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EE 503
HW1:

Thanks,

22) AA^H and A^HA have same set of eigen values?

For any matrix B: $\boxed{\underline{\underline{B}}\underline{x_i} = \lambda_i \underline{x_i}}$ satisfies

Let $B = AA^H$, then $\underline{\underline{B}} x_i = \lambda_i \underline{x_i}$

$\underline{\underline{A}}\underline{\underline{A}}^H \underline{x_i} = \lambda_i x_i$ → eigen vector of AA^H which means that λ_i's are set of eigen values of AA^H

Now by multiplying both sides of eqn. $AA^H x_i = \lambda_i x_i$ with A^H from the left:

$\underline{\underline{A}}^H \cdot \underline{\underline{A}}\underline{\underline{A}}^H \underline{x_i} = \underline{\underline{A}}^H \lambda_i \underline{x_i}$

Let $\underline{\underline{A}}^H \underline{x_i} = \underline{x_J}$

$\underline{\underline{A}}^H \underline{\underline{A}} \cdot \underline{x_J} = \lambda_i x_J$ → eigen vector of A^HA which means that λ_i's are set of eigen values of A^HA.

⇒ Thus $\underline{\underline{A}}\underline{\underline{A}}^H$ and $\underline{\underline{A}}^H\underline{\underline{A}}$ have same set of eigen values.

27) 2.4 from Hayes:

a) $$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{b}$$

$$A^H \cdot A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$(A^H A)^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

The least squares solution $= x_0 = (A^H A)^{-1} A^H \cdot b$

$$= \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \underbrace{\frac{1}{3} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \end{bmatrix}}_{(A^H A)^{-1} A^H} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

b) Projection of vector $b = \hat{b} = A \cdot x_0$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

P_A = Projection matrix $= A \cdot \underbrace{(A^H A)^{-1} A^H}$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \cdot \frac{1}{3} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

c) $\hat{b} = P_A \cdot b = A \cdot x_0 = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ as found in b)

d) $$P_A^{\perp} = I - P_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} +1 & 1 & -1 \\ 1 & +1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

$$b^{\perp} = P_A^{\perp} b = \frac{1}{3} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}$$

$b^{\perp}$ and $\hat{b}$ are orthogonal if $(b^{\perp})^T \hat{b} = 0$: $b^{\perp} \cdot \hat{b} = [0]$ satisfies

$P_A^{\perp}$ represents complementary projector

28) 2.15 from Hayes:

a) $A - \lambda I = \begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{bmatrix}$

$$\det(A - \lambda I) = (1-\lambda)\Big((2-\lambda)\cdot(1-\lambda) - 1\Big) + 1(-1+\lambda)$$
$$= \lambda \cdot (1-\lambda)\cdot(\lambda-3)$$

$\Rightarrow$ eigen values of A:

$\lambda_1 = 1 \quad \lambda_2 = 0 \quad \lambda_3 = 3$

Eigen vectors:

for $\lambda_1 = 1$: $A\vartheta_i = \lambda_i \vartheta_i$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 1 \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow \left.\begin{aligned} a - b &= a \\ -a + 2b - c &= b \\ -b + c &= c \end{aligned}\right\}$$

$b = 0$

$a = -c$

Let ϑ_1 = The normalized eigen vector of A for $\lambda = 1$:

$$\vartheta_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

For $\lambda_3 = 3$:

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} p \\ r \\ s \end{bmatrix} = 3 \begin{bmatrix} p \\ r \\ s \end{bmatrix} \Rightarrow \begin{aligned} p - r &= 3p \\ -p + 2r - s &= 3r \\ -r + s &= 3s \Rightarrow \end{aligned}$$

$r = -2p = -2s$

ϑ_3 = The normalized eigen vector of A for $\lambda_3 = 2$:

$$\vartheta_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

For $\lambda_2 = 0 \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \left.\begin{aligned} x - y &= 0 \\ -x + 2y - z &= 0 \\ -y + z &= 0 \end{aligned}\right\} \Rightarrow x = y = z$

$$\vartheta_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

28) $\Rightarrow$ ctd.

b) $\det(A) = 1.(2-1) + 1.(-1) = 1-1=0$

d) Let A be a matrix with eigen values λ_i and let B be a matrix as follows $B = A + \alpha I$, Then B, A have same eigen vectors and eigen values of B are $\lambda_i + \alpha$

In this question: $B = A + I$, so $\alpha = 1$

Eigen values of $A+I$ are

$\lambda_1 = 1+1=2$

$\lambda_2 = 0+1=1$

$\lambda_3 = 3+1=4$

And Eigen vectors of $A+I$ are same with A.

c) $$\underline{\underline{M}} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix} \qquad \underline{\underline{\Lambda}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

A may be decomposed as:

$$\underline{\underline{A}} = \underline{\underline{M}}\,\underline{\underline{\Lambda}}\,\underline{\underline{M}}^H$$

$$\underline{\underline{A}} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

29) 2.17 from Hayes

Characteristic polynomial of matrices whose eigen values are 1 and 4 $= (\lambda-1)(\lambda-4)$

$$A \cdot v_i = \lambda_i \cdot v_i$$

for $\lambda_1 = 1$, $v_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is given $\Rightarrow$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} \Rightarrow \begin{array}{ll} 3a+b=3 & (1) \\ 3c+d=1 & (2) \end{array}$

for $\lambda_2 = 4$, $v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is given $\Rightarrow$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 4\begin{bmatrix} 2 \\ 1 \end{bmatrix} \Longrightarrow \begin{array}{ll} 2a+b=8 & (3) \\ 3c+d=4 & (4) \end{array}$$

By solving (1),(2),(3),(4) : $\begin{array}{l} 3a+b=3 \\ 2a+b=8 \\ \hline \end{array}$

$\Rightarrow a=-5$, $b=18$

$\begin{array}{l} 3c+d=1 \\ 2c+d=4 \\ \hline \end{array}$

$\Longrightarrow c=-3 \Rightarrow d=10$

$$\Rightarrow A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -5 & 18 \\ -3 & 10 \end{bmatrix}$$

30) If a matrix $\underline{\underline{A}}$ doesn't have a zero eigen value, then $\underline{\underline{A}}$ is invertible.

Let $A = \alpha_1^2 \underline{\underline{P}} + \alpha_2^2 (I - \underline{\underline{P}})$

$$\boxed{\underline{\underline{A}}\, \underline{x_i} = \lambda_i \cdot \underline{x_i}}$$

$$(\alpha_1^2 \underline{\underline{P}} + \alpha_2^2 I - \alpha_2^2 \underline{\underline{P}})\, \underline{x_i} = \lambda_i \cdot \underline{x_i}$$

$$\alpha_1^2 \underline{\underline{P}}\, \underline{x_i} + \alpha_2^2 I\, \underline{x_i} - \alpha_2^2 P \underline{x_i} = \lambda_i \cdot \underline{x_i}$$

$$(\alpha_1^2 - \alpha_2^2)\, \underline{\underline{P}}\, \underline{x_i} = (\lambda_i - \alpha_2^2)\, \underline{x_i}$$

$$\boxed{\underline{\underline{P}}\, \underline{x_i} = \frac{\lambda_i - \alpha_2^2}{\alpha_1^2 - \alpha_2^2} \cdot \underline{x_i}} \Rightarrow \text{where } \frac{\lambda_i - \alpha_2^2}{\alpha_1^2 - \alpha_2^2} \text{ are the eigen values of } \underline{\underline{P}}.$$

Since $\underline{\underline{P}}$ is a projector matrix, Eigen values of $\underline{\underline{P}}$ are 1 OR 0.

case1) : $\dfrac{\lambda_i - \alpha_2^2}{\alpha_1^2 - \alpha_2^2} = 1 \Rightarrow \lambda_i - \cancel{\alpha_2^2} = \alpha_1^2 - \cancel{\alpha_2^2}$

$\Rightarrow \lambda_i = \alpha_1^2 \neq 0$

case2) : $\dfrac{\lambda_i - \alpha_2^2}{\alpha_1^2 - \alpha_2^2} = 0 \Rightarrow \lambda_i = \alpha_2^2 \neq 0$

since α_1, α_2 are non-zero.

Since λ_i ($=$ eigen values of $\alpha_1^2 \underline{\underline{P}} + \alpha_2^2 (I - P)$) are not zero, then it's proven that $\alpha_1^2 \underline{\underline{P}} + \alpha_2^2 (I - P)$ matrix is invertible.