BASIC KNOWLEDGE AND TERMS

Each pattern is created and arranged following a rule or rules. The key for solving pattern problems is to identify the core of the patterns.

Typical AMC 8/Mathcounts pattern problems can be classified as the following:

Growing patterns: Growing patterns have a sequence of elements that increase or decrease systematically when viewed as a recursive pattern.

Sequences pattern: Sequences pattern is a pattern of an ordered set of numbers or mathematical entities.

Repeating patterns: Repeating patterns can be generalized by recognizing pattern families that can look different but have the same core.

Geometric Patterns: A geometric pattern is a pattern that has repeating shapes such as dots, lines, triangles, circle, rectangles, and polygons.

USEFUL FORMULAS

Arithmetic sequence:

If any two consecutive terms in a sequence $a_1, a_2, a_3, \ldots, a_n, \ldots,$ have the same difference, the sequence is an arithmetic sequence.
The difference is called the common difference.

$$d = a_{n+1} - a_n$$

The nth term is expressed as

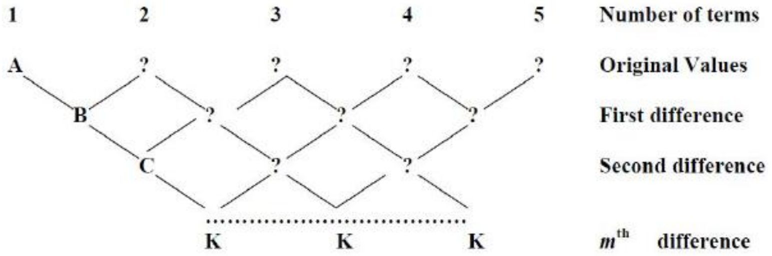
$$a_n = a_1 + (n-1)d$$

The sum of n terms in the sequence:

$$S = na_1 + \frac{(n-1)d}{2}n \quad \text{or} \quad S = \frac{(a_1 + a_n)n}{2}.$$

Newton's Little Formula for n^{th} term:

For an arithmetic sequence of high order:



$$a_n = A\binom{n-1}{0} + B\binom{n-1}{1} + C\binom{n-1}{2} + \cdots\cdots + K\binom{n-1}{m}$$

Geometric sequence

If any two consecutive terms in a sequence $a_1, a_2, a_3, \ldots, a_n, \ldots,$ have the same ratio, the sequence is called a geometric sequence (or geometric progression).

a_1 is the **first term**.

a_n is the **general term** or n^{th} term. $a_n = a_1 \cdot q^{n-1}$

The same ratio is called the **common ratio** (q or r).

The sum of the first n terms is expressed as S_n. For example, S_{12} means the sum of the first twelve terms.

$$S_n = \frac{a_1(1-q^n)}{1-q}.$$

GROWING PATTERNS

Example 1. Consider the following pattern:

$$\sqrt{1+1\cdot 2\cdot 3\cdot 4}=5$$

$$\sqrt{1+2\cdot 3\cdot 4\cdot 5}=11$$

$$\sqrt{1+3\cdot 4\cdot 5\cdot 6}=19$$

$$\sqrt{1+4\cdot 5\cdot 6\cdot 7}=29$$

Find $\sqrt{1+50\cdot 51\cdot 52\cdot 53}$

(A) 2550 (B) 2651 (C) 2652 (D) 2756 (E) 2703

Example 2. If the same pattern is continued, what is the number of 1's in the result of the calculation in the eighth line of the pattern?

1 × 9 + 2 = ________

12 × 9 + 3 = ________

123 × 9 + 4 = ________

(A) 4 (B) 6 (C) 8 (D) 9 (E) 10

Example 3. Look for a pattern in the following and then determine the value of *n*:

$$121 = \frac{22 \times 22}{1+2+1}$$

$$12321 = \frac{333 \times 333}{1+2+3+2+1}$$

$$n =$$

The sum of the digits of *n* is:

(A) 14 (B) 16 (C) 18 (D) 19 (E) 20

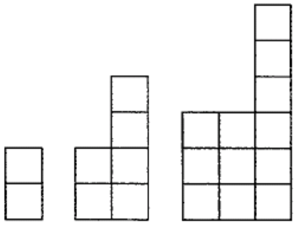
Example 4. Use the pattern given to express 100^2 in the form $a^2 + b^2 - c^2$. What is the value $a + b + c$?

$12^2 = 8^2 + 9^2 - 1^2$
$14^2 = 10^2 + 10^2 - 2^2$
$16^2 = 12^2 + 11^2 - 3^2$
$18^2 = 14^2 + 12^2 - 4^2$

(A) 198 (B) 153 (C) 145 (D) 196 (E) 194

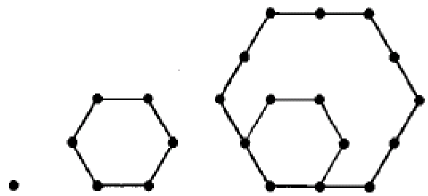
Example 5. The first three towers in a sequence are shown. The *n*th tower is formed by stacking *n* blocks on top of an $n \times n$ square of blocks. How many blocks are in the 99th tower?

(A) 9900 (B) 9816 (C) 9818 (D) 9919 (E) 9801



Example 6. The first three hexagonal numbers are represented as shown. Find the sum of the first five hexagonal numbers.

(A) 44 (B) 45 (C) 48 (D) 39 (E) 50



1 6 15

SEQUENCES PATTERN

☆**Example 7.** Terri produces a sequence of positive integers by following three rules. She starts with a positive integer, then applies the appropriate rule to the result, and continues in this fashion.
Rule 1: If the integer is less than 10, multiply it by 9.
Rule 2: If the integer is even and greater than 9, divide it by 2.
Rule 3: If the integer is odd and greater than 9, subtract 5 from it.

A sample sequence: 23, 18, 9, 81, 76, ...
Find the 198*th* term of the sequence that begins 49, 44. . . .
(A) 54 (B) 6 (C) 22 (D) 49 (E) 11

Example 8. All positive integers appear in the sequence 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, . . ., and each positive integer k appears in the sequence k times. In the sequence, each term after the first is greater than or equal to each of the terms before it. If the integer 12 first appears in the sequence as the nth term, what is the value of n?
(A) 64 (B) 67 (C) 65 (D) 66 (E) 62

Example 9. Complete the pattern: 10, 15, 22.5, 33.75, ———
(A) 44.85 (B) 55.95 (C) 40.675 (D) 50.625 (E) 50

Example 10. The first term of a sequence is 5 and each subsequent term is 5 less than twice the preceding term. What is the eighth term?
(A) 5 (B) 6 (C) 2 (D) 4 (E) 8

Example 11. What is the 50th letter in this pattern: *ABCAABBCCAAABBBCCC* . ...?
(A) *A* (B) *B* (C) *C* (D) *D* (E) *E*

Example 12. A sequence is formed by writing the word COMPETITIONS over and over again. What is the 496^{th} letter in this sequence?

(A) C (B) O (C) M (D) P (E) E

Example 13. The sequence 0, 1, 2, 2, 3, 3, 0, 1, 2, 2, 3, 3, . . .repeats every six terms. The first term is 0. What is the 998^{th} term?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 998

Example 14. The first six terms of a sequence are 1, –2, 3, –4, 5, –6. . . .. The odd-numbered terms are increasing consecutive positive odd integers starting with 1. The even-numbered terms are decreasing consecutive negative even integers starting with –2. What is the sum of the 50^{th} and 51^{st} terms of the sequence?

(A) –101 (B) –1 (C) 0 (D) 1 (E) 101

REPEATING PATTERNS

Repeating patterns can be generalized by recognizing pattern families that can look different but have the same core.

Example 15. What is the 100^{th} digit of the decimal representation of $\frac{1}{7}$?

(A) 1 (B) 4 (C) 2 (D) 8 (E) 7

Example 16. What is the 17^{th} digit after the decimal point in the decimal expansion of $\frac{11}{7}$?

(A) 5 (B) 4 (C) 2 (D) 8 (E) 7

Example 17. What is the 123,999th digit after the decimal in the decimal expansion of $\frac{123}{999}$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 9

Example 18. Starting with a green bead, colored beads are placed on a string according to the pattern green, red, blue, yellow, white, orange. If this pattern is repeated, what is the color of the 51st bead?

(A) Green (B) Red (C) Blue (D) Yellow (E) White

Example 19. The table shown shows Pythagorean triples for which $c = b + 1$. Find the value of c when $a = 15$.

a	b	c
3	4	5
5	12	13
7	24	25
9	40	41

(A) 110 (B) 111 (C) 112 (D) 113 (E) 115

Example 20. The whole numbers are written consecutively in rows as shown. Each row contains two more numbers than the previous row. What is the number of the row in which the number 1,300 is listed?

Row 1					0				
Row 2				1	2	3			
Row 3			8	7	6	5	4		
Row 4		9	10	11	12	13	14	15	
Row 5	24	23	22	21	20	19	18	17	16

(A) 35 (B) 36 (C) 37 (D) 38 (E) 39

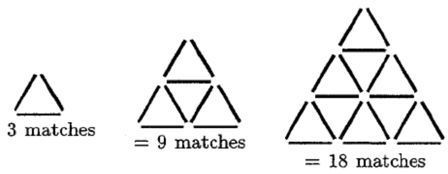
Example 21. The lattice shown is continued for 100 rows. What will be the third number in the 100^{th} row?

Row 1:	1	2	3	4	5	6	7
Row 2:	8	9	10	11	12	13	14
Row 3:	15	16	17	18	19	20	21
Row 4:	22	23	24	25	26	27	28

(A) 696 (B) 695 (C) 697 (D) 694 (E) 99

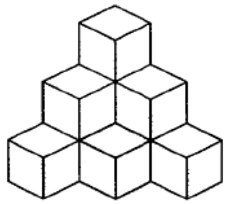
GEOMETRIC PATTERNS

Example 22. Referring to the sketches, it is seen that 3, 9, and 18 matches are required to make the triangular patterns depicted, respectively. How many matches would be needed to construct a similar figure with a ten match-stick base?

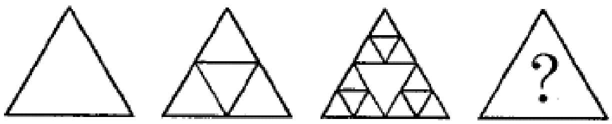


(A) 108 (B) 162 (C) 163 (D) 165 (E) 167

Example 23. The diagram shows an arrangement of 10 cubes in 3 layers. How many cubes will it take to make 8 layers?
(A) 116 (B) 118 (C) 120 (D) 124 (E) 144



Example 24. By continuing the pattern shown, how many non-overlapping triangles would appear in the last figure?



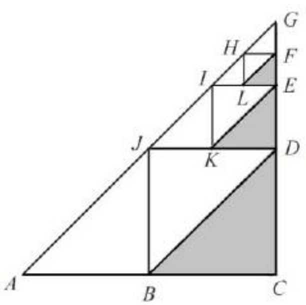
(A) 36 (B) 38 (C) 40 (D) 44 (E) 99

Example 25. Each time a change occurs, the central one-fourth of every white equilateral triangle is shaded. What fractional part of the original equilateral triangle would be shaded after four changes? Express your answer as a common fraction.

(A) (B) $\frac{173}{256}$ (C) $\frac{43}{64}$ (D) $\frac{177}{256}$ (E) $\frac{41}{256}$

Example 26. Thirty-six cannon balls are placed on a flat surface in the shape of a square to form the base of a display beside the cannon. How many additional cannonballs are needed to form a "pyramid" with the given square base?
(A) 55 (B) 91 (C) 40 (D) 36 (E) 99

☆**Example 27.** Points *B*, *D*, and *J* are midpoints of the sides of right triangle *ACG*. Points *K*, *E*, *I* are midpoints of the sides of triangle *JDG*, etc. If the dividing and shading process is done 101 times (the first three are shown) and $AC = CG = 12$, then the total area of the shaded triangles is nearest to
(A) 24 (B) 12 (C) 18 (D) 19 (E) 20



Example 28. A grocer stacks apples in the shape of a square pyramid. The bottom layer is a 10×10 square, the top layer is one apple, and the nth layer is an $n \times n$ square. How many apples does she have in the pyramid?
(A) 368 (B) 385 (C) 340 (D) 440 (E) 399

PROBLEMS

Problem 1. Find the numerical value x_8, if

$x_0 = 1^0$

$x_1 = 2^0 + 2^1$

$x_2 = 4^0 + 4^{\frac{1}{2}} + 4^1$

$x_3 = 8^0 + 8^{\frac{1}{3}} + 8^{\frac{2}{3}} + 8^1$

$x_4 = 16^0 + 16^{\frac{1}{4}} + 16^{\frac{1}{2}} + 16^{\frac{3}{4}} + 16^1$

(A) 512 (B) 511 (C) 256 (D) 1024 (E) 1023

Problem 2. Look for a pattern:

$$11 \times 11 = 121$$
$$111 \times 111 = 12321$$
$$1111 \times 1111 = 1234321$$

Find the value of n: $111111 \times 111111 = n$

Problem 3. Look for a pattern:

$1^3 = 1^2 - 0^2$

$2^3 = 3^2 - 1^2$

$3^3 = 6^2 - 3^2$

$\vdots$

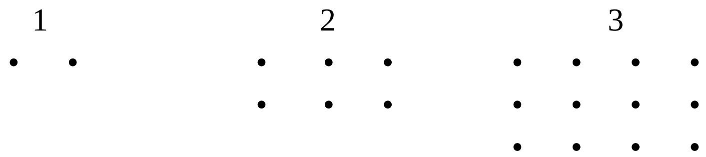
$6^3 = n^2 - m^2$

What is the value of $m + n$?

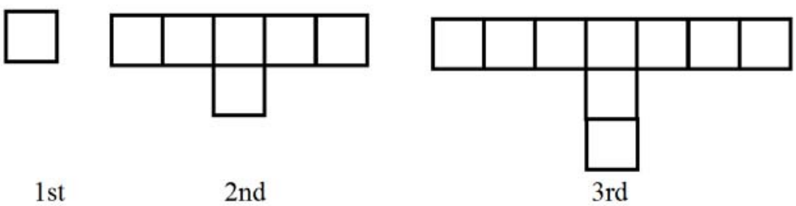
Problem 4. Follow the pattern to determine the value of 8(23456789) + 9.

$$8(2) + 2 = 18$$
$$8(23) + 3 = 187$$
$$8(234) + 4 = 1876$$
$$8(2345) + 5 = 18765$$
$$8(23456) + 6 = 187654$$

Problem 5. The first figure contains 2 dots, the second 6 dots, and the third 12 dots. If the pattern continues, how many dots would the tenth figure contain?



Problem 6. Each arrangement of squares is formed from the preceding arrangement by adding two additional squares to each end of the horizontal row and one square to the vertical column. How many squares will be in the sixth figure in the sequence?



Problem 7. If the pattern continues, what is the next term in the sequence 1, 7, 25, 61, 121, . . . ?

Problem 8. Complete the pattern: 40.5, 9, 2, ______

Problem 9. Find the next decimal term in the sequence:

$0, 0.5, 0.\overline{6}, 0.75, \ldots$

Problem 10. What is the 100^{th} letter in the pattern *ABCABCABC* . . . ?

Problem 11. The sequence shown was formed by writing the first letter of the alphabet followed by writing the first two letters of the alphabet and continuing the pattern by writing one more letter of the alphabet each time. Continuing this pattern, what letter is the 280^{th} letter in this sequence?

A, A, B, A, B, C, A, B, C, D, A, B, C, D, E, . . .

Problem 12. A sequence of letters is formed by writing 1 A, 2 B's, 3 C's, and so forth, increasing the number of letters written by one each time the next letter of the alphabet is written. What is the 200^{th} letter in the sequence?

Problem 13. Begin with the 200-digit number 987654321098765 . . . 543210, which repeats the digits 0-9 in reverse order. From the left, choose every third digit to form a new number. Repeat the same process with the new number. Continue the process repeatedly until the result is a two-digit number. What is the resulting two-digit number?

Problem 14. What is the 1997^{th} digit to the right of the decimal point in the decimal expansion of $\frac{1}{7}$?

Problem 15. What is the 199^{th} digit of the decimal representation of $\frac{3}{37}$?

Problem 16. What is the 125^{th} digit beyond the decimal point in the decimal representation of $\frac{4}{7}$?

Problem 17. The positive odd integers are arranged in 5 columns, A, B, C, D, and E, continuing the pattern shown. In which column will 1599 appear?

A	B	C	D	E
	1	3	5	7
15	13	11	9	
	17	19	21	23
31	29	27	25	
	33	35	37	39
47	45	43	41	
	49	51	. . .	

Problem 18. The multiples of 3 are arranged in the following manner:

Column 1	Column 2	Column 3	Column 4
3	6	9	12
21	18	15	12
21	24	27	30
39	36	33	30
39	42	…	…

In which column will the number 1992 appear?

Problem 19. The natural numbers from 1 to 1,000 are arranged consecutively from left to right in a triangle as shown. Each row contains one more number than the row below. What number is directly above 723?

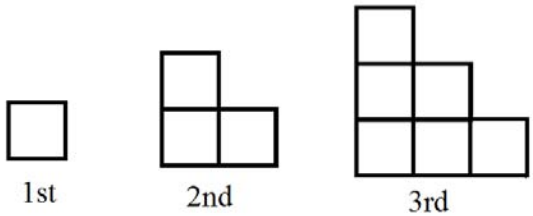
. . .				
11	12	13	14	15
7	8	9	10	
4	5	6		
2	3			
1				

Problem 20. Triangular numbers can be represented by a triangular array. For example, 1, 3, and 6 can be represented as:

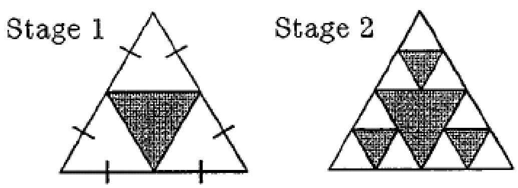
1	3	6
•	•	•
	• •	• •
		• • •

The difference of a pair of consecutive triangular numbers is 12. Find their sum.

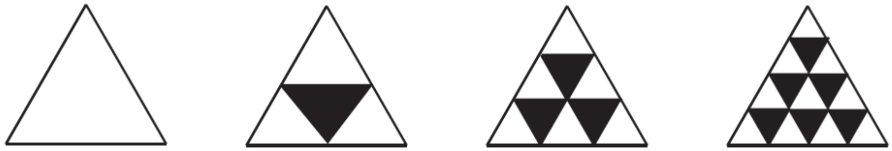
Problem 21. Squares are used to build the following sequence of drawings. If the length of a side of each square is one unit, how many units are in the perimeter of the 8^{th} drawing?



Problem 22. At each stage the midpoints of the sides of each unshaded equilateral triangle are connected and the triangle formed is shaded. Continuing in this process, what is the number of the stage when the shaded area is first larger than 90% of the area of the original equilateral triangle?

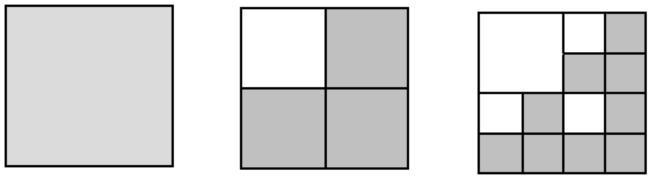


☆**Problem 23.** If the pattern in the diagram continues, what fraction of the interior would be shaded in the ninth triangle?



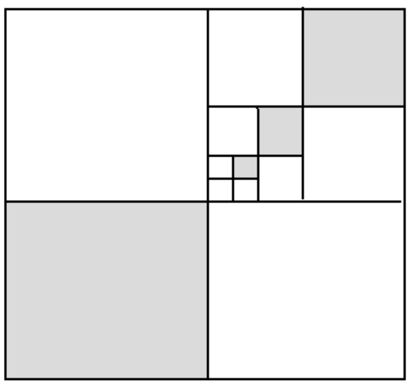
(A) 3/8 (B) 5/27 (C) 79/16 (D) 4/9 (E) 11/45

Problem 24. As you proceed from term to term, each shaded square is divided into four congruent squares and the upper left square of the four is painted white. By continuing the pattern, what fractional part of the tenth figure will be shaded? Express your answer as a common fraction in which the numerator and denominator are expressed in prime factored form using exponents.



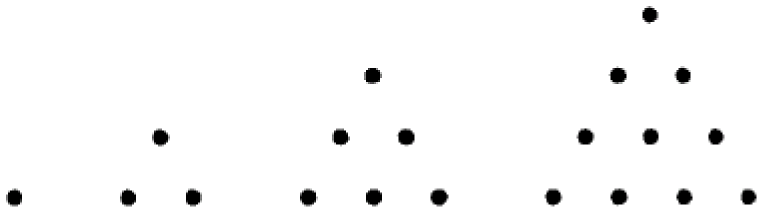
1 2 3

Problem 25. Each of the figures is a square formed by connecting midpoints of opposite sides of a larger square. What fraction of the largest square is shaded?

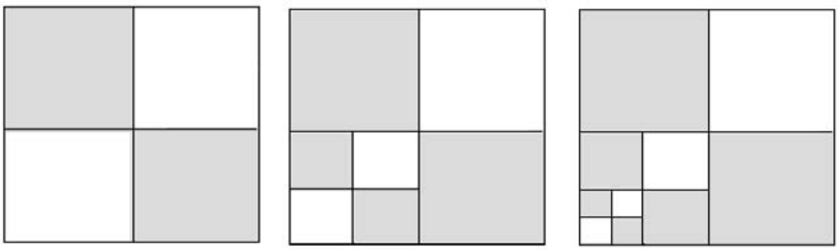


Problem 26. The first four triangular numbers are pictured. The *n*th triangular number is formed by drawing a row of *n* dots below the (*n* – 1)st triangular dot

pattern. The *k*th triangular number is represented by 120 dots. What is the value of *k*?

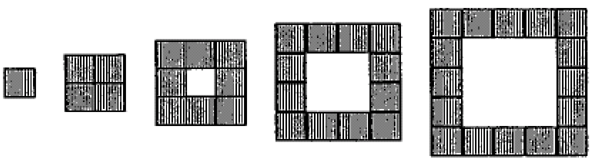


Problem 27. At each stage, the square at the lower left is divided into 4 congruent square regions, 2 of which are shaded. The area of the entire square (including shaded and unshaded parts) is 256 square units. How many square units are in the shaded area at the fifth stage? Express your answer as a decimal.



Stage 1 Stage 2 Stage 3

Problem 28. The "border number" of an $n \times n$ square is defined as the number of unit squares whose edges border the edges of a larger square. The border numbers of 1×1, 2×2, 3×3, 4×4, and 5×5 squares are illustrated. What is the border number of a 20 unit by 20 unit square?



1 4 8 12 16