \qquad 4\times 8=32$

$$\frac{\ }{8} \; ? \; \frac{\ }{11} \quad \Rightarrow \quad 33 > 32 \quad \Rightarrow \quad \frac{3}{8} > \frac{4}{11}$$

Example 1. Mary made two pies that were exactly the same size. The first pie was a cherry pie, which she cut into 6 equal slices. The second was a pumpkin pie, which she cut into 12 equal pieces. Mary takes her pies to a party. People eat 3 slices of cherry pie and 6 slices of pumpkin pie. Did people eat more cherry pie or pumpkin pie?

Example 2. Peter has two cakes that are the same size. The first cake was chocolate, which he cut 12 equal parts. The second cake was marble, which he cut into 6 equal parts. His family eats 5 slices of chocolate cake and 3 slices of marble cake. Did they eat more chocolate cake or marble cake?

☆**Example 3.** (AMC 8) What is the correct ordering of the three numbers 5/19, 7/21, and 9/23, in increasing order?

(A) $\frac{9}{23} < \frac{7}{21} < \frac{5}{19}$ (B) $\frac{5}{19} < \frac{7}{21} < \frac{9}{23}$ (C) $\frac{9}{23} < \frac{5}{19} < \frac{7}{21}$

(D) $\frac{5}{19} < \frac{9}{23} < \frac{7}{21}$ (E) $\frac{7}{21} < \frac{5}{19} < \frac{9}{23}$

3.2. Sum of A Series of Fractions

Useful formulas:

$$\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1} \qquad \Rightarrow \qquad \frac{1}{3(3+1)}=\frac{1}{3}-\frac{1}{3+1}=\frac{1}{3}-\frac{1}{4}$$

$$\frac{1}{n}=\frac{1}{2n}+\frac{1}{2n} \qquad \Rightarrow \qquad \frac{1}{3}=\frac{1}{2\times 3}+\frac{1}{2\times 3}=\frac{1}{6}+\frac{1}{6}$$

$$\frac{1}{n(n+k)}=\frac{1}{k}(\frac{1}{n}-\frac{1}{n+k}) \qquad \Rightarrow \qquad \frac{1}{3(3+2)}=\frac{1}{2}(\frac{1}{3}-\frac{1}{5})$$

Example 4. Find the sum: $\frac{1}{1\times 3}+\frac{1}{3\times 5}+\cdots+\frac{1}{11\times 13}$.

Example 5. Calculate: $\frac{1}{2\times 4}+\frac{1}{4\times 6}+\cdots+\frac{1}{98\times 100}$.

3.3. Continued Fractions

The simple continued fraction representation of a number is given by:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \dots}}}}$$

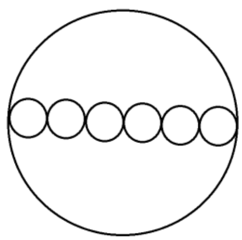
where a_0 is an integer, any other a_i members are positive integers, and n is a non-negative integer.

Example 6. Simplify: $1 + \cfrac{1}{1 + \cfrac{1}{1+1}}$. Express your answer as a common fraction.

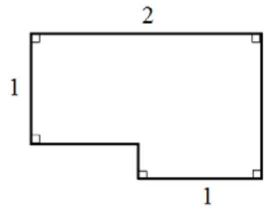
Example 7. Simplify: $\cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}}$. Express your answer as a common fraction.

3.4. Fraction Related to Geometry

☆**Example 8.** Six pepperoni circles will exactly fit across the diameter of a 12-inch pizza when placed as shown. If a total of 20 circles of pepperoni are placed on this pizza without overlap, what fraction of the pizza is covered by pepperoni?
(A) 2/3 (B) 5/9 (C) 5/7 (D) 5/6 (E) 23/36.

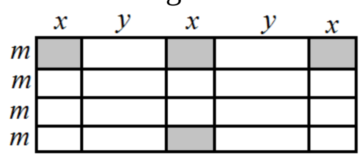


Example 9. The area of the figure shown is $\frac{11}{5}$. What is the perimeter of the figure?



Example 10. In the figure shown, all angles are right angles and $y = 2x$. If m, x, and y are lengths of the segments indicated, what fraction of the figure is shaded?

(A) $\frac{1}{7}$ (B) $\frac{1}{5}$ (C) $\frac{1}{14}$ (D) $\frac{3}{10}$ (E) $\frac{5}{14}$



3.5. Fraction Related to Numbers and Expressions

Example 11. What reduced common fraction is equivalent to $18\frac{1}{3}\%$?

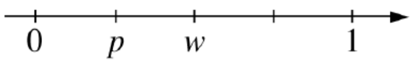
Example 12. If 1/2 + 1/5 + 1/8 > 1/ x + 1/6 + 1/8, then x could not be which of the following?
(A) 1 (B) 2 (C) 3 (D) 4 (E) 8

Example 13. If a and b are integers such that $a + b > 160$ and $a/b = 0.15$, what is the smallest possible value of a?
(A) 140 (B) 21 (C) 24 (D) 3 (E) 15

Example 14. Which of the following numbers is between $\frac{1}{6}$ and $\frac{1}{5}$?
(A) 0.14 (B) 0.15 (C) 0.16 (D) 0.17 (E) 0.26

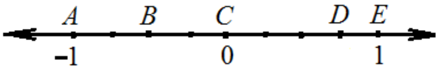
3.6. Fraction Related to Number Lines

Example 15. On the number line shown, the tick marks are equally spaced. What is the value of $w + p$?



(A) 3/4 (B) 2/3 (C) 1/2 (D) 1/3 (E) 1/4

Example 16. Dots are equally spaced on the number line shown. Which of the lettered points has a coordinate equal to $1-(-\frac{1}{2})^2$?



(A) A (B) B (C) C (D) D (E) E

3.7. Fraction Applications

Example 17. An hour-long television program included 20 minutes of commercials. What fraction of the hour-long program was not commercials?

Example 18. A container is $\frac{3}{5}$ full of water. If 16 gallons of the water were removed from the container, it would be $\frac{1}{3}$ full. How many gallons of water does this container hold when it is completely full?

(A) 20 (B) 35 (C) 40 (D) 60 (E) 90.

Example 19. Roy planted corn on $\frac{2}{7}$ of his land. If he planted 60 acres of corn, how many acres of land does he have?

(A) 90 (B) $112\frac{1}{2}$ (C) 135 (D) 210 (E) $337\frac{1}{2}$

☆**Example 20.** Peter's family ordered a 12-slice pizza for dinner. Peter ate two slices and shared another slice equally with his brother Tom. What fraction of the pizza did Peter eat?

(A) $\frac{1}{8}$ (B) $\frac{1}{7}$ (C) $\frac{5}{24}$ (D) $\frac{1}{6}$ (E) $\frac{1}{24}$

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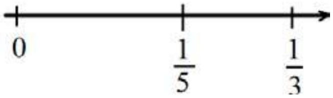
MORE EXAMPLES

Example 21. The fraction halfway between $\frac{1}{5}$ and $\frac{1}{3}$ (on the number line) is



(A) $\frac{1}{4}$ (B) $\frac{2}{15}$ (C) $\frac{4}{15}$ (D) $\frac{53}{200}$ (E) $\frac{8}{15}$

Example 22. How many more degrees of arc are there in $\frac{1}{5}$ of a circle than in $\frac{1}{6}$ of circle?
(A) 9° (B) 12° (C) 24° (D) 30° (E) 36°

Example 23. Write the common fraction equivalent to $2\frac{1}{2}\%$.

Example 24. If $\frac{3}{8}$ of a number is $\frac{21}{2}$, what is $\frac{1}{7}$ of the number?
(A) 3/2 (B) 9/2 (C) 4 (D) 6 (E) 28

Example 25. If $1/4 + 1/5 + 1/6 < 1/5 + 1/6 + 1/y$, then y could be which of the following?
(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

Example 26. Jenny had a pizza that was divided into 8 equal slices. She ate 3 of them. Alex has a pizza that is the same size, but his is divided into 4 equal slices. He ate 3 slices of his pizza. Who ate more pizza?

Example 27. Find the sum: $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots+\frac{1}{49\times 50}$.

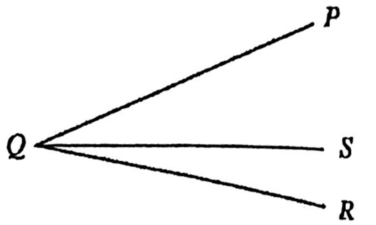
Example 28. Simplify: $\cfrac{1}{8+\cfrac{1}{8+\cfrac{1}{8}}}$.

Example 29. If $0 < a < b$, which of the following is greater than b/a?

(A) 1 (B) a/b (C) $1/(b/a)$ (D) $b/2a$ (E) $2b/a$

PROBLEMS

Problem 1. In the figure below, the measure of $\angle SQR$ is 2/5 the measure of $\angle PQR$. If the measure of $\angle PQR$ is 2/3 the measure of a right angle, what is the measure of $\angle SQP$?



Note: Figure not drawn to scale.

(A) 24° (B) 36° (C) 48° (D) 60° (E) 96°

Problem 2. If 1/7 + 1/8 + 1/9 > 1/ x + 1/8 + 1/9, then x could be which of the following?

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

Problem 3. If 1/4 + 1/5 + 1/6 < 1/5 + 1/y + 1/8, then y could be which of the following?

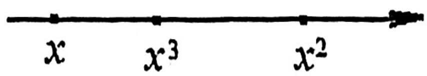
(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

Problem 4. If a and b are integers such that $a + b < 133$ and $a/b = 0.2$, what is the greatest possible value of b?

Problem 5. If $x = \frac{1}{3}$, what is the value of $\frac{1}{x} + \frac{1}{x-1}$?

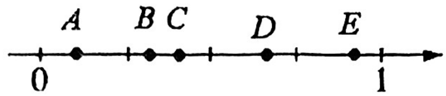
(A) −3/2 (B) 3/2 (C) 1 (D) 2 (E) 3

Problem 6. If x, x^2, and x^3 lie on a number line in the order shown below, which of the following could be the value of x?



(A) −2 (B) $-\frac{1}{2}$ (C) $\frac{2}{3}$ (D) 1 (E) $\frac{3}{2}$

Problem 7. If the tick marks on the number line below are equally spaced, which of the lettered points A through E is between 1/4 and 3/8?



(A) A (B) B (C) C (D) D (E) E

Problem 8. Bob is baking two pans of brownies that are the same size. One pan has nuts in it and the other pan does not. He cuts the pan with nuts into 8 equal pieces. He cuts the pan without nuts into 16 equal pieces. His friends ate 2 brownies with nuts and 3 brownies without nuts. Did they eat more of the brownies with nuts or without nuts?

Problem 9. On a blueprint, $\frac{1}{4}$ inch represents 16 feet, If a driveway is 80 feet long, what is its length, in inches, on the map?

(A) $\frac{3}{4}$ (B) $\frac{5}{8}$ (C) $\frac{5}{4}$ (D) $2\frac{1}{2}$ (E) 20

Problem 10. Of all the students in a high school, on a certain day $\frac{3}{5}$ rode the bus to school, $\frac{3}{10}$ rode in a car, and the remaining students walked. What fraction of the school's students walked to school on that day?

Problem 11. Find the sum $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots+\frac{1}{100\times 101}$.

Problem 12. Calculate: $\frac{3}{1\times 4}+\frac{3}{4\times 7}+\frac{3}{7\times 10}+\cdots+\frac{3}{19\times 22}$.

Problem 13. Find the sum: $\frac{1}{1\times 3}+\frac{1}{3\times 5}+\cdots+\frac{1}{1997\times 1999}$.

Problem 14. Express as a simplified mixed number: $3+\cfrac{2}{3+\cfrac{2}{3+\cfrac{2}{3}}}$.

Problem 15. In a poll, 45 people were in favor of building a new library, 27 people were against it, and 3 people had no opinion. What fraction of those polled were in favor of building a new library?

(A) $\frac{7}{10}$ (B) $\frac{3}{5}$ (C) $\frac{3}{7}$ (D) $\frac{1}{3}$ (E) $\frac{3}{10}$

Problem 16. Express as a common fraction: $\cfrac{1}{1+\cfrac{1}{5}}$.

Problem 17. Express the value of the following expression as a common fraction.

$$1+\cfrac{2}{3+\cfrac{4}{5}}$$

Problem 18. Simplify the following and express as a mixed number. $1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{3}}}$.

Problem 19. By 7:00 A.M., 1/5 of the junior class had arrived at a school dance. By 8:00 A.M., 60 more juniors had arrived, raising attendance to 1/3 of the junior class. How many people are in the junior class?

(A) 90 (B) 120 (C) 180 (D) 380 (E) 450

Problem 20. On a hike, Ian walked downhill $\frac{3}{7}$ of the time and uphill $\frac{4}{7}$ of the time. His downhill walking rate was 5 miles per hour, and his uphill walking rate was 3 miles per hour. The distance that Ian walked downhill was what fraction of the total distance that he walked?

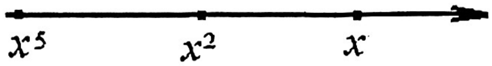
(A) $\frac{4}{7}$ (B) $\frac{3}{7}$ (C) $\frac{2}{5}$ (D) $\frac{5}{9}$ (E) $\frac{7}{9}$

Problem 21. On each of the days Monday through Thursday, Toni spent 2 hour commuting to work and 1 hour commuting back home. What fraction of the total number of hours in these four days did she spend commuting?

(A) $\frac{1}{12}$ (B) $\frac{1}{24}$ (C) $\frac{5}{12}$ (D) $\frac{5}{24}$ (E) $\frac{1}{8}$

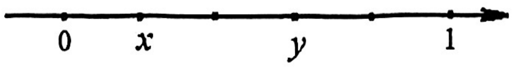
Problem 22. If the value of $1/t + 5$ is twice the value of $1/t - 1$, what is 7/2 of the value of t ?

Problem 23. If x, x^2, and x^5 lie on a number line in the order shown below, which of the following could be the value of x?



(A) –2 (B) $-\frac{1}{2}$ (C) $\frac{2}{3}$ (D) 1 (E) $\frac{3}{2}$

Problem 24. On the number line below, the tick marks are equally spaced. What is the value of $x - y$?



(A) 4/5 (B) 3/4 (C) 2/5 (D) 1/4 (E) 1/5

Problem 25. If a cake is cut into thirds and each third is cut into sevenths, how many pieces of cake are there?

Problem 26. A company sells boxes of balloons in which the balloons are red, green, or blue. Luann purchased a box of balloons in which $\frac{1}{5}$ of them were red. If there were half as many green balloons in the box as red ones and 21 balloons were blue, how many balloons were in the box?

1. BASIC KNOWLEDGE

Even integer:

An integer with the last digit of 0, 2, 4, 6, or 8. General form: $2n$ or $2n + 2$, where n is any integer.

Examples: Even integers: 10, 12, 14, 16, and 18.

Odd integers:

An integer with the last digit of 1, 3, 5, 7, or 9. General form: $2n + 1$ or $2n - 1$.

Examples: Odd integers: 11, 13, 15, 17, and 19.

Parity:

An even number has even parity and an odd number has odd parity.

Properties:

even + even = even.	$12 + 14 = 26$.
even + odd = odd.	$12 + 13 = 25$
odd + odd = even.	$13 + 13 = 26$
odd × odd = odd.	$15 \times 15 = 225$
odd ÷ odd = odd.	$1001 \div 11 = 91$
odd × even = even.	$11 \times 12 = 132$
odd ≠ even.	$1 \neq 2$

2. PROBLEM SOLVING SKILLS

The sum of any even integer and 1 is odd: $4 + 1 = 5$.

The sum of two consecutive integers is odd: $n + (n + 1) = 2n + 1$; $12 + 13 = 25$.

The product of two consecutive integers is even: $n(n + 1)$; $12 \times 13 = 156$.

Any two consecutive integers have opposite parity: 12 is even and 13 odd.
$a + b$ and $a - b$ have the same parity: $15 - 5 = 10$ even; $15 + 5 = 20$ even.

If the product of n positive integers is even,

If the product of n positive integers is odd,

If the number of odd integers is even,

If the number of odd integers is odd,

Example 1. Add any 30 consecutive positive integers together. Is the sum even or odd?

Example 2. 300 is the sum of 15 consecutive even positive integers. What is the greatest even positive integer among them?

Example 3. The sequence 1, 1, 2, 3, 5, 8, 13, 21, … is formed like this: any term is the sum of the two terms before it, starting from the third term. How many are even numbers of the first 63 terms of the sequence?

Example 4. All the positive even integers greater than 0 are arranged in five columns (*A, B, C, D,* and *E*) as shown. Continuing the pattern, in what column will the integer 50 be?

A	B	C	D	E
	2	4	6	8
16	14	12	10	
	18	20	22	24
32	30	28	26	

…………………………………………

Example 5. The sum of all multiples of 3 from 20 to 100 is S. Is S even or odd?

Example 6. Five lamps are arranged in a row as shown in the figure below. Each lamp has its own switch. All five lamps A, B, C, D, and E are now off. Ben starts to turn each switch from A to E and he repeats the pattern (always from A to E in order) until he turns the switches 2012 times. Which lamps are on finally?

Example 7. If x and y are integers and $x^2y^2 + x^3$ is odd, which of the following statements must be true?

I. x^2 is odd.

II. y is odd.

III. $x + y^2$ is odd.

(A) I only (B) III only (C) I and II (D) I and III (E) II and III

MORE EXAMPLES

Example 8. If a and b are positive integers and $a^2 - b^2 = 7$, what is the value of b?
(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

Example 9. The lengths of the sides of a right triangle are consecutive even integers, and the hypotenuse of the right triangle is x. Which of the following equations could be used to find x?
(A) $x + x - 1 = x - 2$ (B) $x^2 + (x - 1)^2 = (x - 2)^2$
(C) $x^2 = (x - 2)^2 + (x - 4)^2$ (D) $x + x + 2 = x + 4$
(E) $x^2 = (x - 2)(x - 4)$

Example 10. If a and b are positive consecutive odd integers, which of the following must be a positive odd integer?
(A) $a + b$ (B) $a - b$ (C) $2a + b$ (D) $2a - 2b$ (E) $\frac{a+b}{2}$

Example 11. If x and y are positive consecutive odd integers, where $y > x$, which of the following is equal to $y^2 - x^2$?
(A) $6y$ (B) $8y$ (C) $4(y - 2)$ (D) $2y - 1$ (E) $4(y - 1)$

Example 12. The sequence 1, 1, 2, 4, 7, 13, 24, … is formed like this: any term is the sum of the three terms before it starting from the fourth term. Is the 100^{th} term even or odd in the sequence?

Example 13. The sequence 1, 1, 2, 3, 5, 8, 13, 21, … is formed like this: any term is the sum of the two terms before it, starting from the third term. How many odd numbers are there in the first 900 terms of the sequence?

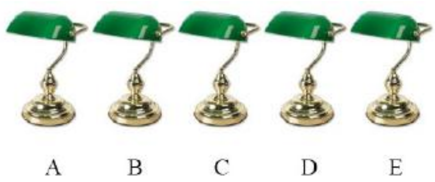
Example 14. Mr. Mathis and his student Peter worked together to solve math problems last week. When each person solved a problem, that person put a marble in his own box. Mr. Mathis's problem solving speed was half of his student. At the end of the problem solving session, Peter had four boxes of marbles and Mr. Mathis had two boxes of marbles. Each box is labeled with the number of marbles inside it. The numbers are 78, 94, 86, 87, 82, and 80, respectively. Which two boxes belong to Mr. Mathis?

Example 15. All the positive integers greater than 1 are arranged in five columns (*A*, *B*, *C*, *D*, *E*) as shown. Continuing the pattern, in what column will the integer 800 be written?

	A	B	C	D	E
Row 1		2	3	4	5
Row 2	9	8	7	6	
Row 3		10	11	12	13
Row 4	17	16	15	14	
Row 5		18	19	20	21
⋮					

(A) *A* (B) *B* (C) *C* (D) *D* (E) *E*

Example 16. Five lamps are arranged in a row as shown in the figure below. Each lamp has its own switch. All five lamps *A, B, C, D*, and *E* are now off. Ben starts to turn each switch from *A* to *E* and he repeats the pattern (always from *A* to *E* in order) until he turns the switches 126 times. Which lamps are on in the end?



PROBLEMS

Problem 1. If a and b are positive integers and $a^2 - b^2 = 143$, what is the value of a?

(A) 1 (B) 11 (C) 12 (D) 13 (E) 14

Problem 2. The lengths of the sides of a right triangle are consecutive even integers, and the length of the longer leg is x. Which of the following equations could be used to find x?

(A) $x + x + 1 = x + 2$ (B) $x^2 + (x + 1)^2 = (x + 2)^2$
(C) $(x - 2)^2 + x^2 = (x + 2)^2$ (D) $x - 2 + x = x + 2$
(E) $x^2 = (x - 2)(x + 2)$

Problem 3. If a and b are positive odd integers, which of the following must be a positive even integer?

(A) $a + b$ (B) $a - b$ (C) $2a + b$ (D) $2a - b$ (E) $\frac{a+b}{2}$

Problem 4. If x and y are positive consecutive even integers, where $y > x$, which of the following is equal to $y^2 - x^2$?

(A) $2x$ (B) $4x$ (C) $2x + 2$ (D) $2x + 4$ (E) $4x + 4$

Problem 5. If x and y are positive consecutive odd integers, where $y > x$, which of the following is equal to $y^2 - x^2 + 8$?

(A) $6x$ (B) $8x$ (C) $2x + 2$ (D) $2x + 4$ (E) $4(x + 3)$

Problem 6. If a and b are odd integers, which of the following must also be an odd integer?

I. $(a + 2)b$ II. $(a + 2) + b$ III. $(a + 2) - b$

(A) I only (B) II only (C) III only (D) I and II (E) II and III

Problem 7. If t represents an odd integer, which of the following expressions represents an even integer?

(A) $t + 4$ (B) $2t - 3$ (C) $3t - 6$ (D) $3t + 8$ (E) $5t + 5$

Problem 8. If $\frac{x+7}{2}$ is an integer. Then x must be
(A) a negative integer (B) a positive integer (C) a multiple of 3
(D) an even integer (E) an odd integer

Problem 9. If k is a positive integer, which of the following must represent an odd integer that is twice the value of an odd integer?
(A) $4k + 3$ (B) $2k + 3$ (C) $2k + 4$ (D) $4k + 1$ (E) $4k + 2$

Problem 10. If k is a negative even integer and n is a positive odd integer, which of the following could be equal to $n - k$?
I. 0 II. 1 III. 3.

(A) I only (B) II only (C) III only (D) I and III only (E) I, II, and III

Problem 11. The sum of the positive odd integers less than 1000 is subtracted from the sum of the positive even integers less than or equal to 1000. What is the resulting difference?

Problem 12. Each of the 75 children in a line was assigned one of the integers from 1 through 99 by counting off in order. Then, standing in the same order, the children counted off in the opposite direction, so that the child who was assigned the number 99 the first time was assigned the number 1 the second time. Which of the following is a pair of numbers assigned to the same child?
(A) 50 and 48 (B) 49 and 50 (C) 66 and 33 (D) 33 and 67 (E) 45 and 32

Problem 13. The counting numbers are arrange in four columns as shown below. Under which column letter will 2012 appear?

A	B	C	D
1	2	3	4
8	7	6	5
9	10	11	12
	…	14	13

Problem 14. Suppose all the counting numbers are arranged in columns as shown below. Under what column-letter will 2012 appear?

A	B	C	D	E	F	G
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	_	_

Problem 15. Is $1 + 2 + 3 + 4 + \ldots + 2011 + 2012$ even or odd?

Problem 16. Is the expression $1\times2+3\times4+5\times6+...+99\times100$ even or odd?

Problem 17. Seven lamps labeled *A* through *G* are arranged in a row. Each lamp has its own switch. Now lamps *A*, *C*, *E*, and *G* are on and other lamps are off. Ben starts to flip each switch from *A* to *G* the following way: if the lamp is on, he turns it off; if the lamp is off, he turns it on. He repeats the pattern until he flips the switches 2011 times. Which lamps are on finally?

1. BASIC KNOWLEDGE

Definitions

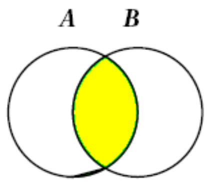
Sets: A set is any well-defined collection of objects. Individual objects are called the elements or members of the set.

Subsets: A subset is a sub-collection of a set. We denote set B as subset of A by the notation $B \subseteq A$.

Proper subset: If B is a subset of A and B is not equal to A, B is a proper subset of A, written as $B \subset A$.

Universal set: denoted by U, is the set that contains all elements considered in a given discussion.

Intersection of Sets



$A \cap B$

The intersection of sets A and B, written as $A \cap B$, is the set of all elements belonging to both A and B.

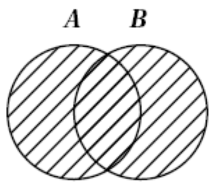
$A \cap B \Rightarrow A$ and B

* One simple way to remember this:

note $\cap$ is like the second letter in the word “and.”

Example 1. Find $A \cap B$ if Let $A = \{1, 2, 3, 4, 5, 6\}$, and $B = \{1, 3, 5, 7, 9\}$.

Union of sets

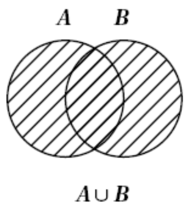


$A \cup B$

The union of sets A and B, written as $A \cup B$, is the set of all elements belonging to either A or B.

*Note the symbol $\cup$ is like the first letter in the word “Union”.

Example 2. Find if Let A = {1, 2, 3, 4, 5, 6}, and B = {1, 3, 5, 7, 9}.

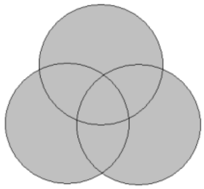


The Union Formula for Two Events

The Union Formula for Three Events

The union of sets A, B, and C, written as $A \bigcup B \bigcup C$, is the set of all elements belonging to A, or B, or C.

$$n(A \bigcup B \bigcup C) = n(A) + n(B) + n(C)$$
$$- n(A \bigcap B) - n(B \bigcap C) - n(C \bigcap A) + n(A \bigcap B \bigcap C)$$



2. PROBLEM SOLVING SKILLS

(1). Calculation of the number of subsets

The number of subsets of a set with n elements is 2^n.

Example 3. How many subsets of {C, H, E, N , P} have an even number of elements?

☆ **Example 4.** (2008 AMC 10A) Two subsets of the set S = {a, b, c, d, e}are to be chosen so that their union is S and their intersection contains exactly two elements. In how many ways can this be done, assuming that the order in which the subsets are chosen does not matter?

(A) 20 (B) 40 (C) 60 (D) 160 (E) 320

(2). Calculation of the number of proper subsets

The number of proper subsets of a set with n elements is $2^n - 1$.

Example 5. How many proper subsets are there for {J, U, L, I, A}?

(3). Calculation of the number of elements