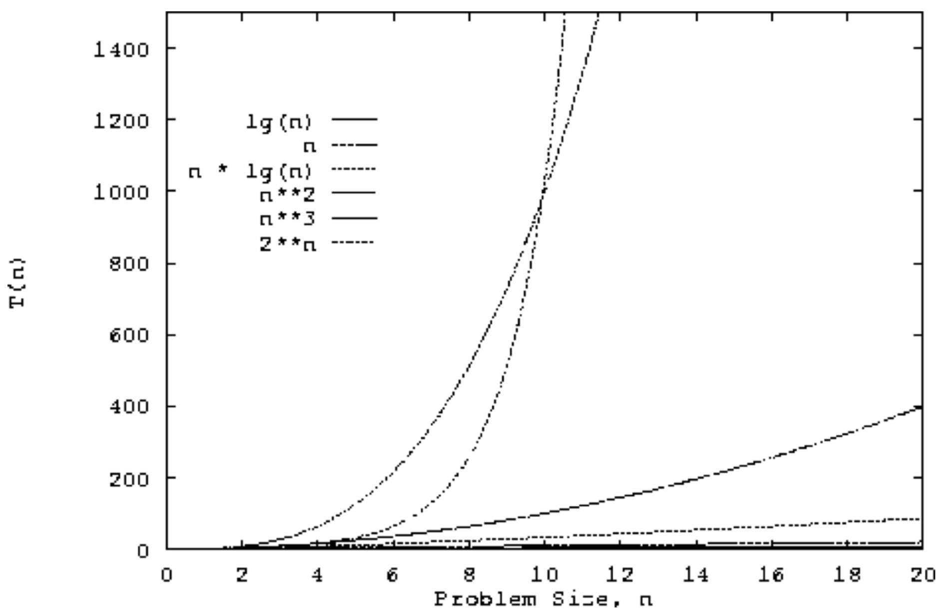
Lecture 6: Big O and Asymptotic Analysis

CS 106B: Programming Abstractions

Autumn 2020, Stanford University Computer Science Department

Lecturers: Chris Gregg and Julie Zelenski



Slide 2

Announcements

- Assignment 2 was released on Saturday.
- Now that you have submitted Assignment 1, your section leaders will be grading them throughout the week and will schedule an Interactive Grading session with you to go over your grade. IG attendance is part of your section participation grade, so please make sure to attend!
- The add/drop deadline is coming up on Friday at 5pm PDT - if you want to chat about what the class is going to look like going forward, feel free to reach out to the course staff and we'd be happy to chat.

Slide 3

Computational Complexity

- How does one go about analyzing programs to compare how the program behaves as it scales? E.g., let's look at a **`vectorMax()`** function:

```
int vectorMax(Vector<int> &v){
  int currentMax = v[0];
  int n = v.size();
  for (int i = 1; i < n; i++){
      if (currentMax < v[i]) {
          currentMax = v[i];
      }
  }
  return currentMax;
}
```

- What is **n**? Why is it important to this function?
- If we want to see how this algorithm behaves as **n** changes, we could do the following:
    1. Code the algorithm in C++
    2. Determine, for each instruction of the compiled program the time needed to execute that instruction (need assembly language)
    3. Determine the number of times each instruction is executed when the program is run.
    4. Sum up all the times we calculated to get a running time.
- Steps 1-4 might work, but it is complicated, especially for today's machines that optimize everything "under the hood." (and reading assembly code takes a certain patience).

Slide 4

Assembly Code for the **`vectorMax`** function

```
   0x000000010014adf0 <+0>:     push   %rbp
   0x000000010014adf1 <+1>:     mov    %rsp,%rbp
   0x000000010014adf4 <+4>:     sub    $0x20,%rsp
   0x000000010014adf8 <+8>:     xor    %esi,%esi
   0x000000010014adfa <+10>:    mov    %rdi,-0x8(%rbp)
   0x000000010014adfe <+14>:    mov    -0x8(%rbp),%rdi
   0x000000010014ae02 <+18>:    callq  0x10014aea0 <std::__1::basic_ostream<char, 
std::__1::char_traits<char> >::operator<<(long)+32>
   0x000000010014ae07 <+23>:    mov    (%rax),%esi
   0x000000010014ae09 <+25>:    mov    %esi,-0xc(%rbp)
   0x000000010014ae0c <+28>:    mov    -0x8(%rbp),%rdi
   0x000000010014ae10 <+32>:    callq  0x10014afb0 <std::__1::basic_ostream<char, 
std::__1::char_traits<char> >::operator<<(long)+304>
   0x000000010014ae15 <+37>:    mov    %eax,-0x10(%rbp)
   0x000000010014ae18 <+40>:    movl   $0x1,-0x14(%rbp)
   0x000000010014ae1f <+47>:    mov    -0x14(%rbp),%eax
   0x000000010014ae22 <+50>:    cmp    -0x10(%rbp),%eax
   0x000000010014ae25 <+53>:    jge    0x10014ae6c <vectorMax(Vector<int>&)+124>
   0x000000010014ae2b <+59>:    mov    -0xc(%rbp),%eax
   0x000000010014ae2e <+62>:    mov    -0x8(%rbp),%rdi
   0x000000010014ae32 <+66>:    mov    -0x14(%rbp),%esi
   0x000000010014ae35 <+69>:    mov    %eax,-0x18(%rbp)
   0x000000010014ae38 <+72>:    callq  0x10014aea0 <std::__1::basic_ostream<char, 
std::__1::char_traits<char> >::operator<<(long)+32>
   0x000000010014ae3d <+77>:    mov    -0x18(%rbp),%esi
   0x000000010014ae40 <+80>:    cmp    (%rax),%esi
   0x000000010014ae42 <+82>:    jge    0x10014ae59 <vectorMax(Vector<int>&)+105>
   0x000000010014ae48 <+88>:    mov    -0x8(%rbp),%rdi
   0x000000010014ae4c <+92>:    mov    -0x14(%rbp),%esi
   0x000000010014ae4f <+95>:    callq  0x10014aea0 <std::__1::basic_ostream<char, 
std::__1::char_traits<char> >::operator<<(long)+32>
   0x000000010014ae54 <+100>:   mov    (%rax),%esi
   0x000000010014ae56 <+102>:   mov    %esi,-0xc(%rbp)
   0x000000010014ae59 <+105>:   jmpq   0x10014ae5e <vectorMax(Vector<int>&)+110>
   0x000000010014ae5e <+110>:   mov    -0x14(%rbp),%eax
   0x000000010014ae61 <+113>:   add    $0x1,%eax
   0x000000010014ae64 <+116>:   mov    %eax,-0x14(%rbp)
   0x000000010014ae67 <+119>:   jmpq   0x10014ae1f <vectorMax(Vector<int>&)+47>
   0x000000010014ae6c <+124>:   mov    -0xc(%rbp),%eax
   0x000000010014ae6f <+127>:   add    $0x20,%rsp
   0x000000010014ae73 <+131>:   pop    %rbp
   0x000000010014ae74 <+132>:   retq
```

Slide 5

Algorithm Analysis: Primitive Operations

- Instead of those complex steps, we can define primitive operations for our C++ code.
    - Assigning a value to a variable
    - Calling a function
    - Arithmetic (e.g., adding two numbers)
    - Comparing two numbers
    - Indexing into a Vector
    - Returning from a function
- We assign "1 operation" to each step. We are trying to gather data so we can compare this to other algorithms.

Slide 6

Algorithm Analysis: Primitive Operations

```
int vectorMax(Vector<int> &v){
    int currentMax = v[0];
    int n = v.size();
    for (int i=1; i < n; i++){
        if (currentMax < v[i]) {
            currentMax = v[i];
        }
    }
    return currentMax;
}
```

executed once (2 ops)
executed once (2 ops)
executed once (1 op)
ex. n times (*n* ops)
executed *n-1* times (2*(*n*-1) ops))
ex. n-1 times (2*(n-1) ops)
ex. at most n-1 times (2*(n-1) ops), but as few as zero times
ex. once (1 op)

- Specifically, we can count up the primitive operations:
  - **`currentMax = v[0]`** has two operations (variable assignment and indexing into a Vector) and is executed once (total operations so far = 2)
  - **`n = v.size()`** has two operations (variable assignment and calling a function) and is executed once (total operations so far: 4)
  - The **`int i = 1`** in the **`for`** loop declaration is executed once, with 1 operation (total = 5)
  - The **`i < n`** in the **`for`** declaration is executed **`n`** times (total: n + 5)
  - The **`i++`** in the **`for`** declaration is executed **`n - 1`** times, and is two operations (arithmetic and assignment) (total: 3n + 3 operations so far)
  - **`if (currentMax < v[i])`** is executed **`n - 1`** times and is two operations (comparison and vector indexing) (total: 5n + 1)
  - **`currentMax = v[i]`** is executed at most **`n - 1`** times, and is two operations (assignment and vector indexing) *but* as few as zero times (if we never have to update the max!) (total: either 7n - 1 operations or 5n + 1 operations)
  - **`return currentMax`** is executed once, and is one operation (returning from a function) (total operations: either 7n or 5n + 2)

Slide 7

Algorithm Analysis: Primitive Operations

- Summary:
  - Primitive operations for vectorMax():
    - at least: **`2 + 2 + 1 + n + 2 * (n - 1) + 2 * (n - 1) + 1 = 5n + 2`**
    - at most: **`2 + 2 + 1 + n + 2 * (n - 1) + 2 * (n - 1) + 2 * (n - 1) + 1 = 7n`**
  - i.e., if there are **n** items in the Vector, there are between **`5n + 2`** operations and **`7n`** operations completed in the function.
- In other words: the *best case* number of operations is **`5n + 2`** and the *worst case* number of operations is *7n*.

Slide 8

Algorithm Analysis: Simplify!

- Do we *really* need this much detail? Nope!
- Let's simplify: we want a *big picture* approach.
- It is enough to know that **`vectorMax()`** grows

  linearly proportional to n
- In other words, as the number of elements increases, the algorithm has to do proportionally more work, and that relationship is linear. 8x more elements? 8x more work.
- The reason we can make this generalization is that we want to discuss the behavior for increasingly large values of **n**. The details of the **`+ 2`** and even the **5** or **7** becomes irrelevant as the numbers get increasingly large.

Slide 9

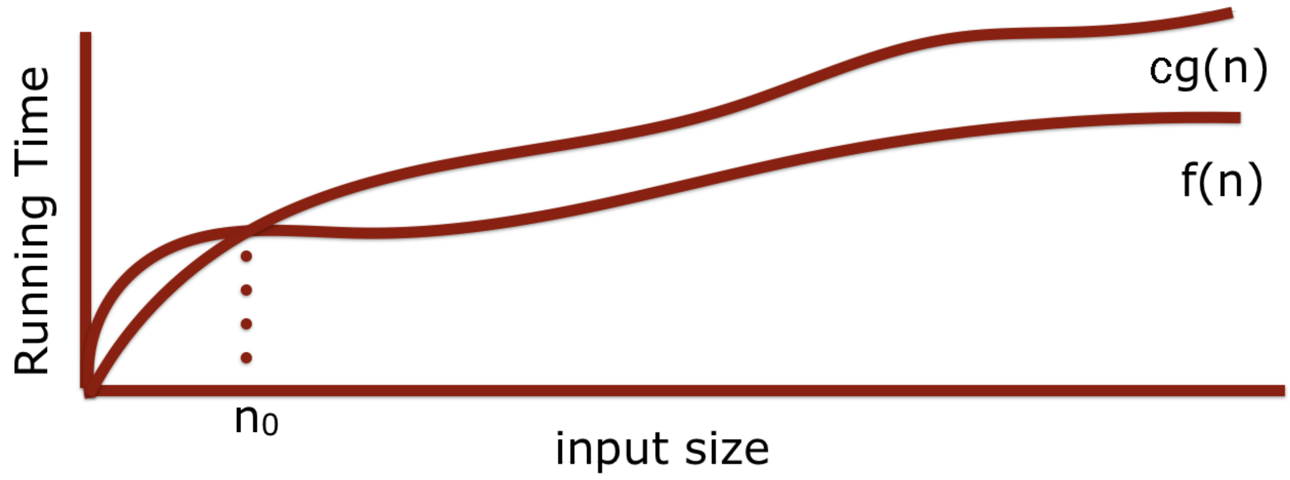
Algorithm Analysis: Big O

- Our simplification uses a mathematical construct known as *Big-O* notation — think "O" as in "on the Order of."
- Wikipedia:

Big-O notation describes the limiting behavior of a function when the argument tends towards a particular value or infinity, usually in terms of simpler functions.

- In mathematical terms:

  Let $f(n)$ and $g(n)$ be functions mapping nonnegative integers to real numbers. We say that $f(n)$ is $O(g(n))$ if there is a real constant $c > 0$ and an integer constant $n_0 \geq 1$, such that $f(n) \leq cg(n)$ for every integer $n \geq n_0$. This definition is often referred to as the "big-Oh" notation. We can also say, "$f(n)$ is *order* $g(n)$."



Slide 10

Algorithm Analysis: Removing constants and less significant factors

- All of the analysis on the previous few slides was actually even more than you need to worry about.
- The dirty little trick for figuring out Big-O: look at the number of steps you calculated, throw out all the constants, find the *biggest factor* and that's your answer:

`5n + 2` is `O(n)`

- Why? Because constants are not important when compared against other functions that we will cover shortly.
- In other words: for very large values of `n`, there is an insignificant difference between `5n + 2` and, simply, `n`. If `n` is 1 billion, the difference between 1 billion and 5 billion and 2 is irrelevant when compared to other functions that we will see soon.

Slide 11

Big O: important functions

- We will care about teh following functions that appear often in data structures and algorithms:

constant	logarithmic	linear	n log n	quadratic	polynomial (other than n^2, (n^2))	exponential
O(1)	O(log n)	O(n)	O(n log n)	O(n^2), O(n^2)	O(n^k), O(n^k) (where k >= 1)	O(a^n), O(a^n)

- When you are deciding what Big-O is for an algorithm or function, simplify until you reach one of these functions, and you will have your answer.
- Factors farther to the right are more important. If two or more factors from the table are in your calculation, leave only the one farthest to the right.
- Practice: what is Big-O for this function?

  $20n^3 + 10n \log n + 5$
  (alternate: `20n^3 + 10n log n + 5`)

- Answer:

  $O(n^3)$
  (alternate: `O(n^3)`)

- First, strip the constants: $n^3 + n \log n$ (n^3 + n log n)

- Then, find the biggest factor: n^3 (n^3)

- Practice: what is Big-O for this function?

```
2000 log n + 7n log n + 5
```

- Answer:

```
O(n log n)
```

- First, strip the constants: log n + n log n
- Then, find the biggest factor: n log n

Slide 12

Algorithm Analysis: Back to **`vectorMax()`**

```
int vectorMax(Vector<int> &v){
    int currentMax = v[0];
    int n = v.size();
    for (int i = 1; i < n; i++){
        if (currentMax < v[i]) {
            currentMax = v[i];
        }
    }
    return currentMax;
}
```

- When you are analyzing an algorithm or code for its *computational complexity* using Big-O notation, you can ignore the primitive operations that would contribute less-important factors to the run-time. Also, you always take the *worst case* behavior for Big-O.
- So, for **`vectorMax()`**: ignore the original two variable initializations, the return statement, the comparison, and the setting of **`currentMax`** in the loop.
- Notice that the important part of the function is the fact that the loop conditions will change with the size of the array: for each extra element, there will be one more iteration. This is a linear relationship, and therefore **`O(n)`**.

Slide 13

Algorithm Analysis: Back to **`vectorMax()`**

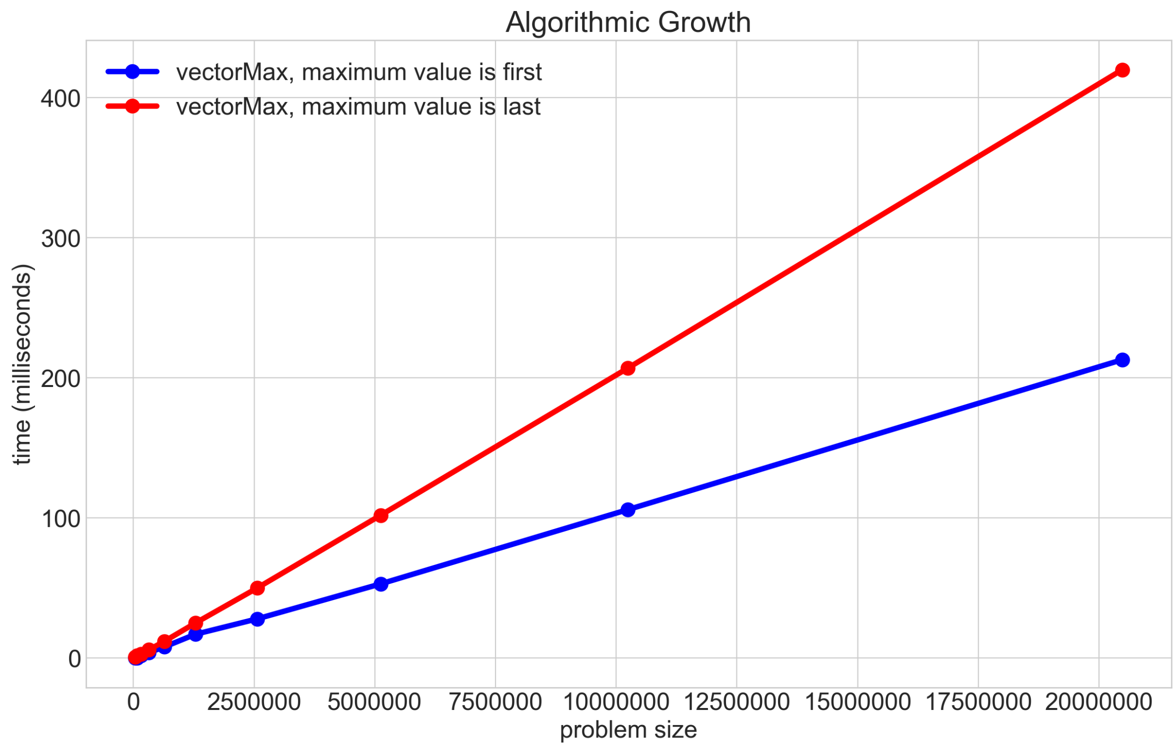
```
int vectorMax(Vector<int> &v){
    int currentMax = v[0];
    int n = v.size();
    for (int i = 1; i < n; i++){
        if (currentMax < v[i]) {
            currentMax = v[i];
        }
    }
    return currentMax;
}
```

- For vectorMax(): ignore the original two variable initializations, the return statement, the comparison, and the setting of currentMax in the loop.
- Notice that the important part of the function is the fact that the loop conditions will change with the size of the array: for each extra element, there will be one more iteration. This is a linear relationship, and therefore **`O(n)`**.

Slide 14

Graphing the Data for **`vectorMax`**

- In the code for today's lecture, you can find a program that times **`vectorMax`** for vector sizes between 40000 and 20,480,000 elements. Additionally, it times for the case where the first element is the maximum, and also when the last element is the maximum (and all prior elements are maximum at each iteration). The graph for those results is shown below:



- Notice that both plots show a linear relationship!

Slide 15

Nested Loops

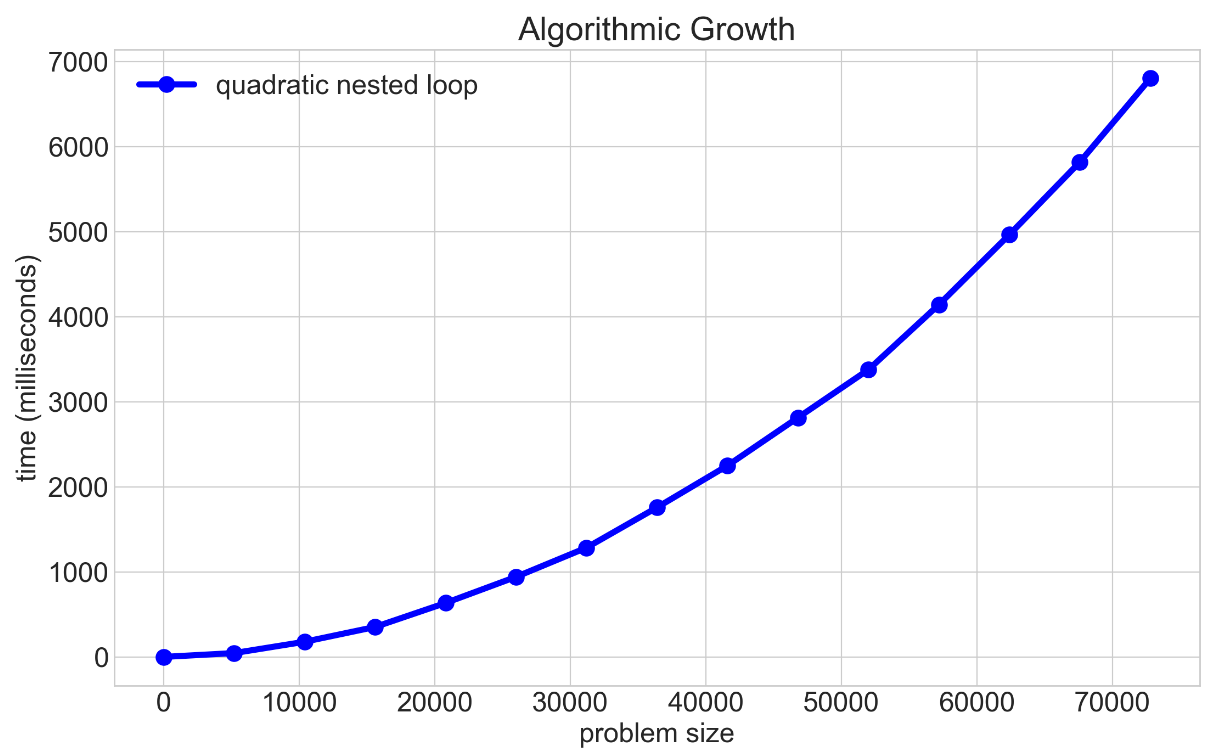
- Take a look at the following function, which has a nested loop:

```
int nestedLoop1(int n){
      int result = 0;
      for (int i = 0; i < n; i++){
              for (int j = 0; j < n; j++){
                      result++;
              }
      }
      return result;
}
```

- The inner loop (variable **`j`**) has a complexity of **`O(n)`**, as our analysis from the previous slides would show. *However*, the entire inner loop happens **`n`** times, as well! This *squares* the number of times **`result`** is incremented.
- Now we have a *quadratic* relationship between **`n`** and the time to complete the function, and this is O(n^2) (alt: O(n^2)) behavior!
- With O(n^2) behavior, if the size of the problem is 10 times bigger, the running time will be 100 times longer. We don't like O(n^2) behavior!
- As an example: let's say an O(n^2) function takes 5 seconds for a container with 100 elements.
  - How much time would it take if we had 1000 elements?
  - Answer: 500 seconds! This is because 10x more elements is (10^2)x more time!
  - If we had 10000 elements, that is 100x more elements, meaning it would take (100^2)x more time, or 10,000 times as long as the original (50,000 seconds, or almost 14 hours), for only 100 times more elements. Quadratic behavior does not *scale* well!

Slide 16

Plotting O(n^2) behavior



Slide 17

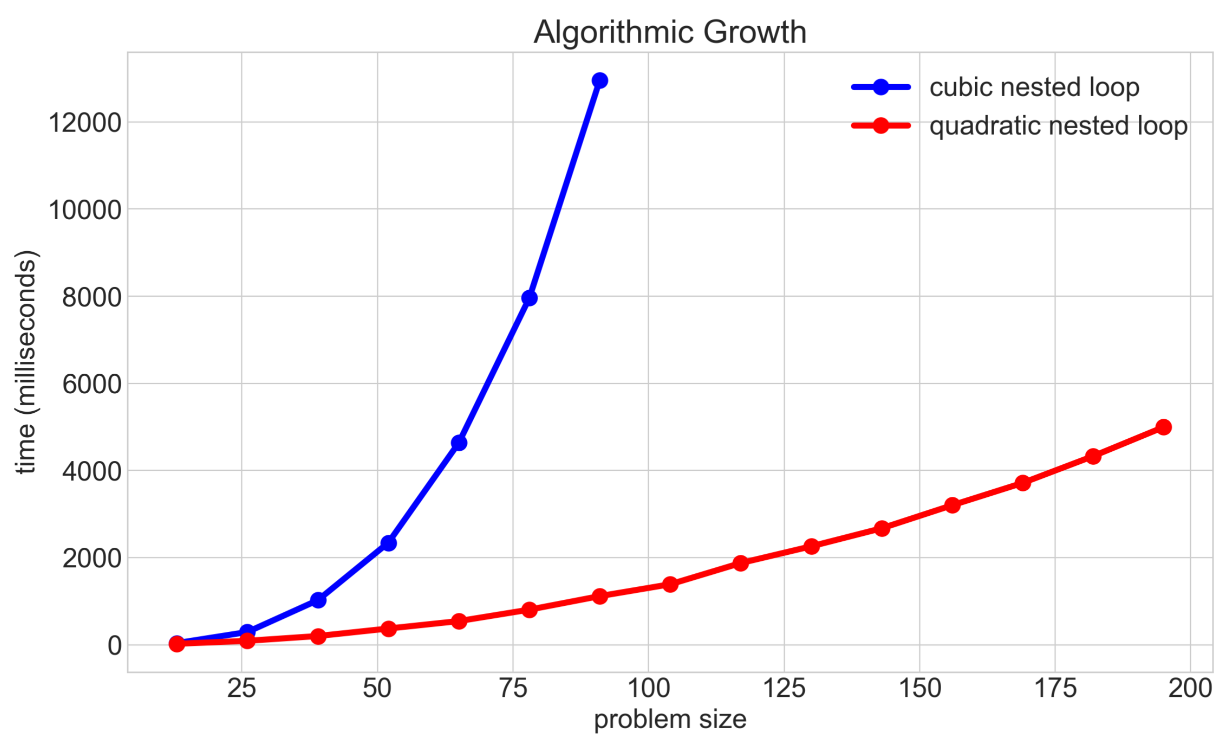
Cubic O(n^3) Behavior

```
int nestedLoop2(int n){
    int result = 0;
    for (int i = 0; i < n; i++){
        for (int j = 0; j < n; j++){
            for (int k = 0; k < n; k++) {
                result++;
            }
        }
    }
    return result;
}
```

- What would the complexity be of a 3-nested loop?
- Answer: O(n^3) (cubic – anything above quadratic is generally called polynomial)
- In real life, this comes up in 3D imaging, video, etc., and it is **slow**!
- Graphics cards are built with hundreds or thousands of processors to tackle this problem
- If it took 10 seconds for `n = 5`, switching to `n = 50` would incur a (10^3) time increase, or 10,000 seconds!

Slide 18

Plotting O(n^2) -vs- O(n^3) behavior



- This graph actually has an additional constant slow-down for the quadratic data, simply so it could be graphed on a single chart.

Slide 19

Hidden loops

- You have to be particularly cognizant of when a data structure might be performing a looping operation without you writing the loop.
- The **Vector** class is a good example: when you insert at the beginning of a vector, what happens?

vec before insert:

index:	0	1	2	3
value:	4	8	15	16

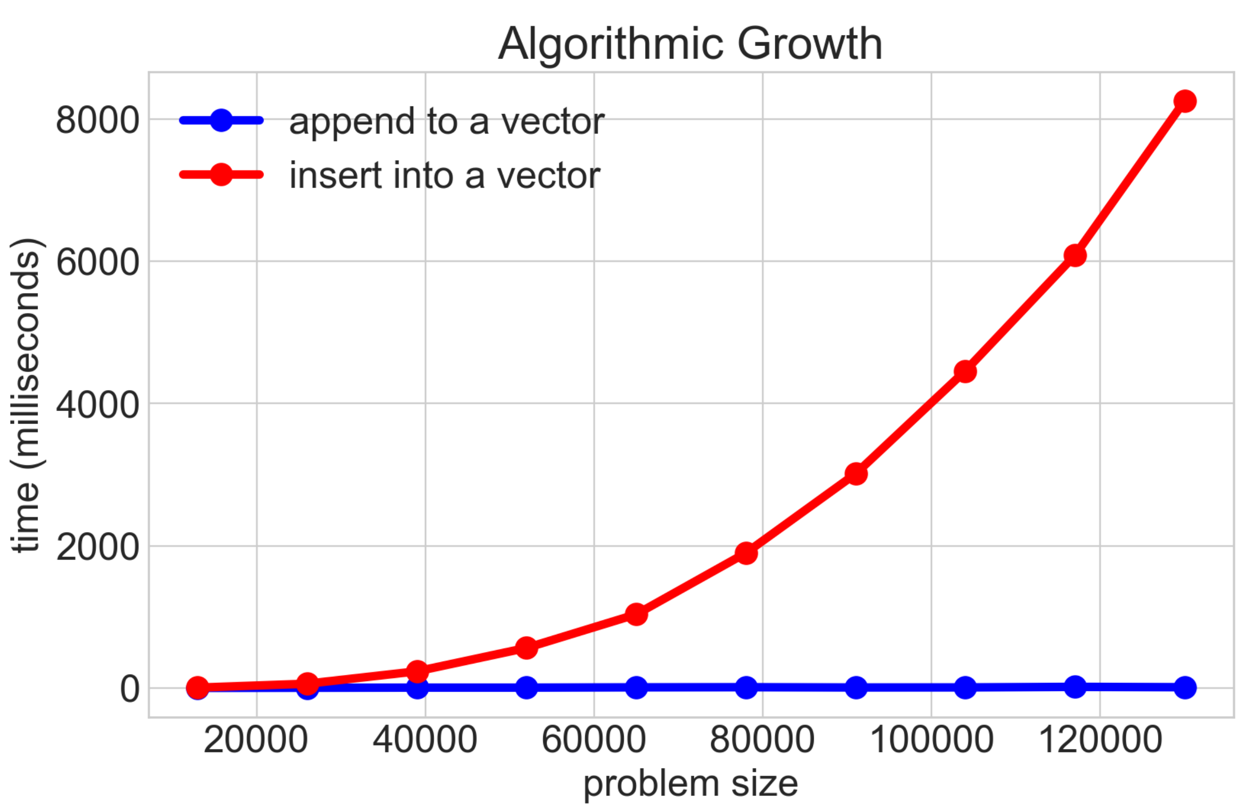
```
vec.insert(0, 100);
```

index:	0	1	2	3	4
value:	100	4	8	15	16

- All elements had to be moved down – how do you think this was done? With a loop inside the **Vector** class!
- What would the complexity (big O) be of the following function?

```
void populateVec(Vector<int>& vec, int n) {
  for (int i = 0; i < n; i++) {
      vec.insert(0, i);
  }
}
```

- Answer: O(n^2), because under the hood, for each insert, the vector has to move *all* of the elements.
- Here is a graph of **populateVec** as above, and also by replacing **vec.insert(i, 0)** with **vec.add(i)**:



Slide 20

Constant Time

- When an algorithm's time is independent of the number of elements in the container it holds, this is constant time complexity, or **O(1)**. We **love** O(1) algorithms! Examples include (for efficiently designed data structures):
  - Adding or removing from the end of a Vector.

- Later in the course:
- Pushing onto a stack or popping off a stack.
- Enqueuing or dequeuing from a queue.
- Inserting or searching for a value in a hash table

Slide 21

Linear Search

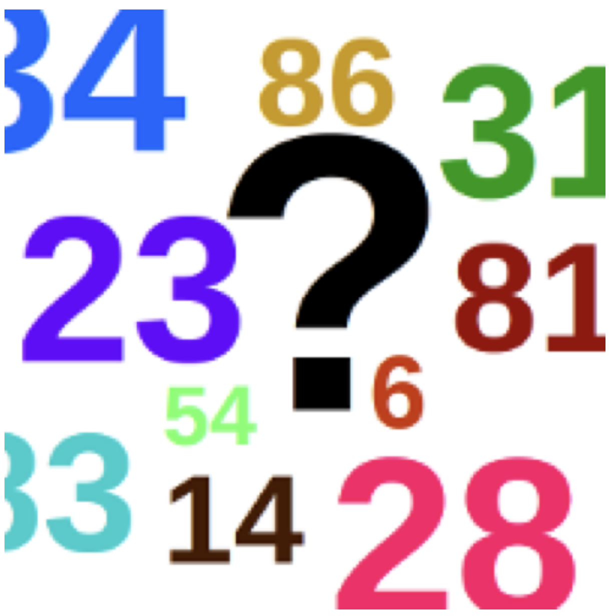
```
void linearSearchVector(Vector<int>& vec, int numToFind){
    int numCompares = 0;
    bool answer = false;
    int n = vec.size();

    for (int i = 0; i < n; i++) {
        numCompares++;
        if (vec[i]==numToFind) {
            answer = true;
            break;
        }
    }
    cout << "Found? " << (answer ? "True" : "False") << ", "
         << "Number of compares: " << numCompares << endl << endl;
}
```

- The code above performs a *linear search* on a vector, to search for a particular value. What is the complexity of the function?
- Well, it depends! Let's say the number we are looking for is the first element? Then, we could say that the complexity is O(1), because it only took a single pass through the loop to find the value. Or, what if the value is the last one in the vector? Then, it would be O(n), because we had to loop through the entire vector.
- Best case: O(1)
- Worst case: O(n)
- We often only report the worst case, because we don't know what the data is, and we want to be informed about what the worst case is.

Slide 22

Binary Search



- There is another type of search that we can perform on a list that is in order: binary search (as you might have seen in 106A)
- If you have ever played a *guess my number* game before, you will have implemented a binary search, if you played the game efficiently!
- The game is played as follows:
  - one player thinks of a number between 0 and 100 (or any other maximum).
  - the second player guesses a number between 1 and 100
  - the first player says "higher" or "lower," and the second player keeps guessing until they guess correctly.

- The most efficient guessing algorithm for the number guessing game is simply to choose a number that is between the high and low that you are currently bound to. Example:

```
bounds: 0, 100
guess: 50 (no, the answer is lower)
new bounds: 0, 49
guess: 25 (no, the answer is higher)
new bounds: 26, 49
guess: 38
etc.
```

- With each guess, the search space is **divided into two**.
- This means that we can *very quickly* converge on a solution. In fact, we can converge on a solution in *logarithmic* time, or O(log n).
- If we played the guessing game with numbers between 1 and 128, how many guesses, maximum, would we need?
    - First, we have 128 to choose from, then 64, then 32, then 16, then 8, then 4, then 2, and finally 1 – this is 8 guesses, or `log_2(128) + 1`. If we *doubled* the range to between 1 and 256, we would only have to make a single extra guess.

Slide 23

Binary Search

Here is some code that performs a binary search:

```
void binarySearchVector(Vector<int> &vec, int numToFind) {
    int low=0;
    int high=vec.size()-1;
    int mid;
    int numCompares = 0;
    bool found=false;
    while (low <= high) {
        numCompares++;
        mid = low + (high - low) / 2; // to avoid overflow
        if (vec[mid] > numToFind) {
            high = mid - 1;
        }
        else if (vec[mid] < numToFind) {
            low = mid + 1;
        }
        else {
            found = true;
            break;
        }
    }
    cout << "Found? " << (found ? "True" : "False") << ", " <<
    "Number of compares: " << numCompares << endl << endl;
}
```

- The worst-case complexity of this code is `O(log n)`. It is technically `O(log_2 n)`, but we don't have to worry about logarithmic bases when looking at complexity.
- The best case is O(1) (if we guess immediately)
- The general rule for determining if something is logarithmic: if the problem is one of "divide and conquer," it is logarithmic.
    - If, at each stage, the problem size is cut in half (or a third, etc.), it is logarithmic.
- We *love* logarithmic time, because we can solve large problems very fast. * Logarithmic time is not quite as fast as O(1) time, but it is almost always good enough, and still very fast.

Slide 24

Exponential Time

- If we love logarithmic and constant time algorithms, we hate *exponential time* algorithms, **`O(a^n)`**.
- We will see examples of exponential time algorithms soon (when we get to recursion, in particular), but one good example is finding every possible subset of a set.
    - For the set {1, 2, 3}, the possible combinations are {}, {1}, {2}, {3}, {1, 2}, {2, 3}, {1, 3}, {1, 2, 3}. For an **n** of 3, we ended up having to enumerate eight different subsets, meaning that it is certainly worse than linear. Indeed, exponential problems grow very, *very* fast, and are often not possible to solve completely, even for relatively small values for **n**.

Slide 25

Ramifications of Big O Differences

- Here are some numbers
- If we have an algorithm that has 1000 elements, and the O(log n) version runs in 10 nanoseconds…

constant	logarithmic	linear	n log n	quadratic	polynomial (other than n^2, (n^2))	exponential
`1 nanoseconds`	`10 nanoseconds`	`1 microsecond`	`10 microseconds`	`1 millisecond`	`1 second`	`10^292 years`

Slide 26

Ramifications of Big O Differences

- If we have an algorithm that has 1000 elements, and the O(log n) version runs in 10 milliseconds…

constant	logarithmic	linear	n log n	quadratic	polynomial (other than n^2, (n^2))	exponential
`1 milliseconds`	`10 milliseconds`	`1 second`	`10 seconds`	`17 minutes`	`277 hours`	`heat death of the universe`

Slide 27

Recap

- Asymptotic Analysis / Big-O / Computational Complexity
- We want a "big picture" assessment of our algorithms and functions
- We can ignore constants and factors that will contribute less to the result!
- We most often care about worst case behavior.
- We love O(1) and O(log n) behaviors!
- Big-O notation is useful for determining how a particular algorithm behaves, but be careful about making comparisons between algorithms – sometimes this is helpful, but it can be misleading.
- Algorithmic complexity can determine the difference between running your program over your lunch break, or waiting until the Sun becomes a Red Giant and swallows the Earth before your program finishes – that's how important it is!