Special nonlocal game with constant alphabet

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1 Components of the new game

1.1 The linear system game

Following ideas from Ref. [**?**], we would like to embed the qudit Pauli group in a solution group and make the d-dependence implicit. The qudit Pauli group is defined as

$$\mathcal{P}_d = \langle x, z, \mathcal{J} : x^d = z^d = \mathcal{J}^d = e, zxz^{-1}x^{-1} = \mathcal{J}, x\mathcal{J}x^{-1}\mathcal{J}^{-1} = z\mathcal{J}z^{-1}\mathcal{J}^{-1} = e\rangle. \tag{1}$$

Firstly, we would like to replace the relation $x^d = z^d = e$ by the following relations

$$u_x x u_x = x^2 \tag{2}$$

$$u_z z u_z = z^2. \tag{3}$$

With constraints imposed by other components of the whole game, the two conditions above imply that the eigenvalues of x and z are $\{\omega_d^k = e^{ik\pi/d}\}_{k=1}^d$ with d odd, prime and $\mathbb{Z}/(d\mathbb{Z})$ has primitive root 2 and the eigenspaces for different eigenvalues are of the same dimension. Later we will set the exponent to be $a \in \{2, 3, 5\}$ which is the primitive root of infinitely many prime numbers and construct a nonlocal game that can self-test infinitely many maximally entangled state. [1]

Secondly, we drop the relation $\mathcal{J}^d = e$ and make the value of $\mathcal{J}$ determined by x and z in the relation $xzx^{-1}z^{-1} = \mathcal{J}$.

It can be easily checked that the qudit Pauli-x and Pauli-z operators of dimension d satisfy the new relations above, where Pauli-x and Pauli-z operators are defined by

$$\sigma_x = \sum_{i=0}^{d-1} |i+1 \pmod d\rangle\langle i| \qquad \sigma_z = \sum_{i=0}^{d-1} \omega_d^i |i\rangle\langle i|. \tag{4}$$

1.2 U and X generate all the $(d-1) \times (d-1)$ matrices

Suppose U and X satisfies the relation $UXU^\dagger = X^2$. Moreover, we know the eigen-decomposition of X is

$$X = \sum_{i=0}^{d-1} \omega_d^i |i\rangle\langle i|, \tag{5}$$

[1]Figuring out what a is will take us one step closer to resolving Artin's Conjecture[**?**].

and the form of U is

$$U = \sum_{i=0}^{d-1} |i/2\rangle\langle i|. \tag{6}$$

Note that all the operations on the eigenvector label i is taken modulo d. We are going to prove that $\{U^k X^l\}$ for $k = 0, 1 \dots d-2$ and $l = 1, 2 \dots d-1$ is linearly independent, when the action of U and X are restricted to the basis $\{|i\rangle\}_{i=1}^{d-1}$ such that

$$U^k X^l = \sum_{i=1}^{d-1} \omega_d^{il} |i/2^k\rangle\langle i|. \tag{7}$$

From now on, U and X mean their actions restricted to the subset $\text{span}(\{|i\rangle\}_{i=1}^{d-1})$.

Suppose there exists a set of complex numbers $\{x_{k,l}\}$ for $k = 0, 1 \dots d-2$ and $l = 1, 2 \dots d-1$ such that

$$M = \sum_{k=0}^{d-2} \sum_{l=1}^{d-1} x_{k,l} U^k X^l = 0. \tag{8}$$

We further assume that we have $2^{k_i} \equiv i \pmod{d}$ for $i = 1, 2 \dots d-1$. The fact that 2 is a primitive root of d guarantees that k_i's are distinct. Then we can group $\{x_{k,l}\}$ into vectors: $|x_{k_1}\rangle, |x_{k_2}\rangle \dots |x_{k_{d-1}}\rangle$, where $|x_{k_i}\rangle = (x_{k_i,1}, x_{k_i,2} \dots x_{k_i,d-1})^\intercal$. We are going to prove that $|x_{k_i}\rangle = 0$ for all i.

Starting with $|x_{k_1}\rangle$. Let's look at the entry $\langle 1|M|1\rangle$ where

$$\langle 1|M|1\rangle = \sum_{k=0}^{d-2} \sum_{l=1}^{d-1} \sum_{i=1}^{d-1} x_{k,l} \omega_d^{il} \langle 1|i/2^k\rangle\langle i|1\rangle. \tag{9}$$

For the term $\langle 1|i/2^k\rangle\langle i|1\rangle \neq 0$ we must have $i = 1$ and $2^k \equiv 1 \pmod{d}$, or equivalently, $k = k_1$. Hence, we can conclude that

$$\langle 1|M|1\rangle = \sum_{l=1}^{d-1} x_{k_1,l} \omega_d^l = 0. \tag{10}$$

Similarly we can determine that for all $j = 1, 2 \dots d-1$,

$$\langle j|M|j\rangle = \sum_{k=0}^{d-2} \sum_{l=1}^{d-1} \sum_{i=1}^{d-1} x_{k,l} \omega_d^{il} \langle j|i/2^k\rangle\langle i|j\rangle = \sum_{l=1}^{d-1} x_{k_1,l} \omega_d^{jl} = 0. \tag{11}$$

Hence we get $d-1$ equations with $d-1$ variables, and the linear system can be represented by

$$W|x_{k_1}\rangle = 0, \tag{12}$$

where $W_{m,n} = \omega_d^{mn}$. Then we define

$$\tilde{W} = \begin{pmatrix} 1 & 1 \\ 1 & W \end{pmatrix}, \tag{13}$$

and we will use the fact that $\tilde{W}$ is non-singular to prove that $|x_{k_1}\rangle = 0$.

First observe that $\tilde{W}$ is a Vandermonde matrix, hence it is non-singular. Now we define $|\tilde{x}_{k_1}\rangle = (0, x_{k_1,1}, \ldots x_{k_1,d-1})^\intercal$ and prove that it satisfies the condition that

$$\tilde{W}|\tilde{x}_{k_1}\rangle = 0, \tag{14}$$

which involves d equations. The last $d-1$ equations come from M. We only need to prove that $\sum_{l=1}^{d} x_{k_1,l} = 0$. It can be seen from

$$0 = \sum_{j=1}^{d-1} \langle j|M|j\rangle = \sum_{j=1}^{d-1}\sum_{l=1}^{d-1} x_{k_1,l}\omega_d^{jl} = \sum_{l=1}^{d-1} x_{k_1,l}(\sum_{j=1}^{d-1}\omega_d^{jl}) = \sum_{l=1}^{d-1} -x_{k_1,l} \tag{15}$$

where we have used the fact that $\sum_{j=1}^{d-1}\omega_d^{jl} = -1$ for all $l = 1, 2 \ldots d-1$. Since $\tilde{W}$ is non-singluar, we know $|\tilde{x}_{k_1}\rangle = 0$ which implies that $|x_{k_1}\rangle = 0$.

For $|x_{k_a}\rangle$, we look at entries $\{\langle j|M|aj\rangle\}_{j=1}^{d-1}$ for $a = 2 \ldots d-1$ and get equations

$$0 = \langle j|M|aj\rangle = \sum_{l=1}^{d-1} x_{k_a,l}\omega_d^{ajl} \tag{16}$$

The corresponding W matrix has value ω_d^{amn} at coordinate m, n, so it is also a submatrix of a Vandermonde matrix. Similar argument gives us that $|x_{k_a}\rangle = 0$.

To summarize, we have proven that $x_{k,l} = 0$ for all k and l, which implies that the elements of the set $\{U^k X^l\}$ are linearly independent and forms a basis for all the $(d-1)\times(d-1)$ matrices.

1.3 The extended weighted CHSH game

We want to force Alice and Bob to reuse observables from the linear system game developed above so that we can make sure that the observables have eigenvalues ω_d and ω_d^{d-1}, and the corresponding eigen-spaces are of dimension 1. The modified weighted CHSH game to force the structure of observable U will be denoted by $CHSH_U^{(d)}$, where the superscript d means that the rules of this game depends on d. Intuitively, we want to combine $CHSH_X^{(d)}$ and the fact that $X \sim X^2$ to achieve the same effect as $(X)^d = \mathbb{1}$.

Assume Bob use observable B_1 and B_2, such that $B_1B_2 = U$. The special condition B_1, B_2 satisfy is that the eigenvalues for U are $\{\omega_d^i\}_{i=0}^{d-1}$. Let $|v_1\rangle$ and $|v_{d-1}\rangle$ be the eigenvectors for eigenvalues ω_d and ω_d^{d-1}. In our case, we can show that when $U = \sigma_x$ or σ_z, such B_1, B_2 exists. I am not sure if such decomposition exists in general.

In another note, I have shown that B_1, B_2 with shared state $\frac{1}{\sqrt{2}}(|u_1\rangle|u_1\rangle + |u_{d-1}\rangle|u_{d-1}\rangle)$ can maximize $\langle I_{-\cot(\pi/2d)}\rangle$ where $|u_1\rangle, |u_{d-1}\rangle \in \text{span}\{|v_1\rangle, |v_{d-1}\rangle\}$.

So Alice and Bob will each get a symbol x,$y \in \{0, 1, *\}$ respectively and they answer with $a, b \in \{0, 1, \diamond, \perp\}$. The scoring rules are

- **Case 1:** $x = y = *$, Alice and Bob should answer with $a, b \in \{\diamond, \perp\}$ and they score only if $a = b$;
- **Case 2a:** $x, y \in \{0, 1\}$ and they answer with $a, b \in \{0, 1\}$, then their answers are scored according to $I_{-\cot(\pi/2d)}$
- **Case 2b:** $x, y \in \{0, 1\}$ and Alice answer with $\perp$, then all possible outputs from Bob are discarded;

- **Case 3:** $x \in 0, 1, y = *$, when Bob answers $\diamond$, Alice should answer with $\{0.1\}$ but not $\perp$, when Bob answers $\perp$ Alice should answer $\perp$ too;

- **Case 4:** other combination of inputs are not scored.

In the ideal strategy Alice and Bob share the state $|\psi\rangle = 1/\sqrt{d}\sum_{i=0}^{d-1}|u_i\rangle|u_i\rangle$. We define two subspaces $V = \text{span}\{|u_1\rangle, |u_{d-1}\rangle\}$ and $V^{\perp} = \mathbb{C}^d \setminus \text{span}\{|u_1\rangle, |u_{d-1}\rangle\}$ amd define Π_V and $\Pi_V^{\perp}$ to be the corresponding projectors. Note that V is the subspace on which they should play the $CHSH_U^{(d)}$ to maximaize $\langle I_{-\cot(\pi/2d)}\rangle$.

We present the measurements along with the ideal correlation. The correlation for the first test

$$P(\diamond\diamond \mid **) = \langle\psi|\Pi_V \otimes \Pi_V|\psi\rangle = \frac{2}{d} \tag{17}$$

$$P(\perp\perp \mid **) = \langle\psi|\Pi_V^{\perp} \otimes \Pi_V^{\perp}|\psi\rangle = \frac{d-2}{d} \tag{18}$$

$$P(\diamond \perp \mid **) = \langle\psi|\Pi_V \otimes \Pi_V^{\perp}|\psi\rangle = 0 \tag{19}$$

$$P(\perp \diamond \mid **) = \langle\psi|\Pi_V^{\perp} \otimes \Pi_V|\psi\rangle = 0 \tag{20}$$

CHSH-type correlations

$$P(00|00) = \langle\psi| \left[|u_1\rangle\langle u_1| \otimes |u_1^{(0)}\rangle\langle u_1^{(0)}| \right] |\psi\rangle = \frac{\cos^2(\pi/4d)}{d} \tag{21}$$

$$P(01|00) = \langle\psi| \left[|u_1\rangle\langle u_1| \otimes |u_{d-1}^{(0)}\rangle\langle u_{d-1}^{(0)}| \right] |\psi\rangle = \frac{\sin^2(\pi/4d)}{d} \tag{22}$$

$$P(10|00) = \langle\psi| \left[|u_{d-1}\rangle\langle u_{d-1}| \otimes |u_1^{(0)}\rangle\langle u_1^{(0)}| \right] |\psi\rangle = \frac{\sin^2(\pi/4d)}{d} \tag{23}$$

$$P(11|00) = \langle\psi| \left[|u_{d-1}\rangle\langle u_{d-1}| \otimes |u_{d-1}^{(0)}\rangle\langle u_{d-1}^{(0)}| \right] |\psi\rangle = \frac{\cos^2(\pi/4d)}{d} \tag{24}$$

$$P(00|01) = \langle\psi| \left[|u_1\rangle\langle u_1| \otimes |u_1^{(1)}\rangle\langle u_1^{(1)}| \right] |\psi\rangle = \frac{\cos^2(\pi/4d)}{d} \tag{25}$$

$$P(01|01) = \langle\psi| \left[|u_1\rangle\langle u_1| \otimes |u_{d-1}^{(1)}\rangle\langle u_{d-1}^{(1)}| \right] |\psi\rangle = \frac{\sin^2(\pi/4d)}{d} \tag{26}$$

$$P(10|01) = \langle\psi| \left[|u_{d-1}\rangle\langle u_{d-1}| \otimes |u_1^{(1)}\rangle\langle u_1^{(1)}| \right] |\psi\rangle = \frac{\sin^2(\pi/4d)}{d} \tag{27}$$

$$P(11|01) = \langle\psi| \left[|u_{d-1}\rangle\langle u_{d-1}| \otimes |u_{d-1}^{(1)}\rangle\langle u_{d-1}^{(1)}| \right] |\psi\rangle = \frac{\cos^2(\pi/4d)}{d} \tag{28}$$

$$P(00|10) = \langle\psi| \left[|u_1^{(+)}\rangle\langle u_1^{(+)}| \otimes |u_1^{(0)}\rangle\langle u_1^{(0)}| \right] |\psi\rangle = \frac{1+\sin(\pi/2d)}{2d} \tag{29}$$

$$P(01|10) = \langle\psi| \left[|u_1^{(+)}\rangle\langle u_1^{(+)}| \otimes |u_{d-1}^{(0)}\rangle\langle u_{d-1}^{(0)}| \right] |\psi\rangle = \frac{1-\sin(\pi/2d)}{2d} \tag{30}$$

$$P(10|10) = \langle\psi| \left[|u_{d-1}^{(+)}\rangle\langle u_{d-1}^{(+)}| \otimes |u_1^{(0)}\rangle\langle u_1^{(0)}| \right] |\psi\rangle = \frac{1-\sin(\pi/2d)}{2d} \tag{31}$$

$$P(11|10) = \langle\psi| \left[|u_{d-1}^{(+)}\rangle\langle u_{d-1}^{(+)}| \otimes |u_{d-1}^{(0)}\rangle\langle u_{d-1}^{(0)}| \right] |\psi\rangle = \frac{1+\sin(\pi/2d)}{2d} \tag{32}$$

$$P(00|11) = \langle\psi| \left[|u_1^{(+)}\rangle\langle u_1^{(+)}| \otimes |u_1^{(1)}\rangle\langle u_1^{(1)}| \right] |\psi\rangle = \frac{1-\sin(\pi/2d)}{2d} \tag{33}$$

$$P(01|11) = \langle\psi| \left[|u_1^{(+)}\rangle\langle u_1^{(+)}| \otimes |u_{d-1}^{(1)}\rangle\langle u_{d-1}^{(1)}| \right] |\psi\rangle = \frac{1+\sin(\pi/2d)}{2d} \tag{34}$$

$$P(10|11) = \langle\psi| \left[|u_{d-1}^{(+)}\rangle\langle u_{d-1}^{(+)}| \otimes |u_1^{(1)}\rangle\langle u_1^{(1)}| \right] |\psi\rangle = \frac{1+\sin(\pi/2d)}{2d} \tag{35}$$

$$P(11|11) = \langle\psi| \left[|u_{d-1}^{(+)}\rangle\langle u_{d-1}^{(+)}| \otimes |u_{d-1}^{(1)}\rangle\langle u_{d-1}^{(1)}| \right] |\psi\rangle = \frac{1-\sin(\pi/2d)}{2d}. \tag{36}$$

Other correlations

$$P(\perp|0)_A = P(\perp|1)_A = \langle\psi|\Pi_V^\perp \otimes \mathbb{1}|\psi\rangle = \frac{d-2}{d} \tag{37}$$

$$P(0\diamond|0*) = \langle\psi|\left[|u_1\rangle\langle u_1| \otimes \Pi_V\right]|\psi\rangle = \frac{1}{d} \tag{38}$$

$$P(1\diamond|0*) = \langle\psi|\left[|u_{d-1}\rangle\langle u_{d-1}| \otimes \Pi_V\right]|\psi\rangle = \frac{1}{d} \tag{39}$$

$$P(0\perp|0*) = \langle\psi|\left[|u_1\rangle\langle u_1| \otimes \Pi_V^\perp\right]|\psi\rangle = 0 \tag{40}$$

$$P(1\perp|0*) = \langle\psi|\left[|u_{d-1}\rangle\langle u_{d-1}| \otimes \Pi_V^\perp\right]|\psi\rangle = 0 \tag{41}$$

$$P(\perp\diamond|0*) = \langle\psi|\Pi_V^\perp \otimes \Pi_V|\psi\rangle = 0 \tag{42}$$

$$P(\perp\perp|0*) = \langle\psi|\Pi_V^\perp \otimes \Pi_V^\perp|\psi\rangle = \frac{d-2}{d}. \tag{43}$$

when the input is $(1,*)$, the correlation is similar as above. We define the states for the measurements as follows

$$|u_1^{(0)}\rangle = \cos(\pi/4d)|u_1\rangle + \sin(\pi/4d)|u_{d-1}\rangle \tag{44}$$

$$|u_{d-1}^{(0)}\rangle = \sin(\pi/4d)|u_1\rangle - \cos(\pi/4d)|u_{d-1}\rangle \tag{45}$$

$$|u_1^{(1)}\rangle = \cos(\pi/4d)|u_1\rangle - \sin(\pi/4d)|u_{d-1}\rangle \tag{46}$$

$$|u_{d-1}^{(1)}\rangle = \sin(\pi/4d)|u_1\rangle + \cos(\pi/4d)|u_{d-1}\rangle \tag{47}$$

$$|u_1^{(+)}\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + \frac{1}{\sqrt{2}}|u_{d-1}\rangle \tag{48}$$

$$|u_{d-1}^{(+)}\rangle = \frac{1}{\sqrt{2}}|u_1\rangle - \frac{1}{\sqrt{2}}|u_{d-1}\rangle. \tag{49}$$

Observation 1. We have some orthogonality requirements as follows,

$$\langle\psi|A_*^\perp B_*^\diamond|\psi\rangle = \langle\psi|A_*^\diamond B_*^\perp|\psi\rangle = 0, \tag{50}$$

$$\langle\psi|A_0^0 B_*^\perp|\psi\rangle = \langle\psi|A_0^1 B_*^\perp|\psi\rangle = 0 \tag{51}$$

$$\langle\psi|A_1^0 B_*^\perp|\psi\rangle = \langle\psi|A_1^1 B_*^\perp|\psi\rangle = 0. \tag{52}$$

We know $A_*^\perp B_*^\diamond|\psi\rangle$, $A_*^\diamond B_*^\perp|\psi\rangle$, $A_0^0 B_*^\perp|\psi\rangle$, $A_0^1 B_*^\perp|\psi\rangle$, $A_1^0 B_*^\perp|\psi\rangle$ and $A_1^1 B_*^\perp|\psi\rangle$are orthogonal to $|\psi\rangle$. We also know that $A_0^0|\psi\rangle$,$A_0^1|\psi\rangle$,$A_1^0|\psi\rangle$,$A_1^1|\psi\rangle$ and $A_*^\diamond|\psi\rangle$ are orthogonal to $B_*^\perp|\psi\rangle$. The last orthogonality relation is that $A_*^\perp|\psi\rangle \perp B_*^\diamond|\psi\rangle$.

Observation 2. We can also observe that

$$\langle\psi|A_*^\diamond B_*^\diamond|\psi\rangle \tag{53}$$

$$=\langle\psi|(A_0^0 + A_0^1)(B_0^0 + B_0^1)|\psi\rangle = \langle\psi|(A_0^0 + A_0^1)(B_1^0 + B_1^1)|\psi\rangle \tag{54}$$

$$=\langle\psi|(A_1^0 + A_1^1)(B_0^0 + B_0^1)|\psi\rangle = \langle\psi|(A_1^0 + A_1^1)(B_1^0 + B_1^1)|\psi\rangle \tag{55}$$

$$=\langle\psi|(A_0^0 + A_0^1)B_*^\diamond|\psi\rangle = \langle\psi|(A_1^0 + A_1^1)B_*^\diamond|\psi\rangle \tag{56}$$

$$=2/d. \tag{57}$$

Since we know $\|B_*^\diamond|\psi\rangle\| = \|(A_0^0 + A_0^1)|\psi\rangle\| = \sqrt{2/d}$, combining with the relation, we find that

$$\frac{\langle\psi|B_*^\diamond(A_0^0 + A_0^1)B_*^\diamond|\psi\rangle}{\|B_*^\diamond|\psi\rangle\|^2} = 1, \tag{58}$$

which means that

$$(A_0^0 + A_0^1)B_*^\diamond|\psi\rangle = B_*^\diamond|\psi\rangle. \tag{59}$$

Since the projective measurement $(A_0^0 + A_0^1)$ commute with $B_*^\diamond$, we can get

$$\frac{\langle\psi|(A_0^0 + A_0^1)B_*^\diamond(A_0^0 + A_0^1)|\psi\rangle}{\|(A_0^0 + A_0^1)|\psi\rangle\|^2} = 1, \tag{60}$$

with similar argument we get

$$B_*^\diamond(A_0^0 + A_0^1)|\psi\rangle = (A_0^0 + A_0^1)|\psi\rangle. \tag{61}$$

The two conclusions above can be chained by commutativity to reach the conclusion that

$$(A_0^0 + A_0^1)|\psi\rangle = B_*^\diamond|\psi\rangle. \tag{62}$$

Following the same line of argument, we can conclude that

$$B_*^\diamond|\psi\rangle = A_*^\diamond|\psi\rangle = (A_0^0 + A_0^1)|\psi\rangle = (A_1^0 + A_1^1)|\psi\rangle. \tag{63}$$

Another conclusion we can draw is that $B_*^\diamond = A_*^\diamond$ and $B_*^\perp = A_*^\perp$. We can assume the Schmidt decomposition of $|\psi\rangle$ is

$$|\psi\rangle = \sum_{i=1}^{d} \lambda_i |i\rangle|i\rangle. \tag{64}$$

If supp $A_*^\diamond \neq$ supp $B_*^\diamond$, then $B_*^\diamond|\psi\rangle \neq A_*^\diamond|\psi\rangle$. Since $B_*^\diamond$ and $A_*^\diamond$ are projectors, By the orthogonality relations, we can also conclude that $A_*^\perp = B_*^\perp$.

Observation 3. We can also observe that

$$\begin{aligned}
&\langle\psi|A_0^\perp|\psi\rangle = \langle\psi|A_1^\perp|\psi\rangle && (65)\\
=&\langle\psi|A_0^\perp B_*^\perp|\psi\rangle = \langle\psi|A_1^\perp B_*^\perp|\psi\rangle && (66)\\
=&\langle\psi|A_*^\perp B_*^\perp|\psi\rangle && (67)\\
=&\frac{d-2}{d}. && (68)
\end{aligned}$$

By similar argument as in Observation 2, we conclude that

$$B_*^\perp|\psi\rangle = A_*^\perp|\psi\rangle = A_0^\perp|\psi\rangle = A_1^\perp|\psi\rangle. \tag{69}$$

Now we re-examine the CHSH-type correlation

$$\begin{aligned}
&\langle\psi|A_0^0 B_0^0|\psi\rangle && (70)\\
=&\langle\psi|(A_*^\diamond + A_*^\perp)A_0^0 B_0^0(A_*^\diamond + A_*^\perp)|\psi\rangle && (71)\\
=&\langle\psi|A_*^\diamond A_0^0 B_0^0 A_*^\diamond|\psi\rangle + \langle\psi|A_*^\diamond A_0^0 B_0^0 A_*^\perp|\psi\rangle && (72)\\
&+ \langle\psi|A_*^\perp A_0^0 B_0^0 A_*^\diamond|\psi\rangle + \langle\psi|A_*^\perp A_0^0 B_0^0 A_*^\perp|\psi\rangle && (73)\\
=&\langle\psi|B_*^\diamond A_0^0 B_0^0 B_*^\diamond|\psi\rangle + \langle\psi|A_*^\diamond A_0^0 B_0^0 A_0^\perp|\psi\rangle && (74)\\
&+ \langle\psi|A_0^\perp A_0^0 B_0^0 A_*^\diamond|\psi\rangle + \langle\psi|A_0^\perp A_0^0 B_0^0 A_0^\perp|\psi\rangle && (75)\\
=&\langle\psi|B_*^\diamond A_0^0 B_0^0 B_*^\diamond|\psi\rangle, && (76)
\end{aligned}$$

where we use the facts that $B_*^\diamond|\psi\rangle = A_*^\diamond|\psi\rangle$, $A_*^\perp|\psi\rangle = A_0^\perp|\psi\rangle$ and that $\mathrm{span}(A_0^0) \cap \mathrm{span}(A_0^\perp) = \emptyset$. This means that if Alice and Bob share state $B_*^\diamond|\psi\rangle / \|B_*^\diamond|\psi\rangle\|$ and apply $A_0^0 B_0^0$ we get conditional probability

$$\frac{\langle\psi|B_*^\diamond A_0^0 B_0^0 B_*^\diamond|\psi\rangle}{\langle\psi|B_*^\diamond B_*^\diamond|\psi\rangle} = \frac{\cos^2(\pi/4d)}{2}. \tag{77}$$

We can rewrite the other correlations in a similar way and get a new set of correlation which match exactly with the optimal correlation of $\langle \mathbb{1}_{-\cot(\pi/2d)} \rangle$ and use the self-testing argument on $\{A_0^0 - A_0^1, B_0^0 - B_0^1, A_1^0 - A_1^1, B_1^0 - B_1^1\}$ and $B_*^\diamond|\psi\rangle / \|B_*^\diamond|\psi\rangle\|$.

1.4 The single value test

By the following test, we want to make sure that $B_1 B_2$ has eigenvalue 1. The key observation is that $B_1 B_2 |psi\rangle = |\psi\rangle$ implies that

$$|\psi\rangle = B_1|\psi\rangle = B_2|\psi\rangle. \tag{78}$$

Alice and Bob will each get a symbol $x,y \in \{0, 1, *\}$ respectively and they answer with $a, b \in \{0, 1, \diamond, \perp\}$. The scoring rules are

- **Case 1:** $x = y = *$, Alice and Bob should answer with $a, b \in \{\diamond, \perp\}$ and they score only if $a = b$;
- **Case 2a:** $x = *$ and $y = 1$, when Alice answer $\diamond$, Bob should answer 0, when Alice answer$\perp$ all answers of Bob are accepted;
- **Case 2b:** $x = *$ and $y = 2$, when Alice answer $\diamond$, Bob should answer 0, when Alice answer$\perp$ all answers of Bob are accepted.

The ideal strategy has projective measurement

$$A_*^\diamond = B_*^\diamond = |u_0\rangle\langle u_0|, \qquad A_*^\perp = B_*^\perp = \mathbb{1} - |u_0\rangle\langle u_0|, \tag{79}$$

and Bob will reuse B_1 B_2 from his strategy to win the linear system game. The shared state is

$$|\psi\rangle = \frac{1}{d} \sum_{i \in [d]} |u_i\rangle|u_i\rangle. \tag{80}$$

So in the ideal correlation, we have

$$P(\diamond\diamond \mid **) = \langle\psi|A_*^\diamond \otimes B_*^\diamond|\psi\rangle = \frac{1}{d} \tag{81}$$

$$P(\perp\perp \mid **) = \langle\psi|A_*^\perp \otimes B_*^\perp|\psi\rangle = \frac{d-1}{d} \tag{82}$$

$$P(\diamond \perp \mid **) = \langle\psi|A_*^\diamond \otimes B_*^\perp|\psi\rangle = 0 \tag{83}$$

$$P(\perp \diamond \mid **) = \langle\psi|A_*^\perp \otimes B_*^\diamond|\psi\rangle = 0 \tag{84}$$

$$P(\diamond 0 \mid *1) = \langle\psi|A_*^\diamond \otimes B_1^0|\psi\rangle = \frac{1}{d} \tag{85}$$

$$P(\diamond 0 \mid *2) = \langle\psi|A_*^\diamond \otimes B_2^0|\psi\rangle = \frac{1}{d} \tag{86}$$

By the marginal distribution we know that $\|A_*^\diamond|\psi\rangle\| = \frac{1}{\sqrt{d}}$. From the condition that $P(\diamond 0|*1) = 1/d$, we know

$$\frac{\langle\psi|A_*^\diamond B_1^0|\psi\rangle}{\|A_*^\diamond|\psi\rangle\|^2} = 1, \tag{87}$$

which implies that

$$B_1^0\frac{A_*^\diamond|\psi\rangle}{\|A_*^\diamond|\psi\rangle\|} = \frac{A_*^\diamond|\psi\rangle}{\|A_*^\diamond|\psi\rangle\|}. \tag{88}$$

With similar reasoning, we get

$$B_2^0\frac{A_*^\diamond|\psi\rangle}{\|A_*^\diamond|\psi\rangle\|} = \frac{A_*^\diamond|\psi\rangle}{\|A_*^\diamond|\psi\rangle\|}. \tag{89}$$

Hence $\frac{A_*^\diamond|\psi\rangle}{\|A_*^\diamond|\psi\rangle\|}$ is in the intersection of supp(B_1^0) and supp(B_2^0). Since supp(B_x^0) and supp(B_x^1) are disjoint, we know

$$B_x\frac{A_*^\diamond|\psi\rangle}{\|A_*^\diamond|\psi\rangle\|} = \frac{A_*^\diamond|\psi\rangle}{\|A_*^\diamond|\psi\rangle\|} \tag{90}$$

for $x = 1, 2$. Therefor, we know $\frac{A_*^\diamond|\psi\rangle}{\|A_*^\diamond|\psi\rangle\|}$ is an eigenvector of B_1B_2 with eigenvalue 1 because

$$B_1B_2\frac{A_*^\diamond|\psi\rangle}{\|A_*^\diamond|\psi\rangle\|} = B_1\frac{A_*^\diamond|\psi\rangle}{\|A_*^\diamond|\psi\rangle\|} = \frac{A_*^\diamond|\psi\rangle}{\|A_*^\diamond|\psi\rangle\|}. \tag{91}$$

In the later section, when we apply this test to make sure some unitary U has eigenvalue 1 with one-dimensional eigen-space, we denote the test by SVT_U.

2 The new game

Assume the linear system game has n variables and m equations. Alice receives $x \in \{1, \dots m+8\}$ and Bob receives $y \in \{1, \dots, n+4\}$. The scoring rules are the following:

- when $x \in \{1, \dots m\}$ and $y \in \{1, \dots n\}$, they are scored according to the linear system game;
- when $x \in \{m+1, m+2, m+3\}$ and $y \in \{1, 2, n+1\}$, they are scored according to $CHSH^{(d)}_{\sigma_x}$, where

$$*_A = m+1, \quad 0_A = m+2, \quad 1_A = m+3, \tag{92}$$
$$*_B = n+1, \quad 0_B = 1, \quad 1_A = 2, \tag{93}$$

  are the inputs for the game $CHSH^{(d)}_{\sigma_x}$ (The intuition behind is that $B_1B_2 = \sigma_x$.);
- when $x \in \{m+4, m+5, m+6\}$ and $y \in \{3, 4, n+2\}$, they are scored according to $CHSH^{(d)}_{\sigma_z}$, where

$$*_A = m+4, \quad 0_A = m+5, \quad 1_A = m+6, \tag{94}$$
$$*_B = n+2, \quad 0_B = 3, \quad 1_A = 4, \tag{95}$$

  are the inputs for the game $CHSH^{(d)}_{\sigma_z}$ (The intuition behind is that $B_3B_4 = \sigma_z$.);

- when $x = m+7$ and $y \in \{1, 2, n+3\}$, they are scored according to SVT_{σ_x}, where

$$*_A = m+7 \tag{96}$$
$$*_B = n+3, \quad 1_B = 1, \quad 2_B = 2 \tag{97}$$

  are the inputs for the game SVT_{σ_x} (The intuition behind is that $B_1B_2 = \sigma_x$.);

- when $x = m+8$ and $y \in \{3, 4, n+4\}$, they are scored according to SVT_{σ_z}, where

$$*_A = m+8 \tag{98}$$
$$*_B = n+4, \quad 3_B = 3, \quad 4_B = 4 \tag{99}$$

  are the inputs for the game SVT_{σ_z} (The intuition behind is that $B_3B_4 = \sigma_z$.);

- otherwise, they score 0.

Note that the dimension d is defined in the rules of $CHSH^{(d)}_{\sigma_z}$ and $CHSH^{(d)}_{\sigma_x}$. Next we are going to prove that the strategy winning this game optimally can self-test d-dimensional EPR pair and d-dimensional σ_x and σ_z.

2.1 Proof Sketch

Now we examine the implications of Alice and Bob winning the linear constraint game perfectly. By Lemma 4.3 of [**?**], we can extract an operator solution from the perfect winning strategy of the linear system game: For each variable $\{x_i\}_{i=1}^n$, Alice and Bob has operators A_i and B_i respectively. The condition that they agree with assignment to variables means that

$$\langle\psi|A_i \otimes \overline{B_i}|\psi\rangle = 1 \Rightarrow A_i \otimes \overline{B_i}|\psi\rangle = |\psi\rangle \tag{100}$$

and the condition that Alice's assignments satisfy the constraint means that

$$\mathrm{Tr}(\rho_A \Pi_{j:A_{ij}\neq 0} A_j) = \mathrm{Tr}(\rho_A) \text{ for all } 1 \leq i \leq m \tag{101}$$

where $\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$. For any $|v\rangle \in \mathrm{supp}(\rho_A)$, we have

$$\Pi_{j:A_{ij}\neq 0} A_j |v\rangle = |v\rangle. \tag{102}$$

Since the relation $u^{-1}xu = x^2$ is embedded in this linear system game, we know

$$\sigma_A(u)^\dagger A_1A_2\sigma_A(u)|v\rangle = (A_1A_2)^2|v\rangle \text{ for all } |v\rangle \in \mathrm{supp}(\rho_A). \tag{103}$$

For simplicity, we define $X = A_1A_2$ and $U = \sigma_A(u)$. The condition becomes

$$UXU^\dagger|v\rangle = X^2|v\rangle \text{ for all } |v\rangle \in \mathrm{supp}(\rho_A). \tag{104}$$

On the subspace $\mathrm{supp}(\rho_A)$, we have the equality

$$UXU^\dagger X^{-2} = \mathbb{1}. \tag{105}$$

We will come back to the implication of this condition later.

Next we look at the implication of winning the $CHSH^{(d)}_{A_1A_2}$ game optimally. It means that there exists isometris V_A on Alice side and V_B on Bob's side such that

$$(V_A \otimes V_B)\frac{A^\diamond_* \otimes B^\diamond_*|\psi\rangle}{\|A^\diamond_* \otimes B^\diamond_*|\psi\rangle\|} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \otimes |extra\rangle \tag{106}$$

$$(V_A \otimes V_B)\frac{B_1 + B_2}{2\cos(\pi/2d)}\frac{A^\diamond_* \otimes B^\diamond_*|\psi\rangle}{\|A^\diamond_* \otimes B^\diamond_*|\psi\rangle\|} = (\mathbb{1} \otimes \sigma_z)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \otimes |extra\rangle \tag{107}$$

$$(V_A \otimes V_B)\frac{B_1 - B_2}{-2\sin(\pi/2d)}\frac{A^\diamond_* \otimes B^\diamond_*|\psi\rangle}{\|A^\diamond_* \otimes B^\diamond_*|\psi\rangle\|} = (\mathbb{1} \otimes \sigma_x)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \otimes |extra\rangle. \tag{108}$$

Suppose $A^\diamond_*$ measures states $|u_0\rangle$ and $|u_1\rangle$. Then we know

$$(B_1 + B_2)|u_0\rangle = 2\cos(\pi/2d)|u_0\rangle \tag{109}$$

$$(B_1 + B_2)|u_1\rangle = -2\cos(\pi/2d)|u_1\rangle \tag{110}$$

$$(B_1 - B_2)|u_0\rangle = -2\sin(\pi/2d)|u_1\rangle \tag{111}$$

$$(B_1 - B_2)|u_1\rangle = -2\sin(\pi/2d)|u_0\rangle \tag{112}$$

which implies that

$$\begin{aligned}
B_1|u_0\rangle &= \cos(\pi/2d)|u_0\rangle - \sin(\pi/2d)|u_1\rangle \\
B_2|u_0\rangle &= \cos(\pi/2d)|u_0\rangle + \sin(\pi/2d)|u_1\rangle \\
B_1|u_1\rangle &= -\sin(\pi/2d)|u_0\rangle - \cos(\pi/2d)|u_1\rangle \\
B_2|u_1\rangle &= \sin(\pi/2d)|u_0\rangle - \cos(\pi/2d)|u_1\rangle \\
B_1B_2|u_0\rangle &= \cos(\pi/2d)B_1|u_0\rangle + \sin(\pi/2d)B_1|u_1\rangle \\
&= \cos(\pi/2d)(\cos(\pi/2d)|u_0\rangle - \sin(\pi/2d)|u_1\rangle) - \sin(\pi/2d)(\sin(\pi/2d)|u_0\rangle + \cos(\pi/2d)|u_1\rangle) \\
&= \cos(\pi/d)|u_0\rangle - \sin(\pi/d)|u_1\rangle \\
B_1B_2|u_1\rangle &= \sin(\pi/2d)B_1|u_0\rangle - \cos(\pi/2d)B_1|u_1\rangle \\
&= \sin(\pi/2d)(\cos(\pi/2d)|u_0\rangle - \sin(\pi/2d)|u_1\rangle) + \cos(\pi/2d)(\sin(\pi/2d)|u_0\rangle + \cos(\pi/2d)|u_1\rangle) \\
&= \sin(\pi/d)|u_0\rangle + \cos(\pi/d)|u_1\rangle.
\end{aligned}$$

We can conclude that

$$B_1B_2(|u_0\rangle + i|u_1\rangle) = e^{i\frac{\pi}{d}}(|u_0\rangle + i|u_1\rangle) \tag{113}$$

$$B_1B_2(|u_0\rangle - i|u_1\rangle) = e^{-i\frac{\pi}{d}}(|u_0\rangle - i|u_1\rangle). \tag{114}$$

Define $|x_1\rangle = 1/\sqrt{2}(|u_0\rangle + i|u_1\rangle)$, then it is the eigenvector of X with eigenvalue $e^{i\pi/d} = \omega_d$. Similarly, define $|x_{d-1}\rangle = 1/\sqrt{2}(|u_0\rangle - i|u_1\rangle)$ such that $X|x_{d-1}\rangle = \omega_d^{d-1}|x_{d-1}\rangle$. We know

$$\tilde{A}^\diamond_*|\tilde{\psi}\rangle = \tilde{A}^\diamond_* \otimes \tilde{B}^\diamond_*|\tilde{\psi}\rangle = \frac{1}{\sqrt{d}}(|u_0\rangle|u_0\rangle + |u_1\rangle|u_1\rangle). \tag{115}$$

If we restrict to Alice's system, we get

$$\mathrm{Tr}_B(\tilde{A}^\diamond_*|\tilde{\psi}\rangle\langle\tilde{\psi}|\tilde{A}^\diamond_*) = \tilde{A}^\diamond_*\rho_A\tilde{A}^\diamond_* = \frac{1}{d}|u_0\rangle\langle u_0| + |u_1\rangle\langle u_1| = \frac{1}{d}\tilde{A}^\diamond_*. \tag{116}$$

Hence ρ_A acts as $\mathbb{1}$ on $\mathrm{supp}(\tilde{A}^\diamond_*)$. We also know that there exists local unitary U_A such that

$$U_A\tilde{A}^\diamond_*U_A^\dagger = |x_1\rangle\langle x_1| + |x_{d-1}\rangle\langle x_{d-1}| \tag{117}$$

and

$$\mathrm{Tr}(|x_1\rangle\langle x_1|\rho_A) = \mathrm{Tr}(U_A|u_0\rangle\langle u_0|U_A^\dagger\rho_A) = \frac{1}{d}\,\mathrm{Tr}(U_A|u_0\rangle\langle u_0|U_A^\dagger) = \frac{1}{d}, \tag{118}$$

and similarly $\mathrm{Tr}(|x_{d-1}\rangle\langle x_{d-1}|\rho_A) = 1/d$.

Based on eq. (105), we know

$$XU^\dagger|x_1\rangle = U^\dagger X^2|x_1\rangle = \omega_d^2 U^\dagger|x_1\rangle, \tag{119}$$

so $U^\dagger|x_1\rangle$ is an eigenvector of X with eigenvalue ω_d^2. By induction, we know $X(U^\dagger)^i|x_1\rangle = \omega_d^{2^i}(U^\dagger)^i|x_1\rangle$. From the set $\{(U^\dagger)^i|x_1\rangle\}_{i=0}^{d-2}$, we can identify $|x_i\rangle$ for $1 \le i \le d-1$ such that $X|x_i\rangle = \omega_d^i|x_i\rangle$. From the single value test, we know

$$\tilde{A}_\triangle^\diamond|\tilde{\psi}\rangle = \tilde{A}_\triangle^\diamond \otimes \tilde{B}_\triangle^\diamond|\tilde{\psi}\rangle = \frac{1}{\sqrt{d}}|x_0\rangle|x_0\rangle \tag{120}$$

where $X|x_0\rangle = |x_0\rangle$. Hence

$$\tilde{A}_\triangle^\diamond = |x_0\rangle\langle x_0| \tag{121}$$

and

$$\tilde{A}_\triangle^\diamond\rho_A\tilde{A}_\triangle^\diamond = \frac{1}{d}|x_0\rangle\langle x_0| = \frac{1}{d}\tilde{A}_\triangle^\diamond. \tag{122}$$

The full eigen-decomposition of X is

$$X = \sum_{i=0}^{d-1}\omega_d|x_i\rangle\langle x_i|. \tag{123}$$

[Can we get the full eigen-decomposition from the unitarity of X?] If we substitute this eigen-decomposition into eq. (105), we have

$$\sum_{i=0}^{d-1}\omega_d^i\,\mathrm{Tr}(U|x_i\rangle\langle x_i|U^{-1}\rho_A) = \sum_{i=0}^{d-1}\omega_d^{2i}\,\mathrm{Tr}(|x_i\rangle\langle x_i|\rho_A), \tag{124}$$

or equivalently

$$\sum_{i=0}^{d-1}\omega_d^i\,\mathrm{Tr}\left[\left(U|x_i\rangle\langle x_i|U^{-1} - |x_{i/2}\rangle\langle x_{i/2}|\right)\rho_A\right] = 0. \tag{125}$$

Define $\alpha_i = \mathrm{Tr}\left[\left(U|x_i\rangle\langle x_i|U^{-1} - |x_{i/2}\rangle\langle x_{i/2}|\right)\rho_A\right] \in [-1,1]$. The fact that $\sum_{i=0}^{d-1}\alpha_i\omega_d^i = 0$ implies that $\alpha_i = 0$ or $\alpha_i = 1$ for all i. Let's look at α_2, which is defined to be

$$\alpha_2 = \mathrm{Tr}(U|x_2\rangle\langle x_2|U^{-1}\rho_A) - \mathrm{Tr}(|x_1\rangle\langle x_1|\rho_A) = \mathrm{Tr}(U|x_2\rangle\langle x_2|U^{-1}\rho_A) - \frac{1}{d} \le 1 - \frac{1}{d}, \tag{126}$$

so $\alpha_2 = 0$ and $\alpha_i = 0$ for all i.

If we trace out Bob's system, we get

$$\begin{aligned}
&\mathrm{Tr}_B\left[(U_A\otimes U_B)(\tilde{A}_*^\diamond)|\tilde{\psi}\rangle\langle\tilde{\psi}|(\tilde{A}_*^\diamond)(U_A\otimes U_B)^\dagger\right] &(127)\\
=&U_A\tilde{A}_*^\diamond\rho_A\tilde{A}_*^\diamond U_A^\dagger &(128)\\
=&\frac{1}{d}(|x_1\rangle\langle x_1| + |x_{d-1}\rangle\langle x_{d-1}|) &(129)
\end{aligned}$$

where $U_A = U_B = |x_1\rangle\langle u_0| + |x_{d-1}\rangle\langle u_1|$ when their actions are restricted to $\mathrm{supp}(\tilde{A}^{\diamond}_{*})$. By eq. (119), we get

$$U^{\dagger} U_A \tilde{A}^{\diamond}_{*} \rho_A \tilde{A}^{\diamond}_{*} U_A^{\dagger} U = \frac{1}{d}(|x_2\rangle\langle x_2| + |x_{d-2}\rangle\langle x_{d-2}|) \tag{130}$$

So if we repeatedly conjugate $U_A \tilde{A}^{\diamond}_{*} \rho_A \tilde{A}^{\diamond}_{*} U_A^{\dagger}$ by $U^{\dagger}$ $d-1$ times and sum the equations up, we get

$$\sum_{i=0}^{d-2} (U^{\dagger})^i U_A \tilde{A}^{\diamond}_{*} \rho_A \tilde{A}^{\diamond}_{*} U_A^{\dagger} U^i = \frac{2}{d} \sum_{i=1}^{d-1} |x_i\rangle\langle x_i|. \tag{131}$$

Note that here we used the fact that d has primitive root 2.

From the single value test, we know

$$\tilde{A}^{\diamond}_{\triangle} |\tilde{\psi}\rangle = \tilde{A}^{\diamond}_{\triangle} \otimes \tilde{B}^{\diamond}_{\triangle} |\tilde{\psi}\rangle = \frac{1}{\sqrt{d}} |x_0\rangle |x_0\rangle \tag{132}$$

where $X|x_0\rangle = |x_0\rangle$. Again, we trace out Bob's system and get that

$$\tilde{A}^{\diamond}_{\triangle} \rho_A \tilde{A}^{\diamond}_{\triangle} = \frac{1}{d} |x_0\rangle\langle x_0|. \tag{133}$$

Define $V = A^{\diamond}_{\triangle} + \sum_{i=0}^{(d-1)/2} (U^{\dagger})^i U_A \tilde{A}^{\diamond}_{*}$, we can check that

$$VV^{\dagger} = A^{\diamond}_{\triangle} + \sum_{i,j=0}^{(d-1)/2} (U^{\dagger})^i U_A \tilde{A}^{\diamond}_{*} U_A^{\dagger} U^j \tag{134}$$

Moreover,

$$\mathrm{Tr}(\tilde{A}^{\diamond}_{\triangle} \rho_A \tilde{A}^{\diamond}_{\triangle}) + \sum_{i=0}^{(d-1)/2} \mathrm{Tr}[(U^{\dagger})^i U_A \tilde{A}^{\diamond}_{*} \rho_A \tilde{A}^{\diamond}_{*} U_A^{\dagger} U^i] \tag{135}$$

$$= \mathrm{Tr}(\tilde{A}^{\diamond}_{\triangle} \rho_A) + \sum_{i=0}^{(d-1)/2} \mathrm{Tr}[\tilde{A}^{\diamond}_{*} U_A (U^{\dagger})^{2i} U_A \tilde{A}^{\diamond}_{*} \rho_A] \tag{136}$$

$$=1 \tag{137}$$

$$= \mathrm{Tr}(\rho_A). \tag{138}$$

A The linear system game

In this section, we are going to embed the group $\mathcal{P}$ into a solution group $\Gamma_{\mathcal{P}}$ of a linear system game.

We first embed the relation $xzx^{-1}z^{-1} = \mathcal{J}$ in a linear system game by ideas from the Magic Square game. It has been shown that the following relations embeds $xzx^{-1}z^{-1} = \mathcal{J}$ [2]. Let $y_3 = z$ and $y_7 = x$ and the linear relations are

$$y_1 y_2 y_3 = e, y_4 y_5 y_6 = e, y_7 y_8 y_9 = e, \tag{139}$$

$$y_1^{-1} y_4^{-1} y_7^{-1} = e, y_2^{-1} y_5^{-1} y_8^{-1} = \mathcal{J}, y_3^{-1} y_6^{-1} y_9^{-1} = e. \tag{140}$$

[2] For example, Fig. 11 of Ref. [?] proves it in a group picture.

We refer to these set of linear relations as the MS_l relations. The magic square game also introduces some commutation relations:

$$[y_1, y_2] = [y_1, y_3] = [y_1, y_4] = [y_1, y_7] = e, \tag{141}$$
$$[y_2, y_3] = [y_2, y_5] = [y_2, y_8] = e, \tag{142}$$
$$[y_3, y_6] = [y_3, y_9] = e, \tag{143}$$
$$[y_4, y_7] = [y_4, y_5] = [y_4, y_6] = e, \tag{144}$$
$$[y_5, y_6] = [y_5, y_8] = e, \tag{145}$$
$$[y_6, y_9] = e, \tag{146}$$
$$[y_7, y_8] = [y_7, y_9] = e, \tag{147}$$
$$[y_8, y_9] = e, \tag{148}$$

which will be referred as MS_c. So we define $\mathcal{P}$ as

$$\mathcal{P} = \langle \{y_i\}_{i=1}^{9}, u, \mathcal{J} : MS_l \cup MS_c \cup \{[\mathcal{J}, y_i] = [\mathcal{J}, u] = e, u y_7 u^{-1} = y_7^2, u^{-1} y_3 u = y_3^2\}\rangle. \tag{149}$$

Then we can start to use a procedure similar to the ones developed in Ref. [?] to embed $\mathcal{P}$ into the solution group of a linear system game.

The first step is to embed $\mathcal{P}$ into a group which almost satisfy the definition of homogeneous-linear-plus-conjugacy group [?].

Replacing y_3. We start by introducing $y_{3,1}$ and $y_{3,2}$ of order 2 such that $y_3 = y_{3,1}y_{3,2}$ and $y_{3,1}$ commutes with y_i, u and $\mathcal{J}$ for $i \neq 3$. Then the relation $uy_3u^{-1} = y_3^2$ is rewritten as $uy_{3,2}u^{-1} = y_{3,2}y_{3,1}y_{3,2}$ so we introduce $y_{3,3} = y_{3,2}y_{3,1}y_{3,2}$. The group $\mathcal{P}$ is embedded in

$$\begin{aligned}\mathcal{P} = \langle y_{3,1}, y_{3,2}, y_{3,3}, x, u, \{y_i\}_{i\neq 3}\mathcal{J} :\ & y_{3,1}, y_{3,2}, y_{3,3} \text{ of order 2;} \\ & \mathcal{J} \text{ commute with all the generators, } [y_i, y_{3,1}] = [u, y_{3,1}] = e \text{ for } i \neq 3, \\ & MS_c \text{ with } y_3 \text{ replaced by } y_{3,2}, MS_l \text{ with } y_3 \text{ replaced by } y_{3,1}y_{3,2}, \\ & u^{-1}y_7u = y_7^2, y_{3,3} = y_{3,2}y_{3,1}y_{3,2}, uy_{3,2}u^{-1} = y_{3,3}\rangle\end{aligned} \tag{150}$$

Replace y_7. We introduce $y_{7,1}$ and $y_{7,2}$ of order 2 such that $y_7 = y_{7,1}y_{7,2}$ and $y_{7,1}$ commutes with u, $\mathcal{J}$ and y_i for $i \neq 3, 7$. Then the relation $u^{-1}y_7u = y_7^2$ is rewritten as $u^{-1}y_{7,2}u = y_{7,2}y_{7,1}y_{7,2}$ so we introduce $y_{7,3} = y_{7,2}y_{7,1}y_{7,2}$. The relation $y_7y_{3,1}y_7^{-1} = y_{3,1}$ is rewritten as $y_{7,2}y_{3,1}y_{7,2} = y_{7,1}y_{3,1}y_{7,1}$ so we introduce $y_{7,4} = y_{7,1}y_{3,1}y_{7,1}$. The group $\mathcal{P}$ is embedded as

$$\begin{aligned}\mathcal{P} = \langle \{y_{3,i}\}_{i=1}^{3}, \{y_{7,i}\}_{i=1}^{4}, \{y_i\}_{i\neq 3,7}, u, \mathcal{J} :\ & \{y_{3,i}\}_{i=1}^{3}\{y_{7,i}\}_{i=1}^{4} \text{ of order 2;} \\ & \mathcal{J} \text{ commutes with all the generators, } [u, y_{3,1}] = [u, y_{7,1}] = e, \\ & y_{3,1}, y_{7,1} \text{ commute with } y_i,\ i \neq 3, 7, \\ & MS_c \text{ with } y_3, y_7 \text{ replaced by } y_{3,2}, y_{7,2}, \\ & MS_l \text{ with } y_3,\ y_7 \text{ replaced by } y_{3,1}y_{3,2} \text{ and } y_{7,1}y_{7,2} \\ & y_{3,3} = y_{3,2}y_{3,1}y_{3,2}, uy_{3,2}u^{-1} = y_{3,3}, \\ & y_{7,3} = y_{7,2}y_{7,1}y_{7,2}, u^{-1}y_{7,2}u = y_{7,3}, \\ & y_{7,4} = y_{7,1}y_{3,1}y_{7,1}, y_{7,2}y_{3,1}y_{7,2} = y_{7,4}\rangle\end{aligned} \tag{151}$$

Replacing u. We introduce u_1, u_2 of order 2 such that $u = u_1u_2$ and u_1 commutes with y_i for $i \neq 3, 7$ and $\mathcal{J}$. The conjugacy relations involving u are $uy_{3,2}u^{-1} = y_{3,3}$, $u^{-1}y_{7,2}u = y_{7,3}$, $uy_{3,1}u^{-1} = y_{3,1}$

and $uy_{7,1}u^{-1} = y_{7,1}$. So we need to introduce u_3, u_4, u_5, u_6 such that

$$u_3 = u_2 y_{3,2} u_2 = u_1 y_{3,3} u_1, \tag{152}$$
$$u_4 = u_1 y_{7,2} u_1 = u_2 y_{7,3} u_2, \tag{153}$$
$$u_5 = u_2 y_{3,1} u_2 = u_1 y_{3,1} u_1, \tag{154}$$
$$u_6 = u_2 y_{7,1} u_2 = u_1 y_{7,1} u_1. \tag{155}$$

At this stage $\mathcal{P}$ has generators $\{y_{3,i}\}_{i=1}^{3}, \{y_{7,i}\}_{i=1}^{4}, \{y_i\}_{i\neq 3,7}, \{u_i\}_{i=1}^{6}$ with 6 linear relations and 44 conjugacy relations.

Replacing y_i for $i \neq 3, 7$. The conjugacy relations involving y_1 are

$$y_1 y_{3,1} y_1^{-1} = y_{3,1}, y_1 y_{7,1} y_1^{-1} = y_{7,1}. \tag{156}$$

and 4 commutation relations from MS_c. So we need to introduce $y_{1,i}$ for $i = 1, 2 \ldots 8$ such that they all are of order 2 and

$$y_1 = y_{1,1} y_{1,2} \tag{157}$$
$$y_{1,3} = y_{1,2} y_{3,1} y_{1,2} = y_{1,1} y_{3,1} y_{1,1}, \tag{158}$$
$$y_{1,4} = y_{1,2} y_{7,1} y_{1,2} = y_{1,1} y_{7,1} y_{1,1}, \tag{159}$$
$$y_{1,5} = y_{1,2} y_2 y_{1,2} = y_{1,1} y_2 y_{1,1}, \tag{160}$$
$$y_{1,6} = y_{1,2} y_{3,2} y_{1,2} = y_{1,1} y_{3,2} y_{1,1}, \tag{161}$$
$$y_{1,7} = y_{1,2} y_4 y_{1,2} = y_{1,1} y_4 y_{1,1}, \tag{162}$$
$$y_{1,8} = y_{1,2} y_{7,2} y_{1,2} = y_{1,1} y_{7,2} y_{1,1}. \tag{163}$$

The new commutation relations are $y_{1,1}$ commutes with all the remaining y_j's and commutation relations from MS_c involving y_1 with y_1 replaced by $y_{1,2}$. Then we repeat this process with y_2. In summary, replacing $y_1, y_2, y_4, y_5, y_6, y_8, y_9$ introduces $8 + 9 + 10 + 11 + 12 + 13 + 14 = 77$ new variables and $12 + 14 + 16 + 18 + 20 + 22 + 24 = 126$ new conjugacy relations. In total $\mathcal{P}$ has 90 variables excluding $\mathcal{J}$, 6 linear relations and 170 conjugacy relations.

Then following the recipe given in Proposition 4.2 and Lemma 4.4 of Ref. [**?**], we embed $\mathcal{P}$ into the solution group of a linear system game having 2351 variables and 1916 linear relations. Alice's output alphabet is of size 64 and of size 2 for Bob.

Such linear system game has at the biggest size when $a = 5$ or the special relation is $uxu^{-1} = x^5$. The biggest game has 2465 variables and 2006 equations. We leave the derivation for curious readers.

B Slofstra's Binary Constraint game

We start with an extended homogeneous-linear-plus-conjugacy group

$$K = \langle x, y, a, b, c : a^2 = b^2 = c^2 = e, abc = e, yay^{-1} = a, yby^{-1} = c, xyx^{-1} = y^2 \rangle \tag{164}$$

and construct a solution group corresponding to a nonlocal game.

We first embed K into another homogeneous-linear-plus-conjugacy group, K', with x, y replaced by elements of order 2. By Proposition 4.8 of [**?**], we first introduce z, w such that $z^2 = w^2 = e$,

$y = zw$ and $xz = zx$ then

$$K' = \langle a, b, c, z, w, a', b', z', x : abc = e, \tag{165}$$
$$waw = a', wbw = b', wz'w = z \tag{166}$$
$$za'z = a, zb'z = c, \tag{167}$$
$$xwx^{-1} = z', xzx^{-1} = z\rangle. \tag{168}$$

Next, we introduce u, v such that $u^2 = v^2 = e$ and $x = uv$, then

$$\begin{aligned} K' = \langle a, b, c, z, w, a', b', z', u, v, z_v : abc &= e, \\ waw &= a', wbw = b', wz'w = z \\ za'z &= a, zb'z = c, \\ vwv &= w', vzv = z_v, \\ uw'u &= z', uz_vu = z\rangle \end{aligned}$$

Note that we skipped the relations that all elements are of order 2. To easier introduce new elements in the following construction, we relabel the elements as

$$\begin{aligned} &x_1 = w,\ x_2 = a,\ x_3 = a',\ x_4 = b,\ x_5 = b',\ x_6 = z,\ x_7 = z' \\ &x_8 = c,\ x_9 = v,\ x_{10} = u,\ x_{11} = w',\ x_{12} = z_v, \end{aligned}$$

so K' can also be written as

$$\begin{aligned} K' = \langle \{x_i\}_{i=1}^{12} : x_i^2 &= e, x_2x_4x_8 = e, \\ & x_1x_2x_1 = x_3,\ x_1x_4x_1 = x_5,\ x_1x_7x_1 = x_6, \\ & x_6x_3x_6 = x_2,\ x_6x_5x_6 = x_8, \\ & x_9x_1x_9 = x_{11}, x_9x_6x_9 = x_{12} \\ & x_{10}x_{11}x_{10} = x_7, x_{10}x_{12}x_{10} = x_6\rangle. \end{aligned} \tag{169}$$

Here the special element is $x_2 = a$. Then we add another two order-2 element, t Z, to add a linear relation to it

$$\hat{K} = \langle K', t, Z : t^2 = Z^2 = e,\ tx_2t = Z.\ Zx_2 = J\rangle_{\mathbb{Z}_2}. \tag{170}$$

We rename $t = x_{13}$ and $Z = x_{14}$, then $\hat{K}$ contains linear relations

$$x_2x_4x_8 = e,\ x_{14}x_2 = J$$

and conjugacy relations

$$\begin{aligned} & x_1x_2x_1 = x_3,\ x_1x_4x_1 = x_5,\ x_1x_7x_1 = x_6, \\ & x_6x_3x_6 = x_2,\ x_6x_5x_6 = x_8, \\ & x_9x_1x_9 = x_{11}, x_9x_6x_9 = x_{12} \\ & x_{10}x_{11}x_{10} = x_7, x_{10}x_{12}x_{10} = x_6 \\ & x_{13}x_2x_{13} = x_{14}. \end{aligned}$$

We collect the subscript of x_i's in the conjugacy relations and define

$$C = \{(1,2,3),(1,4,5),(1,7,6),(6,3,2),(6,5,8), \tag{171}$$
$$(9,1,11),(9,6,12),(10,11,7),(10,12,6),(13,2,14)\} \tag{172}$$

such that

$$(i,j,k) \in C \iff x_i x_j x_i = x_k.$$

We also define

$$\Gamma = \langle \{x_i\}_{i=1}^{14} : x_i^2 = e, x_2 x_4 x_8 = e,\ x_{14} x_2 = J \rangle. \tag{173}$$

In the next part, we are going to convert the conjugacy relations to linear relations and make Γ the solution group we want. So we embed $\hat{K}$ in $\overline{K}$ where

$$\begin{aligned} \overline{K} = \langle \Gamma, \{w_i, y_i, j_i\}_{i=1}^{14}, f \text{ :all elements are order 2,} & \\ x_i = y_i z_i = f w_i,\ f y_i f = z_i \quad \text{where } 1 \le i \le 14 & \\ y_j z_k = z_k y_j,\ w_i y_j w_i = z_k \quad \text{for all } (i,j,k) \in C \rangle. & \end{aligned}$$

We convert new relations in $\overline{K}$ to linear relation or conjugacy relation by introduce element g_{jk} for all $(i,j,k) \in C$, such that $g_{jk}^2 = e$ and $g_{jk} = y_j z_k$. The new form of $\overline{K}$ is

$$\begin{aligned} \overline{K} = \langle \Gamma, \{w_i, y_i, j_i\}_{i=1}^{14}, f, \{g_{jk}\}_{(i,j,k)\in C} \text{ :all elements are order 2,} & \\ x_i y_i z_i = e,\ x_i f w_i = e \text{ for all } 1 \le i \le 14 & \\ g_{jk} y_j z_k = e \text{ for all } (i,j,k) \in C & \\ f y_i f = z_i \text{ for all } 1 \le i \le 14 & \\ w_i y_j w_i = z_k \text{ for all } (i,j,k) \in C \rangle & \end{aligned} \tag{174}$$

The last step is to convert the conjugacy relations in $\overline{K}$to linear relations.

We let Γ absorb the new elements and new linear relations by first relabelling

$$\begin{aligned} & x_{i+14} = w_i, \quad x_{i+28} = y_i, \quad x_{i+42} = z_i \quad 1 \le i \le 14 \\ & x_{57} = f, \\ & x_{58} = g_{23}, \quad x_{59} = g_{45}, \quad x_{60} = g_{76}, \quad x_{61} = g_{32}, \quad x_{62} = g_{58}, \\ & x_{63} = g_{1,11}, \quad x_{64} = g_{8,12}, \quad x_{65} = g_{11,7}, \quad x_{66} = g_{12,6}, \quad x_{67} = g_{2,13}. \end{aligned}$$

then

$$\begin{aligned} \Gamma = \langle \{x_i\}_1^{67} :& x_i^2 = e, x_2 x_4 x_8 = e,\ x_{14} x_2 = J, \\ & x_i x_{i+28} x_{i+42} = e;\ x_i x_{57} x_{i+14} = e \ \text{ for } 1 \le i \le 14, \\ & x_{58} x_{30} x_{45} = e, x_{59} x_{32} x_{47} = e, x_{60} x_{35} x_{48} = e, \\ & x_{61} x_{31} x_{44} = e, x_{62} x_{33} x_{50} = e, x_{63} x_{29} x_{53} = e, \\ & x_{64} x_{34} x_{54} = e, x_{65} x_{39} x_{49} = e, x_{66} x_{40} x_{48} = e, x_{67} x_{30} x_{56} = e \rangle \end{aligned}$$

We also change C to cover new conjugacy relations

$$\begin{aligned} C = & \{(57, i+28, i+42)\}_{i=1}^{14} \cup \\ & \{(15,30,45), (15,32,47), (15,35,48), (20,31,44), (20,33,50), (23,29,53), \\ & (23,34,54), (24,39,49), (24,40,48), (27,30,56)\}. \end{aligned}$$

For each $I = (i,j,k) \in C$, we introduce seven new variables $\{y_{Ii}\}_{i=1}^{7}$ such that

$$x_i y_{I1} y_{I2} = x_i y_{I5} y_{I6} = x_j y_{I2} y_{I3} = x_k y_{I6} y_{I7} = y_{I3} y_{I4} y_{I5} = y_{I1} y_{I4} y_{I7} = e.$$

We add such relations to Γ and get the final form of Γ which is

$$\begin{aligned}\Gamma = \langle \{x_i\}_{i=1}^{67} \cup \{\{y_{Ii}\}_{i=1}^{7}\}_{I\in C} :\{ & x_i^2 = y_{Ij}^2 = e, x_2x_4x_8 = e,\ x_{14}x_2 = J, \\ & x_ix_{i+26}x_{i+39} = e;\ x_ix_{53}x_{i+13} = e \ \text{ for } 1 \le i \le 14, \\ & x_{58}x_{30}x_{45} = e, x_{59}x_{32}x_{47} = e, x_{60}x_{35}x_{48} = e, \\ & x_{61}x_{31}x_{44} = e, x_{62}x_{33}x_{50} = e, x_{63}x_{29}x_{53} = e, \\ & x_{64}x_{34}x_{54} = e, x_{65}x_{39}x_{49} = e, x_{66}x_{40}x_{48} = e, x_{67}x_{30}x_{56} = e\} \\ & \cup \{x_iy_{I1}y_{I2} = x_iy_{I5}y_{I6} = x_jy_{I2}y_{I3} = x_ky_{I6}y_{I7} = e\}_{I=(i,j,k)\in C} \\ & \cup \{y_{I3}y_{I4}y_{I5} = y_{I1}y_{I4}y_{I7} = e\}_{I\in C}\rangle.\end{aligned} \tag{175}$$

The solution group Γ has 235 variables and 184 equations, which match what is given in [**?**].

B.1 The single value test

By the following test, we want to make sure that the operator X has eigenvalue 1. In this test $\mathcal{X} = \{\Delta\}$, $\mathcal{Y} = \{0, 1, \Delta\}$, $\mathcal{A} = \{\diamond, \perp\}$ and $\mathcal{B} = \{0, 1, \diamond, \perp\}$. As before, we first give intuitions about how Alice and Bob should behave to achieve the optimal correlation, then we give the optimal correlation and the corresponding optimal strategy.

- **Case 1**: when $x = y = \Delta$, Alice and Bob should answer with $a, b \in \{\diamond, \perp\}$ and their answers should agree;
- **Case 2**: when $x = \Delta$ and $y = 0$, if Alice answers with $\diamond$, Bob should answer 0, and if Alice answers $\perp$, Bob can answer with any $b \in \mathcal{B}$;
- **Case 3**: when $x = \Delta$ and $y = 1$, if Alice answers with $\diamond$, Bob should answer 0, and if Alice answers with $\perp$, Bob can answer with any $b \in \mathcal{B}$.

The optimal correlation and strategy. As in the previous test, Alice and Bob should share the entangled state $|\psi\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|u_i\rangle|u_i\rangle \sim |EPR(d)\rangle$. Define the subspace $V = \text{span}\{|u_0\rangle\}$. The ideal strategy has projective measurement

$$\begin{aligned} A_\Delta^\diamond &= B_\Delta^\diamond = |u_0\rangle\langle u_0|, & A_\Delta^\perp &= B_\Delta^\perp = \mathbb{1} - |u_0\rangle\langle u_0|, \\ B_0^0|_V &= B_1^0|_V = |u_0\rangle\langle u_0|, & B_0^1|_V &= B_1^1|_V = 0. \end{aligned}$$

Note that it is possible to construct observables B_0 and B_1 from LS that can be used in both the extended weighted CHSH test and the single value test. Then we can conclude that B_0B_1 has eigenvalue 1.

It is easy to calculate the ideal correlation, which is

		$y = \Delta$		$y = 0$		$y = 1$	
		$b = \diamond$	$b = \perp$	$b = 0$	$b = 1$	$b = 0$	$b = 1$
$x = \Delta$	$a = \diamond$	$1/d$	0	$1/d$	0	$1/d$	0
	$a = \perp$	0	$(d-1)/d$	$P(\perp 0\|\Delta 0)$	$\frac{d-1}{d} - P(\perp 0\|\Delta 0)$	$P(\perp 0\|\Delta 1)$	$\frac{d-1}{d} - P(\perp 0\|\Delta 0)$

Table 1: Ideal correlation of the single value test.

The enforcement imposed by the single value test is summarized in the following lemma.

Lemma 1. *If a quantum strategy* $(\{\tilde{A}_\Delta^a\}_a, \{\{\tilde{B}_y^b\}_b\}_y, |\tilde{\psi}\rangle$ *has the same behaviour as the optimal one, then* $(\tilde{B}_0^0 - \tilde{B}_0^1)(\tilde{B}_1^0 - \tilde{B}_1^1)$ *has eigenvalue* 1.

Proof. By the marginal distribution we know that $\|A_\Delta^\diamond|\psi\rangle\| = \frac{1}{\sqrt{d}}$. From the condition that $P(\diamond 0|\Delta 0) = 1/d$, we know

$$\frac{\langle\psi|\tilde{A}_\Delta^\diamond \tilde{B}_0^0|\psi\rangle}{\|\tilde{A}_\Delta^\diamond|\psi\rangle\|^2} = 1, \tag{176}$$

which implies that

$$\tilde{B}_0^0 \frac{\tilde{A}_\Delta^\diamond|\psi\rangle}{\|\tilde{A}_\Delta^\diamond|\psi\rangle\|} = \frac{\tilde{A}_\Delta^\diamond|\psi\rangle}{\|\tilde{A}_\Delta^\diamond|\psi\rangle\|}. \tag{177}$$

With similar reasoning, we get

$$\tilde{B}_1^0 \frac{\tilde{A}_\Delta^\diamond|\psi\rangle}{\|\tilde{A}_\Delta^\diamond|\psi\rangle\|} = \frac{\tilde{A}_\Delta^\diamond|\psi\rangle}{\|\tilde{A}_\Delta^\diamond|\psi\rangle\|}. \tag{178}$$

Hence $\frac{\tilde{A}_\Delta^\diamond|\psi\rangle}{\|\tilde{A}_\Delta^\diamond|\psi\rangle\|}$ is in the intersection of $\mathrm{supp}(\tilde{B}_0^0)$ and $\mathrm{supp}(\tilde{B}_1^0)$. Since for any $y \in [2]$, $\mathrm{supp}(\tilde{B}_y^0)$ and $\mathrm{supp}(\tilde{B}_y^1)$ are disjoint, we know

$$(\tilde{B}_y^0 - \tilde{B}_y^1)\frac{\tilde{A}_\Delta^\diamond|\psi\rangle}{\|\tilde{A}_\Delta^\diamond|\psi\rangle\|} = \frac{\tilde{A}_\Delta^\diamond|\psi\rangle}{\|\tilde{A}_\Delta^\diamond|\psi\rangle\|} - 0 = \frac{\tilde{A}_\Delta^\diamond|\psi\rangle}{\|\tilde{A}_\Delta^\diamond|\psi\rangle\|}.$$

Therefor, we know $\frac{\tilde{A}_\Delta^\diamond|\psi\rangle}{\|\tilde{A}_\Delta^\diamond|\psi\rangle\|}$ is an eigenvector of $\tilde{B}_0\tilde{B}_1$ with eigenvalue 1 because

$$\tilde{B}_0\tilde{B}_1\frac{\tilde{A}_\Delta^\diamond|\psi\rangle}{\|\tilde{A}_\Delta^\diamond|\psi\rangle\|} = \tilde{B}_0\frac{\tilde{A}_\Delta^\diamond|\psi\rangle}{\|\tilde{A}_\Delta^\diamond|\psi\rangle\|} = \frac{\tilde{A}_\Delta^\diamond|\psi\rangle}{\|\tilde{A}_\Delta^\diamond|\psi\rangle\|}. \tag{179}$$

□

In the next section, when we apply this test to make sure the operator X has eigenvalue 1, and we denote the test by SVT_X.