Special nonlocal game with constant alphabet

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1 Components of the new game

1.1 The linear system game

Following ideas from Ref. [CS17], we would like to embed the qudit Pauli group in a solution group and make the d-dependence implicit. The qudit Pauli group is defined as

$$\mathcal{P}_d = \langle x, z, \mathcal{J} : x^d = z^d = \mathcal{J}^d = e, zxz^{-1}x^{-1} = \mathcal{J}, x\mathcal{J}x^{-1}\mathcal{J}^{-1} = z\mathcal{J}z^{-1}\mathcal{J}^{-1} = e\rangle. \tag{1}$$

Firstly, we would like to replace the relation $x^d = z^d = e$ by the following relations

$$u_x x u_x = x^2 \tag{2}$$

$$u_z z u_z = z^2. \tag{3}$$

With constraints imposed by other components of the whole game, the two conditions above imply that the eigenvalues of x and z are $\{\omega_d^k = e^{ik\pi/d}\}_{k=1}^d$ with d odd, prime and $\mathbb{Z}/(d\mathbb{Z})$ has primitive root 2 and the eigenspaces for different eigenvalues are of the same dimension. Later we will set the exponent to be $a \in \{2, 3, 5\}$ which is the primitive root of infinitely many prime numbers and construct a nonlocal game that can self-test infinitely many maximally entangled state. [1]

Secondly, we drop the relation $\mathcal{J}^d = e$ and make the value of $\mathcal{J}$ determined by x and z in the relation $xzx^{-1}z^{-1} = \mathcal{J}$.

It can be easily checked that the qudit Pauli-x and Pauli-z operators of dimension d satisfy the new relations above, where Pauli-x and Pauli-z operators are defined by

$$\sigma_x = \sum_{i=0}^{d-1} |i+1 \pmod{d}\rangle\langle i| \qquad \sigma_z = \sum_{i=0}^{d-1} \omega_d^i |i\rangle\langle i|. \tag{4}$$

Moreover, if $U_x\sigma_x U_x^\dagger = \sigma_x^2$ and $U_z\sigma_z U_z^\dagger = \sigma_z^2$, we can verify that

$$U_x U_z = \mathbb{1} = U_z U_x, \tag{5}$$

[1]Figuring out what a is will take us one step closer to resolving Artin's Conjecture[Mur88].

Hence, we define the new group by

$$\begin{aligned}\mathcal{P} = \langle x, z, u, \mathcal{J} : & zxz^{-1}x^{-1} = \mathcal{J}, [x, \mathcal{J}] = [z, \mathcal{J}] = [u, \mathcal{J}] = e, \\ & uxu^{-1} = x^2, u^{-1}zu = z^2 \rangle.\end{aligned} \tag{6}$$

Since $\mathcal{P}_d$'s relations satisfy the relations of $\mathcal{P}$, can we say $\mathcal{P}_d$ is a subgroup of $\mathcal{P}$? This group will be embedded in a solution group, $\Gamma_{\mathcal{P}}$, following Slofstra's embedding techniques. See Appendix A for details.

1.2 The extended weighted CHSH game

We want to force Alice and Bob to reuse observables from the linear system game developed above so that we can make sure that the observables have eigenvalues ω_d and ω_d^{d-1}, and the corresponding eigen-spaces are of dimension 1. The modified weighted CHSH game to force the structure of observable U will be denoted by $CHSH_U^{(d)}$, where the superscript d means that the rules of this game depends on d. Intuitively, we want to combine $CHSH_X^{(d)}$ and the fact that $X \sim X^2$ to achieve the same effect as $(X)^d = \mathbb{1}$.

Assume Bob use observable B_1 and B_2, such that $B_1 B_2 = U$. The special condition B_1, B_2 satisfy is that the eigenvalues for U are $\{\omega_d^i\}_{i=0}^{d-1}$. Let $|v_1\rangle$ and $|v_{d-1}\rangle$ be the eigenvectors for eigenvalues ω_d and ω_d^{d-1}. In our case, we can show that when $U = \sigma_x$ or σ_z, such B_1, B_2 exists. I am not sure if such decomposition exists in general.

In another note, I have shown that B_1, B_2 with shared state $\frac{1}{\sqrt{2}}(|u_1\rangle|u_1\rangle + |u_{d-1}\rangle|u_{d-1}\rangle)$ can maximize $\langle I_{-\cot(\pi/2d)}\rangle$ where $|u_1\rangle, |u_{d-1}\rangle \in \mathrm{span}\{|v_1\rangle, |v_{d-1}\rangle\}$.

So Alice and Bob will each get a symbol $x,y \in \{0, 1, *\}$ respectively and they answer with $a, b \in \{0, 1, \diamond, \perp\}$. The scoring rules are

- **Case 1:** $x = y = *$, Alice and Bob should answer with $a, b \in \{\diamond, \perp\}$ and they score only if $a = b$;
- **Case 2a:** $x, y \in \{0, 1\}$ and they answer with $a, b \in \{0, 1\}$, then their answers are scored according to $I_{-\cot(\pi/2d)}$
- **Case 2b:** $x, y \in \{0, 1\}$ and Alice answer with $\perp$, then all possible outputs from Bob are discarded;
- **Case 3:** $x \in 0, 1, y = *$, when Bob answers $\diamond$, Alice should answer with $\{0.1\}$ but not $\perp$, when Bob answers $\perp$ Alice should answer $\perp$ too;
- **Case 4:** other combination of inputs are not scored.

In the ideal strategy Alice and Bob share the state $|\psi\rangle = 1/\sqrt{d}\sum_{i=0}^{d-1}|u_i\rangle|u_i\rangle$. We define two subspaces $V = \mathrm{span}\{|u_1\rangle, |u_{d-1}\rangle\}$ and $V^{\perp} = \mathbb{C}^d \setminus \mathrm{span}\{|u_1\rangle, |u_{d-1}\rangle\}$ amd define Π_V and $\Pi_V^{\perp}$ to be the corresponding projectors. Note that V is the subspace on which they should play the $CHSH_U^{(d)}$ to maximaize $\langle I_{-\cot(\pi/2d)}\rangle$.

We present the measurements along with the ideal correlation. The correlation for the first test

$$P(\diamond\diamond\,|\,**) = \langle\psi|\Pi_V \otimes \Pi_V|\psi\rangle = \frac{2}{d} \tag{7}$$
$$P(\perp\perp\,|\,**) = \langle\psi|\Pi_V^{\perp} \otimes \Pi_V^{\perp}|\psi\rangle = \frac{d-2}{d} \tag{8}$$
$$P(\diamond\perp\,|\,**) = \langle\psi|\Pi_V \otimes \Pi_V^{\perp}|\psi\rangle = 0 \tag{9}$$
$$P(\perp\diamond|\,**) = \langle\psi|\Pi_V^{\perp} \otimes \Pi_V|\psi\rangle = 0 \tag{10}$$

CHSH-type correlations

$$P(00|00) = \langle\psi|\left[|u_1\rangle\langle u_1| \otimes |u_1^{(0)}\rangle\langle u_1^{(0)}|\right]|\psi\rangle = \frac{\cos^2(\pi/4d)}{d} \tag{11}$$
$$P(01|00) = \langle\psi|\left[|u_1\rangle\langle u_1| \otimes |u_{d-1}^{(0)}\rangle\langle u_{d-1}^{(0)}|\right]|\psi\rangle = \frac{\sin^2(\pi/4d)}{d} \tag{12}$$
$$P(10|00) = \langle\psi|\left[|u_{d-1}\rangle\langle u_{d-1}| \otimes |u_1^{(0)}\rangle\langle u_1^{(0)}|\right]|\psi\rangle = \frac{\sin^2(\pi/4d)}{d} \tag{13}$$
$$P(11|00) = \langle\psi|\left[|u_{d-1}\rangle\langle u_{d-1}| \otimes |u_{d-1}^{(0)}\rangle\langle u_{d-1}^{(0)}|\right]|\psi\rangle = \frac{\cos^2(\pi/4d)}{d} \tag{14}$$
$$P(00|01) = \langle\psi|\left[|u_1\rangle\langle u_1| \otimes |u_1^{(1)}\rangle\langle u_1^{(1)}|\right]|\psi\rangle = \frac{\cos^2(\pi/4d)}{d} \tag{15}$$
$$P(01|01) = \langle\psi|\left[|u_1\rangle\langle u_1| \otimes |u_{d-1}^{(1)}\rangle\langle u_{d-1}^{(1)}|\right]|\psi\rangle = \frac{\sin^2(\pi/4d)}{d} \tag{16}$$
$$P(10|01) = \langle\psi|\left[|u_{d-1}\rangle\langle u_{d-1}| \otimes |u_1^{(1)}\rangle\langle u_1^{(1)}|\right]|\psi\rangle = \frac{\sin^2(\pi/4d)}{d} \tag{17}$$
$$P(11|01) = \langle\psi|\left[|u_{d-1}\rangle\langle u_{d-1}| \otimes |u_{d-1}^{(1)}\rangle\langle u_{d-1}^{(1)}|\right]|\psi\rangle = \frac{\cos^2(\pi/4d)}{d} \tag{18}$$
$$P(00|10) = \langle\psi|\left[|u_1^{(+)}\rangle\langle u_1^{(+)}| \otimes |u_1^{(0)}\rangle\langle u_1^{(0)}|\right]|\psi\rangle = \frac{1+\sin(\pi/2d)}{2d} \tag{19}$$
$$P(01|10) = \langle\psi|\left[|u_1^{(+)}\rangle\langle u_1^{(+)}| \otimes |u_{d-1}^{(0)}\rangle\langle u_{d-1}^{(0)}|\right]|\psi\rangle = \frac{1-\sin(\pi/2d)}{2d} \tag{20}$$
$$P(10|10) = \langle\psi|\left[|u_{d-1}^{(+)}\rangle\langle u_{d-1}^{(+)}| \otimes |u_1^{(0)}\rangle\langle u_1^{(0)}|\right]|\psi\rangle = \frac{1-\sin(\pi/2d)}{2d} \tag{21}$$
$$P(11|10) = \langle\psi|\left[|u_{d-1}^{(+)}\rangle\langle u_{d-1}^{(+)}| \otimes |u_{d-1}^{(0)}\rangle\langle u_{d-1}^{(0)}|\right]|\psi\rangle = \frac{1+\sin(\pi/2d)}{2d} \tag{22}$$
$$P(00|11) = \langle\psi|\left[|u_1^{(+)}\rangle\langle u_1^{(+)}| \otimes |u_1^{(1)}\rangle\langle u_1^{(1)}|\right]|\psi\rangle = \frac{1-\sin(\pi/2d)}{2d} \tag{23}$$
$$P(01|11) = \langle\psi|\left[|u_1^{(+)}\rangle\langle u_1^{(+)}| \otimes |u_{d-1}^{(1)}\rangle\langle u_{d-1}^{(1)}|\right]|\psi\rangle = \frac{1+\sin(\pi/2d)}{2d} \tag{24}$$
$$P(10|11) = \langle\psi|\left[|u_{d-1}^{(+)}\rangle\langle u_{d-1}^{(+)}| \otimes |u_1^{(1)}\rangle\langle u_1^{(1)}|\right]|\psi\rangle = \frac{1+\sin(\pi/2d)}{2d} \tag{25}$$
$$P(11|11) = \langle\psi|\left[|u_{d-1}^{(+)}\rangle\langle u_{d-1}^{(+)}| \otimes |u_{d-1}^{(1)}\rangle\langle u_{d-1}^{(1)}|\right]|\psi\rangle = \frac{1-\sin(\pi/2d)}{2d}. \tag{26}$$

Other correlations

$$P(\perp|0)_A = P(\perp|1)_A = \langle\psi|\Pi_V^{\perp}\otimes\mathbb{1}|\psi\rangle = \frac{d-2}{d} \tag{27}$$

$$P(0\diamond|0*) = \langle\psi|\left[|u_1\rangle\langle u_1|\otimes\Pi_V\right]|\psi\rangle = \frac{1}{d} \tag{28}$$

$$P(1\diamond|0*) = \langle\psi|\left[|u_{d-1}\rangle\langle u_{d-1}|\otimes\Pi_V\right]|\psi\rangle = \frac{1}{d} \tag{29}$$

$$P(0\perp|0*) = \langle\psi|\left[|u_1\rangle\langle u_1|\otimes\Pi_V^{\perp}\right]|\psi\rangle = 0 \tag{30}$$

$$P(1\perp|0*) = \langle\psi|\left[|u_{d-1}\rangle\langle u_{d-1}|\otimes\Pi_V^{\perp}\right]|\psi\rangle = 0 \tag{31}$$

$$P(\perp\diamond|0*) = \langle\psi|\Pi_V^{\perp}\otimes\Pi_V|\psi\rangle = 0 \tag{32}$$

$$P(\perp\perp|0*) = \langle\psi|\Pi_V^{\perp}\otimes\Pi_V^{\perp}|\psi\rangle = \frac{d-2}{d}. \tag{33}$$

when the input is $(1,*)$, the correlation is similar as above. We define the states for the measurements as follows

$$|u_1^{(0)}\rangle = \cos(\pi/4d)|u_1\rangle + \sin(\pi/4d)|u_{d-1}\rangle \tag{34}$$

$$|u_{d-1}^{(0)}\rangle = \sin(\pi/4d)|u_1\rangle - \cos(\pi/4d)|u_{d-1}\rangle \tag{35}$$

$$|u_1^{(1)}\rangle = \cos(\pi/4d)|u_1\rangle - \sin(\pi/4d)|u_{d-1}\rangle \tag{36}$$

$$|u_{d-1}^{(1)}\rangle = \sin(\pi/4d)|u_1\rangle + \cos(\pi/4d)|u_{d-1}\rangle \tag{37}$$

$$|u_1^{(+)}\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + \frac{1}{\sqrt{2}}|u_{d-1}\rangle \tag{38}$$

$$|u_{d-1}^{(+)}\rangle = \frac{1}{\sqrt{2}}|u_1\rangle - \frac{1}{\sqrt{2}}|u_{d-1}\rangle. \tag{39}$$

Observation 1. We have some orthogonality requirements as follows,

$$\langle\psi|A_*^{\perp}B_*^{\diamond}|\psi\rangle = \langle\psi|A_*^{\diamond}B_*^{\perp}|\psi\rangle = 0, \tag{40}$$

$$\langle\psi|A_0^0B_*^{\perp}|\psi\rangle = \langle\psi|A_0^1B_*^{\perp}|\psi\rangle = 0 \tag{41}$$

$$\langle\psi|A_1^0B_*^{\perp}|\psi\rangle = \langle\psi|A_1^1B_*^{\perp}|\psi\rangle = 0. \tag{42}$$

We know $A_*^{\perp}B_*^{\diamond}|\psi\rangle$, $A_*^{\diamond}B_*^{\perp}|\psi\rangle$, $A_0^0B_*^{\perp}|\psi\rangle$, $A_0^1B_*^{\perp}|\psi\rangle$, $A_1^0B_*^{\perp}|\psi\rangle$ and $A_1^1B_*^{\perp}|\psi\rangle$are orthogonal to $|\psi\rangle$. We also know that $A_0^0|\psi\rangle$,$A_0^1|\psi\rangle$,$A_1^0|\psi\rangle$,$A_1^1|\psi\rangle$ and $A_*^{\diamond}|\psi\rangle$ are orthogonal to $B_*^{\perp}|\psi\rangle$. The last orthogonality relation is that $A_*^{\perp}|\psi\rangle \perp B_*^{\diamond}|\psi\rangle$.

Observation 2. We can also observe that

$$\begin{aligned}
&\langle\psi|A_*^{\diamond}B_*^{\diamond}|\psi\rangle & (43)\\
=&\langle\psi|(A_0^0+A_0^1)(B_0^0+B_0^1)|\psi\rangle = \langle\psi|(A_0^0+A_0^1)(B_1^0+B_1^1)|\psi\rangle & (44)\\
=&\langle\psi|(A_1^0+A_1^1)(B_0^0+B_0^1)|\psi\rangle = \langle\psi|(A_1^0+A_1^1)(B_1^0+B_1^1)|\psi\rangle & (45)\\
=&\langle\psi|(A_0^0+A_0^1)B_*^{\diamond}|\psi\rangle = \langle\psi|(A_1^0+A_1^1)B_*^{\diamond}|\psi\rangle & (46)\\
=&2/d. & (47)
\end{aligned}$$

Since we know $\|B_*^{\diamond}|\psi\rangle\| = \|(A_0^0+A_0^1)|\psi\rangle\| = \sqrt{2/d}$, combining with the relation, we find that

$$\frac{\langle\psi|B_*^{\diamond}(A_0^0+A_0^1)B_*^{\diamond}|\psi\rangle}{\|B_*^{\diamond}|\psi\rangle\|^2} = 1, \tag{48}$$

which means that

$$(A_0^0 + A_0^1)B_*^\diamond|\psi\rangle = B_*^\diamond|\psi\rangle. \tag{49}$$

Since the projective measurement $(A_0^0 + A_0^1)$ commute with $B_*^\diamond$, we can get

$$\frac{\langle\psi|(A_0^0 + A_0^1)B_*^\diamond(A_0^0 + A_0^1)|\psi\rangle}{\|(A_0^0 + A_0^1)|\psi\rangle\|^2} = 1, \tag{50}$$

with similar argument we get

$$B_*^\diamond(A_0^0 + A_0^1)|\psi\rangle = (A_0^0 + A_0^1)|\psi\rangle. \tag{51}$$

The two conclusions above can be chained by commutativity to reach the conclusion that

$$(A_0^0 + A_0^1)|\psi\rangle = B_*^\diamond|\psi\rangle. \tag{52}$$

Following the same line of argument, we can conclude that

$$B_*^\diamond|\psi\rangle = A_*^\diamond|\psi\rangle = (A_0^0 + A_0^1)|\psi\rangle = (A_1^0 + A_1^1)|\psi\rangle. \tag{53}$$

Another conclusion we can draw is that $B_*^\diamond = A_*^\diamond$ and $B_*^\perp = A_*^\perp$. We can assume the Schmidt decomposition of $|\psi\rangle$ is

$$|\psi\rangle = \sum_{i=1}^{d} \lambda_i|i\rangle|i\rangle. \tag{54}$$

If supp $A_*^\diamond \neq$ supp $B_*^\diamond$, then $B_*^\diamond|\psi\rangle \neq A_*^\diamond|\psi\rangle$. Since $B_*^\diamond$ and $A_*^\diamond$ are projectors, By the orthogonality relations, we can also conclude that $A_*^\perp = B_*^\perp$.

Observation 3. We can also observe that

$$\begin{aligned}
&\langle\psi|A_0^\perp|\psi\rangle = \langle\psi|A_1^\perp|\psi\rangle && (55)\\
=&\langle\psi|A_0^\perp B_*^\perp|\psi\rangle = \langle\psi|A_1^\perp B_*^\perp|\psi\rangle && (56)\\
=&\langle\psi|A_*^\perp B_*^\perp|\psi\rangle && (57)\\
=&\frac{d-2}{d}. && (58)
\end{aligned}$$

By similar argument as in Observation 2, we conclude that

$$B_*^\perp|\psi\rangle = A_*^\perp|\psi\rangle = A_0^\perp|\psi\rangle = A_1^\perp|\psi\rangle. \tag{59}$$

Now we re-examine the CHSH-type correlation

$$\begin{aligned}
&\langle\psi|A_0^0 B_0^0|\psi\rangle && (60)\\
=&\langle\psi|(A_*^\diamond + A_*^\perp)A_0^0 B_0^0(A_*^\diamond + A_*^\perp)|\psi\rangle && (61)\\
=&\langle\psi|A_*^\diamond A_0^0 B_0^0 A_*^\diamond|\psi\rangle + \langle\psi|A_*^\diamond A_0^0 B_0^0 A_*^\perp|\psi\rangle && (62)\\
&+ \langle\psi|A_*^\perp A_0^0 B_0^0 A_*^\diamond|\psi\rangle + \langle\psi|A_*^\perp A_0^0 B_0^0 A_*^\perp|\psi\rangle && (63)\\
=&\langle\psi|B_*^\diamond A_0^0 B_0^0 B_*^\diamond|\psi\rangle + \langle\psi|A_*^\diamond A_0^0 B_0^0 A_0^\perp|\psi\rangle && (64)\\
&+ \langle\psi|A_0^\perp A_0^0 B_0^0 A_*^\diamond|\psi\rangle + \langle\psi|A_0^\perp A_0^0 B_0^0 A_0^\perp|\psi\rangle && (65)\\
=&\langle\psi|B_*^\diamond A_0^0 B_0^0 B_*^\diamond|\psi\rangle, && (66)
\end{aligned}$$

where we use the facts that $B_*^\diamond|\psi\rangle = A_*^\diamond|\psi\rangle$, $A_*^\perp|\psi\rangle = A_0^\perp|\psi\rangle$ and that $\mathrm{span}(A_0^0) \cap \mathrm{span}(A_0^\perp) = \emptyset$. This means that if Alice and Bob share state $B_*^\diamond|\psi\rangle/\|B_*^\diamond|\psi\rangle\|$ and apply $A_0^0 B_0^0$ we get conditional probability

$$\frac{\langle\psi|B_*^\diamond A_0^0 B_0^0 B_*^\diamond|\psi\rangle}{\langle\psi|B_*^\diamond B_*^\diamond|\psi\rangle} = \frac{\cos^2(\pi/4d)}{2}. \tag{67}$$

We can rewrite the other correlations in a similar way and get a new set of correlation which match exactly with the optimal correlation of $\langle \mathbb{1}_{-\cot(\pi/2d)}\rangle$ and use the self-testing argument on $\{A_0^0 - A_0^1, B_0^0 - B_0^1, A_1^0 - A_1^1, B_1^0 - B_1^1\}$ and $B_*^\diamond|\psi\rangle/\|B_*^\diamond|\psi\rangle\|$.

1.3 The single value test

By the following test, we want to make sure that $B_1 B_2$ has eigenvalue 1. The key observation is that $B_1 B_2 |psi\rangle = |\psi\rangle$ implies that

$$|\psi\rangle = B_1|\psi\rangle = B_2|\psi\rangle. \tag{68}$$

Alice and Bob will each get a symbol x,$y \in \{0, 1, *\}$ respectively and they answer with $a, b \in \{0, 1, \diamond, \perp\}$. The scoring rules are

- **Case 1:** $x = y = *$, Alice and Bob should answer with $a, b \in \{\diamond, \perp\}$ and they score only if $a = b$;
- **Case 2a:** $x = *$ and $y = 1$, when Alice answer $\diamond$, Bob should answer 0, when Alice answer$\perp$ all answers of Bob are accepted;
- **Case 2b:** $x = *$ and $y = 2$, when Alice answer $\diamond$, Bob should answer 0, when Alice answer$\perp$ all answers of Bob are accepted.

The ideal strategy has projective measurement

$$A_*^\diamond = B_*^\diamond = |u_0\rangle\langle u_0|, \qquad A_*^\perp = B_*^\perp = \mathbb{1} - |u_0\rangle\langle u_0|, \tag{69}$$

and Bob will reuse B_1 B_2 from his strategy to win the linear system game. The shared state is

$$|\psi\rangle = \frac{1}{d}\sum_{i\in[d]} |u_i\rangle|u_i\rangle. \tag{70}$$

So in the ideal correlation, we have

$$P(\diamond\diamond \mid **) = \langle\psi|A_*^\diamond \otimes B_*^\diamond|\psi\rangle = \frac{1}{d} \tag{71}$$

$$P(\perp\perp \mid **) = \langle\psi|A_*^\perp \otimes B_*^\perp|\psi\rangle = \frac{d-1}{d} \tag{72}$$

$$P(\diamond \perp \mid **) = \langle\psi|A_*^\diamond \otimes B_*^\perp|\psi\rangle = 0 \tag{73}$$

$$P(\perp \diamond \mid **) = \langle\psi|A_*^\perp \otimes B_*^\diamond|\psi\rangle = 0 \tag{74}$$

$$P(\diamond 0 \mid *1) = \langle\psi|A_*^\diamond \otimes B_1^0|\psi\rangle = \frac{1}{d} \tag{75}$$

$$P(\diamond 0 \mid *2) = \langle\psi|A_*^\diamond \otimes B_2^0|\psi\rangle = \frac{1}{d} \tag{76}$$

By the marginal distribution we know that $\|A_*^\diamond|\psi\rangle\| = \frac{1}{\sqrt{d}}$. From the condition that $P(\diamond 0|*1) = 1/d$, we know

$$\frac{\langle\psi|A_*^\diamond B_1^0|\psi\rangle}{\|A_*^\diamond|\psi\rangle\|^2} = 1, \tag{77}$$

which implies that

$$B_1^0 \frac{A_*^\diamond|\psi\rangle}{\|A_*^\diamond|\psi\rangle\|} = \frac{A_*^\diamond|\psi\rangle}{\|A_*^\diamond|\psi\rangle\|}. \tag{78}$$

With similar reasoning, we get

$$B_2^0 \frac{A_*^\diamond|\psi\rangle}{\|A_*^\diamond|\psi\rangle\|} = \frac{A_*^\diamond|\psi\rangle}{\|A_*^\diamond|\psi\rangle\|}. \tag{79}$$

Hence $\frac{A_*^\diamond|\psi\rangle}{\|A_*^\diamond|\psi\rangle\|}$ is in the intersection of $\mathrm{supp}(B_1^0)$ and $\mathrm{supp}(B_2^0)$. Since $\mathrm{supp}(B_x^0)$ and $\mathrm{supp}(B_x^1)$ are disjoint, we know

$$B_x \frac{A_*^\diamond|\psi\rangle}{\|A_*^\diamond|\psi\rangle\|} = \frac{A_*^\diamond|\psi\rangle}{\|A_*^\diamond|\psi\rangle\|} \tag{80}$$

for $x = 1, 2$. Therefor, we know $\frac{A_*^\diamond|\psi\rangle}{\|A_*^\diamond|\psi\rangle\|}$ is an eigenvector of $B_1 B_2$ with eigenvalue 1 because

$$B_1 B_2 \frac{A_*^\diamond|\psi\rangle}{\|A_*^\diamond|\psi\rangle\|} = B_1 \frac{A_*^\diamond|\psi\rangle}{\|A_*^\diamond|\psi\rangle\|} = \frac{A_*^\diamond|\psi\rangle}{\|A_*^\diamond|\psi\rangle\|}. \tag{81}$$

In the later section, when we apply this test to make sure some unitary U has eigenvalue 1 with one-dimensional eigen-space, we denote the test by SVT_U.

2 The new game

Assume the linear system game has n variables and m equations. Alice receives $x \in \{1, \ldots m+8\}$ and Bob receives $y \in \{1, \ldots, n+4\}$. The scoring rules are the following:

- when $x \in \{1, \ldots m\}$ and $y \in \{1, \ldots n\}$, they are scored according to the linear system game;
- when $x \in \{m+1, m+2, m+3\}$ and $y \in \{1, 2, n+1\}$, they are scored according to $CHSH_{\sigma_x}^{(d)}$, where

  $$*_A = m+1, \quad 0_A = m+2, \quad 1_A = m+3, \tag{82}$$
  $$*_B = n+1, \quad 0_B = 1, \quad 1_A = 2, \tag{83}$$

  are the inputs for the game $CHSH_{\sigma_x}^{(d)}$ (The intuition behind is that $B_1 B_2 = \sigma_x$.);
- when $x \in \{m+4, m+5, m+6\}$ and $y \in \{3, 4, n+2\}$, they are scored according to $CHSH_{\sigma_z}^{(d)}$, where

  $$*_A = m+4, \quad 0_A = m+5, \quad 1_A = m+6, \tag{84}$$
  $$*_B = n+2, \quad 0_B = 3, \quad 1_A = 4, \tag{85}$$

  are the inputs for the game $CHSH_{\sigma_z}^{(d)}$ (The intuition behind is that $B_3 B_4 = \sigma_z$.);

- when $x = m+7$ and $y \in \{1, 2, n+3\}$, they are scored according to SVT_{σ_x}, where

$$*_A = m+7 \tag{86}$$
$$*_B = n+3, \quad 1_B = 1, \quad 2_B = 2 \tag{87}$$

  are the inputs for the game SVT_{σ_x} (The intuition behind is that $B_1B_2 = \sigma_x$.);

- when $x = m+8$ and $y \in \{3, 4, n+4\}$, they are scored according to SVT_{σ_z}, where

$$*_A = m+8 \tag{88}$$
$$*_B = n+4, \quad 3_B = 3, \quad 4_B = 4 \tag{89}$$

  are the inputs for the game SVT_{σ_z} (The intuition behind is that $B_3B_4 = \sigma_z$.);

- otherwise, they score 0.

Note that the dimension d is defined in the rules of $CHSH^{(d)}_{\sigma_z}$ and $CHSH^{(d)}_{\sigma_x}$. Next we are going to prove that the strategy winning this game optimally can self-test d-dimensional EPR pair and d-dimensional σ_x and σ_z.

2.1 Proof Sketch

Now we examine the implications of Alice and Bob winning the linear constraint game perfectly. By Lemma 4.3 of [CS17], we can extract an operator solution from the perfect winning strategy of the linear system game: For each variable $\{x_i\}_{i=1}^n$, Alice and Bob has operators A_i and B_i respectively and $\mathcal{J}$ is mapped to $\sigma_A(\mathcal{J})$ in Alice's solution and $\sigma_B(\mathcal{J})$ in Bob's solution. The condition that they agree with assignment to variables means that

$$\langle\psi|A_i \otimes \overline{B_i}|\psi\rangle = 1 \Rightarrow A_i \otimes \overline{B_i}|\psi\rangle = |\psi\rangle \tag{90}$$

and the condition that Alice's assignments satisfy the constraint means that

$$\mathrm{Tr}(\rho_A \Pi_{i:H(l,i)\neq 0} A_i) = \mathrm{Tr}(\rho_A \sigma_A(\mathcal{J})^{b(l)}). \tag{91}$$

where $\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$, l is an equation label and $H(l,i)$ is the coefficient of x_i in equation l. For any $|v\rangle \in \mathrm{supp}(\rho_A)$, we have

$$\Pi_{i:H(l,i)\neq 0} A_i|v\rangle = \sigma_A(\mathcal{J})^{b(l)}|v\rangle. \tag{92}$$

Since the relation $u^{-1}xu = x^2$ is embedded in this linear system game, we know

$$\sigma_A(u)^\dagger A_1A_2\sigma_A(u)|v\rangle = (A_1A_2)^2|v\rangle \text{ for all } |v\rangle \in \mathrm{supp}(\rho_A). \tag{93}$$

So when A_1A_2 is restricted to $\mathrm{supp}(\rho_A)$, it is similar to $(A_1A_2)^2$. From now on, by A_1A_2 we mean $A_1A_2|_{\mathrm{supp}(\rho_A)}$. We will come back to the implication of the condition $A_1A_2 \sim (A_1A_2)^2$ later.

Next we look at the implication of winning the $CHSH^{(d)}_{\sigma_x}$ game optimally. It means that there exists isometris V_A on Alice side and V_B on Bob's side such that

$$(V_A \otimes V_B)\frac{A_*^\diamond \otimes B_*^\diamond|\psi\rangle}{\|A_*^\diamond \otimes B_*^\diamond|\psi\rangle\|} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \otimes |extra\rangle \tag{94}$$

$$(V_A \otimes V_B)\frac{B_1 + B_2}{2\cos(\pi/2d)}\frac{A_*^\diamond \otimes B_*^\diamond|\psi\rangle}{\|A_*^\diamond \otimes B_*^\diamond|\psi\rangle\|} = (\mathbb{1} \otimes \sigma_z)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \otimes |extra\rangle \tag{95}$$

$$(V_A \otimes V_B)\frac{B_1 - B_2}{-2\sin(\pi/2d)}\frac{A_*^\diamond \otimes B_*^\diamond|\psi\rangle}{\|A_*^\diamond \otimes B_*^\diamond|\psi\rangle\|} = (\mathbb{1} \otimes \sigma_x)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \otimes |extra\rangle. \tag{96}$$

Suppose $A_*^\diamond$ measures states $|u_0\rangle$ and $|u_1\rangle$. Then we know

$$(B_1 + B_2)|u_0\rangle = 2\cos(\pi/2d)|u_0\rangle \tag{97}$$
$$(B_1 + B_2)|u_1\rangle = -2\cos(\pi/2d)|u_1\rangle \tag{98}$$
$$(B_1 - B_2)|u_0\rangle = -2\sin(\pi/2d)|u_1\rangle \tag{99}$$
$$(B_1 - B_2)|u_1\rangle = -2\sin(\pi/2d)|u_0\rangle \tag{100}$$

which implies that

$$\begin{aligned}
B_1|u_0\rangle &= \cos(\pi/2d)|u_0\rangle - \sin(\pi/2d)|u_1\rangle \\
B_2|u_0\rangle &= \cos(\pi/2d)|u_0\rangle + \sin(\pi/2d)|u_1\rangle \\
B_1|u_1\rangle &= -\sin(\pi/2d)|u_0\rangle - \cos(\pi/2d)|u_1\rangle \\
B_2|u_1\rangle &= \sin(\pi/2d)|u_0\rangle - \cos(\pi/2d)|u_1\rangle \\
B_1B_2|u_0\rangle &= \cos(\pi/2d)B_1|u_0\rangle + \sin(\pi/2d)B_1|u_1\rangle \\
&= \cos(\pi/2d)(\cos(\pi/2d)|u_0\rangle - \sin(\pi/2d)|u_1\rangle) - \sin(\pi/2d)(\sin(\pi/2d)|u_0\rangle + \cos(\pi/2d)|u_1\rangle) \\
&= \cos(\pi/d)|u_0\rangle - \sin(\pi/d)|u_1\rangle \\
B_1B_2|u_1\rangle &= \sin(\pi/2d)B_1|u_0\rangle - \cos(\pi/2d)B_1|u_1\rangle \\
&= \sin(\pi/2d)(\cos(\pi/2d)|u_0\rangle - \sin(\pi/2d)|u_1\rangle) + \cos(\pi/2d)(\sin(\pi/2d)|u_0\rangle + \cos(\pi/2d)|u_1\rangle) \\
&= \sin(\pi/d)|u_0\rangle + \cos(\pi/d)|u_1\rangle.
\end{aligned}$$

We can conclude that

$$B_1B_2(|u_0\rangle + i|u_1\rangle) = e^{i\frac{\pi}{d}}(|u_0\rangle + i|u_1\rangle) \tag{101}$$
$$B_1B_2(|u_0\rangle - i|u_1\rangle) = e^{-i\frac{\pi}{d}}(|u_0\rangle - i|u_1\rangle). \tag{102}$$

Define $|x_1\rangle = 1/\sqrt{2}(|u_0\rangle + i|u_1\rangle)$, then it is the eigenvector of B_1B_2 with eigenvalue $e^{i\pi/d} = \omega_d$. Similarly, we define $|x_{d-1}\rangle = 1/\sqrt{2}(|u_0\rangle - i|u_1\rangle)$ and it is the eigenvector of B_1B_2 with eigenvalue ω_d^{d-1}. Moreover, from the self-testing derivation, we know

$$\|(|x_1\rangle\langle x_1| + |x_{d-1}\rangle\langle x_{d-1}|)|\psi\rangle\|^2 = 2/d. \tag{103}$$

The condition that $B_1B_2 \sim (B_1B_2)^2$ implies that $\omega_d, \omega_d^2, \omega_d^4 \dots$ are all the eigenvalues of B_1B_2 and the eigen-space of each eigenvalue are of dimension 1. Since 2 is a primitive root of d, we know $\omega_d^i = \omega_d^{2^{k_i}}$ for some k_i and $i \in \{1, 2 \dots, d-1\}$, or equivalently, $2^{k_i} \equiv i \pmod d$, which means that B_1B_2 is finite-dimensional and we have states $|x_i\rangle$ for $i = 1 \dots d-1$ such that

$$B_1B_2|x_i\rangle = \omega_d^i|x_i\rangle \tag{104}$$

and

$$\|(\sum_{i=1}^{d-1} |x_i\rangle\langle x_i|)|\psi\rangle\|^2 = \frac{d-1}{d}. \tag{105}$$

The implication of the SVT_{σ_x} is that there exists state $|x_0\rangle$ such that

$$B_1B_2|x_0\rangle = |x_0\rangle \text{ and } \||x_0\rangle\langle x_0||\psi\rangle\|^2 = 1/d. \tag{106}$$

Hence $\sum_{i=0}^{d-1} |x_i\rangle\langle x_i|$ is the projector onto $\mathrm{supp}(\rho_B)$ which means the dimension of $\mathrm{supp}(\rho_B)$ is d and the same as the rank of B_1B_2. So far, we have recovered the full eigen-decompostion of B_1B_2 as

$$B_1B_2 = \sum_{i\in[d]} \omega_d^i |x_i\rangle\langle x_i|. \tag{107}$$

Similarly, $CHSH_{\sigma_z}^{(d)}$ and SVT_{σ_z} give us the full eigen-decomposition of B_3B_4 as

$$B_3B_4 = \sum_{i\in[d]} \omega_d^i |z_i\rangle\langle z_i|. \tag{108}$$

The reasoning above also holds for A_1, A_2, A_3, A_4 because B_1, B_2, B_3, B_4 are the conjugate operator solution. In the rest of the proof, we will work with A_1, A_2, A_3, A_4 since they are the operator solution.

Thm. 4 of Ref. [CLS17] states that when we have a perfect strategy, $\mathcal{J} \neq e$ in the solution group $\Gamma_{\mathcal{P}}$ and x doesn't commute with z. Since $\mathcal{J}$ is in the center of $\Gamma_{\mathcal{P}}$ and both A_1A_2 and A_3A_4 have full rank, we know $\sigma_A(\mathcal{J}) = \alpha\mathbb{1}$ with $\alpha \neq 1$. The relation $zx = \mathcal{J}xz$ implies that in the operator solution

$$\begin{aligned} A_3A_4A_1A_2|z_1\rangle =&\sigma(\mathcal{J})A_1A_2A_3A_4|z_1\rangle \\ =&\alpha\omega_d A_1A_2|z_1\rangle \end{aligned} \tag{109}$$

where we use the fact that $|z_1\rangle \in \mathrm{supp}(\rho_A)$.

Then by induction, we can show that $(A_1A_2)^i|z_1\rangle$ is the eigenvector of A_3A_4 with eigenvalue $\alpha^{-i}\omega_d$, which gives us another set of eigenvalues, $\{\omega_d\alpha^i\}_{i\in[d]} = \{\omega_d^i\}_{i\in[d]}$. If we assume $\alpha\omega_d = \omega_d^l$ for some l, then we know $\alpha^d = 1$.

Hence, $\{(A_1A_2), (A_3A_4), \sigma_A(\mathcal{J})\}$ is a finite-dimensional representation of $\mathcal{P}_d$ and it can be extended to a finite-dimensional representation of $\mathcal{P}_d^{\otimes 2}$, which gives an operator solution of the d-dimensional Magic Square game. Then Lemma 4.3 of [CS17] tells us that $\alpha = \omega_d$. Going back to eq. (109), we can conclude that on the basis $\{(A_1A_2)^i|z_1\rangle\}_{i\in[d]}$, A_1A_2 acts as the σ_x operator and A_3A_4 acts as the σ_z operator.

Built on the results so far, the consistency criterion tells us that

$$A_1A_2 \otimes \overline{B_1B_2}|\psi\rangle = |\psi\rangle \Rightarrow X_A \otimes \overline{X_B}|\psi\rangle = |\psi\rangle, \tag{110}$$

$$A_3A_4 \otimes B_3B_4|\psi\rangle = |\psi\rangle \Rightarrow Z_A \otimes \overline{Z_B}|\psi\rangle = |\psi\rangle. \tag{111}$$

Considering all $i, j \in [d]$, we have

$$X_A^i Z_A^j \otimes \overline{X_B}^i \overline{Z_B}^j |\psi\rangle = |\psi\rangle \tag{112}$$

$$\Rightarrow \exists \text{ isometries } V_A', V_B' \text{ such that } V_A' \otimes V_B'|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i\in[d]} |ii\rangle \tag{113}$$

which is a standard result from Ref. [Got99].

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A The linear system game

In this section, we are going to embed the group $\mathcal{P}$ into a solution group $\Gamma_{\mathcal{P}}$ of a linear system game.

We first embed the relation $xzx^{-1}z^{-1} = \mathcal{J}$ in a linear system game by ideas from the Magic Square game. It has been shown that the following relations embeds $xzx^{-1}z^{-1} = \mathcal{J}$ [2]. Let $y_3 = z$ and $y_7 = x$ and the linear relations are

$$y_1y_2y_3 = e, y_4y_5y_6 = e, y_7y_8y_9 = e, \tag{114}$$

$$y_1^{-1}y_4^{-1}y_7^{-1} = e, y_2^{-1}y_5^{-1}y_8^{-1} = \mathcal{J}, y_3^{-1}y_6^{-1}y_9^{-1} = e. \tag{115}$$

We refer to these set of linear relations as the MS_l relations. The magic square game also introduces some commutation relations:

$$[y_1, y_2] = [y_1, y_3] = [y_1, y_4] = [y_1, y_7] = e, \tag{116}$$

$$[y_2, y_3] = [y_2, y_5] = [y_2, y_8] = e, \tag{117}$$

$$[y_3, y_6] = [y_3, y_9] = e, \tag{118}$$

$$[y_4, y_7] = [y_4, y_5] = [y_4, y_6] = e, \tag{119}$$

$$[y_5, y_6] = [y_5, y_8] = e, \tag{120}$$

$$[y_6, y_9] = e, \tag{121}$$

$$[y_7, y_8] = [y_7, y_9] = e, \tag{122}$$

$$[y_8, y_9] = e, \tag{123}$$

which will be referred as MS_c. So we define $\mathcal{P}$ as

$$\mathcal{P} = \langle \{y_i\}_{i=1}^{9}, u, \mathcal{J} : MS_l \cup MS_c \cup \{[\mathcal{J}, y_i] = [\mathcal{J}, u] = e, uy_7u^{-1} = y_7^2, u^{-1}y_3u = y_3^2\}\rangle. \tag{124}$$

Then we can start to use a procedure similar to the ones developed in Ref. [Slo17] to embed $\mathcal{P}$ into the solution group of a linear system game.

The first step is to embed $\mathcal{P}$ into a group which almost satisfy the definition of homogeneous-linear-plus-conjugacy group [Slo17].

Replacing y_3. We start by introducing $y_{3,1}$ and $y_{3,2}$ of order 2 such that $y_3 = y_{3,1}y_{3,2}$ and $y_{3,1}$ commutes with y_i, u and $\mathcal{J}$ for $i \neq 3$. Then the relation $uy_3u^{-1} = y_3^2$ is rewritten as $uy_{3,2}u^{-1} =$

[2] For example, Fig. 11 of Ref. [CS17] proves it in a group picture.

$y_{3,2}y_{3,1}y_{3,2}$ so we introduce $y_{3,3} = y_{3,2}y_{3,1}y_{3,2}$. The group $\mathcal{P}$ is embedded in

$$\begin{aligned}\mathcal{P} = \langle y_{3,1}, y_{3,2}, y_{3,3}, x, u, \{y_i\}_{i\neq 3}\mathcal{J} :& y_{3,1}, y_{3,2}, y_{3,3} \text{ of order 2;} \\ & \mathcal{J} \text{ commute with all the generators, } [y_i, y_{3,1}] = [u, y_{3,1}] = e \text{ for } i \neq 3, \\ & MS_c \text{ with } y_3 \text{ replaced by } y_{3,2}, MS_l \text{ with } y_3 \text{ replaced by } y_{3,1}y_{3,2}, \\ & u^{-1}y_7u = y_7^2, y_{3,3} = y_{3,2}y_{3,1}y_{3,2}, uy_{3,2}u^{-1} = y_{3,3}\rangle\end{aligned} \tag{125}$$

Replace y_7**.** We introduce $y_{7,1}$ and $y_{7,2}$ of order 2 such that $y_7 = y_{7,1}y_{7,2}$ and $y_{7,1}$ commutes with u, $\mathcal{J}$ and y_i for $i \neq 3, 7$. Then the relation $u^{-1}y_7u = y_7^2$ is rewritten as $u^{-1}y_{7,2}u = y_{7,2}y_{7,1}y_{7,2}$ so we introduce $y_{7,3} = y_{7,2}y_{7,1}y_{7,2}$. The relation $y_7y_{3,1}y_7^{-1} = y_{3,1}$ is rewritten as $y_{7,2}y_{3,1}y_{7,2} = y_{7,1}y_{3,1}y_{7,1}$ so we introduce $y_{7,4} = y_{7,1}y_{3,1}y_{7,1}$. The group $\mathcal{P}$ is embedded as

$$\begin{aligned}\mathcal{P} = \langle \{y_{3,i}\}_{i=1}^3, \{y_{7,i}\}_{i=1}^4, \{y_i\}_{i\neq 3,7}, u, \mathcal{J} :& \{y_{3,i}\}_{i=1}^3\{y_{7,i}\}_{i=1}^4 \text{ of order 2;} \\ & \mathcal{J} \text{ commutes with all the generators, } [u, y_{3,1}] = [u, y_{7,1}] = e, \\ & y_{3,1}, y_{7,1} \text{ commute with } y_i,\ i \neq 3, 7, \\ & MS_c \text{ with } y_3, y_7 \text{ replaced by } y_{3,2}, y_{7,2}, \\ & MS_l \text{ with } y_3, y_7 \text{ replaced by } y_{3,1}y_{3,2} \text{ and } y_{7,1}y_{7,2} \\ & y_{3,3} = y_{3,2}y_{3,1}y_{3,2}, uy_{3,2}u^{-1} = y_{3,3}, \\ & y_{7,3} = y_{7,2}y_{7,1}y_{7,2}, u^{-1}y_{7,2}u = y_{7,3}, \\ & y_{7,4} = y_{7,1}y_{3,1}y_{7,1}, y_{7,2}y_{3,1}y_{7,2} = y_{7,4}\rangle\end{aligned} \tag{126}$$

Replacing u**.** We introduce u_1, u_2 of order 2 such that $u = u_1u_2$ and u_1 commutes with y_i for $i \neq 3, 7$ and $\mathcal{J}$. The conjugacy relations involving u are $uy_{3,2}u^{-1} = y_{3,3}$, $u^{-1}y_{7,2}u = y_{7,3}$, $uy_{3,1}u^{-1} = y_{3,1}$ and $uy_{7,1}u^{-1} = y_{7,1}$. So we need to introduce u_3, u_4, u_5, u_6 such that

$$u_3 = u_2y_{3,2}u_2 = u_1y_{3,3}u_1, \tag{127}$$
$$u_4 = u_1y_{7,2}u_1 = u_2y_{7,3}u_2, \tag{128}$$
$$u_5 = u_2y_{3,1}u_2 = u_1y_{3,1}u_1, \tag{129}$$
$$u_6 = u_2y_{7,1}u_2 = u_1y_{7,1}u_1. \tag{130}$$

At this stage $\mathcal{P}$ has generators $\{y_{3,i}\}_{i=1}^3, \{y_{7,i}\}_{i=1}^4, \{y_i\}_{i\neq 3,7}, \{u_i\}_{i=1}^6$ with 6 linear relations and 44 conjugacy relations.

Replacing y_i **for** $i \neq 3, 7$**.** The conjugacy relations involving y_1 are

$$y_1y_{3,1}y_1^{-1} = y_{3,1}, y_1y_{7,1}y_1^{-1} = y_{7,1}. \tag{131}$$

and 4 commutation relations from MS_c. So we need to introduce $y_{1,i}$ for $i = 1, 2 \ldots 8$ such that they all are of order 2 and

$$y_1 = y_{1,1}y_{1,2} \tag{132}$$
$$y_{1,3} = y_{1,2}y_{3,1}y_{1,2} = y_{1,1}y_{3,1}y_{1,1}, \tag{133}$$
$$y_{1,4} = y_{1,2}y_{7,1}y_{1,2} = y_{1,1}y_{7,1}y_{1,1}, \tag{134}$$
$$y_{1,5} = y_{1,2}y_2y_{1,2} = y_{1,1}y_2y_{1,1}, \tag{135}$$
$$y_{1,6} = y_{1,2}y_{3,2}y_{1,2} = y_{1,1}y_{3,2}y_{1,1}, \tag{136}$$
$$y_{1,7} = y_{1,2}y_4y_{1,2} = y_{1,1}y_4y_{1,1}, \tag{137}$$
$$y_{1,8} = y_{1,2}y_{7,2}y_{1,2} = y_{1,1}y_{7,2}y_{1,1}. \tag{138}$$

The new commutation relations are $y_{1,1}$ commutes with all the remaining y_j's and commutation relations from MS_c involving y_1 with y_1 replaced by $y_{1,2}$. Then we repeat this process with y_2. In summary, replacing $y_1, y_2, y_4, y_5, y_6, y_8, y_9$ introduces $8+9+10+11+12+13+14 = 77$ new variables and $12+14+16+18+20+22+24 = 126$ new conjugacy relations. In total $\mathcal{P}$ has 90 variables excluding $\mathcal{J}$, 6 linear relations and 170 conjugacy relations.

Then following the recipe given in Proposition 4.2 and Lemma 4.4 of Ref. [Slo17], we embed $\mathcal{P}$ into the solution group of a linear system game having 2351 variables and 1916 linear relations. Alice's output alphabet is of size 64 and of size 2 for Bob.

Such linear system game has at the biggest size when $a = 5$ or the special relation is $uxu^{-1} = x^5$. The biggest game has 2465 variables and 2006 equations. We leave the derivation for curious readers.

B Slofstra's Binary Constraint game

We start with an extended homogeneous-linear-plus-conjugacy group

$$K = \langle x, y, a, b, c : a^2 = b^2 = c^2 = e, abc = e, yay^{-1} = a, yby^{-1} = c, xyx^{-1} = y^2 \rangle \tag{139}$$

and construct a solution group corresponding to a nonlocal game.

We first embed K into another homogeneous-linear-plus-conjugacy group, K', with x, y replaced by elements of order 2. By Proposition 4.8 of [Slo17], we first introduce z, w such that $z^2 = w^2 = e$, $y = zw$ and $xz = zx$ then

$$\begin{aligned} K' = \langle a, b, c, z, w, a', b', z', x :& abc = e, && (140)\\ & waw = a', wbw = b', wz'w = z && (141)\\ & za'z = a, zb'z = c, && (142)\\ & xwx^{-1} = z', xzx^{-1} = z \rangle. && (143) \end{aligned}$$

Next, we introduce u, v such that $u^2 = v^2 = e$ and $x = uv$, then

$$\begin{aligned} K' = \langle a, b, c, z, w, a', b', z', u, v, z_v :& abc = e,\\ & waw = a', wbw = b', wz'w = z\\ & za'z = a, zb'z = c,\\ & vwv = w', vzv = z_v,\\ & uw'u = z', uz_vu = z \rangle \end{aligned}$$

Note that we skipped the relations that all elements are of order 2. To easier introduce new elements in the following construction, we relabel the elements as

$$\begin{aligned} & x_1 = w,\ x_2 = a,\ x_3 = a',\ x_4 = b,\ x_5 = b',\ x_6 = z,\ x_7 = z'\\ & x_8 = c,\ x_9 = v,\ x_{10} = u,\ x_{11} = w',\ x_{12} = z_v, \end{aligned}$$

so K' can also be written as

$$\begin{aligned} K' = \langle \{x_i\}_{i=1}^{12} :& x_i^2 = e, x_2x_4x_8 = e,\\ & x_1x_2x_1 = x_3,\ x_1x_4x_1 = x_5,\ x_1x_7x_1 = x_6,\\ & x_6x_3x_6 = x_2,\ x_6x_5x_6 = x_8,\\ & x_9x_1x_9 = x_{11}, x_9x_6x_9 = x_{12}\\ & x_{10}x_{11}x_{10} = x_7, x_{10}x_{12}x_{10} = x_6 \rangle. \end{aligned} \tag{144}$$

Here the special element is $x_2 = a$. Then we add another two order-2 element, $t\ Z$, to add a linear relation to it

$$\hat{K} = \langle K', t, Z : t^2 = Z^2 = e,\ tx_2t = Z.\ Zx_2 = J\rangle_{\mathbb{Z}_2}. \tag{145}$$

We rename $t = x_{13}$ and $Z = x_{14}$, then $\hat{K}$ contains linear relations

$$x_2x_4x_8 = e,\ x_{14}x_2 = J$$

and conjugacy relations

$$\begin{aligned}
&x_1x_2x_1 = x_3,\ x_1x_4x_1 = x_5,\ x_1x_7x_1 = x_6,\\
&x_6x_3x_6 = x_2,\ x_6x_5x_6 = x_8,\\
&x_9x_1x_9 = x_{11}, x_9x_6x_9 = x_{12}\\
&x_{10}x_{11}x_{10} = x_7, x_{10}x_{12}x_{10} = x_6\\
&x_{13}x_2x_{13} = x_{14}.
\end{aligned}$$

We collect the subscript of x_i's in the conjugacy relations and define

$$\begin{aligned}
C = \{&(1,2,3),(1,4,5),(1,7,6),(6,3,2),(6,5,8), && (146)\\
&(9,1,11),(9,6,12),(10,11,7),(10,12,6),(13,2,14)\} && (147)
\end{aligned}$$

such that

$$(i,j,k) \in C \iff x_ix_jx_i = x_k.$$

We also define

$$\Gamma = \langle \{x_i\}_{i=1}^{14} : x_i^2 = e, x_2x_4x_8 = e,\ x_{14}x_2 = J\rangle. \tag{148}$$

In the next part, we are going to convert the conjugacy relations to linear relations and make Γ the solution group we want. So we embed $\hat{K}$ in $\overline{K}$ where

$$\begin{aligned}
\overline{K} = \langle \Gamma, \{w_i, y_i, j_i\}_{i=1}^{14}, f\ &\text{:all elements are order 2,}\\
&x_i = y_iz_i = fw_i,\ fy_if = z_i \quad \text{where } 1 \le i \le 14\\
&y_jz_k = z_ky_j,\ w_iy_jw_i = z_k \quad \text{for all } (i,j,k) \in C\rangle.
\end{aligned}$$

We convert new relations in $\overline{K}$ to linear relation or conjugacy relation by introduce element g_{jk} for all $(i,j,k) \in C$, such that $g_{jk}^2 = e$ and $g_{jk} = y_jz_k$. The new form of $\overline{K}$ is

$$\begin{aligned}
\overline{K} = \langle \Gamma, \{w_i, y_i, j_i\}_{i=1}^{14}, f, \{g_{jk}\}_{(i,j,k)\in C}\ &\text{:all elements are order 2,}\\
&x_iy_iz_i = e,\ x_ifw_i = e \text{ for all } 1 \le i \le 14\\
&g_{jk}y_jz_k = e \text{ for all } (i,j,k) \in C\\
&fy_if = z_i \text{ for all } 1 \le i \le 14\\
&w_iy_jw_i = z_k \text{ for all } (i,j,k) \in C\rangle
\end{aligned} \tag{149}$$

The last step is to convert the conjugacy relations in $\overline{K}$to linear relations.

We let Γ absorb the new elements and new linear relations by first relabelling

$$\begin{gathered}
x_{i+14} = w_i, \quad x_{i+28} = y_i, \quad x_{i+42} = z_i \quad 1 \le i \le 14 \\
x_{57} = f, \\
x_{58} = g_{23}, \quad x_{59} = g_{45}, \quad x_{60} = g_{76}, \quad x_{61} = g_{32}, \quad x_{62} = g_{58}, \\
x_{63} = g_{1,11}, \quad x_{64} = g_{8,12}, \quad x_{65} = g_{11,7}, \quad x_{66} = g_{12,6}, \quad x_{67} = g_{2,13}.
\end{gathered}$$

then

$$\begin{aligned}
\Gamma = \langle \{x_i\}_1^{67} : & x_i^2 = e, x_2x_4x_8 = e,\ x_{14}x_2 = J, \\
& x_i x_{i+28} x_{i+42} = e;\ x_i x_{57} x_{i+14} = e \ \text{ for } 1 \le i \le 14, \\
& x_{58}x_{30}x_{45} = e, x_{59}x_{32}x_{47} = e, x_{60}x_{35}x_{48} = e, \\
& x_{61}x_{31}x_{44} = e, x_{62}x_{33}x_{50} = e, x_{63}x_{29}x_{53} = e, \\
& x_{64}x_{34}x_{54} = e, x_{65}x_{39}x_{49} = e, x_{66}x_{40}x_{48} = e, x_{67}x_{30}x_{56} = e \rangle
\end{aligned}$$

We also change C to cover new conjugacy relations

$$\begin{aligned}
C = & \{(57, i+28, i+42)\}_{i=1}^{14} \cup \\
& \{(15,30,45), (15,32,47), (15,35,48), (20,31,44), (20,33,50), (23,29,53), \\
& (23,34,54), (24,39,49), (24,40,48), (27,30,56)\}.
\end{aligned}$$

For each $I = (i,j,k) \in C$, we introduce seven new variables $\{y_{Ii}\}_{i=1}^{7}$ such that

$$x_i y_{I1} y_{I2} = x_i y_{I5} y_{I6} = x_j y_{I2} y_{I3} = x_k y_{I6} y_{I7} = y_{I3} y_{I4} y_{I5} = y_{I1} y_{I4} y_{I7} = e.$$

We add such relations to Γ and get the final form of Γ which is

$$\begin{aligned}
\Gamma = \langle \{x_i\}_{i=1}^{67} \cup \{\{y_{Ii}\}_{i=1}^{7}\}_{I\in C} : \{ & x_i^2 = y_{Ij}^2 = e, x_2x_4x_8 = e,\ x_{14}x_2 = J, \\
& x_i x_{i+26} x_{i+39} = e;\ x_i x_{53} x_{i+13} = e \ \text{ for } 1 \le i \le 14, \\
& x_{58}x_{30}x_{45} = e, x_{59}x_{32}x_{47} = e, x_{60}x_{35}x_{48} = e, \\
& x_{61}x_{31}x_{44} = e, x_{62}x_{33}x_{50} = e, x_{63}x_{29}x_{53} = e, \\
& x_{64}x_{34}x_{54} = e, x_{65}x_{39}x_{49} = e, x_{66}x_{40}x_{48} = e, x_{67}x_{30}x_{56} = e\} \\
& \cup \{x_i y_{I1} y_{I2} = x_i y_{I5} y_{I6} = x_j y_{I2} y_{I3} = x_k y_{I6} y_{I7} = e\}_{I=(i,j,k)\in C} \\
& \cup \{y_{I3} y_{I4} y_{I5} = y_{I1} y_{I4} y_{I7} = e\}_{I\in C} \rangle.
\end{aligned} \tag{150}$$

The solution group Γ has 235 variables and 184 equations, which match what is given in [Slo17].