Notes

[1]Dubois, M., Shi, C. Zhu, X., Wang, Y., Zhang, X. Observation of acoustic Dirac-like cone and double zero refractive index *Nature Communications.* **8**, 14871 (2017).

[2]Haberman, M, Guild, M. Acoustic Metamaterials. *Physics Today.* **69**, 42 (2016).

[3]"Positive phase speed" means that the intensity and the wave vector are parallel; "Negative phase speed" means that intensity and the wave vector are anti-parallel.

[4]Blackstock, D. Fundamentals of Physical Acoustics. *Wiley*, 67 (2000).

[5]Blackstock, D. Fundamentals of Physical Acoustics. *Wiley*, 50 (2000).

[6]Blackstock, D. Fundamentals of Physical Acoustics. *Wiley*, 110 (2000).

[7]DZ AMMs do not have this problem as long as both ρ and χ vanish at similar rates as functions of frequency.The normalized plots of $\rho(f)$ and $\chi(f)$ show that both vanish at very similar rates. See Dubois et al., Supplementary Figure 2.

[8]Photons are governed by Maxwell's equations, which, in the absence of charges and currents, read $\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0, \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$, and $\boldsymbol{\nabla} \times \boldsymbol{B} = \mu\epsilon\frac{\partial \boldsymbol{E}}{\partial t}$. These four first-order equations can be combined into two second-order coupled PDEs by first taking the curl of the curl equations:

$$\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{E} &= -\boldsymbol{\nabla} \times \frac{\partial \boldsymbol{B}}{\partial t} \\
\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{B} &= \mu\epsilon \left(\boldsymbol{\nabla} \times \frac{\partial \boldsymbol{E}}{\partial t}\right)
\end{aligned}$$

Applying the identity $\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{P} = \boldsymbol{\nabla}\left(\boldsymbol{\nabla} \cdot \boldsymbol{P}\right) - \nabla^2 \boldsymbol{P}$ the left-hand-side becomes

$$\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\boldsymbol{E}\right)-\nabla^2\boldsymbol{E}=-\boldsymbol{\nabla}\times\frac{\partial\boldsymbol{B}}{\partial t}$$
$$\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\boldsymbol{B}\right)-\nabla^2\boldsymbol{B}=\mu\epsilon\left(\boldsymbol{\nabla}\times\frac{\partial\boldsymbol{E}}{\partial t}\right)$$

But invoking the divergence equations gives

$$\nabla^2\boldsymbol{E}=\boldsymbol{\nabla}\times\frac{\partial\boldsymbol{B}}{\partial t}$$
$$-\nabla^2\boldsymbol{B}=\mu\epsilon\left(\boldsymbol{\nabla}\times\frac{\partial\boldsymbol{E}}{\partial t}\right)$$

We can simplify further, noting that

$$\begin{aligned}\boldsymbol{\nabla}\times\frac{\partial\boldsymbol{B}}{\partial t}&=\frac{\partial}{\partial t}\left(\boldsymbol{\nabla}\times B\right)\\&=\mu\epsilon\frac{\partial^2\boldsymbol{E}}{\partial t^2}\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{\nabla}\times\frac{\partial\boldsymbol{E}}{\partial t}&=\frac{\partial}{\partial t}\left(\boldsymbol{\nabla}\times E\right)\\&=-\frac{\partial^2\boldsymbol{B}}{\partial t^2}\end{aligned}$$

The vector Laplacian equations then become

$$\mu\epsilon\frac{\partial^2\boldsymbol{E}}{\partial t^2}-\nabla^2\boldsymbol{E}=0$$
$$\mu\epsilon\frac{\partial^2\boldsymbol{B}}{\partial t^2}-\nabla^2\boldsymbol{B}=0$$

[9]Huang, X., Lai, Y., Hang, Z. H., Zheng, H. Chan, C. T. Dirac cones induced by accidental degeneracy in photonic crystals and zero-refractive-index materials. *Nat. Mater.* **10**, 582–586 (2011).

[10]$\boldsymbol{q}$ is the so-called "crystal momentum."

[11]We have assumed that the Hamiltonian equals the total energy, i.e., that the fields are conservative. Dimensionally, our result makes sense: $\hbar$ is in Joule-seconds, the gradient with respect to the wave vector factors in meters, and energy is in Joules. The dimensions of $\frac{1}{\hbar}\boldsymbol{\nabla}_q E$ is therefore $[\mathrm{J}^{-1}\cdot \mathrm{s}^{-1}][m][J] = [m/s]$, which are the SI units of velocity.

[12]Wang, L., Wang, Z., Zhang, J. et al. Realization of Dirac point with double cones in optics *Optics Letters*, **34**, 2009.

[13]The massless Dirac equation can be formulated by first taking the Fourier transform of equation (7). Note that we are assuming polarity in the z-direction to make the analogy to the acoustical system in the Dubois et. al paper more direct.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) E_z(x, y, \omega) + k^2(\omega) E_z(x, y, \omega) = 0 \tag{20}$$

Recall that the x- and y-Pauli spin matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}$$

The Laplacian is factored in the style of P.A.M. Dirac. The Laplacian in the spin-$\frac{1}{2}$ basis equals Dirac's factored Laplacian:

$$\left(\tfrac{\partial^2}{\partial x^2}+\tfrac{\partial^2}{\partial y^2}\right)\mathbf{1} \stackrel{?}{=} \left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\tfrac{\partial}{\partial x}+\begin{pmatrix}0&-j\\j&0\end{pmatrix}\tfrac{\partial}{\partial y}\right)\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\tfrac{\partial}{\partial x}+\begin{pmatrix}0&-j\\j&0\end{pmatrix}\tfrac{\partial}{\partial y}\right)$$

$$= \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\frac{\partial^2}{\partial^2 x}+\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&-j\\j&0\end{pmatrix}\frac{\partial^2}{\partial x\partial y}+$$

$$+\begin{pmatrix}0&-j\\j&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\frac{\partial^2}{\partial x\partial y}+\begin{pmatrix}0&-j\\j&0\end{pmatrix}\begin{pmatrix}0&-j\\j&0\end{pmatrix}\frac{\partial^2}{\partial^2 y}$$

$$= \begin{pmatrix}1&0\\0&1\end{pmatrix}\frac{\partial^2}{\partial^2 x}+\begin{pmatrix}j&0\\0&-j\end{pmatrix}\frac{\partial^2}{\partial x\partial y}+\begin{pmatrix}-j&0\\0&j\end{pmatrix}\frac{\partial^2}{\partial x\partial y}+\begin{pmatrix}1&0\\0&1\end{pmatrix}\frac{\partial^2}{\partial^2 y}$$

$$= \begin{pmatrix}1&0\\0&1\end{pmatrix}\frac{\partial^2}{\partial^2 x}+\begin{pmatrix}1&0\\0&1\end{pmatrix}\frac{\partial^2}{\partial^2 y} \qquad (\checkmark)$$

Therefore

$$\left(\tfrac{\partial^2}{\partial x^2}+\tfrac{\partial^2}{\partial y^2}\right)\mathbf{1} = \left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\tfrac{\partial}{\partial x}+\begin{pmatrix}0&-j\\j&0\end{pmatrix}\tfrac{\partial}{\partial y}\right)\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\tfrac{\partial}{\partial x}+\begin{pmatrix}0&-j\\j&0\end{pmatrix}\tfrac{\partial}{\partial y}\right)$$

Equation (20) then becomes

$$\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\tfrac{\partial}{\partial x}+\begin{pmatrix}0&-j\\j&0\end{pmatrix}\tfrac{\partial}{\partial y}\right)\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\tfrac{\partial}{\partial x}+\begin{pmatrix}0&-j\\j&0\end{pmatrix}\tfrac{\partial}{\partial y}\right)\begin{pmatrix}E_{z,1}\\E_{z,2}\end{pmatrix} = -k^2(\omega)\begin{pmatrix}E_{z,1}\\E_{z,2}\end{pmatrix}$$

Since each of the matrix operators above (so-called “Dirac operators”) is a functional square root of the Laplacian, each one generates an eigenvalue of $\sqrt{-k^2(\omega)} = jk(\omega)$. Taking the functional square root of the above gives

$$\frac{1}{j}\begin{pmatrix} 0 & \frac{\partial}{\partial x} - j\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} + j\frac{\partial}{\partial y} & 0 \end{pmatrix}\begin{pmatrix} E_{z,1} \\ E_{z,2} \end{pmatrix} = k(\omega)\begin{pmatrix} E_{z,1} \\ E_{z,2} \end{pmatrix}$$

$$\begin{pmatrix} 0 & -j\left(\frac{\partial}{\partial x} - j\frac{\partial}{\partial y}\right) \\ -j\left(\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}\right) & 0 \end{pmatrix}\begin{pmatrix} E_{z,1} \\ E_{z,2} \end{pmatrix} = k(\omega)\begin{pmatrix} E_{z,1} \\ E_{z,2} \end{pmatrix} \qquad \text{(massless Dirac equation)}$$

[14]Adapted from the *Supplementary Information* of the Huang et. al paper.

[15]In this section, we will switch to the $e^{i(kx-\omega t)}$ convention to match the paper by Huang et. al. The final result is entirely real, so comparison to the results in the previous section should be straightforward.

[16]The supplementary information includes important derivations and details about the experiment.

[17]Solving $\frac{\omega}{c} = \sqrt{\frac{\omega^2}{c_0^2} - \frac{\pi^2}{h^2}}$ for c, the phase speed of the lowest-order waveguide mode,

$$c = \frac{\omega}{k_x} = \left(\sqrt{\frac{1}{c_0^2} - \frac{\pi^2}{\omega^2 h^2}}\right)^{-1} \qquad \text{(Dubois et. al equation 1)}$$

This note was included for the sake of reproducing the only equation that appears in the Dubois et. al paper.