Exercise on Planck Quantities

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In today's lecture we discussed the importance of three numbers—$\hbar$, c, and G—and how they determine the relevance of various regimes of physics (e.g., Newtonian mechanics, quantum field theory, etc.) over given scales.

Three fundamental quantities arise: the Planck mass M_p, the Planck length l_p, and the Planck time t_p. I will derive these quantities from considerations of dimensional analysis given the dimensions of $\hbar$, c, and G (where the dimension of some quantity a is denoted by $[a]$):

$$[\hbar] = \text{J s} = \frac{\text{kg m}^2}{\text{s}}$$

$$[c] = \frac{\text{m}}{\text{s}}$$

$$[G] = \frac{\text{m}^3}{\text{kg s}^2}$$

1 Planck mass M_p

Suppose we want $[M_p] = \text{kg}$. Multiplying $\hbar$ and c gives $\frac{\text{kg m}^3}{\text{s}^2}$. Dividing by $[G]$ leaves us with kg^2. Taking the square root finally gives the desired kg. This suggests that the Planck mass is given by

$$M_p = \sqrt{\frac{\hbar c}{G}}$$

In the classical approximation of quantum mechanics ($\hbar \to 0$), $M_p \to 0$ too (i.e., no notion of "smallest mass"). But in the classical approximation of special relativity ($c \to \infty$), $M_p \to \infty$. It is interesting that these limiting cases offer divergent approximations of the Planck mass.

2 Planck length l_p

Suppose we want $[l_p] = \mathrm{m}$. Multiplying $[\hbar]$ and $[G]$ cancels mass and gives $\frac{\mathrm{m}^5}{\mathrm{s}^3}$. Dividing by c^3 gives dimensions of m^2. Taking the square root gives the desired unit of meters. This suggests the Planck length should be

$$l_p = \sqrt{\frac{G\hbar}{c^3}}$$

In the classical approximation of quantum mechanics ($\hbar \to 0$) and special relativity ($c \to \infty$), $l_p \to 0$ (i.e., no notion of "smallest length") as expected.

3 Planck time t_p

Suppose we want $[t_p] = \mathrm{s}$. Multiplying $[\hbar]$ and $[G]$ cancels mass and gives $\frac{\mathrm{m}^5}{\mathrm{s}^3}$. Now dividing by c^5 gives s^2. Taking the square root gives the desired unit of seconds. This suggests the Planck time should be

$$t_p = \sqrt{\frac{\hbar G}{c^5}}$$

In the classical approximation of quantum mechanics ($\hbar \to 0$), $t_p \to 0$. The same limiting case is achieved in the classical approximation of special relativity ($c \to \infty$): in both approximations there is no notion of "shortest time."