(a) Find the matrix $\mathbf{W}$.

(b) What properties does the matrix $\mathbf{W}$ have?

(c) What is the inverse of $\mathbf{W}$?

2.2. Prove or disprove each of the following statements:

(a) The product of two upper triangular matrices is upper triangular.

(b) The product of two Toeplitz matrices is Toeplitz.

(c) The product of two centrosymmetric matrices is centrosymmetric.

2.3. Find the minimum norm solution to the following set of underdetermined linear equations,

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ -1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

2.4. Consider the set of inconsistent linear equations $\mathbf{Ax} = \mathbf{b}$ given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

(a) Find the least squares solution to these equations.

(b) Find the projection matrix $\mathbf{P}_A$.

(c) Find the best approximation $\hat{\mathbf{b}} = \mathbf{P}_A\mathbf{b}$ to $\mathbf{b}$.

(d) Consider the matrix

$$\mathbf{P}_A^\perp = \mathbf{I} - \mathbf{P}_A$$

Find the vector $\mathbf{b}^\perp = \mathbf{P}_A^\perp\mathbf{b}$ and show that it is orthogonal to $\hat{\mathbf{b}}$. What does the matrix $\mathbf{P}_A^\perp$ represent?

2.5. Consider the problem of trying to model a sequence $x(n)$ as the sum of a constant plus a complex exponential of frequency ω_0 [5]

$$\hat{x}(n) = c + ae^{jn\omega_0} \quad ; \quad n = 0, 1, \ldots, N-1$$

where c and a are unknown. We may express the problem of finding the values for c and a as one of solving a set of overdetermined linear equations

$$\begin{bmatrix} 1 & 1 \\ 1 & e^{j\omega_0} \\ \vdots & \vdots \\ 1 & e^{j(N-1)\omega_0} \end{bmatrix} \begin{bmatrix} c \\ a \end{bmatrix} = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

(a) Find the least squares solution for c and a.

(b) If N is even and $\omega_0 = 2\pi k/N$ for some integer k, find the least squares solution for c and a.

(a) Find the eigenvalues and eigenvectors of **A**.

(b) Are the eigenvectors unique? Are they linearly independent? Are they orthogonal?

(c) Diagonalize **A**, i.e., find **V** and **D** such that

$$\mathbf{V}^H\mathbf{A}\mathbf{V} = \mathbf{D}$$

where **D** is a diagonal matrix.

2.14. Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

2.15. Consider the following 3×3 symmetric matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

(a) Find the eigenvalues and eigenvectors of **A**.

(b) Find the determinant of **A**.

(c) Find the spectral decomposition of **A**.

(d) What are the eigenvalues of $\mathbf{A} + \mathbf{I}$ and how are the eigenvectors related to those of **A**?

2.16. Suppose that an $n \times n$ matrix **A** has eigenvalues $\lambda_1, \ldots, \lambda_n$ and eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

(a) What are the eigenvalues and eigenvectors of $\mathbf{A}^2$?

(b) What are the eigenvalues and eigenvectors of $\mathbf{A}^{-1}$?

2.17. Find a matrix whose eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 4$ with eigenvectors $\mathbf{v}_1 = [3, 1]^T$ and $\mathbf{v}_2 = [2, 1]^T$.

2.18. Gerschgorin's circle theorem states that every eigenvalue of a matrix **A** lies in at least one of the circles $C_1, \ldots, C_N$ in the complex plane where C_i has center at the diagonal entry a_{ii} and its radius is $r_i = \sum_{j \neq i} |a_{ij}|$.

1. Prove this theorem by using the eigenvalue equation $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ to write

$$(\lambda - a_{ii})x_i = \sum_{j \neq i} a_{ij}x_j$$

$|x_i| > |x_j|, \forall j \neq i$

and then use the triangular inequality,

$|(\lambda - a_{ii})| \le \sum |a_{ij}| \frac{|x_j|}{|x_i|}$

$$\left|\sum_{j \neq i} a_{ij}x_j\right| \le \sum_{j \neq i} |a_{ij}x_j|$$

$\le \sum |a_{ij}|$

2. Use this theorem to establish the bound on λ_{max} given in Property 7.

3. The matrix

$$\begin{bmatrix} 4 & 1 & 2 \\ 2 & 3 & 0 \\ 3 & 2 & 6 \end{bmatrix}$$