STATS C161/C261: Homework 4
Due May 30 at 4pm (brought to class or submitted electronically on CCLE)

Send questions regarding problem submission to Rosie Jia (ruoxuan@ucla.edu).
In the problems below, we will assume a neural network of the form,

$$\text{Hidden layer:} \quad z_j^{\mathrm{H}} = \sum_{k=1}^{N_i} W_{kj}^{\mathrm{H}} x_k + b_j^{\mathrm{H}}, \quad u_j^{\mathrm{H}} = g_{\mathrm{act}}(z_j^{\mathrm{H}}), \quad j = 1, \ldots, N_h \tag{1a}$$

$$\text{Output layer:} \quad z_j^{\mathrm{O}} = \sum_{k=1}^{N_h} W_{kj}^{\mathrm{O}} u_k^{\mathrm{H}} + b_j^{\mathrm{O}}, \quad u^{\mathrm{O}} = g_{\mathrm{out}}(\mathbf{z}^{\mathrm{O}}). \quad j = 1, \ldots, N_o. \tag{1b}$$

1. Consider the neural network for binary classification with a scalar input $x$ and parameters

$$W^{\mathrm{H}} = [1, 1], \quad b^{\mathrm{H}} = [-1, -3] \quad W^{\mathrm{O}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad b^{\mathrm{O}} = 0.5,$$

using a hard thresholds for the activation and output functions,

$$g_{\mathrm{act}}(z) = \begin{cases} 1 & \text{if } z \geq 0, \\ 0 & \text{if } z < 0, \end{cases} \quad g_{\mathrm{out}}(z) = \begin{cases} 1 & \text{if } z \geq 0, \\ 0 & \text{if } z < 0. \end{cases} \tag{2}$$

   (a) What is $N_h$, the number of hidden units? What is $N_o$, the number of output units?
   (b) Write $\mathbf{z}^{\mathrm{H}}$ in terms of $x$. Draw the functions for each component $z_j^{\mathrm{H}}$.
   (c) Write $\mathbf{u}^{\mathrm{H}}$ in terms of $x$. Draw the functions for each component $u_j^{\mathrm{H}}$.
   (d) Draw $z^{\mathrm{O}}$ in terms of $x$.
   (e) Draw $\hat{y} = u^{\mathrm{O}}$ in terms of $x$. Note that since we are using a hard threshold output $g_{\mathrm{out}}(z)$, $u^{\mathrm{O}}$ can be interpreted as a hard decision estimate on the class (not a soft probability).

2. Re-do problem 1 for the continuous-valued prediction case where we use a identity output map. Use a ReLU activation,

$$g_{\mathrm{act}}(z) = \max\{0, z\}, \quad g_{\mathrm{out}}(z) = z.$$

3. *Two-dimensional example:* Consider a neural network on a 2-dimensional input $\mathbf{x} = (x_1, x_2)$ with weights and biases:

$$W^{\mathrm{H}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad b^{\mathrm{H}} = [0, 0, -1], \quad W^{\mathrm{O}} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad b^{\mathrm{O}} = -1.5.$$

Assume the network uses hard threshold activation function and sigmoid output function,

$$g_{\mathrm{act}}(z) = \begin{cases} 1 & \text{if } z \geq 0, \\ 0 & \text{if } z < 0, \end{cases} \quad g_{\mathrm{out}}(z) = \frac{1}{1 + e^{-z}}. \tag{3}$$

   (a) Write the components of $\mathbf{z}^{\mathrm{H}}$ and $\mathbf{u}^{\mathrm{H}}$ as a function of $(x_1, x_2)$. For each component $j$, indicate where in the $(x_1, x_2)$ plane $u_j^{\mathrm{H}} = 1$.
   (b) Write $z^{\mathrm{O}}$ as a function of $(x_1, x_2)$.
   (c) Assuming $P(y = 1|x) = u^{\mathrm{O}}$, what is the maximum of $P(y = 1|x)$ and for what values $x$ does the highest probability occur.