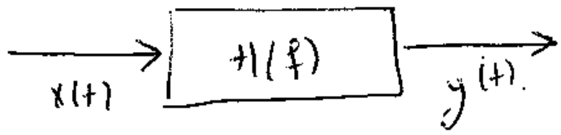
Review of DSP Topics:



In many applications, either the input $x(t)$ or the system $H(f)$ is band-limited; that is the function has a maximum allowable frequency.

$$X(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt.$$

$X(f)$ is an alternative representation to the time-domain representation. The canonical basis for time domain representation can be written as:

$$x(t) = \int_{-\infty}^{\infty} \underbrace{x(t_0)}_{\text{expansion coef.}} \overbrace{\delta(t_0 - t_0)}^{\text{Basis function}}\, dt_0.$$

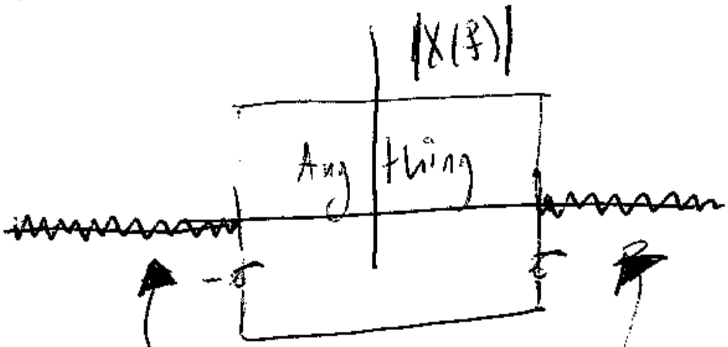
Note: In discrete time, the same relation is

$$x(n) = \sum_{n_0=-\infty}^{\infty} \underbrace{x(n_0)}_{\text{expansion coef.}} \overbrace{\delta(n-n_0)}^{\text{Basis function}}$$

$$\hookrightarrow e_{n_0} = [\,\underset{-\infty}{\cdot}\, 0 \ldots 0 \ldots 0 \quad \underset{\uparrow \atop n_0}{1} \quad \underset{\infty}{0}\,]^T$$

$$x(t) = \int_{-\infty}^{\infty} \underbrace{x(t_0)}_{\text{expansion coef's}} \underbrace{\delta(t-t_0)}_{\text{Basis functions}} \, dt_0 = \int_{-\infty}^{\infty} \underbrace{X(f)}_{\text{expansion coef's}} \underbrace{e^{+j2\pi f t}}_{\text{Basis functions}} \, df.$$

If $X(f)$ is band-limited to σ, then. $\boxed{X(f) = 0 \quad \text{for } f \geq \sigma}$



zero outside the box

If $X(f)$ or $H(f)$ is band-limited $\longrightarrow$ $Y(f)$ is also limited to same band.

Since $\boxed{Y(f) = X(f)\,H(f)}$

The fundamental relation making Discrete time signal processing a viable processing technique is the sampling relation.

③

If

$X(f)$ is bandlimited to ϵ,

Then.

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\mathrm{sinc}\left(\frac{t-nT}{T}\right) \quad \text{if } \underbrace{\frac{1}{T}}_{\text{sampling freq.}} \geq 2\,\underbrace{\epsilon}_{\text{freq limitation}}$$

where $\mathrm{sinc}(x) = \dfrac{\sin(\pi x)}{\pi x}$;

Assume that both $x(t)$ and $h(t)$ are expressed in terms of their samples and sinc function.

$$x(t) = \sum_{n=} x(nT)\,\mathrm{sinc}\left(\frac{t}{T}-n\right) \quad ; \quad \& \; h(t) = \sum_{m} h(mT)\,\mathrm{sinc}\left(\frac{t}{T}-m\right)$$

Then.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(t-\tau)\,x(\tau)\,d\tau.$$

$$= \sum_{n,m} x(nT)\,h(mT) \int_{-\infty}^{\infty} \mathrm{sinc}\left(\frac{t-\tau}{T}-m\right) \mathrm{sinc}\left(\frac{\tau}{T}-n\right) d\tau$$

$$\overset{①}{=} \sum_{n,m} x(nT)\,h(mT) \cdot \underbrace{\left(\mathrm{sinc}\left(\frac{t-\tau-mT}{T}\right), \mathrm{sinc}\left(\frac{\tau-nT}{T}\right) \right)_{\tau}}_{\text{inner product w.r.t } \tau.}$$

$$\overset{②}{=} \sum_{n,m} x(nT)\,h(mT) \cdot \left(\mathcal{F}\left\{ \mathrm{sinc}\left(\frac{\tau-(t-mT)}{T}\right) \right\}, \mathcal{F}\left\{ \mathrm{sinc}\left(\frac{\tau-nT}{T}\right) \right\} \right)$$

$$\overset{③}{=} \sum x(nT)\,h(mT) \left(\underbrace{\mathrm{rect}(fT)\, e^{-j2\pi f(t-mT)}}_{\mathcal{F}\{\mathrm{sinc}(\cdots)\}},\ \underbrace{\mathrm{rect}(fT)\, e^{-j2\pi f nT}}_{\mathcal{F}\{\mathrm{sinc}(\cdots)\}} \right)_f \quad ④$$

$$\overset{④}{=} \sum x(nT)\,h(mT) \underbrace{\int_{-\infty}^{\infty} e^{+j2\pi f(\cancel{t}(m+n)T-t)} \cdot \mathrm{rect}(fT)\, df}$$

$$\overset{⑤}{=} \sum x(nT)\,h(mT) \cdot \delta((m-n)T-t) * \mathrm{sinc}\left(\frac{t}{T}\right)$$

$$\overset{⑥}{=} \sum_{n,m} x(nT)\,h(mT) \cdot \mathrm{sinc}\left(\frac{t-(m+n)T}{T}\right)$$

$$\overset{⑦}{=} \sum_{n,k=-\infty}^{\infty} x(nT)\,h((k-n)T)\,\mathrm{sinc}\left(\frac{t}{T}-k\right)$$

$(k=m+n)$

$$= \sum_{k=-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} x(nT)\,h((k-n)T) \right) \mathrm{sinc}\left(\frac{t}{T}-k\right)$$

$$\overset{⑧}{=} \sum \left(x[n] * h[n] \right) \mathrm{sinc}\left(\frac{t}{T}-k\right)$$

$x[n] = x(nT)$

$h[n] = h(nT)$

In (1) the integral is expressed as a inner product.

In (2) Fourier transform is applied to the arguments of inner product. Both inner products are the same since Fourier transform is orthonormal.

In (3) Fourier transforms of sinc functions are written.

In (4) The inner product is written

Note that

$$\left(X(f), Y(f)\right) = \int X(f)\, Y^*(f)\, df.$$

when $X(f), Y(f)$ can take complex values

In (5) The integral in (4) is evaluated by interpretting the integral as a $\mathcal{F}^{-1}\{\}$ relation

In (6) Shift property

In (7) Re-arrangement of summation.

In (8) The output is written in terms of samples of $x(t)$ and $h(t)$.

The relation (8) shows that the exact value of channel output can be calculated by the discrete convolution of samples.

So there is no loss of information by discrete time processing (convolution) of samples.

Z-Tranform:

The samples of $x(n)$ can be expressed alternatively as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

(the coefficient of z^{-n} stores $x(n)$)

For example: if $x[n] = \left(\frac{1}{2}\right)^n u(n)$

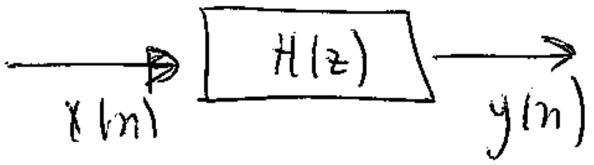
then $X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2z}\right)^n = \frac{1}{1-2z}$ if $\left|\frac{1}{2z}\right| < 1$

As a result, the coefficients of $\frac{1}{1-2z}$ are the $x[n]$ values.

Review your z-transform notes on convergence and the effect of left-right sidedness of the sequences.

* Z-tranform is useful to determine the properties of sequences. The sequence can be the impulse response of a system. Then the properties of the sytem is determined by the z-tranform expression. (Stability of the system, Stability of the inverse system, etc.)

Processing of Samples:

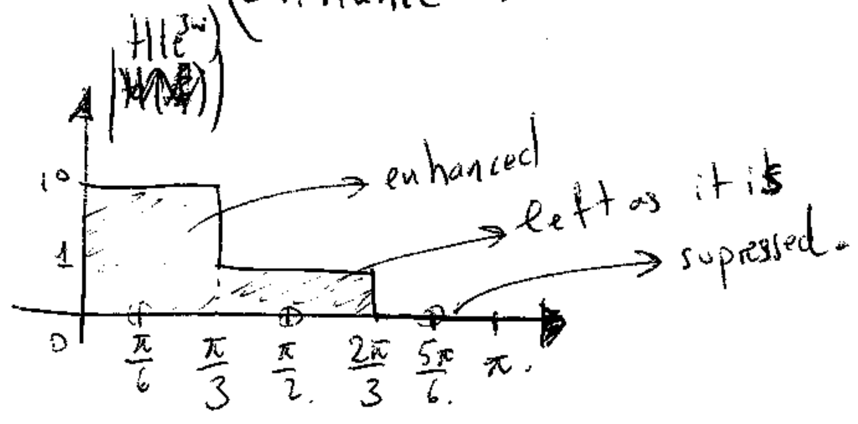


$H(z)$: A calculator implementing relations like

$$y(n)=\sum_{k=1}^{P} a_k y(n-k) + \sum_{k=0}^{Q} b_k x(n-k)$$

for $$H(z) = \frac{\sum_{k=0}^{Q} b_k z^{-k}}{1-\sum_{k=1}^{P} a_k z^{-k}}.$$

① Filtering: $\left(\begin{array}{l}\text{Suppress some frequencies} \\ \text{Enhance some others}\end{array}\right)$



That is if $x(n)$ has the following definition.

$$x(n) = e^{+j\frac{\pi}{6}n} + e^{j\frac{\pi}{2}n} + e^{j\frac{5\pi}{6}n} \longrightarrow y(n)=h(n)*x(n)$$

$$\rightarrow \quad y(n) = \underbrace{10\cdot e^{j\frac{\pi}{6}n}}_{\text{enhanced}} + \underbrace{e^{j\frac{\pi}{2}n}}_{\text{as it is}} + \underbrace{0}_{\text{supressed}}$$

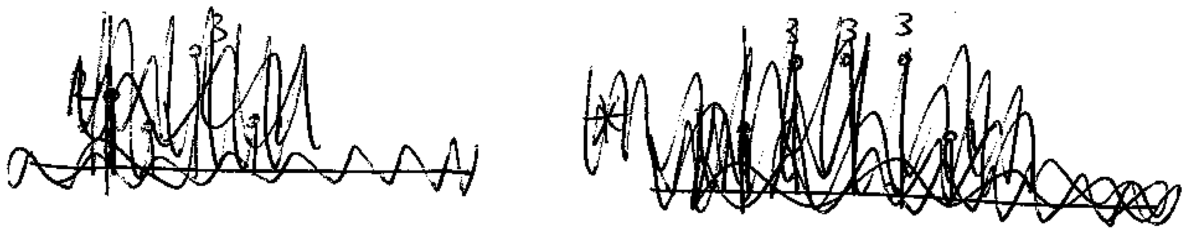
Details of $y(n)$ calculation:

$$y(n) = x(n) * h(n)$$
$$= \sum h(k) \cdot \left[x(n-k) \right]$$
$$= \sum h(k) \left[e^{j\frac{\pi}{6}(n-k)} + e^{j\frac{\pi}{2}(n-k)} + e^{j\frac{5\pi}{6}(n-k)} \right]$$
$$= \left(\sum h(k) e^{-j\frac{\pi}{6}k} \right) e^{j\frac{\pi}{6}n} + \left(\sum h(k) e^{-j\frac{\pi}{2}k} \right) e^{j\frac{\pi}{2}n}$$
$$+ \left(\sum h(k) e^{-j\frac{5\pi}{6}k} \right) e^{j\frac{5\pi}{6}n}.$$
$$= \underbrace{H(e^{j\pi/6})}_{10} e^{j\frac{\pi}{6}n} + \underbrace{H(e^{j\frac{\pi}{2}})}_{1} e^{j\frac{\pi}{2}n} + \underbrace{H(e^{j\frac{5\pi}{6}})}_{0.} e^{j\frac{5\pi}{6}n}.$$

enhanced freq.

Supressed freq.

Filtering: (Matrices interpetation)



$$\xrightarrow{x(n)} \boxed{H(z)} \xrightarrow{y(n)}$$

Let $x(n) = x_0 \delta(n) + x_1 \delta(n-1) + \text{———} + x_N \delta(n-N)$

$h(n) = h_0 \delta(n) + h_1 \delta(n-1) + \text{——} + h_p \delta(n-p)$

(p is the number of taps of the filter).

$$y(n) = x(n) * h(n)$$

$$y(n) = x_0 h(n) + x_1 h(n-1) + \text{——} \; x_N h(n-N)$$

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ \vdots \\ y(N) \end{bmatrix} = \begin{array}{c} \begin{bmatrix} h_0 & 0 & 0 & & 0 \\ h_1 & h_0 & 0 & & 0 \\ \vdots & h_1 & h_0 & & \vdots \\ h_p & \vdots & h_1 & \ddots & \\ & h_p & \vdots & & h_0 \\ & 0 & h_p & & h_1 \\ & 0 & 0 & & \\ & 0 & 0 & & \end{bmatrix} \\ \begin{array}{ccc} h(n) & h(n-1) & h(n-2) \end{array} \end{array} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ \vdots \\ x_N \end{bmatrix}$$

or

$$y(n) = h_0 x(n) + h_1 x(n-1) + \text{——} + h_p x(n-p)$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} x_0 & 0 & 0 & & \\ x_1 & x_0 & 0 & & \\ x_2 & x_1 & x_0 & & \\ \vdots & \vdots & x_1 & & \\ \vdots & \vdots & & \ddots & x_0 \\ & & & & x_1 \\ x_N & x_{N-1} & x_{N-2} & & \vdots \\ 0 & x_N & x_{N-1} & & \vdots \\ 0 & 0 & x_N & & \vdots \\ 0 & 0 & 0 & & \vdots \\ 0 & 0 & 0 & & x_N \\ 0 & 0 & 0 & & 0 \end{bmatrix}}_{\text{Shifted inputs}} \underbrace{\begin{bmatrix} h_0 \\ \vdots \\ h_p \end{bmatrix}}_{\text{filter coefs!}}$$

$$\underline{y} = \underline{\underline{\text{Toeplitz}}}\{\underline{x}\} \cdot \underline{h} = \text{Toeplitz}\{\underline{\underline{h}}\} \cdot \underline{x}$$

Toeplitz is the constant diagonal matrix. It is also called convolution matrix.

Shift Matrix: A special Toeplitz matrix that shifts the input sequence at the output.

$$\underline{\underline{S}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad ; \quad \underline{\underline{S}}\,\underline{x} = \begin{bmatrix} 0 \\ \underline{x} \\ \\ \end{bmatrix}$$

Any ~~Toeplitz~~ convolution matrix can be written in terms of shift matrices.

$$\underline{\underline{H}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \\ 0 & 4 & 3 & 2 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

matrix with 1's along the main diagonal.

$$\underline{\underline{H}} = \underline{\underline{I}} + 2\underline{\underline{S}} + 3\underline{\underline{S}}^2 + 4\underline{\underline{S}}^3$$

It is clear that $\underline{\underline{H}}\,\underline{\underline{S}}^k = \underline{\underline{S}}^k\,\underline{\underline{H}}$, which implies the following:

$$\underline{\underline{H}}\,\underbrace{\underline{\underline{S}}^k\,\underline{x}(n)}_{} = \underline{\underline{S}}^k\,\underbrace{\underline{\underline{H}}\,\underline{x}(n)}_{y(n)}$$

$$\underline{\underline{H}}\,\underline{x}(n-k) = \underline{y}(n-k)$$

Delayed input

Delayed output.

This shows or illustrates the time invariancy of the system.

<u>Diagonalization of Convolution Matrices:</u> (For infinite dimensional matrices)

$$\underline{\underline{H}} = h(0)\,\underline{\underline{I}} + h(1)\,\underline{\underline{S}} + h(2)\,\underline{\underline{S}}^2 + \cdots$$

If we can find the eigenvectors of $\underline{\underline{S}}$, they are also the eigenvectors of $\underline{\underline{H}}$.

<u>eigenvectors of $\underline{\underline{S}}$:</u> $S\,x(n) \overset{\text{def}}{=} x(n-1) = \lambda\, x(n)$: eigen vector. $x(n)$ is defined in $-\infty < n < \infty$

Take $\underline{x}(n) = e^{jwn}$ then

$$\underline{\underline{S}}\,\underline{x}(n) = e^{jw(n-1)} = \underbrace{e^{-jw}}_{\lambda}\,\underbrace{e^{jwn}}_{\underline{x}(n)}.$$

So $\underline{x}(n) = e^{jwn}$ is an eigenvector of S with eigenvalue e^{-jw}.

<u>eigenvectors of $\underline{\underline{H}}$:</u> eigenvector with eigenvalue λ is.

$$\underline{\underline{H}}\,\underline{x}(n) = \lambda\,\underline{x}(n)$$

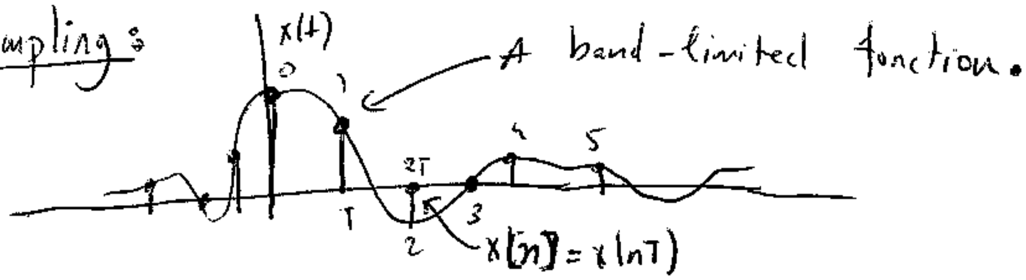
Take $\underline{x}(n) = e^{jwn} \longrightarrow \underline{\underline{H}}\,x(n) = \underbrace{\left(\sum h(k)\,e^{-jwk}\right)}_{\lambda}\underbrace{e^{jwn}}_{\underline{x}(n)}$

<u>Conclusion</u>: Eigenvectors are complex exponentials. and corresponding eigenvalues are the Fourier transform of $h[n]$ sequence.

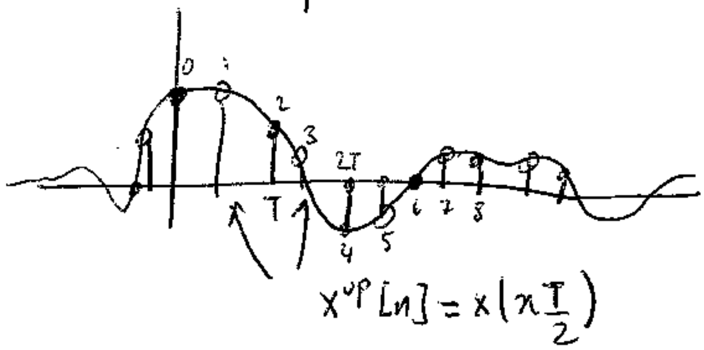
Processing of Samples (cont'd)

② Upsampling / Downsampling:

Upsampling:



If we take more samples



Given the samples $x[n]$; we can reconstruct $x(t)$ and sample it with half of the first sampling period to generate $x^{up}[n]$.

But note that

$x^{up}[2n] = x[n]$. $\longrightarrow$ $x^{up}[2n+1]$ can be calculated by sinc interpolation.

Define

$$\hat{x}[n] = \begin{cases} x[n/2] & n: \text{even} \\ 0 & n: \text{odd.} \end{cases}$$

$$x(n) = x_0\,\delta(n) + x_1\,\delta(n-1) + \cdots$$

$$\hat{x}(n) = x_0\,\delta(n) + 0\cdot\delta(n-1) + x_1\,\delta(n-2) + 0\cdot\delta(n-3) + \cdots$$

$$\hat{X}(z) = \sum_n \hat{x}(n)\,z^{-n} = \sum_{n,\,\text{even}} \hat{x}(n)\,z^{-n} = \sum_{\ell=-\infty}^{\infty} \hat{x}(2\ell)\,z^{-2\ell}$$

$$= \sum x(\ell)\left(z^{+2}\right)^{-\ell}$$

$$= X\left(z^{+2}\right)$$

So $\hat{X}(z) = X(z^2)$.

Matrices intergretation:

$$\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N) \end{bmatrix} \longrightarrow \hat{x}(n) = \begin{array}{cc} \begin{bmatrix} x(0) \\ 0 \\ x(1) \\ 0 \\ x(2) \\ \vdots \\ 0 \\ x(N) \end{bmatrix} & \begin{array}{l} \leftarrow 0 \\ \\ \leftarrow 2 \\ \\ \\ \\ \\ \leftarrow 2N \end{array} \end{array}$$

$$\underline{\underline{U}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

rows of zeros

$\underline{\underline{U}}$ is the ~~op~~ matrix for the up-sampling operation.

Note: Up-sampling by 2 discussion can be generalized, trivially.

Down - Sampling:

$$x^{down}(n) = x(Mn) \qquad M \text{ is an integer.}$$

$$\left.\begin{aligned} x^{down}(0) &= x(0) \\ x^{down}(1) &= x(M) \\ x^{down}(2) &= x(2M) \\ &\vdots \end{aligned}\right\} \rightarrow \text{every } M^{th} \text{ sample is taken.}$$

(also called decimation)

$X^{down}(z) = ?$ The z-domain definition for down-sampling is a little tricky. Aliasing components appear in z-domain.

Take $M=2$, (down sample by 2)

$$X^{down}_{by\,2}(z) = x(0) + x(2)z^{-1} + x(4)z^{-2} + \dots$$

Let's calculate

$$X(z) + X(-z) = \overbrace{x_0 + x_1 z^{-1} + x_2 z^{-2} + \dots}^{X(z)} + \underbrace{x_0 + (-x_1)z^{-1} + x_2 z^{-2} + \dots}_{X(-z)} = 2x_0 + 2x(2)z^{-2} + 2x(4)z^{-4} + \dots$$

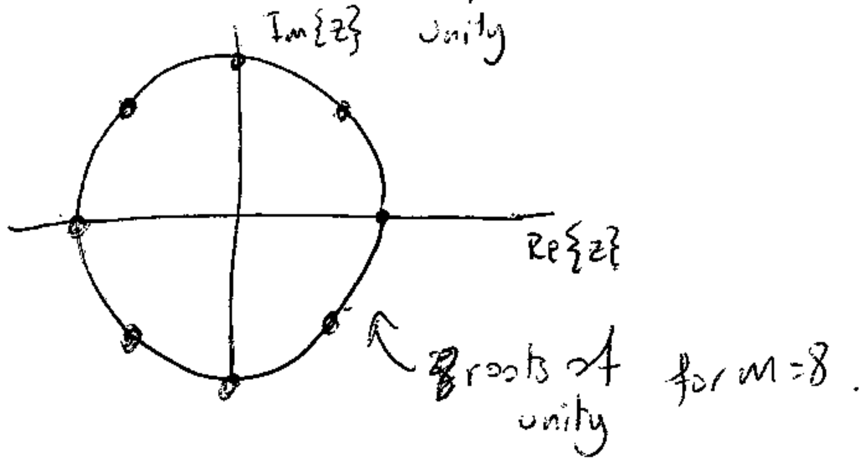
Then

$$X^{down}_{by\,2}(z) = \left.\frac{X(w) + X(-w)}{2}\right|_{w = z^{1/2}}$$

$$\boxed{X^{down}(z) = \frac{X(z^{1/2}) + X(-z^{1/2})}{2}}$$

In general

$$X_{\substack{\text{down} \\ \text{by} \\ M}}(z) = \frac{1}{M} \sum_{k=0}^{M-1} X\Big(\underbrace{e^{j\frac{2\pi}{M}k}}_{\substack{\text{Roots} \\ \text{of} \\ \text{unity}}} z^{1/M}\Big)$$



<u>Matrices Interpretation:</u>

$$\underline{\underline{D}}_2 = \begin{pmatrix} \text{Down} \\ \text{by} \\ 2 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ & & & & & \end{bmatrix}$$

$$\underline{\underline{D}}_2 \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{2N} \end{bmatrix} = \begin{bmatrix} x_0 \\ x_2 \\ x_4 \\ \vdots \\ x_N \end{bmatrix}$$

Note that Down sampling matrix is the transpose of Up sampling matrix.