1. BASIC KNOWLEDGE

Definitions

Sets: A set is any well-defined collection of objects. Individual objects are called the elements or members of the set.

Subsets: A subset is a sub-collection of a set. We denote set B as subset of A by the notation $B \subseteq A$.

Proper subset: If B is a subset of A and B is not equal to A, B is a proper subset of A, written as $B \subset A$.

Universal set: denoted by U, is the set that contains all elements considered in a given discussion.

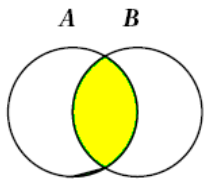
Intersection of Sets

The intersection of sets A and B, written as $A \cap B$, is the set of all elements belonging to both A and B.

$A \cap B \Rightarrow A$ and B

* One simple way to remember this:

note $\cap$ is like the second letter in the word "and."



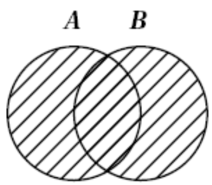
$A \cap B$

Example 1. Find $A \cap B$ if Let $A = \{1, 2, 3, 4, 5, 6\}$, and $B = \{1, 3, 5, 7, 9\}$.

Union of sets

The union of sets A and B, written as $A \cup B$, is the set of all elements belonging to either A or B.

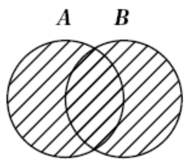
*Note the symbol $\cup$ is like the first letter in the word "Union".



$A \cup B$

Example 2. Find if Let $A = \{1, 2, 3, 4, 5, 6\}$, and $B = \{1, 3, 5, 7, 9\}$.

The Union Formula for Two Events

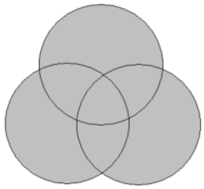


$A \cup B$

The Union Formula for Three Events

The union of sets *A*, *B*, and *C*, written as $A \bigcup B \bigcup C$, is the set of all elements belonging to *A*, or *B*, or *C*.

$$n(A \bigcup B \bigcup C) = n(A) + n(B) + n(C)$$
$$- n(A \bigcap B) - n(B \bigcap C) - n(C \bigcap A) + n(A \bigcap B \bigcap C)$$



2. PROBLEM SOLVING SKILLS

(1). Calculation of the number of subsets

The number of subsets of a set with *n* elements is 2^n.

Example 3. How many subsets of {C, H, E, N , P} have an even number of elements?

☆ **Example 4.** (2008 AMC 10A) Two subsets of the set $S = \{a, b, c, d, e\}$are to be chosen so that their union is *S* and their intersection contains exactly two elements. In how many ways can this be done, assuming that the order in which the subsets are chosen does not matter?

(A) 20 (B) 40 (C) 60 (D) 160 (E) 320

(2). Calculation of the number of proper subsets

The number of proper subsets of a set with n elements is $2^n - 1$.

Example 5. How many proper subsets are there for {J, U, L, I, A}?

(3). Calculation of the number of elements

Example 6. Consider the set $A = (1, 2, 3, \ldots 57)$. How many are neither divisible by 3 nor 5?

(A) 30 (B) 27 (C) 25 (D) 21 (E) 20

(4). Problems Involved Two Events

Example 7. Every member of a math club is taking algebra or geometry and 8 are taking both. If there are 17 taking algebra and 13 taking geometry, how many members are in the club?

(A) 30 (B) 25 (C) 22 (D) 21 (E) 20

☆**Example 8.** At Asheville Middle High School, 30% of the students in the Math Club are in the Science Club, and 80% of the students in the Science Club are in the Math Club. There are 15 students in the Science Club. How many students are in both Clubs?

(A)40 (B) 12 (C) 30 (D) 36 (E) 43

(5). Problems Involving Three Events

Example 9. Of the 400 eighth-graders at Pascal Middle school, 117 take algebra, 109 take advanced computer, 114 take industrial technology. Furthermore, 70 take both algebra and advanced computer, 34 take both algebra and industrial technology, and 29 take both advanced computer and industrial technology. Finally, 164 students take none of these courses. How many students take all three courses?

(A) 28 (B) 29 (C) 34 (D) 35 (E) 63

Example 10. Out of 22 students surveyed on ice cream flavors, 12 like chocolate, 5 like only strawberry, and 6 liked vanilla. If 3 liked chocolate and vanilla, how many students did not like any of these flavors?

(A) 2 (B) 9 (C) 3 (D) 5 (E) 6

Example 11. There are 100 5^{th} graders in Hope Middle School. 58 like English, 38 like Math, and 52 like Spanish. 6 students like Math and English only (not Spanish), 4 students like Math and Spanish only (not English), and 12 students like all three subjects. Each student likes at least one subject. How many students only like English?

(A) 18 (B) 19 (C) 24 (D) 25 (E) 26

☆ **Example 12.** (AMC 10) How many numbers between 1 and 2005 are integer multiples of 3 or 4 but not 12?
(A) 501 (B) 668 (C) 835 (D) 1002 (E) 1169

(6). Problems Involving "At Least" Or "At Most"

Example 13. Of the 200 students at Lakeview High School, 170 are taking science and 175 are taking mathematics. What is the fewest number of students at Lakeview who could be taking both?
(A) 245 (B) 370 (C) 145 (D) 30 (E) 116

Example 14. 8^{th} grade classes are surveyed. 78% of the students like swimming, 80% like computer games, 84% like playing chess and 88% like reading books. At least how many percent of students like all four activities?

(A) 30 (B) 66 (C) 83 (D) 12 (E) 22

Example 15. There are 100 students in a class. 75 of them like to play basketball. 80 like to play chess. 92 like to sing. 85 like to swim. At least how many students like all the 4 activities?

(A) 55 (B) 77 (C) 35 (D) 32 (E) 19

MORE EXAMPLES

Example 16. Set M has m elements and set N has n elements. Set W consists of all elements that are in either set M or set N with the exception of the k common elements ($k > 0$). Which of the following represents the number of elements in set W?

(A) $x + y + 2k$ (B) $x + y - k$ (C) $x + y + 2k$ (D) $x + y - 2k$ (E) $2x + 2y - 2k$

Example 17. Set S consists of the positive multiples of 6 that are less than 100, and set T consists of the positive multiples of 8 that are less than 100. How many numbers do sets S and T have in common?

(A) None (B) One (C) Two (D) Four (E) Eight

Example 18. Set S consists of m integers, and the difference between the greatest integer in S and the least integer in S is 800. A new set of m integers, set T, is formed by multiplying each integer in S by 3 and then adding 10 to the product. What is the difference between the greatest integer in T and the least integer in T?

(A) 800 (B) 2400 (C) 2500 (D) 1600 (E) 2403

Example 19. In a class of 42 students, 18 students are in the Math Club, 5 students are in both the Math Club and the Science Club, and 14 are in neither. How many students are in the Science Club?

(A) 8 (B) 14 (C) 15 (D) 16 (E) 24

Example 20. *A* and *B* are sets. If *A* contains 6 elements, *B* contains 8, and together *A* and *B* contain 10, how many elements in *A* are also in *B*?

(A) 4 (B) 2 (C) 5 (D) 6 (E) 9

Example 21. Ninety-six girls were surveyed at Euclid High School. There were 48 softball players and 45 track athletes. If $\frac{1}{6}$ of the girls surveyed did not play either sport, how many girls played both sports?

(A) 12 (B) 13 (C) 8 (D) 4 (E) 5

Example 22. One hundred students at a certain school take science or math. If 73 students take science and 90 take math, how many take math but not science?
(A) 63 (B) 27 (C) 23 (D) 17 (E) 5

Example 23. In a class of 500 students, every student liked at least one for three kinds of music.
260 liked classical music

260 liked jazz music
75 liked classical and rock music
115 liked rock and jazz music
130 liked classical and jazz music
45 liked classical, jazz, and rock music
How many of the students in this class liked only rock music?
(A) 190 (B) 110 (C) 255 (D) 390 (E) 45

1. BASIC KNOWLEDGE

1. Terms

A permutation is an arrangement or a listing of things in which order is important.

A combination is an arrangement or a listing of things in which order is not important

2. Definition

The symbol ! (factorial) is defined as follows:

$$0! = 1,$$

and for integers $n \geq 1$,

$$n! = n \cdot (n-1) \cdots 1.$$

$1! = 1$,
$2! = 2 \cdot 1 = 2$,
$3! = 3 \cdot 2 \cdot 1 = 6$,
$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$,
$5! = 5 \cdot 4 \cdot 3 \cdot 1 \cdot 1 = 120$,
$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$.

3. Permutations

(1). Different elements, with no repetition. Take r elements each time from n distinct elements ($1 \leq r \leq n$).

Number of permutations $P(n,r) = \dfrac{n!}{(n-r)!}$

(2). n distinct objects can be permutated in $n!$ permutations.

We let $n = r$ in (1) to get $P(n, n) = n!$

Example 1. In how many ways can the letters of the word MATH be arranged?
(A) 4 (B) 8 (C) 12 (D) 24 (E) 1

4. Combinations

Definition Let n, r be non-negative integers such that $0 \le r \le n$. The symbol $\binom{n}{r}$ (read "n choose m") is defined and denoted by

$$\binom{n}{r} = \frac{P(n,r)}{P(r,r)} = \frac{n!}{r!(n-r)!}$$

Remember: $\binom{n}{0} = 1$, $\binom{n}{1} = n$, and $\binom{n}{n} = 1$

Since $n - (n - r) = r$, we have $\binom{n}{r} = \binom{n}{n-r}$

Unlike Permutations, Combinations are used when order does not matter. If we have n different elements, and it doesn't matter which order we take the elements, the number of ways to take m elements where $1 \le m \le n$, is $\binom{n}{m}$.

Example 2. In a group of 2 cats, 3 dogs, and 10 pigs in how many ways can we choose a committee of 6 animals?
(A) 1001 (B) 2002 (C) 3003 (D) 4004 (E) 5005

☆**Example 3.** Consider the set of 5-digit positive integers. How many have at least one 7 in their decimal representation?
(A) 52488 (B) 52484 (C) 37512 (D) 90000 (E) 45000

5. The Sum Rule

If an event E_1 can happen in n_1 ways, event E_2 can happen in n_2 ways, event E_k can happen in n_k ways, and if any event E_1, E_2,.. or E_k happens, the job is done, then the total ways to do the job is $n_1 + n_2 + \cdots + n_k$.

Example 4. In how many ways can one book be selected from a book shelf of 5 paperback books and 3 hardcover books?
(A) 5 (B) 3 (C) 8 (D) 11 (E) 19

6. The Product Rule (Fundamental Counting Principle)

When a task consists of k separate parts, if the first part can be done in n_1 ways, the second part can be done in n_2 ways, and so on through the k^{th} part, which can be done in n_k ways, then total number of possible results for completing the task is given by the product:

$$n_1 \times n_2 \times n_3 \times \cdots \times n_k$$

Example 5. How many ways are there to arrange 5 people in a row of 5 seats?
(A) 5 (B) 10 (C) 12 (D) 120 (E) 720

Example 6. A girl has 5 shirts, 4 skirts, and 3 pairs of shoes. How many different outfits can she create?
(A) 120 (B) 60 (C) 15 (D) 12 (E) 20

☆**Example 7.** How many triples of three positive integers (x, y, z) have the product of 24?

(A) 20 (B) 24 (C) 27 (D) 30 (E) 21

2. PROBLEM SOLVING SKILLS

(2.1). Counting Using Charts

Example 8. Two dice are rolled. How many ways are there to roll a sum of 5?

(2.2). Counting Using Tree Diagram

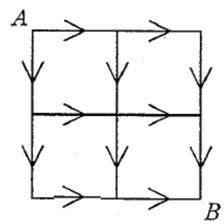
You can use a tree diagram to list all the possible outcomes of events.

Example 9. A designer has 3 fabric colors he may use for a dress: red, green, and blue. Four different patterns are available for the dress. If each dress design requires one color and one pattern, how many different dress designs are possible?

(A) 10 (B) 24 (C) 12 (D) 14 (E) 20

(2.3). Counting using water pipes

Example 10: Using only the line segments given in the indicated direction, how many paths are there from *A* to *B*?



☆**Example 11.** (2013 AMC 8 problem 20) Samantha lives 2 blocks west and 1 block south of the southwest corner of City Park. Her school is 2 blocks east and 2 blocks north of the northeast corner of City Park. On school days she bikes on streets to the southwest corner of City Park, then takes a diagonal path through the park to the northeast corner, and then bikes on streets to school. If her route is as short as possible, how many different routes can she take?

(A) 3 (B) 6 (C) 9 (D) 12 (E) 18

(2.4). Counting with restriction

Example 12. In how many ways can 5 books be arranged on a shelf if two of the books must remain together, but may be interchanged?

(A) 12 (B) 24 (C) 48 (D) 96 (E) 5

-

Example 13. How many ways can 5 distinct paperback books and 1 hardcover book be arranged on a shelf if the hardcover must be the rightmost book on the shelf?

(A) 20 (B) 60 (C) 90 (D) 120 (E) 180

☆**Example 14.** How many four –digit whole numbers do not contain the digit 1? No digit is allowed to use more than once in any such 4-digti number.
(A) 4032 (B) 2016 (C) 1024 (D) 8064 (E) 2688

(2.5) . Grouping

THEOREM 1: Let the number of different objects be n. Divide n into r groups $A_1, A_2, ..., A_r$ such that there are n_1 objects in group A_1, n_2 objects in group A_2, ..., n_r objects in the group A_r, where $n_1 + n_2 + \cdot\cdot\cdot + n_r = n$. The number of ways to do so is

$$\frac{n!}{n_1!n_2!\cdots n_r!}$$

THEOREM 2: Let there be r types of objects: n_1 of type 1, n_2 of type 2; etc. The number of ways in which these $n_1 + n_2 + \cdot\cdot\cdot + n_r = n$ objects can be rearranged is

$$\frac{n!}{n_1!n_2!\cdots n_r!}$$

Example 15. In how many ways may we distribute 6 different books to Alex, Bob, and Catherine such that each person gets 2 books?
(A) 720 (B) 120 (C) 90 (D) 60 (E) 24

Example 16. In how many ways may we distribute 7 different books to Alex, Bob, and Catherine such that Alex and Bob each get 2 books and Catherine gets 3 books?

(A) 2520 (B) 840 (C) 630 (D) 210 (E) 120

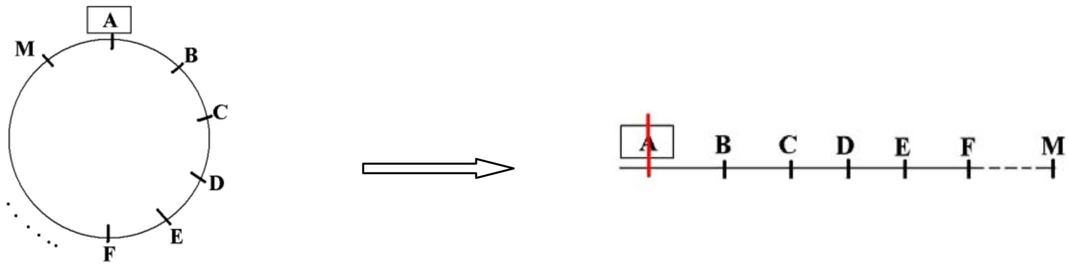
Example 17. In how many different ways can all the letters in INDIANA be arranged in a line? Assume that duplicate letters are indistinguishable.

A. 5040 B. 2520 C. 1260 D. 630 E. none of these

(2.6) . Circular Permutations

THEOREM 3: The number of circular permutations (arrangements in a circle) of n distinct objects is $(n - 1)!$.

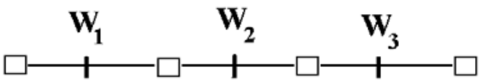
We can think of this as n people being seated at a round table. Since a rotation of the table does not change an arrangement, we can put person A in one fixed place and then consider the number of ways to seat all the others. Person B can be treated as the first person to seat and M the last person to seat. The number of ways to arrange persons A to M is the same as the number of ways to arrange persons B to M in a row. So the number of ways is $(n - 1)!$.



Example 18. In how many ways is it possible to seat seven people at a round table?

Example 19. In how many ways is it possible to seat seven people at a round table if Alex and Bob must not sit in adjacent seats?
(A) 720 (B) 480 (C) 120 (D) 60 (E) 90

Example 20. In how many ways can four men and four women be seated at a round table if no two men are to be in adjacent seats?
(A) 120 (B) 144 (C) 121 (D) 60 (E) 72



Example 21. In how many ways can a family of six people be seated at a round table if the youngest kid must sit between the parents?
(A) 6 (B) 7 (C) 10 (D) 12 (E) 24

(2.7). Combinations with Repetitions

THEOREM 4: Let n be a positive integer. The number of positive integer solutions to $x_1 + x_2 + \cdots + x_r = n$ is $\binom{n-1}{r-1}$.

☆**Example 22.** Three friends have a total of 6 identical pencils, and each one has at least one pencil. In how many ways can this happen?

(A) 6 (B) 7 (C) 10 (D) 12 (E) 24

Example 23. In how many ways may we write the number 9 as the sum of three positive integer summands? Here order counts, so, for example, 1 + 7 + 1 is to be regarded different from 7 + 1 + 1.

(A) 16 (B) 27 (C) 30 (D) 14 (E) 28

THEOREM 5: Let n be a positive integer. The number of non-negative integer solutions to $y_1 + y_2 + \cdots + y_r = n$ is $\binom{n+r-1}{n}$ or $\binom{n+r-1}{r-1}$.

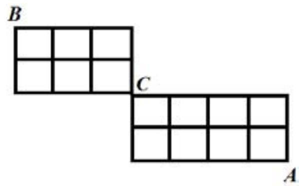
☆**Example 24.** Three friends have a total of 6 identical pencils. In how many ways can this happen?

(A) 10 (B) 12 (C) 28 (D) 24 (E) 16

THEOREM 6: The number of ways to walk from one corner to another corner of an *m* by *n* grid can be calculated by the following formula: $N = \binom{m+n}{n}$, where *m* is the number of rows and *n* the number of column.

Example 25. How many ways are there to get from *A* to *B* if you can only go north or west?

(A) 120 (B) 130 (C) 140 (D) 150 (E) 160



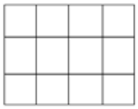
☆**Example 26.** (2013 AMC 8 problem 20) Samantha lives 2 blocks west and 1 block south of the southwest corner of City Park. Her school is 2 blocks east and 2 blocks north of the northeast corner of City Park. On school days she bikes on streets to the southwest corner of City Park, then takes a diagonal path through the

park to the northeast corner, and then bikes on streets to school. If her route is as short as possible, how many different routes can she take?
(A) 3 (B) 6 (C) 9 (D) 12 (E) 18

THEOREM 7: Counting How Many Rectangles

For a rectangular grid with *m* vertical lines and *n* horizontal lines, the total number of rectangles that can be counted is $N = \binom{m}{2} \times \binom{n}{2}$.

Example 27. Consider the figure shown below. How many rectangles are there?



(A) 30 (B) 60 (C) 90 (D) 120 (E) 180

THEOREM 8: Rising (Increasing) Number

A rising number, such as 34689, is a positive integer where each digit is larger than the one to its left.

The number of integers with digits in increasing order can be calculated by $\binom{9}{n}$.

Example 28. How many 3-digit increasing numbers are there?
(A) 34 (B) 64 (C) 84 (D) 92 (E) 120

THEOREM 9: Falling (Decreasing) Number

A falling number is an integer whose decimal representation has the property that each digit except the units digit is larger than the one to its right. For example, 96521 is a falling number but 89642 is not. The number of integers with digits in decreasing order can be calculated: $\binom{10}{n}$.

Example 29. How many 3-digit falling numbers are there?
(A) 60 (B) 90 (C) 120 (D) 240 (E) 180

THEOREM 10: Palindrome Numbers

A **palindrome number** is a number that is the same when written forwards or backwards. The first few palindrome numbers are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 22, 33, 44, 55, 66, 77, 88, 99, 101, 111, and 121.

The number of palindromes with n digits can be found by $9\times10^{\left\lfloor\frac{n-1}{2}\right\rfloor}$ for $n > 1$.
$\lfloor x\rfloor$ is the floor function.

Example 30. How many 3-digit palindromes are there?
(A) 30 (B) 60 (C) 90 (D) 120 (E) 180

3. PROBLEMS

Problem 1. Any 2 points determine a line. If there are 7 points in a plane, no 3 of which lie on the same line, how many lines are determined by pairs of these 7 points?
(A) 15 (B) 18 (C) 21 (D) 30 (E) 36

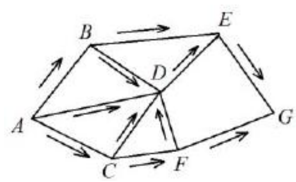
Problem 2. An electrician is testing 7 different wires. For each test, the electrician chooses 2 of the wires and connects them. What is the least number of tests that must be done so that every possible pair of wires is tested?
(A) 7 (B) 14 (C) 16 (D) 18 (E) 21

Problem 3. A snack machine has buttons arranged as shown above. If a selection is made by choosing 2 letters followed by 3 digits, what is the greatest number of different selections that could be made?

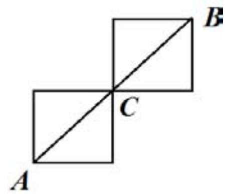
A	D		1	4	7
B	E		2	5	8
C	F		3	6	9

(A) 15 (B) 84 (C) 120 (D) 1260 (E) 1024

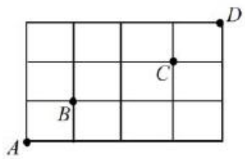
Problem 4. In the figure above, each line segment represents a one-way road with travel permitted only in the direction indicated by the arrow. How many different routes from A to G are possible?
(A) One (B) Six (C) Three (D) Four (E) Seven



Problem 5. A boy is walking along the line starting from point A to point B. Any point of intersection and line cannot be walked twice in one trip.
How many ways are there?
(A) 15 (B) 12 (C) 10 (D) 9 (E) 6

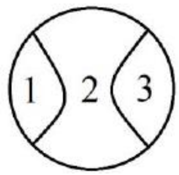


Problem 6. In the figure shown, a path from point A to point D is determined by moving upward or to the right along the grid lines. How many different paths can be drawn from A to D that must include both B and C?



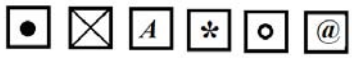
(A) 12 (B) 16 (C) 10 (D) 20 (E) 6

Problem 7. As shown above, a circular shape is divided into 3 regions. A certain design is to be painted such that two regions next to each other must have the different colors. If 6 different colors are available for the design, how many differently pained designs are possible?



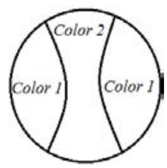
(A) 120 (B) 80 (C) 100 (D) 150 (E) 160

Problem 8. If the 6 cards shown above are placed in a row so that the card with a A on it is never at either end, how many different arrangements are possible?



(A) 2400 (B) 480 (C) 960 (D) 720 (E) 460

Problem 9. As shown above, a certain design is to be painted using 2 different colors. If 6 different colors are available for the design, how many differently pained designs are possible?

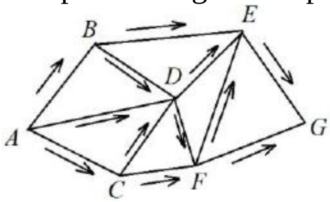


(A) 10 (B) 20 (C) 25 (D) 30 (E) 120

Problem 10. How many different even natural numbers each containing three distinct digits can be written using just the digits 0, 2, 3, 5, and 8?
(A) 6 (B) 12 (C) 20 (D) 30 (E) 36

Problem 11. In how many ways can four married couples be seated at a round table if no two men, as well as no husband and wife are to be in adjacent seats?
(A) 6 (B) 12 (C) 24 (D) 48 (E) 96

Problem 12. In the figure above, each line segment represents a one-way road with travel permitted only in the direction indicated by the arrow. How many different routes from A to G that pass through E are possible?



(A) Four (B) Five (C) Six (D) Seven (E) Eight

Problem 13. Ten points are placed on a circle. What is the greatest number of different lines that can be drawn so that each line passes through two of these points?
(A) 15 (B) 25 (C) 35 (D) 45 (E) 90

Problem 14. Four lines are drawn in a plane so that there are exactly three different intersection points. Into how many nonoverlapping regions do these lines divide the plane?
(A) Seven (B) Nine (C) Eleven (D) Thirteen (E) Fifteen

Problem 15. The four distinct points P, Q, R and S lie on a line l: the five distinct points T, U, V, W , X, and Y lie on a different line that is parallel to line l. What is the total number of different lines that can be drawn so that each line contains exactly two of the seven points?
(A) 9. (B) 12. (C) 10. (D) 20. (E) 40.

Problem 16. A number is called increasing if each of its digits is greater than the digit immediately to its left, if there is one. How many increasing numbers are there between 100 and 200?
(A) 100 (B) 101 (C) 20 (D) 28 (E) 30

Problem 17. In a volleyball league with 5 teams, each team plays exactly 3 games with each of the other 4 teams in the league. What is the total number of games played in this league?
(A) 15 (B) 20 (C) 12 (D) 25 (E) 30

1. BASIC KNOWLEDGE

"Divisible by" means "when you divide one number by another number, the result is a whole number." "Divisible by" and "can be evenly divided by" mean the same thing.

The expressions $\overline{abc}$, $\underline{abc}$, and *abc* are the same. $\overline{abc} = \underline{abc} = 100a + 10b + c$. They represent a three-digit number such as $\overline{234} = \underline{234} = 234$.

2.1. Divisibility rule for 2, 4, 8, and 16:

A number is divisible by 2 if the last digit of the number is divisible by 2 (2^1).
A number is divisible by 4 if the last two digits of the number are divisible by 4 (2^2).
A number is divisible by 8 if the last three digits of the number are divisible by 8 (2^3).
A number is divisible by 16 if the last four digits of the number are divisible by 16 (2^4).

2.2. Divisibility rule for 5, 25, 125, and 225:

A number is divisible by 5 if the last digit of the number is divisible by 5 **(5^1)**.
A number is divisible by 25 if the last two digits of the number form a number that is divisible by 25 **(5^2)**.
A number is divisible by 125 if the last three digits of the number form a number that is divisible by 125 **(5^3)**.
A number is divisible by 625 if the last four digits of the number form a number that is divisible by 625 **(5^4)**.

2.3. Divisibility rule for 3 and 9

A number is divisible by 3 if the sum of the digits of the number is divisible by 3.
A number is divisible by 9 if the sum of the digits of the number is divisible by 9.

2.4. Divisibility rule for 7, 11, and 13:

(1) If you double the last digit and subtract it from the rest of the number and the answer is divisible by 7, the number is divisible by 7. You can apply this rule to that answer again if necessary.

(2) To find out if a number is divisible by 11, add every other digit, and call that sum "x." Add together the remaining digits, and call that sum "y." Take the positive difference of x and y. If the difference is zero or a multiple of eleven, then the original number is a multiple of eleven.
(3) Delete the last digit from the number, and then subtract 9 times the deleted digit from the remaining number. If what is left is divisible by 13, then so is the original number. Repeat the rule if necessary.

(4) If the positive difference of the last three digits and the rest of the digits is divisible by 7, 11, or 13, then the number is divisibly by 7, 11, or 13, respectively.

2.5. Divisibility rule for 6, 10, 12, 14, 15, 18, 24, and 36:

A number is divisible by 6 if the number is divisible by both 2 and 3.
A number is divisible by 10 if the number is divisible by both 2 and 5.
A number is divisible by 12 if the number is divisible by both 3 and 4.
A number is divisible by 14 if the number is divisible by both 2 and 7.
A number is divisible by 15 if the number is divisible by both 3 and 5.
A number is divisible by 18 if the number is divisible by both 2 and 9.
A number is divisible by 24 if the number is divisible by both 3 and 8.
A number is divisible by 36 if the number is divisible by both 4 and 9.

NOTE: If a number is divisible by two numbers that are relatively prime, then it is divisible by the product of those two numbers. "Relatively prime" means two numbers have no common factor other than 1. For example, 3 and 4 are relatively prime.

2. PROBLEM SOLVING SKILLS

(1). Divisibility rule for 2, 4, 8, and 16:

Example 1. If the three-digit number $\overline{78N}$ is divisible by 4, how many possible values of N are there?

(A) 3 (B) 6 (C) 5 (D) 4 (E) 8

(2). Divisibility rule for 5, 25, 125, and 225:

Example 2. ☆ A three-digit integer contains one of each of the digits 3, 4 and 5. What is the probability that the integer is divisible by 5?

(A) 1/6 (B) 1/3 (C) 1/2 (D) 2/3 (E) 5/6

Example 3. The six-digit number $\overline{713EF5}$ is divisible by 125. How many such six-digit numbers are there?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

(3). Divisibility rule for 3 and 9

Example 4. What is the sum of all possible digits which could fill the blank in 47,__21 so that the resulting five-digit number is divisible by 3?

(A) 3 (B) 6 (C) 9 (D) 12 (E) 18

Example 5. Find the least possible value of digit d so that $\underline{437,d03}$ is divisible by 9.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

(4). Divisibility rule for 7, 11, and 13:

Example 6. Which number is not divisible by 7? 616, 567, 798, or 878.

Example 7. What is the largest integer less than 100 and evenly divisible by 7?

Example 8. Which digit should replace a in the units place so that 9867542a is divisible by 11?

(A) 3 (B) 6 (C) 9 (D) 4 (E) 8

(5). Divisibility rule for 6, 10, 12, 14, 15, 18, 24, and 36:

A number is divisible by 6 if the number is divisible by both 2 and 3.
A number is divisible by 10 if the number is divisible by both 2 and 5.
A number is divisible by 12 if the number is divisible by both 3 and 4.
A number is divisible by 14 if the number is divisible by both 2 and 7.
A number is divisible by 15 if the number is divisible by both 3 and 5.
A number is divisible by 18 if the number is divisible by both 2 and 9.
A number is divisible by 24 if the number is divisible by both 3 and 8.
A number is divisible by 36 if the number is divisible by both 4 and 9.

NOTE: If a number is divisible by n numbers that are relatively prime, then it is divisible by the product of those n numbers.

Example 9. What is the greatest three-digit number that is divisible by 6?
(A) 999 (B) 998 (C) 997 (D) 996 (E) 995

Example 10. Given the 4-digit base-ten number $\underline{77A4}$. For what value of the nonzero digit A will this 4-digit number be divisible by 3 and by 4?
(A) 3 (B) 6 (C) 9 (D) 1 (E) 7

Example 11. Find the value of x such that the four-digit number $\underline{x15x}$ is divisible by 18.
(A) 1 (B) 2 (C) 4 (D) 6 (E) 8

Example 12. Find distinct digits A and B such that $\underline{A47B}$ is as large as possible and divisible by 36. Name the number.
(A) 5472 (B) 6471 (C) 5470 (D) 3474 (E) 6470

Example 13. If k is a positive integer divisible by 3, and if $k < 80$, what is the greatest possible value of k?
(A) 75 (B) 76 (C) 77 (D) 78 (E) 79

Example 14. which of the following numbers can be used to show that the statement below is FALSE?
<u>All numbers that are divisible by both 2 and 6 are also divisible by 12.</u>
(A) 8 (B) 12 (C) 18 (D) 24 (E) 36

Example 15. On a square gameboard that is divided into n rows of n squares each, k of these squares not lie along the boundary of the gameboard. If k is one of the four numbers below, what is the possible value for n?
(I) 10 (II) 25 (III) 34 (IV) 52

(A) 10 (B) 12 (C) 14 (D) 16 (E) 18

Example 16. A student practices the four musical notes as shown, starting with the note furthest left and continuing in order from left to right. If the student plays these notes over and over according to this pattern and stops immediately after playing the shaded note, which of the following could be the total number of notes played?

(A) 50 (B) 51 (C) 52 (D) 53 (E) 64

Example 17. Which of the following statement can be used to determine if a number is divisible by 54 or not?
I. The number must be divisible by both 6 and 9.
II. The number must be divisible by both 3 and 18.
III. The number must be divisible by both 2 and 27.

(A) I only (B) III only (C) I and II only (D) I and III only (E) I, II, and III

Example 18. For what digit(s) x will the 7-digit number $3xx6xx2$ be divisible by 4?
(A) 3 (B) 6 (C) 2 (D) 5 (E) 8

Example 19. What is the largest digit which can replace b to make the number 437,b32 divisible by 3?

(A) 3 (B) 6 (C) 2 (D) 5 (E) 8

Example 20. The three-digit number 2a3 is added to the number 326 to give the three-digit number 5b9. If 5b9 is divisible by 9, then $a + b$ equal:

(A) 2 (B) 4 (C) 6 (D) 8 (E) 9

Example 21. A and B are non-zero digits for which $\underline{A468B05}$ is divisible by 11. What is $A + B$?

(A) 6 (B) 8 (C) 10 (D) 14 (E) 12

Example 22. If the 4-digit number $\underline{273X}$ is divisible by 12, what is the value of X ?

(A) 3 (B) 6 (C) 5 (D) 4 (E) 8

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Example 23. What value can a have to make $\underline{a74a}$ divisible by 36?

(A) 2 (B) 4 (C) 6 (D) 8 (E) 9

3. PROBLEMS

Problem 1. If k is a positive integer divisible by 9, and if $k < 200$, what is the greatest possible value of k?
(A) 99 (B) 189 (C) 197 (D) 198 (E) 199

Problem 2. which of the following numbers can be used to show that the statement below is FALSE?
All numbers that are divisible by both 6 and 9 are also divisible by 54.

(A) 162 (B) 108 (C) 9 (D) 72 (E) 54

Problem 3. On a rectangular gameboard that is divided into n rows of m squares each, k of these squares not lie along the boundary of the gameboard. Which of the following is a possible value for k?
(A) 15 (B) 25 (C) 35 (D) 49 (E) 52

Problem 4. When the positive integer s is divided 12, the remainder is 6. When the positive integer t is divided by 12, the remainder is 9. What is the remainder when the product st is divided by 9?
(A) 1 (B) 3 (C) 5 (D) 7 (E) 0

Problem 5. If x is an integer and $3x$ is divisible by 15, which of the following must be true?
I. x is divisible by 15.
II. x is divisible by 5.
III. x is an odd number.

(A) I only (B) II only (C) III only (D) I and II only (E) I, II, and III

Problem 6. If x is divisible by 7 and y is divisible by 8. Which of the following must be divisible by 56?
I. xy II. $7x + 8y$ III. $8x + 7y$

(A) I only (B) III only (C) I and II only (D) I and III only (E) I, II, and III

Problem 7. If a, b, c and d are different positive integers such that a is divisible by b, b is divisible by c, and c is divisible by d, which of the following statements must be true?
I. a is divisible by cd. II. a has at least 4 positive factors. III. $a = bcd$

(A) I only (B) II only (C) I and II (D) I and III only (E) I, II, and III

Problem 8. The four-digit number $\overline{6BB5}$ is divisible by 25. How many such four-digit numbers are there?
(A) 0 (B) 1 (C) 3 (D) 2 (E) 4

Problem 9. The five-digit number $\underline{31d26}$ is divisible by 3. Find the sum of all possible values of d.
(A) 18 (B) 16 (C) 15 (D) 14 (E) 8

Problem 10. The three-digit number $\underline{6x4}$ is divisible by 7. What is the value of x?
(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 11. When Rachel divides her favorite number by 7, she gets a remainder of 5. What will the remainder be if she multiplies her favorite number by 5 and then divides by 7?
(A) 4 (B) 3 (C) 2 (D) 1 (E) 0

Problem 12. For what digit n is the five-digit number $3n85n$ divisible by 6?
(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Problem 13. What digit should replace the tens digit d so that the seven-digit number $5{,}376{,}5d4$ is divisible by 24?
(A) 4 (B) 3 (C) 2 (D) 1 (E) 0

Problem 14. How many 2-digit numbers are not divisible by 13?
(A) 90 (B) 83 (C) 13 (D) 7 (E) 84

Problem 16. An item was placed on sale in January for 30% less than its original price. A final close-out sale was offered in February, and the January sale price was reduced by 40%. What percent of the original price was the final reduced price?

Problem 17. An item was placed on sale in January for 30% less than its original price. A final close-out sale was offered in February, and the January sale price was reduced by 40%. What percent of the original price was the final reduced price?

Problem 18. How many dollars would be paid in simple interest if $200 is borrowed at 12% per year for 5 months?

Problem 19. Two high school classes took the same test. One class of 20 students made an average grade of 80%; the other class of 30 students made and average grade of 70% . The average grade for all students in both classes is:
(A) 75% (B) 74% (C) 72% (D) 77% (E) none of these

Problem 20. A house and store were sold for $12,000 each. The house was sold at a loss of 20% of the cost, and the store at a gain of 20% of the cost. The entire transaction resulted in :
(A) no loss or gain (B) loss of $1000 (C) gain of $1000
(D) gain of $2000 (E) none of these

Problem 21. A housewife saved $2.50 in buying a dress on sale. If she spent $25 for the dress, she saved about:
(A) 8% (B) 9% (C) 10% (D) 11% (E) 12%

Problem 22. How much water should be added to 8 liters of 90% alcohol to make a 40% alcohol solution?

Problem 23. How many liters of water should be added to 10 liter of a 20% saline (salt) solution to make a 5% saline solution?

Problem 24. How many grams of 5% salt solution should be added to 500 grams of a 20% salt solution to make a 15% salt solution?

Problem 25. A 30% alcohol solution is added *x* grams pure water to make a 24% alcohol solution. If *x* gram pure water is added again to the solution, what was the percentage of the resulting solution?
(A) 30% (B) 20% (C) 10% (D) 8% (E) 5%

Problem 26. A sort of coals weighed 100 kg contained 14.5% water. After some time evaporating, the water was 10%. What was the ratio of the weight of the coals now to the weight of coals before evaporation?
(A) 19/20 (B) 20/19 (C) 9/10 (D) 8/10 (E) 171/180

Problem 27. A box of staples contains 4,600 staples that are either silver, black, or red. If 46 percent of the staples are silver and 46 percent are black, how many red staples are there?
(A) 2116 (B) 2484 (C) 368 (D) 828 (E) 920

Problem 28. Alex and Betsy are both salespeople. Alex's weekly compensation consists of $800 plus 30 percent of his sales. Betsy's weekly compensation consists of $600 plus 35 percent of her sales. If they both had the same amount of sales and the same compensation for a particular week, what was that compensation, in dollars? (Disregard the dollar sign when recording your answer).

Problem 29. Twenty-four is $8\frac{1}{3}\%$ of what number?

Problem 30. Find $12\frac{1}{2}\%$ of 160.

Problem 31. If q and r are positive numbers, what percent of $(q + r)$ is r ?
(A) $\frac{1}{100r(q+r)}\%$ (B) $\frac{q+r}{100r}\%$ (C) $\frac{100(q+r)}{r}\%$ (D) $(\frac{100r}{q}+r)\%$ (E) $\frac{100r}{q+r}\%$

Problem 32. On a $10,000 order a merchant has a choice between three successive discounts of 20%, 20%, and 10% and three successive discounts of 40%, 5%, and 5%. By choosing the better offer, he can save:
(A) nothing at all (B) $400 (C) $330 (D) $345 (E) $360

Problem 33. Ann borrowed $750.00 at a simple interest rate of 7.5% per year. How much will Ann owe after eight months?
(A) 37.50 (B) 795.50 (C) 56.25 (D) 787.50 (E) 800.00

Problem 34. A pharmacist wants to dilute a 10% hydrogen peroxide solution to 3%. How much distilled water must he add to make 10 liters of 3% solution?
(A) 5 (B) 7 (C) 8 (D) 10 (E) 2