A Computationally Efficient Fine Frequency Estimation Method For Real-Valued Sinusoids (Supplementary Document)

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Abstract

This document aims to clarify the derivation and properties of the proposed computationally efficient fine frequency estimation method for the real-valued sinusoids given in [1]. Document contains, detailed derivation of the estimator, the fusion rule and its mean square error (MSE) analysis. In addition, the derivation of the simplified expression for the maximum likeliood estimator given in the manuscript is presented. The Cramer-Rao lower bound (CRB) and its hybrid version is discussed to illustrate the minor, but important, differences in their usage.

CONTENTS

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I. Preliminaries

A real sinusoid signal of unknown amplitude, phase and frequency is observed under white real Gaussian noise:

$$r[n] = A\cos(2\pi f n + \phi) + w[n], \quad n = \{0, \ldots, N-1\}. \tag{1}$$

where N is number of samples, A and ϕ are the amplitude and the phase of the real sinusoid signal. The frequency variable f in (1) is the normalized frequency defined in $[0, 0.5)$ and can be expressed as $f = \frac{(k_p+\delta)}{2N}$ in terms of DFT bins where k_p is an integer in $[0, N-1]$ and δ is a real number in $[-0.5, 0.5]$. It is assumed that noise $w[n]$ is Gaussian distributed noise with zero mean and σ_w^2 variance, $w[n] \sim \mathcal{N}(0, \sigma_w^2)$. The signal-to-noise ratio (SNR) definition used in this work is the input SNR which is $\text{SNR} = A^2/2\sigma_w^2$.

II. Maximum Likelihood Estimation (MLE)

The likelihood expression for the unknown parameters can be given as

$$p(x;\theta) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n] - A\cos(2\pi n f/N + \phi))^2\right], \tag{2}$$

where $A > 0$ and $0 < f < N/2$. The MLE can be found by minimizing

$$\begin{aligned} J(A, f, \phi) &= \sum_{n=0}^{N-1}(x[n] - A\cos(2\pi n f/N + \phi))^2 \\ &= \sum_{n=0}^{N-1}(x[n] - A\cos(\phi)\cos(2\pi n f/N) + A\sin(\phi)\sin(2\pi n f/N))^2 \end{aligned} \tag{3}$$

or equivalently,

$$\begin{aligned} J'(\alpha_1, \alpha_2, f) =& (\mathbf{x} - \alpha_1\mathbf{c} - \alpha_2\mathbf{s})^T(\mathbf{x} - \alpha_1\mathbf{c} - \alpha_2\mathbf{s}) \\ J'(\alpha_1, \alpha_2, f) =& (\mathbf{x} - \mathbf{H}\boldsymbol{\alpha})^T(\mathbf{x} - \mathbf{H}\boldsymbol{\alpha}) \end{aligned} \tag{4}$$

where $\boldsymbol{\alpha} = [\alpha_1 \ \alpha_2] = [A\cos(\phi) \ \ -A\sin(\phi)]$ and $\mathbf{H} = [\mathbf{c} \ \mathbf{s}]$ where

$$\begin{aligned} \mathbf{c} =& [1 \ \ \cos(2\pi f/N) \ldots \cos(2\pi f(N-1)/N]^T \\ \mathbf{s} =& [0 \ \ \sin(2\pi f/N) \ldots \sin(2\pi f(N-1)/N]^T \end{aligned} \tag{5}$$

By minimizing the function in (3) over $\boldsymbol{\alpha}$, we get

$$\widehat{\boldsymbol{\alpha}} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{x} \tag{6}$$

Then upon substituting $\widehat{\boldsymbol{\alpha}}$ expression given in (6) into the objective function, we get

$$\begin{aligned} J'(\hat{\alpha_1}, \hat{\alpha_2}, f) =& (\mathbf{x} - \mathbf{H}\widehat{\boldsymbol{\alpha}})^T(\mathbf{x} - \mathbf{H}\widehat{\boldsymbol{\alpha}}) \\ =& (\mathbf{x} - \mathbf{H}(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{x})^T(\mathbf{x} - \mathbf{H}(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{x}) \\ =& \mathbf{x}^T(\mathbf{I} - \mathbf{H}(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T)^T(\mathbf{I} - \mathbf{H}(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T)\mathbf{x} \\ =& ||\mathbf{P}_{\mathrm{H}}^{\perp}\mathbf{x}||^2 \end{aligned} \tag{7}$$

where $\mathbf{P}_{\mathrm{H}} = \mathbf{H}(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T$ is the projection matrix to the range space of $\mathbf{H}$ and $\mathbf{P}_{\mathrm{H}}^{\perp} = \mathbf{I} - \mathbf{H}(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T$ is the projector to its orthogonal complement.

Instead of using $\mathbf{H} = [\mathbf{c}\ \mathbf{s}]$ matrix, we can equivalently use the orthonormalized version of $\mathbf{H}$ called $\mathbf{M}$ i.e. range$\{\mathbf{H}\}$ = range$\{\mathbf{M}\}$ and $\mathbf{M}^T\mathbf{M} = \mathbf{I}$, to express the projector to the range space. Remembering that orthogonal projection matrices satisfy **P**=$\mathbf{P}^T$ and $\mathbf{P}^2$=**P**, we have

$$\begin{aligned}
\hat{f}_{ML} &= \arg\min_f ||\mathbf{P}_{\mathrm{H}}^{\perp}\mathbf{x}||^2 \\
&= \arg\min_f ||\mathbf{P}_{\mathrm{M}}^{\perp}\mathbf{x}||^2 \\
&= \arg\min_f ||\mathbf{P}_{\mathrm{M}}^{\perp}\mathbf{x}||^2 \\
&= \arg\min_f ||(\mathbf{I} - \mathbf{P}_{\mathrm{M}})\mathbf{x}||^2 \\
&= \arg\min_f \mathbf{x}^T(\mathbf{I} - \mathbf{P}_{\mathrm{M}})^2\mathbf{x} \\
&= \arg\min_f \mathbf{x}^T(\mathbf{I} - \mathbf{P}_{\mathrm{M}})\mathbf{x} \\
&= \arg\min_f \mathbf{x}^T\mathbf{x} - \mathbf{x}^T\mathbf{P}_{\mathrm{M}}\mathbf{x} \\
&= \arg\max_f ||\mathbf{P}_{\mathrm{M}}\mathbf{x}||^2 \\
&= \arg\max_f [\mathbf{x}^T\mathbf{M}(\mathbf{M}^T\mathbf{M})^{-1}\mathbf{M}^T\mathbf{x}] \\
&= \arg\max_f ||\mathbf{M}^T\mathbf{x}||^2
\end{aligned} \tag{8}$$

Let's define $\widehat{\mathbf{s}} = \mathbf{s}_{\mathrm{u}} + \mathbf{c}_{\mathrm{u}}$ and $\widehat{\mathbf{c}} = \mathbf{s}_{\mathrm{u}} - \mathbf{c}_{\mathrm{u}}$ where $\mathbf{s}_{\mathrm{u}} = \frac{\mathbf{s}}{||\mathbf{s}||}$, $\mathbf{c}_{\mathrm{u}} = \frac{\mathbf{c}}{||\mathbf{c}||}$. The orthonormalized version of $\mathbf{H}$ can be obtained as $\mathbf{M} = [\widehat{\mathbf{s}}_{\mathrm{u}}\ \widehat{\mathbf{c}}_{\mathrm{u}}]$ where $\widehat{\mathbf{s}}_{\mathrm{u}} = \frac{\widehat{\mathbf{s}}}{||\widehat{\mathbf{s}}||}$, $\widehat{\mathbf{c}}_{\mathrm{u}} = \frac{\widehat{\mathbf{c}}}{||\widehat{\mathbf{c}}||}$ due to $\widehat{\mathbf{s}}^T\widehat{\mathbf{c}} = ||\mathbf{s}_{\mathrm{u}}||^2 - ||\mathbf{c}_{\mathrm{u}}||^2 = 0$. By using new variables we defined, the result in (8) can be written as

$$\begin{aligned}
\hat{f}_{ML} &= \arg\max_f ||\mathbf{M}^T\mathbf{x}||^2 \\
&= \arg\max_f \left[\left(\frac{\widehat{\mathbf{s}}^T\mathbf{x}}{||\widehat{\mathbf{s}}||}\right)^2 + \left(\frac{\widehat{\mathbf{c}}^T\mathbf{x}}{||\widehat{\mathbf{c}}||}\right)^2\right] \\
&= \arg\max_f \left[\frac{(\mathbf{s}_{\mathrm{u}}^T\mathbf{x} + \mathbf{c}_{\mathrm{u}}^T\mathbf{x})^2}{2(1 + \mathbf{s}_{\mathrm{u}}^T\mathbf{c}_{\mathrm{u}})} + \frac{(\mathbf{s}_{\mathrm{u}}^T\mathbf{x} - \mathbf{c}_{\mathrm{u}}^T\mathbf{x})^2}{2(1 - \mathbf{s}_{\mathrm{u}}^T\mathbf{c}_{\mathrm{u}})}\right] \\
&\quad \text{where } ||\widehat{\mathbf{s}}||^2 = 2(1 + \mathbf{s}_{\mathrm{u}}^T\mathbf{c}_{\mathrm{u}}) \text{ and } ||\widehat{\mathbf{c}}||^2 = 2(1 - \mathbf{s}_{\mathrm{u}}^T\mathbf{c}_{\mathrm{u}}) \\
&= \arg\max_f \left[\frac{(\mathbf{s}_{\mathrm{u}}^T\mathbf{x})^2 + (\mathbf{c}_{\mathrm{u}}^T\mathbf{x})^2 - 2(\mathbf{s}_{\mathrm{u}}^T\mathbf{x})(\mathbf{c}_{\mathrm{u}}^T\mathbf{x})(\mathbf{s}_{\mathrm{u}}^T\mathbf{c}_{\mathrm{u}})}{1 - (\mathbf{s}_{\mathrm{u}}^T\mathbf{c}_{\mathrm{u}})^2}\right] \\
&= \arg\max_f \left[\frac{(\mathbf{s}_{\mathrm{u}}^T\mathbf{x})^2 + (\mathbf{c}_{\mathrm{u}}^T\mathbf{x})^2 - 2(\mathbf{s}_{\mathrm{u}}^T\mathbf{x})(\mathbf{c}_{\mathrm{u}}^T\mathbf{x})(\mathbf{s}_{\mathrm{u}}^T\mathbf{c}_{\mathrm{u}})}{1 - (\mathbf{s}_{\mathrm{u}}^T\mathbf{c}_{\mathrm{u}})^2}\frac{||\mathbf{s}||^2||\mathbf{c}||^2}{||\mathbf{s}||^2||\mathbf{c}||^2}\right] \\
&= \arg\max_f \left[\frac{(\mathbf{s}^T\mathbf{x})^2||\mathbf{c}||^2 + (\mathbf{c}^T\mathbf{x})^2||\mathbf{s}||^2 - 2(\mathbf{s}^T\mathbf{x})(\mathbf{c}^T\mathbf{x})(\mathbf{s}^T\mathbf{c})}{||\mathbf{s}||^2||\mathbf{c}||^2 - (\mathbf{s}^T\mathbf{c})^2}\right]
\end{aligned} \tag{9}$$

The expression in (9) remind the Fourier transform of the observed signal $x[n]$. To associate the fourier transform of $x[n]$ with the expression in (9) define

$$\begin{aligned} R + jX &= \sum_{n=0}^{N-1} e^{j2wn} \\ &= \sum_{n=0}^{N-1} \cos(2wn) + j\sin(2wn) \\ &= e^{jw(N-1)}\frac{\sin(wN)}{\sin(w)} \end{aligned} \tag{10}$$

where $w = 2\pi f/N$, f is the frequency bin and N is the DFT size. By using the the trigonometric identities, R and X can be expressed as $||\mathbf{c}||^2 - ||\mathbf{s}||^2$ and $2\mathbf{s}^T\mathbf{c}$ respectively. Also $||\mathbf{c}||^2 + ||\mathbf{s}||^2$ is equal to N. The expression in (9) can be written by using the new variables $||\mathbf{c}||^2 = (N+R)/2$, $||\mathbf{s}||^2 = (N-R)/2,\ \mathbf{s}^T\mathbf{c} = X/2$.

$$\begin{aligned} \hat{f}_{ML} &= \arg\max_f \left[\frac{(\mathbf{s}^T\mathbf{x})^2||\mathbf{c}||^2 + (\mathbf{c}^T\mathbf{x})^2||\mathbf{s}||^2 - 2(\mathbf{s}^T\mathbf{x})(\mathbf{c}^T\mathbf{x})(\mathbf{s}^T\mathbf{c})}{||\mathbf{s}||^2||\mathbf{c}||^2 - (\mathbf{s}^T\mathbf{c})^2}\right] \\ &= \arg\max_f \left[\frac{2(\mathbf{s}^T\mathbf{x})^2(N+R) + 2(\mathbf{c}^T\mathbf{x})^2(N-R) - 4X(\mathbf{s}^T\mathbf{x})(\mathbf{c}^T\mathbf{x})}{N^2 - R^2 - X^2}\right] \\ &= \arg\max_f \left[\frac{2N((\mathbf{s}^T\mathbf{x})^2 + (\mathbf{c}^T\mathbf{x})^2) + 2R((\mathbf{s}^T\mathbf{x})^2 - (\mathbf{c}^T\mathbf{x})^2) - 4X(\mathbf{s}^T\mathbf{x})(\mathbf{c}^T\mathbf{x})}{N^2 - (R^2 + X^2)}\right] \\ &= \arg\max_f \left[\frac{2N|R_k|^2 + 2\text{Real}\{R_k^2(R+jX)\}}{N^2 - (R^2+X^2)}\right] \end{aligned} \tag{11}$$

where R_k is the Fourier transform (DTFT) of x[n] and also R_k is equal to $(\mathbf{s}^T\mathbf{x}) + j(\mathbf{c}^T\mathbf{x})$. By using the result in (10) , the final expression becomes

$$\begin{aligned} \hat{f}_{ML} &= \arg\max_f \frac{N|R_k(e^{jw})|^2 + \frac{\sin(wN)}{\sin(w)}\text{Real}\{R_k^2(e^{jw})e^{jw(N-1)}\}}{N^2 - \frac{\sin^2(wN)}{\sin^2(w)}} \\ &= \arg\max_f \frac{|R_k(e^{jw})|^2 + \frac{\sin(wN)}{N\sin(w)}\text{Real}\{R_k^2(e^{jw})e^{jw(N-1)}\}}{N - \frac{\sin^2(wN)}{N\sin^2(w)}} \end{aligned}. \tag{12}$$

III. Cramer-Rao Lower Bound

The Cramer-Rao Bound (CRB) is, by definition, a lower bound for the mean square error estimation of non-random parameters for unbiased estimators, [2]. It is derived by the calculation of Fisher information matrix, followed by its inversion. These bounds are known are quadratic bounds, in general.

To get the asymptotic expression given in the manuscript, the limit of the Fisher information matrix (FIM) as $N \to \infty$ is calculated. This matrix is called the asymptotic FIM.

The elements of the information matrix can be expressed as

$$
\begin{aligned}
\mathrm{I}(\boldsymbol{\Theta})_{11} &= \frac{1}{\sigma^2}\sum_{n=0}^{N-1}\cos^2(2\pi f_0 n+\phi) \quad \stackrel{N\to\infty}{\longrightarrow} \frac{N}{2\sigma^2} \\
\mathrm{I}(\boldsymbol{\Theta})_{12} = \mathrm{I}(\boldsymbol{\Theta})_{21} &= \frac{-\pi A}{\sigma^2}\sum_{n=0}^{N-1} n\sin(2(2\pi f_0 n+\phi)) \quad \stackrel{N\to\infty}{\longrightarrow} 0 \\
\mathrm{I}(\boldsymbol{\Theta})_{13} = \mathrm{I}(\boldsymbol{\Theta})_{31} &= \frac{-A}{2\sigma^2}\sum_{n=0}^{N-1} \sin(2(2\pi f_0 n+\phi)) \quad \stackrel{N\to\infty}{\longrightarrow} 0 \\
\mathrm{I}(\boldsymbol{\Theta})_{22} &= \frac{(2\pi A)^2}{\sigma^2}\sum_{n=0}^{N-1} n^2\sin^2(2\pi f_0 n+\phi) \quad \stackrel{N\to\infty}{\longrightarrow} \frac{(2\pi A)^2}{2\sigma^2}\sum_{n=0}^{N-1} n^2 \\
\mathrm{I}(\boldsymbol{\Theta})_{23} = \mathrm{I}(\boldsymbol{\Theta})_{32} &= \frac{2\pi A^2}{\sigma^2}\sum_{n=0}^{N-1} n\sin^2(2\pi f_0 n+\phi) \quad \stackrel{N\to\infty}{\longrightarrow} \frac{2\pi A^2}{2\sigma^2}\sum_{n=0}^{N-1} n \\
\mathrm{I}(\boldsymbol{\Theta})_{33} &= \frac{A^2}{\sigma^2}\sum_{n=0}^{N-1} \sin^2(2\pi f_0 n+\phi) \quad \stackrel{N\to\infty}{\longrightarrow} \frac{NA^2}{2\sigma^2}
\end{aligned}
\tag{13}
$$

where $\boldsymbol{\Theta} = [A \ f_0 \ \phi]$. A is the amplitude, f_0 is the frequency and ϕ is the phase angle. As noted in [2, p.57], upon the inversion of the FIM matrix, the 2nd row and 2nd column entry becomes the CRB. This asymptotic case is called asymptotic Cramer Rao Bound (ACRB).

The Hybrid Cramer Rao Bound (HCRB) is calculated as follows: The expectation of the deterministic FIM matrix given by (13) is calcuated with respect to the nuisance parameter first and then the matrix is inverted, as in usual CRB calculations, [3]. In the present problem, the nuisance parameter is phase ϕ and it is assumed to be uniformly distributed in $[0, 2\pi)$. Since we have $E\{\sin(A+\phi)\} = 0$ and $E\{\sin^2(A+\phi)\} = \frac{1}{2}$, the expectation of each entry of FIM matrix given in (13) is identical to the entries of the asymptotic FIM. Hence, the CRB for the non-random parameter setting as $N \to \infty$ and HCRB with uniformly distributed ϕ is identically the same.

The ACRB (or HCRB) expression is readily given by [2, eq.(3.41)],

$$
\mathrm{var}(\hat{f}) \geq \frac{12}{(2\pi)^2 N(N^2-1)\mathrm{SNR}} \tag{14}
$$

To convert this bound to the unit of $2N$ point DFT bins, we need to multiply by (14) by $(2N)^2$, since $f = 1 \leftrightarrow \omega = 2\pi$ and $\omega = 2\pi$ corresponds to the $2N$'th DFT bin.

Figure 1 shows the CRB and some other bounds for $N = 16$ case. As can be seen from Figure 1, for short data records, the CRB is a function of the unknown parameter ϕ. On the other hand, ACRB or HCRB coincide to a value below the average value of the observed CRB oscillation. This is somewhat expected considering the convexity of $1/x$ function for $x > 0$.

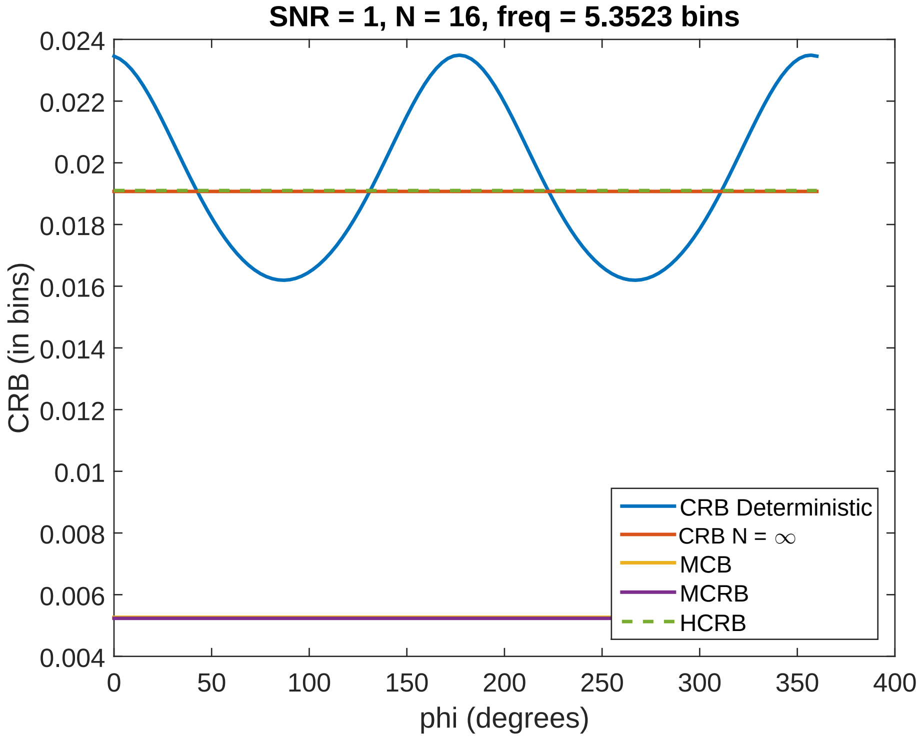


Fig. 1. Cramer-Rao Bound and other bounds

IV. Detailed Derivation of the Estimator

The first stage of the proposed estimator is 2N-point DFT of the input signal $r[n]$. In the suggested estimator, the frequency bin with the maximum amplitude denoted as $\widehat{k_p}$. In high SNR case, this notation gives the same result with k_p. That's why, in the following parts of the derivation k_p is used for the sake of simplicity.

$$
\begin{aligned}
R[k] =& A\sum_{n=0}^{N-1}\cos\left(\frac{2\pi}{2N}n(k_p+\delta)+\phi\right)e^{-j\frac{2\pi}{2N}nk}+\sum_{n=0}^{N-1}w[n]e^{-j\frac{2\pi}{2N}nk}\\
=&\frac{Ae^{j\phi}}{2}\sum_{n=0}^{N-1}e^{j\frac{2\pi}{2N}(k_p-k+\delta)n}+\frac{Ae^{-j\phi}}{2}\sum_{n=0}^{N-1}e^{-j\frac{2\pi}{2N}(k_p+k+\delta)n}+W[k]\\
R[k] =&\frac{Ae^{j\phi}}{2}\frac{1-e^{j\pi(l+\delta)}}{1-e^{j\pi(l+\delta)/N}}+\frac{Ae^{-j\phi}}{2}\frac{1-e^{-j\pi(2k_p-l+\delta)}}{1-e^{-j\pi(2k_p-l+\delta)/N}}+W[k]\\
R[k] =&\frac{Ae^{j\phi}}{2}e^{j\frac{\pi}{2N}(l+\delta)(N-1)}\frac{\sin(\frac{\pi}{2}(l+\delta))}{\sin(\frac{\pi}{2N}(l+\delta))}+\\
&\frac{Ae^{-j\phi}}{2}e^{-j\frac{\pi}{2N}(2k_p-l+\delta)(N-1)}\frac{\sin(\frac{\pi}{2}(2k_p-l+\delta))}{\sin(\frac{\pi}{2N}(2k_p-l+\delta))}+W[k]\\
R[k] =&\frac{Ae^{j\widetilde{\phi}}}{2}e^{-j\frac{\pi}{2N}k(N-1)}\frac{\sin(\frac{\pi}{2}(l+\delta))}{\sin(\frac{\pi}{2N}(l+\delta))}+\frac{Ae^{-j\widetilde{\phi}}}{2}e^{-j\frac{\pi}{2N}k(N-1)}\frac{\sin(\frac{\pi}{2}(2k_p-l+\delta))}{\sin(\frac{\pi}{2N}(2k_p-l+\delta))}+W[k]
\end{aligned}
\tag{15}
$$

where $l = k_p - k$ and $\widetilde{\phi} = \phi + \frac{\pi}{2N}(k_p + \delta)(N-1)$.

To simplify the expressions, we define

$$\begin{aligned}
\widehat{R}[k] =& R[k]e^{j\frac{\pi}{2N}k(N-1)} \\
\widehat{R}[k] =& \frac{A\cos(\widetilde{\phi})}{2}\Big\{\frac{\sin(\frac{\pi}{2}(k_p - k + \delta))}{\sin(\frac{\pi}{2N}(k_p - k + \delta))} + \frac{\sin(\frac{\pi}{2}(k_p + k + \delta))}{\sin(\frac{\pi}{2N}(k_p + k + \delta))}\Big\} \\
&+ j\frac{A\sin(\widetilde{\phi})}{2}\Big\{\frac{\sin(\frac{\pi}{2}(k_p - k + \delta))}{\sin(\frac{\pi}{2N}(k_p - k + \delta))} - \frac{\sin(\frac{\pi}{2}(k_p + k + \delta))}{\sin(\frac{\pi}{2N}(k_p + k + \delta))}\Big\} + W[k]e^{j\frac{\pi}{2N}k(N-1)} \\
\widehat{R}[k] =& \widehat{R}_{re}[k] + j\widehat{R}_{im}[k]
\end{aligned} \tag{16}$$

By using the three DFT outpus in frequency bins $\{k_p - 1, k_p, k_p + 1\}$, $Ratio_{re}$ and $Ratio_{im}$ are defined as follows:

$$\begin{aligned}
Ratio_{re} &= \frac{\widehat{R}_{re}[k_p + 1] - \widehat{R}_{re}[k_p - 1]}{2\widehat{R}_{re}[k_p] - \widehat{R}_{re}[k_p - 1] - \widehat{R}_{re}[k_p + 1]} \\
Ratio_{im} &= \frac{\widehat{R}_{im}[k_p + 1] - \widehat{R}_{im}[k_p - 1]}{2\widehat{R}_{im}[k_p] - \widehat{R}_{im}[k_p - 1] - \widehat{R}_{im}[k_p + 1]}
\end{aligned} \tag{17}$$

At this point, the purpose is by using the ratios in (17) to estimate the δ. Due to nonlinear relation between δ and results in (17), we apply Taylor series expansion on $\widehat{R}_{re}[k]$ and $\widehat{R}_{re}[k]$ about the point $\delta = \delta_{pre}$ under no noise assumption and take first two terms for the estimation.

$$\begin{aligned}
\widehat{R}_{re}[k] =& \quad \frac{A\cos(\widetilde{\phi})}{2}\Big\{\frac{\sin(\frac{\pi}{2}(k_p - k + \delta))}{\sin(\frac{\pi}{2N}(k_p - k + \delta))} + \frac{\sin(\frac{\pi}{2}(k_p + k + \delta))}{\sin(\frac{\pi}{2N}(k_p + k + \delta))}\Big\} \\
\approx& \quad M_{re}\Big\{f[-k] + (\delta - \delta_{pre})\dot{f}[-k] + f[k] + (\delta - \delta_{pre})\dot{f}[k]\Big\} \\
\widehat{R}_{im}[k] =& \quad \frac{A\sin(\widetilde{\phi})}{2}\Big\{\frac{\sin(\frac{\pi}{2}(k_p - k + \delta))}{\sin(\frac{\pi}{2N}(k_p - k + \delta))} - \frac{\sin(\frac{\pi}{2}(k_p + k + \delta))}{\sin(\frac{\pi}{2N}(k_p + k + \delta))}\Big\} \\
\approx& \quad M_{im}\Big\{f[-k] + (\delta - \delta_{pre})\dot{f}[-k] - f[k] - (\delta - \delta_{pre})\dot{f}[k]\Big\}
\end{aligned} \tag{18}$$

where $M_{re} = \frac{A\cos(\widetilde{\phi})}{2}$, $M_{im} = \frac{A\sin(\widetilde{\phi})}{2}$, $f[k] = \frac{\sin(\frac{\pi}{2}(k_p - k + \delta))}{\sin(\frac{\pi}{2N}(k_p - k + \delta))}$ and $\dot{f}[k]$ is the derivative of $f[k]$ with respect to δ. After approximation in (18) ,

$$\begin{aligned}
Ratio_{re} &\approx \frac{\overline{B} + (\delta - \delta_{pre})\widehat{B}}{\overline{C} + (\delta - \delta_{pre})\widehat{C}} \\
&= \frac{\overline{B} + (\delta - \delta_{pre})\widehat{B}}{\overline{C} + (\delta - \delta_{pre})\widehat{C}} \; \frac{\overline{C} - (\delta - \delta_{pre})\widehat{C}}{\overline{C} - (\delta - \delta_{pre})\widehat{C}} \\
&\approx \frac{\overline{BC} + (\widehat{B}\overline{C} - \overline{B}\widehat{C})(\delta - \delta_{pre})}{\overline{C}^2} \\
&\approx \frac{\overline{B}}{\overline{\overline{C}}} + \frac{\widehat{B}\overline{C} - \overline{B}\widehat{C}}{\overline{C}^2}(\delta - \delta_{pre})
\end{aligned} \tag{19}$$

$$
\begin{aligned}
Ratio_{im} &\approx \frac{\overline{D}+(\delta-\delta_{pre})\widehat{D}}{\overline{E}+(\delta-\delta_{pre})\widehat{E}} \\
&= \frac{\overline{D}+(\delta-\delta_{pre})\widehat{D}}{\overline{E}+(\delta-\delta_{pre})\widehat{E}}\ \frac{\overline{E}-(\delta-\delta_{pre})\widehat{E}}{\overline{E}-(\delta-\delta_{pre})\widehat{E}} \\
&\approx \frac{\overline{DE}+(\widehat{D}\overline{E}-\overline{D}\widehat{E})(\delta-\delta_{pre})}{\overline{E}^2} \\
&\approx \frac{\overline{D}}{\overline{E}}+\frac{\widehat{D}\overline{E}-\overline{B}\widehat{E}}{\overline{E}^2}(\delta-\delta_{pre})
\end{aligned}
\tag{20}
$$

where the new variables explicity given as

$\overline{B} = f[-k_p-1]+f[k_p+1]-f[-k_p+1]-f[k_p-1]$,

$\widehat{B} = \dot{f}[-k_p-1]+\dot{f}[k_p+1]-\dot{f}[-k_p+1]-\dot{f}[k_p-1]$,

$\overline{C} = 2f[-k_p]+2f[k_p]-f[-k_p+1]-f[k_p-1]-f[-k_p-1]-f[k_p+1]$

$\widehat{C} = 2\dot{f}[-k_p]+2\dot{f}[k_p]-\dot{f}[-k_p+1]-\dot{f}[k_p-1]-\dot{f}[-k_p-1]-\dot{f}[k_p+1]$

$\overline{D} = f[-k_p-1]-f[k_p+1]-f[-k_p+1]+f[k_p-1]$,

$\widehat{D} = \dot{f}[-k_p-1]-\dot{f}[k_p+1]-\dot{f}[-k_p+1]+\dot{f}[k_p-1]$,

$\overline{E} = 2f[-k_p]-2f[k_p]-f[-k_p+1]+f[k_p-1]-f[-k_p-1]+f[k_p+1]$

$\widehat{E} = 2\dot{f}[-k_p]-2\dot{f}[k_p]-\dot{f}[-k_p+1]+\dot{f}[k_p-1]-\dot{f}[-k_p-1]+\dot{f}[k_p+1]$

We should note that another important assumption is $(\delta-\delta_{pre})^2$ is close the zero and the terms contain the multiplier $(\delta-\delta_{pre})^2$ in (19) are neglected. Under high SNR, iterative estimation of δ makes this assumption meaningful. At the next step of the estimator, the exact ratios in (17) and the approximate versions in (19) are combined to estimate δ. Two independent estimates can be generated as shown in (21). The independence of the estimates and their fusion is examined later on.

$$
\begin{aligned}
\widehat{\delta}_{real} &= c_{N_{real}}(Ratio_{re}-\overline{B}/\overline{C})+\delta_{pre} \\
\widehat{\delta}_{imag} &= c_{N_{imag}}(Ratio_{im}-\overline{D}/\overline{E})+\delta_{pre} \\
\text{where}\quad c_{N_{real}} &= \frac{\overline{C}^2}{\widehat{B}\overline{C}-\overline{B}\widehat{C}} \quad \text{and} \quad c_{N_{imag}} = \frac{\overline{E}^2}{\widehat{D}\overline{E}-\overline{D}\widehat{E}}
\end{aligned}
\tag{21}
$$

V. MSE Expression for the $\widehat{\delta}_{re}$, $\widehat{\delta}_{im}$ and $\widehat{\delta}_{final}$

A. MSE derivation

In the first iteration , δ_{pre} is chosen as a constant (0.25 in [1]), which is the value used in the numerical results section of the manuscript. After some iterations δ_{pre} become very close to the real δ under high SNR. Then the $Ratio_{re}$ is approximated to the $\overline{B}/\overline{C}$ and $Ratio_{im}$ is approximated to the $\overline{D}/\overline{E}$. At this point, to calculate the theoretical MSE of the estimator, noise terms are added to the ratio.

$$\begin{aligned}
Ratio_{re(w/n)} &= \frac{\frac{A\cos(\widetilde{\phi})}{2}\overline{B} + w^{re}_{num}}{\frac{A\cos(\widetilde{\phi})}{2}\overline{C} + w^{re}_{denum}} = \frac{\frac{\overline{B}}{\overline{C}} + \frac{w^{re}_{num}}{\frac{A\cos(\widetilde{\phi})}{2}\overline{C}}}{1 + \frac{w^{re}_{denum}}{\frac{A\cos(\widetilde{\phi})}{2}\overline{C}}} \\
&= \frac{\frac{\overline{B}}{\overline{C}} + \widehat{w}^{re}_{num}}{1 + \widehat{w}^{re}_{denum}} \frac{1 - \widehat{w}^{re}_{denum}}{1 - \widehat{w}^{re}_{denum}} \\
&\approx \frac{\overline{B}}{\overline{C}} + \widehat{w}^{re}_{num} - \frac{\overline{B}}{\overline{C}}\widehat{w}^{re}_{denum} \\
Ratio_{im(w/n)} &\approx \frac{\overline{D}}{\overline{E}} + \widehat{w}^{im}_{num} - \frac{\overline{D}}{\overline{E}}\widehat{w}^{im}_{denum}
\end{aligned} \tag{22}$$

where $\widehat{w}^{re}_{num} = \frac{w^{re}_{num}}{\frac{A\cos(\widetilde{\phi})}{2}\overline{C}}$, $\widehat{w}^{re}_{denum} = \frac{w^{re}_{denum}}{\frac{A\cos(\widetilde{\phi})}{2}\overline{C}}$, $\widehat{w}^{im}_{num} = \frac{w^{im}_{num}}{\frac{A\cos(\widetilde{\phi})}{2}\overline{C}}$, $\widehat{w}^{im}_{denum} = \frac{w^{im}_{denum}}{\frac{A\cos(\widetilde{\phi})}{2}\overline{C}}$. From the result in (22), MSEs of $Ratio_{re}$ and $Ratio_{im}$ can be written as "$\text{var}(\widehat{w}^{re}_{num}) + \frac{\overline{B}^2}{\overline{C}^2}\text{var}(\widehat{w}^{re}_{denum}) - 2\frac{\overline{B}}{\overline{C}}E\{\widehat{w}^{re}_{num}\widehat{w}^{re}_{denum}\}$" and "$\text{var}(\widehat{w}^{im}_{num}) + \frac{\overline{D}^2}{\overline{E}^2}\text{var}(\widehat{w}^{im}_{denum}) - 2\frac{\overline{D}}{\overline{E}}E\{\widehat{w}^{im}_{num}\widehat{w}^{im}_{denum}\}$" respectively. So, need to calculate variance of $\widehat{w}^{re}_{num}$, $\widehat{w}^{re}_{denum}$, $\widehat{w}^{im}_{num}$ and $\widehat{w}^{im}_{denum}$and their correlation to finalize the MSE calculation. This should be repeated for both real and imaginary parts.

$$\begin{aligned}
\text{var}(w^{re}_{num}) &= \text{var}(n_1 - n_{-1}) \\
\text{var}(w^{re}_{denum}) &= \text{var}(2n_0 - n_1 - n_{-1}) \\
E\{w^{re}_{num}w^{re}_{denum}\} &= E\{(n_1 - n_{-1})(2n_0 - n_1 - n_{-1})\}
\end{aligned} \tag{23}$$

where $n_l = \text{Real}\left\{W[k]e^{j\frac{\pi}{2N}k(N-1)}\right\} = \text{Real}\left\{\alpha^{k_p+l}\sum_{n=0}^{N-1}\left(e^{-\frac{j2\pi(k_p+l)n}{2N}}w[n]\right)\right\}$, $l = k - k_p$, $\alpha = e^{\frac{j\pi(N-1)}{2N}}$ and $S[l] = \alpha^{k_p+l}\sum_{n=0}^{N-1}\left(e^{-j\frac{2\pi(k_p+l)n}{2N}}w[n]\right)$. Also we know that $n_l = (S[l] + S^*[l])/2$. Before calculating the related variances for the MSE calculation, the statistical properties of the $S[l]$ is investigated as follows.

$$\begin{aligned}
E\{S[l]S[m]\} &= E\left\{\alpha^{k_p+l}\sum_{n=0}^{N-1}\left(e^{-\frac{j2\pi(k_p+l)n}{2N}}w[n]\right)\alpha^{k_p+m}\sum_{q=0}^{N-1}\left(e^{-\frac{j2\pi(k_p+m)q}{2N}}w[q]\right)\right\} \\
&= \alpha^{2k_p+l+m}\sum_{n=0}^{N-1}\left(e^{-\frac{j2\pi(2k_p+l+m)n}{2N}}E\{w^2[n]\}\right) \\
&= \sigma^2_w\alpha^{2k_p+l+m}\frac{1 - e^{-j\pi(2k_p+l+m)}}{1 - e^{\frac{-j\pi(2k_p+l+m)}{N}}} \\
&= \sigma^2_w\alpha^{2k_p+l+m}e^{\frac{-j\pi(N-1)(2k_p+l+m)}{2N}}\frac{\sin(\frac{\pi(2k_p+l+m)}{2})}{\sin(\frac{\pi(2k_p+l+m)}{2N})} \\
&= \sigma^2_w\frac{\sin(\frac{\pi(2k_p+l+m)}{2})}{\sin(\frac{\pi(2k_p+l+m)}{2N})}
\end{aligned} \tag{24}$$

$$
\begin{aligned}
E\{S[l]S^*[m]\} &= E\left\{\alpha^{k_p+l}\sum_{n=0}^{N-1}\left(e^{-\frac{j2\pi(k_p+l)n}{2N}}w[n]\right)\alpha^{-k_p-m}\sum_{q=0}^{N-1}\left(e^{\frac{j2\pi(k_p+m)q}{2N}}w[q]\right)\right\} \\
&= \sigma_w^2\alpha^{l-m}\frac{1-e^{-j\pi(l-m)}}{1-e^{\frac{-j\pi(l-m)}{N}}} \\
&= \sigma_w^2\frac{\sin(\frac{\pi(l-m)}{2})}{\sin(\frac{\pi(l-m)}{2N})} \\
&= \begin{cases} 0 & \text{if } \ l-m \text{ is even} \\ \frac{\sigma_w^2}{\sin(\frac{\pi(l-m)}{2N})} & \text{if } \ l-m=4k+1\ , k \text{ is an integer} \\ \frac{-\sigma_w^2}{\sin(\frac{\pi(l-m)}{2N})} & \text{if } \ l-m=4k+3\ , k \text{ is an integer} \end{cases}
\end{aligned} \tag{25}
$$

By using the results in (24) , $\widehat{w}_{num}^{re}$, $\widehat{w}_{denum}^{re}$ and $E\{\widehat{w}_{num}^{im}\widehat{w}_{denum}^{im}\}$ can be easily calculated.

$$
\begin{aligned}
\text{var}(w_{num}^{re}) &= \text{var}(n_1-n_{-1}) \\
&= E\{n_1^2\}+E\{n_{-1}^2\}-2E\{n_{-1}n_1\} \\
&= E\left\{\frac{(S[1]+S^*[1])^2}{4}\right\}+E\left\{\frac{(S[-1]+S^*[-1])^2}{4}\right\} \\
&\qquad\qquad\qquad\qquad - E\left\{\frac{(S[-1]+S^*[-1])(S[1]+S^*[1])}{4}\right\} \\
&= \frac{N\sigma_w^2}{2}+\frac{N\sigma_w^2}{2}+0 \\
&= N\sigma_w^2 \\
\text{var}(w_{denum}^{re}) &= \text{var}(2n_0-n_{-1}-n_1) \\
&= 4E\{n_0^2\}+E\{(n_{-1}+n_1)^2\}-4E\{n_0(n_{-1}+n_1)\} \\
&= 2N\sigma_w^2+N\sigma_w^2-4E\left\{\left[\frac{(S[0]+S^*[0])}{2}\right]\left[\frac{(S[-1]+S^*[-1]+S[1]+S^*[1])}{2}\right]\right\} \\
&= \sigma_w^2\left[3N-\frac{4}{\sin(\frac{\pi}{2N})}-2(-1)^{k_p}\left(\frac{1}{\sin(\frac{\pi(2k_p+1)}{2N})}-\frac{1}{\sin(\frac{\pi(2k_p-1)}{2N})}\right)\right] \\
E\{w_{num}^{re}w_{denum}^{re}\} &= E\{(n_1-n_{-1})(2n_0-n_1-n_{-1})\} \\
&= \frac{1}{4}E\{(S[1]+S^*[1]-S[-1]-S^*[-1]) \\
&\qquad\qquad\qquad \times(2S[0]+2S^*[0]-S[1]-S^*[1]-S[-1]-S^*[-1])\} \\
&= \sigma_w^2(-1)^{k_p}\left(\frac{1}{\sin(\frac{\pi(2k_p+1)}{2N})}+\frac{1}{\sin(\frac{\pi(2k_p-1)}{2N})}\right)
\end{aligned} \tag{26}
$$

For the imaginary part of the $\widehat{R}[k]$, MSE calculation can be done in the same manner.

$$
\begin{aligned}
Ratio_{im(w/n)} &= \frac{\frac{A\sin(\widetilde{\phi})}{2}\overline{D} + w^{im}_{num}}{\frac{A\sin(\widetilde{\phi})}{2}\overline{E} + w^{im}_{denum}} = \frac{\frac{\overline{D}}{\overline{E}} + \frac{w^{im}_{num}}{\frac{A\sin(\widetilde{\phi})}{2}\overline{E}}}{1 + \frac{w^{im}_{denum}}{\frac{A\sin(\widetilde{\phi})}{2}\overline{E}}} \\
&= \frac{\frac{\overline{D}}{\overline{E}} + \widehat{w}^{im}_{num}}{1 + \widehat{w}^{im}_{denum}} \frac{1 - \widehat{w}^{im}_{denum}}{1 - \widehat{w}^{im}_{denum}} \approx \frac{\overline{D}}{\overline{E}} + \widehat{w}^{im}_{num} - \frac{\overline{D}}{\overline{E}}\widehat{w}^{im}_{denum}
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
\text{var}(w^{im}_{num}) &= \text{var}(n_1 - n_{-1}) \\
\text{var}(w^{im}_{denum}) &= \text{var}(2n_0 - n_1 - n_{-1}) \\
E\{w^{im}_{num} w^{im}_{denum}\} &= E\{(n_1 - n_{-1})(2n_0 - n_1 - n_{-1})\}
\end{aligned} \tag{28}
$$

where $n_l = \text{Imag}\left\{W[k]e^{j\frac{\pi}{2N}k(N-1)}\right\} = \text{Imag}\left\{\alpha^{k_p+l}\sum_{n=0}^{N-1}\left(e^{-\frac{j2\pi(k_p+l)n}{2N}}w[n]\right)\right\}$, $l = k - k_p$, $\alpha = e^{\frac{j\pi(N-1)}{2N}}$ and $S[l] = \alpha^{k_p+l}\sum_{n=0}^{N-1}\left(e^{-j\frac{2\pi(k_p+l)n}{2N}}w[n]\right)$. Also we know that $n_l = (S[l] - S^*[l])/2$. By using the results in (24) ,

$$
\begin{aligned}
\text{var}(w^{im}_{num}) &= \text{var}(n_1 - n_{-1}) \\
&= E\{n_1^2\} + E\{n_{-1}^2\} - 2E\{n_{-1}n_1\} \\
&= E\left\{\frac{(S[1] - S^*[1])^2}{(2j)^2}\right\} + E\left\{\frac{(S[-1] - S^*[-1])^2}{(2j)^2}\right\} \\
&\qquad - E\left\{\frac{(S[-1] - S^*[-1])(S[1] - S^*[1])}{(2j)^2}\right\} \\
&= \frac{N\sigma_w^2}{2} + \frac{N\sigma_w^2}{2} + 0 \\
&= N\sigma_w^2 \\
\text{var}(w^{im}_{denum}) &= \text{var}(2n_0 - n_{-1} - n_1) \\
&= 4E\{n_0^2\} + E\{(n_{-1} + n_1)^2\} - 4E\{n_0(n_{-1} + n_1)\} \\
&= 2N\sigma_w^2 + N\sigma_w^2 - 4E\left\{\left[\frac{(S[0] - S^*[0])}{2j}\right]\left[\frac{(S[-1] - S^*[-1] + S[1] - S^*[1])}{2j}\right]\right\} \\
&= \sigma_w^2\left[3N - \frac{4}{\sin(\frac{\pi}{2N})} + 2(-1)^{k_p}\left(\frac{1}{\sin(\frac{\pi(2k_p+1)}{2N})} - \frac{1}{\sin(\frac{\pi(2k_p-1)}{2N})}\right)\right] \\
E\{w^{im}_{num} w^{im}_{denum}\} &= E\{(n_1 - n_{-1})(2n_0 - n_1 - n_{-1})\} \\
&= \frac{1}{(2j)^2}E\{(S[-1] - S^*[-1] - S[1] + S^*[1]) \\
&\qquad \times (2S[0] - 2S^*[0] - S[-1] + S^*[-1] - S[1] + S^*[1])\} \\
&= -\sigma_w^2(-1)^{k_p}\left(\frac{1}{\sin(\frac{\pi(2k_p+1)}{2N})} + \frac{1}{\sin(\frac{\pi(2k_p-1)}{2N})}\right) \\
&= -E\{w^{re}_{num} w^{re}_{denum}\}
\end{aligned} \tag{29}
$$

By using results in (21) and (22), the mean squared errors of $\widehat{\delta}_{real}$ and $\widehat{\delta}_{imag}$ can be written as $c^2_{N_{real}}(\text{var}(\widehat{w}^{re}_{num})+\frac{\overline{B}^2}{\overline{C}^2}\text{var}(\widehat{w}^{re}_{denum})-2\frac{\overline{B}}{\overline{C}}E\{\widehat{w}^{re}_{num}\widehat{w}^{re}_{denum}\})$ and $c^2_{N_{imag}}(\text{var}(\widehat{w}^{im}_{num})+\frac{\overline{D}^2}{\overline{E}^2}\text{var}(\widehat{w}^{im}_{denum})-2\frac{\overline{D}}{\overline{E}}E\{\widehat{w}^{im}_{num}\widehat{w}^{im}_{denum}\})$.

$$\begin{aligned}
E[(\delta-\widehat{\delta}_{real})^2|\underline{\widetilde{\phi}}=\widetilde{\phi}] &= \frac{\overline{C}^4}{(\widehat{B}\overline{C}-\overline{B}\widehat{C})^2}\left[\sigma_w^2\frac{(N+\frac{\overline{B}^2\sigma^2_{\text{re}}}{\overline{C}^2}-2\frac{\overline{B}}{\overline{C}}\rho)}{\frac{A^2\cos^2(\widetilde{\phi})}{4}\overline{C}^2}\right] \\
&= \frac{2(N\overline{C}^2-2\overline{B}\ \overline{C}\rho+\sigma^2_{\text{re}}\overline{B}^2)}{(\widehat{B}\overline{C}-\overline{B}\widehat{C})^2\cos^2(\widetilde{\phi})\text{SNR}} \\
E[(\delta-\widehat{\delta}_{imag})^2|\underline{\widetilde{\phi}}=\widetilde{\phi}] &= \frac{\overline{E}^4}{(\widehat{D}\overline{E}-\overline{D}\widehat{E})^2}\left[\sigma_w^2\frac{(N+\frac{\overline{D}^2\sigma^2_{\text{im}}}{\overline{E}^2}+2\frac{\overline{D}}{\overline{E}}\rho)}{\frac{A^2\sin^2(\widetilde{\phi})}{4}\overline{E}^2}\right] \\
&= \frac{2(N\overline{E}^2+2\overline{D}\ \overline{E}\rho+\sigma^2_{\text{im}}\overline{D}^2)}{(\widehat{D}\overline{E}-\overline{D}\widehat{E})^2\sin^2(\widetilde{\phi})\text{SNR}}
\end{aligned} \tag{30}$$

where

$$\begin{aligned}
\sigma^2_{\text{re}} &= 3N-4\gamma_1-2(-1)^{k_p}(\gamma_{2k_p+1}-\gamma_{2k_p-1}) \\
\sigma^2_{\text{im}} &= 3N-4\gamma_1+2(-1)^{k_p}(\gamma_{2k_p+1}-\gamma_{2k_p-1}) \\
\rho &= (-1)^{k_p}(\gamma_{2k_p+1}+\gamma_{2k_p-1})
\end{aligned}$$

and $\gamma_k=\sin\left(\frac{\pi}{2N}k\right)$.

The final step of the suggested estimator is the fusion of the two different estimation results in (21). The reasoning for the choice of the fusion coefficient is explained the next part of this paper. Assume a fusion rule as follows $\widehat{\delta}_{final}=\cos^2(\widetilde{\phi})\widehat{\delta}_{real}+\sin^2(\widetilde{\phi})\widehat{\delta}_{imag}$, then MSE becomes;

$$\begin{aligned}
E[(\delta-\widehat{\delta}_{final})^2|\underline{\widetilde{\phi}}=\widetilde{\phi}] &= \frac{2[\cos^2(\widetilde{\phi})f_1+\sin^2(\widetilde{\phi})f_2]}{\text{SNR}} \\
E[(\delta-\widehat{\delta}_{final})^2] &= \frac{2[E\{\cos^2(\widetilde{\phi})\}f_1+E\{\sin^2(\widetilde{\phi})\}f_2]}{\text{SNR}} \\
E[(\delta-\widehat{\delta}_{final})^2)] &= \frac{f_{re}+f_{im}}{\text{SNR}}
\end{aligned} \tag{31}$$

where $f_{\text{re}}=\frac{N\overline{C}^2-2\overline{B}\ \overline{C}\rho+\sigma^2_{\text{re}}\overline{B}^2}{(\widehat{B}\overline{C}-\overline{B}\widehat{C})^2}$, $f_{\text{im}}=\frac{N\overline{E}^2+2\overline{D}\ \overline{E}\rho+\sigma^2_{\text{im}}\overline{D}^2}{(\widehat{D}\overline{E}-\overline{D}\widehat{E})^2}$.

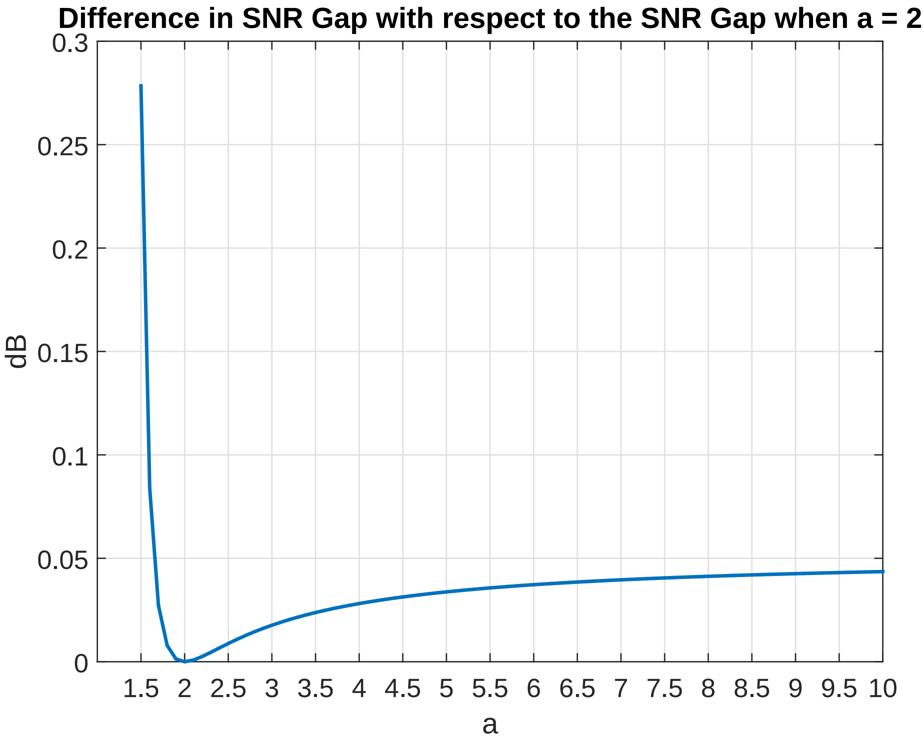
The phase parameter of the real sinusoid signal is a random variable with the distribution of $U[0,2\pi)$. The parameter $\widetilde{\phi}$ is the shifted version of the random phase which is also $U[0,2\pi)$, due to modulo 2π property of the phase, or its periodicity property. Hence, we have $E[\cos^2(\widetilde{\phi})]=E[\sin^2(\widetilde{\phi})]=1/2$.

B. On the selection of invariant function parameter a

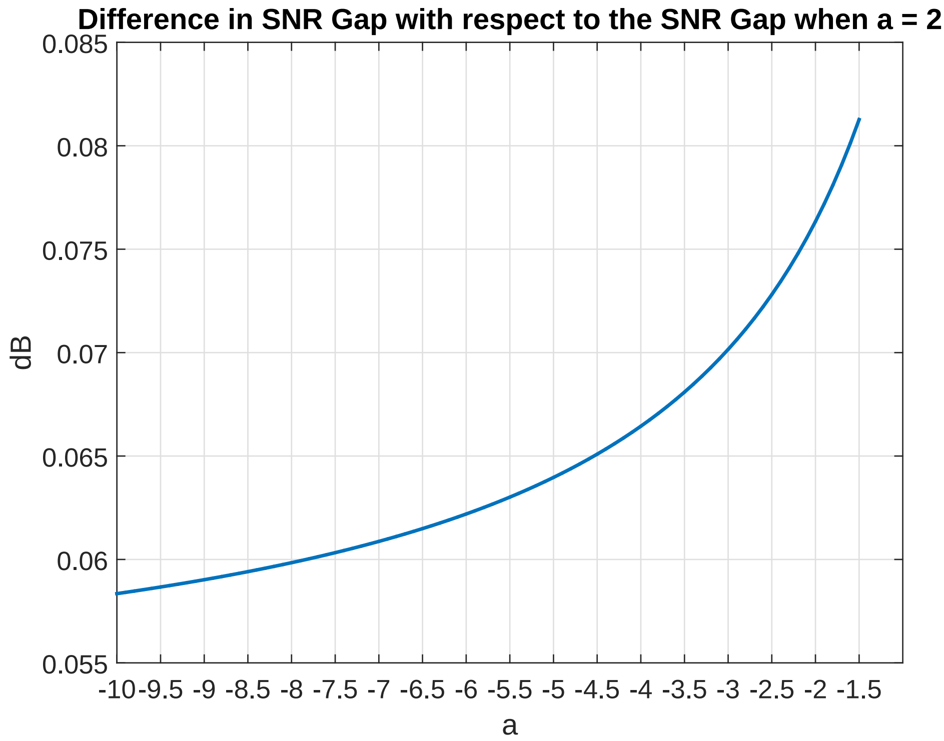
The derivation for the MSE, $E[(\delta-\widehat{\delta}_{final})^2)]$, is specific for the estimator given in (17). It is trivially easy to modify the existing MSE derivation to the following ratio with a generic a parameter,

$$Ratio = \frac{\widehat{R}_{re}[k_p+1]-\widehat{R}_{re}[k_p-1]}{a\widehat{R}_{re}[k_p]-\widehat{R}_{re}[k_p-1]-\widehat{R}_{re}[k_p+1]}. \tag{32}$$

Once this is done and the resulting MSE as a function of a is plotted and we have noticed that the choice of $a=2$ is the MSE minimizing choice for all a values. Figures 2(a) and 2(b) show the additional SNR required as a function of a to achieve the MSE value of the case $a=2$.



(a) for $a > 0$



(b) for $a < 0$

Fig. 2. Additional SNR required as a function of a to achieve the MSE value of the case $a = 2$

VI. Fusion of the Estimates

A. The Cross-Covariance between the estimates δ_{re} and δ_{im}

The mean values of the $\widehat{R}_{re}[k]$ and $\widehat{R}_{im}[k]$ have no effect on the statistical properties of random variables. So, the mean values can be ignored, or taken as zero, to simplify the derivation.

$$\begin{aligned} \widehat{R}_{re}[k] - \mu_{\widehat{R}_{re}} &= \text{Real}\left\{\alpha^{k_p+l}\sum_{n=0}^{N-1}\left(e^{-\frac{j2\pi(k_p+l)n}{2N}}w[n]\right)\right\} \\ \widehat{R}_{im}[k] - \mu_{\widehat{R}_{im}} &= \text{Imag}\left\{\alpha^{k_p+l}\sum_{n=0}^{N-1}\left(e^{-\frac{j2\pi(k_p+l)n}{2N}}w[n]\right)\right\} \end{aligned} \tag{33}$$

where $l = k - k_p$, $m = k' - k_p$, $\alpha = e^{\frac{j\pi(N-1)}{2N}}$, $S[l] = \alpha^{k_p+l}\sum_{n=0}^{N-1}\left(e^{-j\frac{2\pi(k_p+l)n}{2N}}w[n]\right)$ and μ is the mean value of the associated random variable.

$$\begin{aligned} C_{\widehat{R}_{re}[k]\widehat{R}_{im}[k']} &= E\left\{(\widehat{R}_{re}[k]-\mu_{\widehat{R}_{re}})(\widehat{R}_{im}[k']-\mu_{\widehat{R}_{im}})\right\} \\ &= E\left\{\frac{(S[l]+S^*[l])(S[m]-S^*[m])}{4j}\right\} \\ &= \sigma_w^2\frac{\sin(\frac{\pi(2k_p+l+m)}{2})}{\sin(\frac{\pi(2k_p+l+m)}{2N})} + \sigma_w^2\frac{\sin(\frac{\pi(m-l)}{2})}{\sin(\frac{\pi(m-l)}{2N})} - \sigma_w^2\frac{\sin(\frac{\pi(l-m)}{2})}{\sin(\frac{\pi(l-m)}{2N})} - \sigma_w^2\frac{\sin(\frac{\pi(2k_p+l+m)}{2})}{\sin(\frac{\pi(2k_p+l+m)}{2N})} \\ &= 0 \end{aligned} \tag{34}$$

The result in (34) says that the estimates $\widehat{\delta}_{re}$ and $\widehat{\delta}_{im}$ are independent random variables, since they are derived from $\widehat{R}_{re}[k]$ and $\widehat{R}_{im}[k]$ which are independent Gaussian random variables. Therefore the linear unbiased fusion rule $\widehat{\delta}_{final} = \alpha\widehat{\delta}_{re} + (1-\alpha)\widehat{\delta}_{im}$ is well matched for this problem.

B. Fusion Coefficient

In the manuscript, the suggested fusion coefficient is given as

$$\alpha \triangleq \frac{\widetilde{R}^2_{\text{re}}[0]}{\widetilde{R}^2_{\text{re}}[0]+\widetilde{R}^2_{\text{im}}[0]} = \left(1+\left(\frac{\widetilde{R}_{\text{im}}[0]}{\widetilde{R}_{\text{re}}[0]}\right)^2\right)^{-1}.$$

In the paragraph following equation (11) of the manuscript, the explanation provided for the justification of this choice is stated as:

$$\frac{\widetilde{R}_{\text{im}}[0]}{\widetilde{R}_{\text{re}}[0]} = \tan(\tilde{\phi})\underbrace{\left[\tan\left(\frac{\pi k_p}{2N}\right)\cot\left(\frac{\pi(k_p+\delta)}{2N}\right)\right]^p}_{\approx 1} \approx \tan(\tilde{\phi}),$$

where $p = (-1)^{\widehat{k}_p}$ is the parity of $\widehat{k}_p$, taking the value of 1 or -1 depending on $\widehat{k}_p$ being an even or odd integer. Here, we present the details for the derivation of the equality $\frac{\widetilde{R}_{\text{im}}[0]}{\widetilde{R}_{\text{re}}[0]} = \tan(\tilde{\phi})\tan\left(\frac{\pi k_p}{2N}\right)\cot\left(\frac{\pi(k_p+\delta)}{2N}\right)$.

By substituting the definitions for $\widetilde{R}_{\text{re}}$ and $\widetilde{R}_{\text{im}}$ from equation (4) of the manuscript, into $\frac{\widetilde{R}_{\text{im}}[0]}{\widetilde{R}_{\text{re}}[0]}$, we get

$$\frac{\widetilde{R}_{\text{im}}[0]}{\widetilde{R}_{\text{re}}[0]} = \tan(\tilde{\phi})\frac{\sin(\frac{B}{N})\sin(A) - \sin(\frac{A}{N})\sin(B)}{\sin(\frac{B}{N})\sin(A) + \sin(\frac{A}{N})\sin(B)} \tag{35}$$

with $A = \frac{\pi}{2}\delta$ and $B = \frac{\pi}{2}(2k_p + \delta)$. We note that $\sin(B) = \sin(\pi k_p + \frac{\pi}{2}\delta)) = (-1)^{k_p}\sin(A)$. Upon substitution of $\sin(B) = (-1)^{k_p}\sin(A)$, into (35), we get:

$$\frac{\widetilde{R}_{\text{im}}[0]}{\widetilde{R}_{\text{re}}[0]}\cot(\tilde{\phi}) = \frac{\sin(\frac{B}{N}) - (-1)^{k_p}\sin(\frac{A}{N})}{\sin(\frac{B}{N}) + (-1)^{k_p}\sin(\frac{A}{N})} \tag{36}$$

Assuming, for now, k_p is an even number, the equation (36) reduces to

$$\frac{\widetilde{R}_{\text{im}}[0]}{\widetilde{R}_{\text{re}}[0]}\cot(\tilde{\phi}) = \frac{\sin(\frac{B}{N}) - \sin(\frac{A}{N})}{\sin(\frac{B}{N}) + \sin(\frac{A}{N})} = \frac{\sin(\frac{B-A}{2N})\cos(\frac{B+A}{2N})}{\sin(\frac{B+A}{2N})\cos(\frac{B-A}{2N})} = \tan\left(\frac{B-A}{2N}\right)\cot\left(\frac{B+A}{2N}\right) \tag{37}$$

Inserting $\frac{B-A}{2N} = \frac{\pi k_p}{2N}$ and $\frac{B+A}{2N} = \frac{\pi(k_p+\delta)}{2N}$ into (37), results in

$$\frac{\widetilde{R}_{\text{im}}[0]}{\widetilde{R}_{\text{re}}[0]} = \tan(\tilde{\phi})\tan\left(\frac{\pi k_p}{2N}\right)\cot\left(\frac{\pi(k_p+\delta)}{2N}\right). \tag{38}$$

When k_p is an odd number, it can be seen that the numerator of the ratio on the right side of (36) is swapped with its the denominator. Hence, for this case, we have $\frac{\widetilde{R}_{\text{im}}[0]}{\widetilde{R}_{\text{re}}[0]} = \tan(\tilde{\phi})\left[\tan\left(\frac{\pi k_p}{2N}\right)\cot\left(\frac{\pi(k_p+\delta)}{2N}\right)\right]^{-1}$. Both cases can be summarized with the equation given in the paper as

$$\frac{\widetilde{R}_{\text{im}}[0]}{\widetilde{R}_{\text{re}}[0]} = \tan(\tilde{\phi})\underbrace{\left[\tan\left(\frac{\pi k_p}{2N}\right)\cot\left(\frac{\pi(k_p+\delta)}{2N}\right)\right]^{p}}_{\approx 1} \approx \tan(\tilde{\phi}).$$

Here $p = (-1)^{\widehat{k}_p}$ is the parity of $\widehat{k}_p$ (first stage output), taking the value of 1 or -1 depending on $\widehat{k}_p$ being an even or odd integer.

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