1. BASIC KNOWLEDGE

1. Terms

A permutation is an arrangement or a listing of things in which order is important.

A combination is an arrangement or a listing of things in which order is not important

2. Definition

The symbol ! (factorial) is defined as follows:

$$0! = 1,$$

and for integers $n \geq 1$,

$$n! = n \cdot (n-1) \cdots 1.$$

$1! = 1,$
$2! = 2 \cdot 1 = 2,$
$3! = 3 \cdot 2 \cdot 1 = 6,$
$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24,$
$5! = 5 \cdot 4 \cdot 3 \cdot 1 \cdot 1 = 120,$
$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720.$

3. Permutations

(1). Different elements, with no repetition. Take r elements each time from n distinct elements ($1 \leq r \leq n$).

Number of permutations $P(n,r) = \dfrac{n!}{(n-r)!}$

(2). n distinct objects can be permutated in $n!$ permutations.

We let $n = r$ in (1) to get $P(n, n) = n!$

Example 1. In how many ways can the letters of the word MATH be arranged?
(A) 4 (B) 8 (C) 12 (D) 24 (E) 1

4. Combinations

Definition Let n, r be non-negative integers such that $0 \le r \le n$. The symbol $\binom{n}{r}$ (read "n choose m") is defined and denoted by

$$\binom{n}{r} = \frac{P(n,r)}{P(r,r)} = \frac{n!}{r!(n-r)!}$$

Remember: $\binom{n}{0} = 1$, $\binom{n}{1} = n$, and $\binom{n}{n} = 1$

Since $n - (n - r) = r$, we have $\binom{n}{r} = \binom{n}{n-r}$

Unlike Permutations, Combinations are used when order does not matter. If we have n different elements, and it doesn't matter which order we take the elements, the number of ways to take m elements where $1 \le m \le n$, is $\binom{n}{m}$.

Example 2. In a group of 2 cats, 3 dogs, and 10 pigs in how many ways can we choose a committee of 6 animals?
(A) 1001 (B) 2002 (C) 3003 (D) 4004 (E) 5005

☆**Example 3.** Consider the set of 5-digit positive integers. How many have at least one 7 in their decimal representation?
(A) 52488 (B) 52484 (C) 37512 (D) 90000 (E) 45000

5. The Sum Rule

If an event E_1 can happen in n_1 ways, event E_2 can happen in n_2 ways, event E_k can happen in n_k ways, and if any event E_1, E_2,.. or E_k happens, the job is done, then the total ways to do the job is $n_1 + n_2 + \cdots + n_k$.

Example 4. In how many ways can one book be selected from a book shelf of 5 paperback books and 3 hardcover books?
(A) 5 (B) 3 (C) 8 (D) 11 (E) 19

6. The Product Rule (Fundamental Counting Principle)

When a task consists of k separate parts, if the first part can be done in n_1 ways, the second part can be done in n_2 ways, and so on through the k^{th} part, which can be done in n_k ways, then total number of possible results for completing the task is given by the product:

$$n_1 \times n_2 \times n_3 \times \cdots \times n_k$$

Example 5. How many ways are there to arrange 5 people in a row of 5 seats?
(A) 5 (B) 10 (C) 12 (D) 120 (E) 720

Example 6. A girl has 5 shirts, 4 skirts, and 3 pairs of shoes. How many different outfits can she create?
(A) 120 (B) 60 (C) 15 (D) 12 (E) 20

☆**Example 7.** How many triples of three positive integers (x, y, z) have the product of 24?
(A) 20 (B) 24 (C) 27 (D) 30 (E) 21

2. PROBLEM SOLVING SKILLS

(2.1). Counting Using Charts

Example 8. Two dice are rolled. How many ways are there to roll a sum of 5?

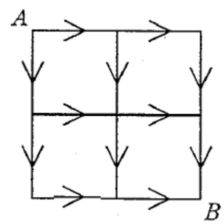
(2.2). Counting Using Tree Diagram

You can use a tree diagram to list all the possible outcomes of events.

Example 9. A designer has 3 fabric colors he may use for a dress: red, green, and blue. Four different patterns are available for the dress. If each dress design requires one color and one pattern, how many different dress designs are possible?
(A) 10 (B) 24 (C) 12 (D) 14 (E) 20

(2.3). Counting using water pipes

Example 10: Using only the line segments given in the indicated direction, how many paths are there from *A* to *B*?



☆**Example 11.** (2013 AMC 8 problem 20) Samantha lives 2 blocks west and 1 block south of the southwest corner of City Park. Her school is 2 blocks east and 2 blocks north of the northeast corner of City Park. On school days she bikes on streets to the southwest corner of City Park, then takes a diagonal path through the park to the northeast corner, and then bikes on streets to school. If her route is as short as possible, how many different routes can she take?

(A) 3 (B) 6 (C) 9 (D) 12 (E) 18

(2.4). Counting with restriction

Example 12. In how many ways can 5 books be arranged on a shelf if two of the books must remain together, but may be interchanged?

(A) 12 (B) 24 (C) 48 (D) 96 (E) 5

Example 13. How many ways can 5 distinct paperback books and 1 hardcover book be arranged on a shelf if the hardcover must be the rightmost book on the shelf?

(A) 20 (B) 60 (C) 90 (D) 120 (E) 180

☆**Example 14.** How many four –digit whole numbers do not contain the digit 1? No digit is allowed to use more than once in any such 4-digti number.
(A) 4032 (B) 2016 (C) 1024 (D) 8064 (E) 2688

(2.5) . Grouping

THEOREM 1: Let the number of different objects be n. Divide n into r groups $A_1, A_2, ..., A_r$ such that there are n_1 objects in group A_1, n_2 objects in group A_2, ..., n_r objects in the group A_r, where $n_1 + n_2 + \cdots + n_r = n$. The number of ways to do so is

$$\frac{n!}{n_1!n_2!\cdots n_r!}$$

THEOREM 2: Let there be r types of objects: n_1 of type 1, n_2 of type 2; etc. The number of ways in which these $n_1 + n_2 + \cdots + n_r = n$ objects can be rearranged is

$$\frac{n!}{n_1!n_2!\cdots n_r!}$$

Example 15. In how many ways may we distribute 6 different books to Alex, Bob, and Catherine such that each person gets 2 books?
(A) 720 (B) 120 (C) 90 (D) 60 (E) 24

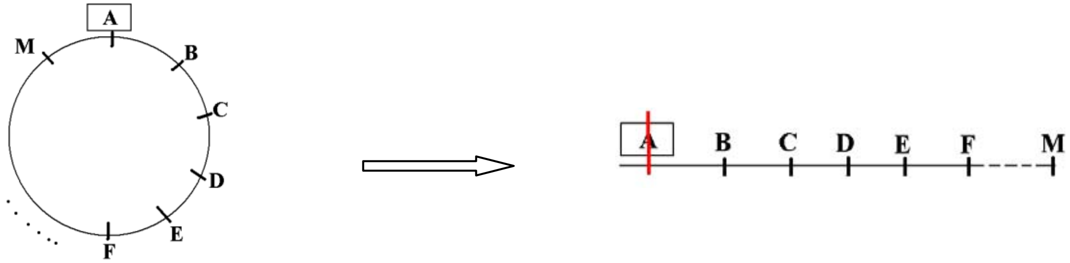
Example 16. In how many ways may we distribute 7 different books to Alex, Bob, and Catherine such that Alex and Bob each get 2 books and Catherine gets 3 books?
(A) 2520 (B) 840 (C) 630 (D) 210 (E) 120

Example 17. In how many different ways can all the letters in INDIANA be arranged in a line? Assume that duplicate letters are indistinguishable.
A. 5040 B. 2520 C. 1260 D. 630 E. none of these

(2.6) . Circular Permutations

THEOREM 3: The number of circular permutations (arrangements in a circle) of n distinct objects is $(n-1)!$.

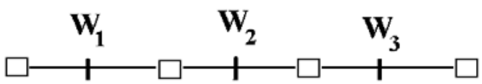
We can think of this as n people being seated at a round table. Since a rotation of the table does not change an arrangement, we can put person A in one fixed place and then consider the number of ways to seat all the others. Person B can be treated as the first person to seat and M the last person to seat. The number of ways to arrange persons A to M is the same as the number of ways to arrange persons B to M in a row. So the number of ways is $(n-1)!$.



Example 18. In how many ways is it possible to seat seven people at a round table?

Example 19. In how many ways is it possible to seat seven people at a round table if Alex and Bob must not sit in adjacent seats?
(A) 720 (B) 480 (C) 120 (D) 60 (E) 90

Example 20. In how many ways can four men and four women be seated at a round table if no two men are to be in adjacent seats?
(A) 120 (B) 144 (C) 121 (D) 60 (E) 72



Example 21. In how many ways can a family of six people be seated at a round table if the youngest kid must sit between the parents?
(A) 6 (B) 7 (C) 10 (D) 12 (E) 24

(2.7). Combinations with Repetitions

THEOREM 4: Let n be a positive integer. The number of positive integer solutions to $x_1 + x_2 + \cdots + x_r = n$ is $\binom{n-1}{r-1}$.

☆**Example 22.** Three friends have a total of 6 identical pencils, and each one has at least one pencil. In how many ways can this happen?

(A) 6 (B) 7 (C) 10 (D) 12 (E) 24

Example 23. In how many ways may we write the number 9 as the sum of three positive integer summands? Here order counts, so, for example, 1 + 7 + 1 is to be regarded different from 7 + 1 + 1.

(A) 16 (B) 27 (C) 30 (D) 14 (E) 28

THEOREM 5: Let n be a positive integer. The number of non-negative integer solutions to $y_1 + y_2 + \cdots + y_r = n$ is $\binom{n+r-1}{n}$ or $\binom{n+r-1}{r-1}$.

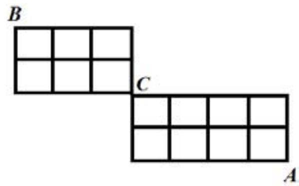
☆**Example 24.** Three friends have a total of 6 identical pencils. In how many ways can this happen?

(A) 10 (B) 12 (C) 28 (D) 24 (E) 16

THEOREM 6: The number of ways to walk from one corner to another corner of an m by n grid can be calculated by the following formula: $N = \binom{m+n}{n}$, where m is the number of rows and n the number of column.

Example 25. How many ways are there to get from A to B if you can only go north or west?

(A) 120 (B) 130 (C) 140 (D) 150 (E) 160



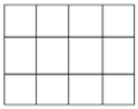
☆**Example 26.** (2013 AMC 8 problem 20) Samantha lives 2 blocks west and 1 block south of the southwest corner of City Park. Her school is 2 blocks east and 2 blocks north of the northeast corner of City Park. On school days she bikes on streets to the southwest corner of City Park, then takes a diagonal path through the

park to the northeast corner, and then bikes on streets to school. If her route is as short as possible, how many different routes can she take?
(A) 3 (B) 6 (C) 9 (D) 12 (E) 18

THEOREM 7: Counting How Many Rectangles

For a rectangular grid with m vertical lines and n horizontal lines, the total number of rectangles that can be counted is $N = \binom{m}{2} \times \binom{n}{2}$.

Example 27. Consider the figure shown below. How many rectangles are there?



(A) 30 (B) 60 (C) 90 (D) 120 (E) 180

THEOREM 8: Rising (Increasing) Number

A rising number, such as 34689, is a positive integer where each digit is larger than the one to its left.

The number of integers with digits in increasing order can be calculated by $\binom{9}{n}$.

Example 28. How many 3-digit increasing numbers are there?
(A) 34 (B) 64 (C) 84 (D) 92 (E) 120

THEOREM 9: Falling (Decreasing) Number

A falling number is an integer whose decimal representation has the property that each digit except the units digit is larger than the one to its right. For example, 96521 is a falling number but 89642 is not. The number of integers with digits in decreasing order can be calculated: $\binom{10}{n}$.

Example 29. How many 3-digit falling numbers are there?
(A) 60 (B) 90 (C) 120 (D) 240 (E) 180

THEOREM 10: Palindrome Numbers

A **palindrome number** is a number that is the same when written forwards or backwards. The first few palindrome numbers are $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 22, 33, 44, 55, 66, 77, 88, 99, 101, 111$, and 121.

The number of palindromes with n digits can be found by $9\times10^{\left\lfloor \frac{n-1}{2} \right\rfloor}$ for $n > 1$.
$\lfloor x \rfloor$ is the floor function.

Example 30. How many 3-digit palindromes are there?
(A) 30 (B) 60 (C) 90 (D) 120 (E) 180

3. PROBLEMS

Problem 1. Any 2 points determine a line. If there are 7 points in a plane, no 3 of which lie on the same line, how many lines are determined by pairs of these 7 points?
(A) 15 (B) 18 (C) 21 (D) 30 (E) 36

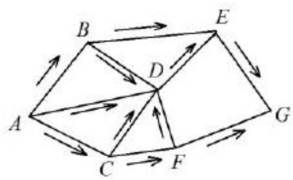
Problem 2. An electrician is testing 7 different wires. For each test, the electrician chooses 2 of the wires and connects them. What is the least number of tests that must be done so that every possible pair of wires is tested?
(A) 7 (B) 14 (C) 16 (D) 18 (E) 21

Problem 3. A snack machine has buttons arranged as shown above. If a selection is made by choosing 2 letters followed by 3 digits, what is the greatest number of different selections that could be made?

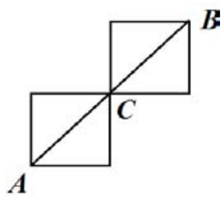
A	D		1	4	7
B	E		2	5	8
C	F		3	6	9

(A) 15 (B) 84 (C) 120 (D) 1260 (E) 1024

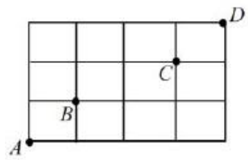
Problem 4. In the figure above, each line segment represents a one-way road with travel permitted only in the direction indicated by the arrow. How many different routes from A to G are possible?
(A) One (B) Six (C) Three (D) Four (E) Seven



Problem 5. A boy is walking along the line starting from point A to point B. Any point of intersection and line cannot be walked twice in one trip.
How many ways are there?
(A) 15 (B) 12 (C) 10 (D) 9 (E) 6

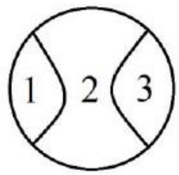


Problem 6. In the figure shown, a path from point *A* to point *D* is determined by moving upward or to the right along the grid lines. How many different paths can be drawn from *A* to *D* that must include both *B* and *C*?



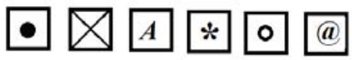
(A) 12 (B) 16 (C) 10 (D) 20 (E) 6

Problem 7. As shown above, a circular shape is divided into 3 regions. A certain design is to be painted such that two regions next to each other must have the different colors. If 6 different colors are available for the design, how many differently pained designs are possible?



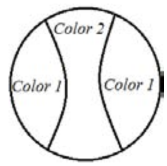
(A) 120 (B) 80 (C) 100 (D) 150 (E) 160

Problem 8. If the 6 cards shown above are placed in a row so that the card with a *A* on it is never at either end, how many different arrangements are possible?



(A) 2400 (B) 480 (C) 960 (D) 720 (E) 460

Problem 9. As shown above, a certain design is to be painted using 2 different colors. If 6 different colors are available for the design, how many differently pained designs are possible?

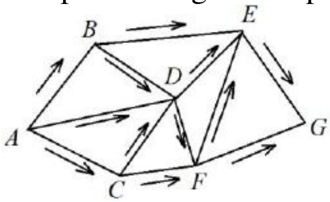


(A) 10 (B) 20 (C) 25 (D) 30 (E) 120

Problem 10. How many different even natural numbers each containing three distinct digits can be written using just the digits 0, 2, 3, 5, and 8?
(A) 6 (B) 12 (C) 20 (D) 30 (E) 36

Problem 11. In how many ways can four married couples be seated at a round table if no two men, as well as no husband and wife are to be in adjacent seats?
(A) 6 (B) 12 (C) 24 (D) 48 (E) 96

Problem 12. In the figure above, each line segment represents a one-way road with travel permitted only in the direction indicated by the arrow. How many different routes from A to G that pass through E are possible?



(A) Four (B) Five (C) Six (D) Seven (E) Eight

Problem 13. Ten points are placed on a circle. What is the greatest number of different lines that can be drawn so that each line passes through two of these points?
(A) 15 (B) 25 (C) 35 (D) 45 (E) 90

Problem 14. Four lines are drawn in a plane so that there are exactly three different intersection points. Into how many nonoverlapping regions do these lines divide the plane?
(A) Seven (B) Nine (C) Eleven (D) Thirteen (E) Fifteen

Problem 15. The four distinct points P, Q, R and S lie on a line l: the five distinct points T, U, V, W , X, and Y lie on a different line that is parallel to line l. What is the total number of different lines that can be drawn so that each line contains exactly two of the seven points?
(A) 9. (B) 12. (C) 10. (D) 20. (E) 40.

Problem 16. A number is called increasing if each of its digits is greater than the digit immediately to its left, if there is one. How many increasing numbers are there between 100 and 200?
(A) 100 (B) 101 (C) 20 (D) 28 (E) 30

Problem 17. In a volleyball league with 5 teams, each team plays exactly 3 games with each of the other 4 teams in the league. What is the total number of games played in this league?
(A) 15 (B) 20 (C) 12 (D) 25 (E) 30