Theoretical & finite-element analysis of ultrasonic vortex beam generation with single-element transducer & phase plate

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Geometry

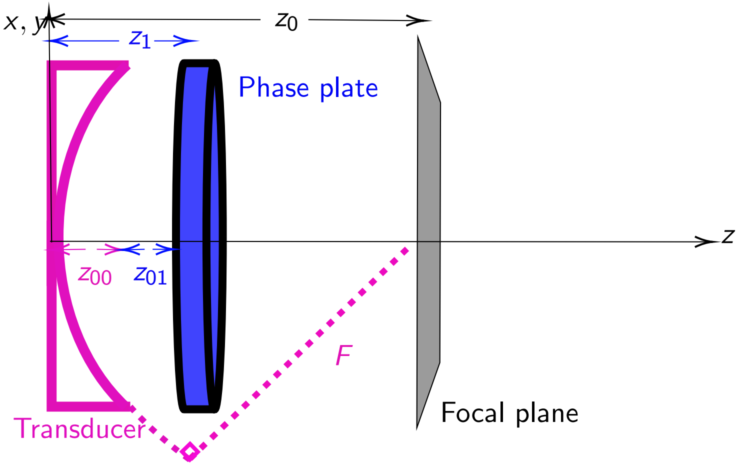


Figure: Positioning of the transducer, phase plate, and focal plane. Subscripts $_{\mathrm{T}}$, $_{1}$, and $_{0}$ are used to denote the transducer, phase plate, and focal plane, respectively.

First & Second Rayleigh integrals

First Rayleigh integral:

$$p(x_1, y_1, z_1^-) = -i\frac{\rho_0 c_0 k}{2\pi} \oiint_{S_T} \frac{V e^{ikR_\mathrm{T}}}{R_\mathrm{T}}\, \mathrm{d}S_\mathrm{T}.$$

Multiply result by phase factor. . .

$$p(x_1, y_1, z_1^+) = p(x_1, y_1, z_1^-) e^{3i\varphi}$$

. . . and insert into second Rayleigh integral:

$$p(x_0, y_0, z_0) = \frac{1}{2\pi} \oiint_{S_1} p(x_1, y_1, z_1^+) \frac{z_0 - z_1}{R_0} \Big[-\frac{ik}{R_0} + \frac{1}{R_0^2} \Big] e^{ikR_0}\, \mathrm{d}S_1$$

Rayleigh integral results

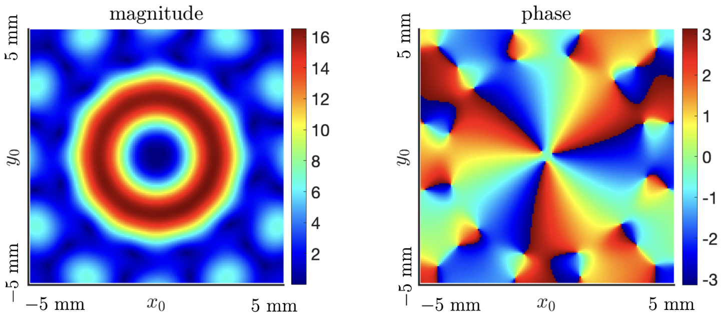


Figure: Normalized pressure magnitude $|(p(x_0, y_0, z_0)| / \rho_0 c_0 u_0$ (left) and phase $\angle p(x_0, y_0, z_0)$ (right) at the focal plane. The pressure magnitude vanishes on-axis, and the three helicoids corresponding to $l = 3$ are seen in the phase, confirming vortex-beam behavior.

Axisymmetric form of Fresnel limit

- No analytical solutions in the literature
- Provides physical intuition
- Faster than Rayleigh integral

Fresnel approximation:

$$p(x,y,z) \simeq \frac{-ik\rho_0 c_0}{2\pi}\frac{e^{ikz}}{z}\times$$
$$\times \iint_{\infty}^{\infty} u_0(x_0,y_0)\exp\left[\frac{ik}{2z}\left((x-x_0)^2+(x-x_0)^2\right)\right]\mathrm{d}x_0\,\mathrm{d}y_0.$$

The axisymmetric form:

$$p(\sigma,z,\varphi) \simeq \frac{-ik\rho_0 c_0}{2\pi}\frac{e^{ikz}}{z}e^{ik\sigma^2/2z}\int_0^{\infty} u_0(\sigma_0,\phi_0)\exp\left(\frac{ik}{2z}\sigma_0^2\right)\sigma_0\times$$
$$\times\int_0^{2\pi}\exp\left[\frac{-ik\sigma\sigma_0}{z}\cos(\varphi_0+\varphi)\right]\mathrm{d}\varphi_0\,\mathrm{d}\sigma_0.$$

Account for focusing and vorticity

$$p(\sigma, z, \varphi) \simeq \frac{-ik\rho_0 c_0}{2\pi} \frac{e^{ikz}}{z} e^{ik\sigma^2/2z} \int_0^\infty u_0(\sigma_0, \phi_0) \exp\left(\frac{ik}{2z}\sigma_0^2\right) \sigma_0 \times$$
$$\times \int_0^{2\pi} \exp\left[\frac{-ik\sigma\sigma_0}{z} \cos(\varphi_0 + \varphi)\right] \mathrm{d}\varphi_0 \, \mathrm{d}\sigma_0.$$

- Curvature of the transducer contributes a factor of $\exp\left(-ik\sigma_0^2/2F\right)$
- Phase plate contributes a factor of $e^{i\Phi}$, where $\Phi = l\varphi$ for $0 \leqslant \varphi < 2\pi$.

Semi-analytical & analytical solutions

Semi-analytical:

$$p(\sigma,\varphi,z) \simeq k\rho_0 c_0 u_0 \frac{e^{ikz}}{z} e^{ik\sigma^2/2z} e^{-3i\varphi} \times \\ \times \int_0^a \exp\left[\frac{ik}{2}\left(\frac{1}{z}-\frac{1}{F}\right)\sigma_0^2\right] J_3(k\sigma\sigma_0/z)\sigma_0 \, \mathrm{d}\sigma_0$$

Analytical:

$$p(\sigma,\varphi,F) \simeq \rho_0 c_0 u_0 \frac{F}{k\sigma^2} e^{ikF} e^{ik\sigma^2/2F} e^{3i\varphi} \times \\ \times \left[\left(\frac{3}{2}\pi \frac{ka\sigma}{F} \mathbf{H}_0(ka\sigma/F) - 8\right) J_1(ka\sigma/F) + \right. \\ \left. + \left(\frac{4ka\sigma}{F} - \frac{3\pi ka\sigma}{2F} \mathbf{H}_1(ka\sigma/F)\right) J_0(ka\sigma/F) \right]$$

Fresnel vs. Rayleigh surf plots

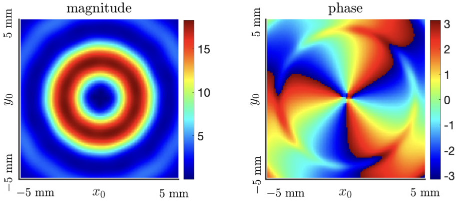


Figure: Magnitude (left) and phase (right) of the Rayleigh solutions in the focal plane

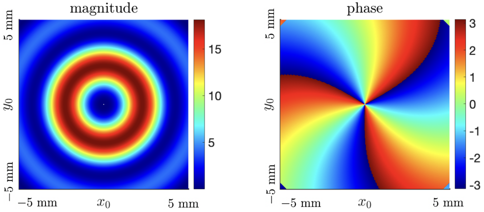


Figure: Magnitude (left) and phase (right) of the Fresnel analytical solution in the focal plane

Comparison to Rayleigh solution

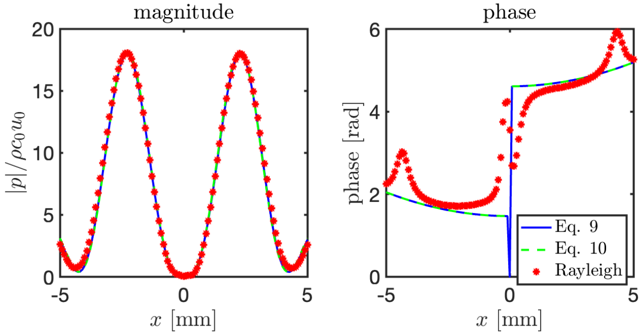


Figure: Magnitudes (left) and phases (right) of the exact Rayleigh and Fresnel solutions along the line $y = 0$ in the focal plane for $l = 3$.

Fourier approach

The problem is split into two. First, equation (1) calculates the pressure incident on the phase plate:

$$p(x_1, y_1, z_1) = \rho_0 c_0 F^{-1}\left\{ \frac{k}{k_z} \hat{u}_{\mathrm{T}}(k_x, k_y) e^{ik_z z} \right\} \qquad (1)$$

Second, a phase factor is included in the field at the phase plate to account for the vorticity. The field is then propagated by inserting the solution from equation (1) into

$$p(x_0, y_0, z_0) = F^{-1}\{F\{p(x_1, y_1, 7.5\ \mathrm{mm}) e^{3il}\} e^{i\Phi(\varphi)}\}. \qquad (2)$$

Geometry for Fourier approach

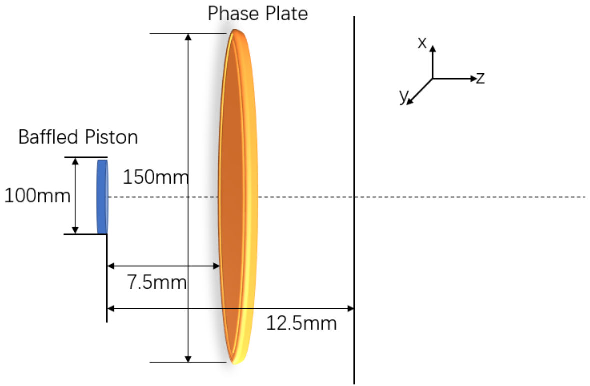
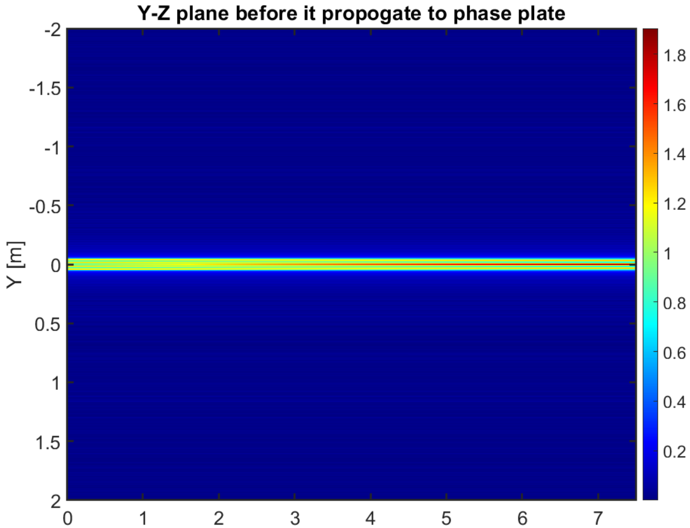


Figure: Simulation Setup illustration

Propagate pressure field

First, perform a FT of the velocity field and propagate to the position of the phase plate. Propagated field in the y-z plane is shown below:



Generate vorticity

Then the phase shift is applied to the pressure field at the position of the phase plate.

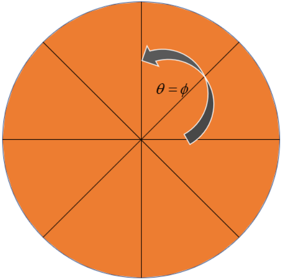


Figure: Top view of the phase plate

At the phase plate, a phase delay of $e^{jl\phi}$ is applied onto every point of the phase plate.

Behavior near the phase plate

The amplitude and phase of the pressure field after applying the phase shift is shown below. Note the phase distribution of the vortex beam is added instantaneously and artificially. The on-axis pressure amplitude is non-zero.

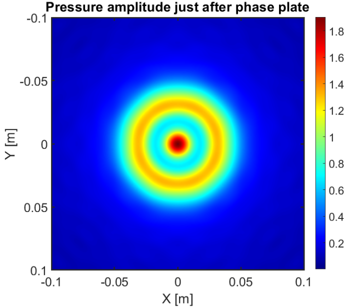


Figure: amplitude at surface of phase plate

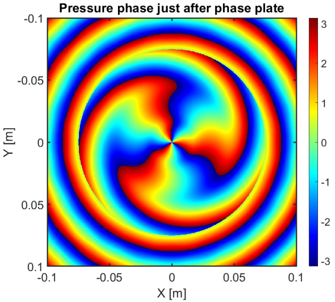
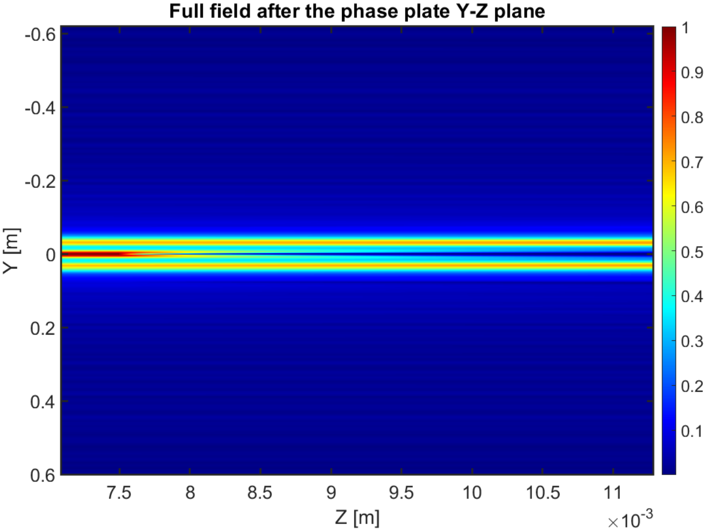


Figure: phase at surface of phase plate

Behavior near the phase plate

As the beam propagate, the spinning of the beam destructively interferes on axis:



Pressure at some distance beyond the phase plate

The phase and amplitude distribution after the vortex beam have propagated for 5mm are shown below:

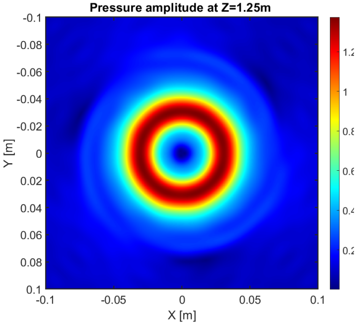


Figure: amplitude beyond phase plate

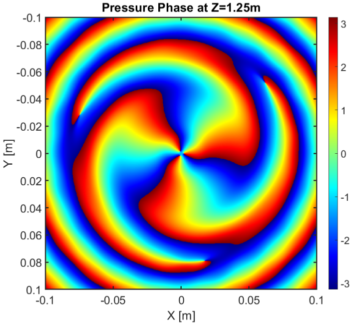


Figure: phase beyond the phase plate

Full journey of the vortex beam

By juxtaposing the field before and after the phase plate, the full field can be seen below:

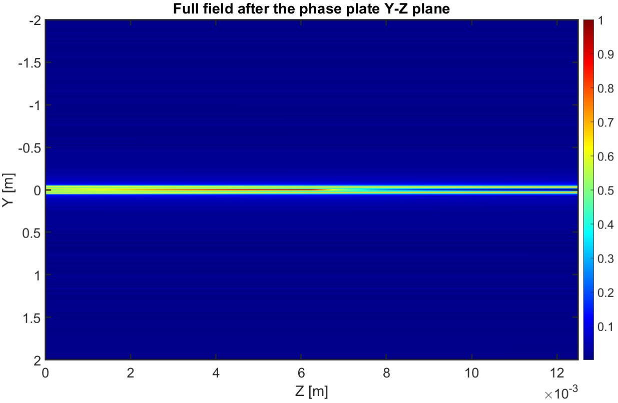


Figure: Full journey of the amplitude of the pressure field in the y-z plane

Geometry for FEM analysis

- Lower frequency of 20 kHz for computational ease
- Phase plate in COMSOL is discretized into 12 partitions, each having phase delay of $\frac{2\pi l}{12}$, with $l = 3$
- material properties of the phase plate achieve impedance match $\rho_1 c_1 = \rho_0 c_0$.

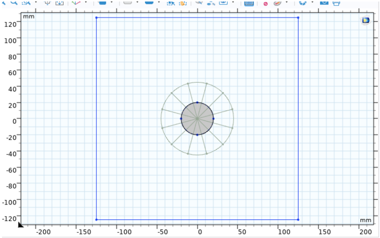


Figure: Top view of the setup

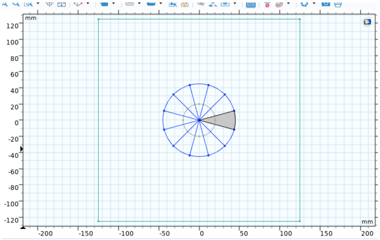


Figure: Illustration of one partition

FEM setup

The source under the phase plate is a circular piston with uniform normal velocity on the axis of the phase plate. The 3D geometry in COMSOL is shown below:

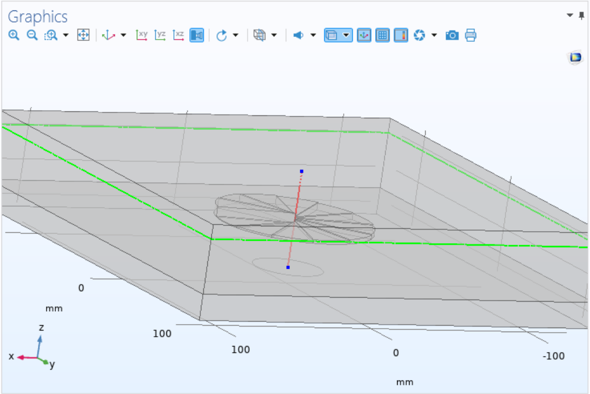


Figure: 3D illustration of the setup

Results

The FEM analysis gives the resulting phase plot in a plane beyond the partitioned phase plate. Observing the center of the phase distribution, the $l = 3$ vorticity can be clearly identified.

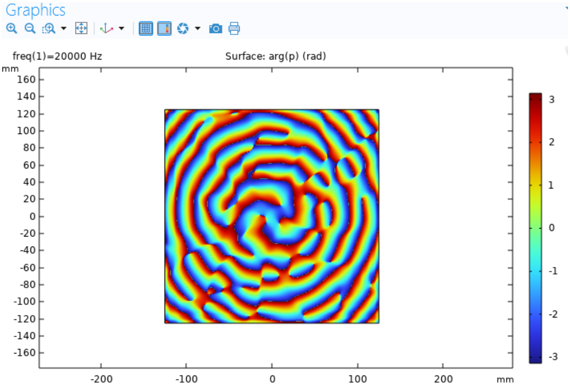


Figure: Phase distribution showing the $l = 3$ vorticity

Conclusion

- Original work identically replicated using Rayleigh integrals
- Semi-analytical and analytical solutions found in Fresnel limit
- Spatial evolution of pressure field analyzed by Fourier acoustics
- Vortex-generation mechanism investigated in FEM analysis
- Applications in particle manipulation, biomedical acoustics, underwater acoustics

Supplemental Notes & References

Questions/Discussion