EE 503
Midterm #2

(Duration: 135 minutes)

1. (25 pts) Suppose we are given a zero mean process $x[n]$ with the autocorrelation

$$r_x[k] = 10\left(\frac{1}{2}\right)^{|k|} + 3\left(\frac{1}{2}\right)^{|k-1|} + 3\left(\frac{1}{2}\right)^{|k+1|}$$

a) Find a filter which, when driven by unit variance white noise, will yield a random process with this autocorrelation.

b) Find a stable and causal filter which, when excited by $x[n]$, will produce zero mean, unit variance white noise.

Hint: $Z\{\alpha^{|n|}\} = \dfrac{1-\alpha^2}{(1-\alpha z^{-1})(1-\alpha z)}$ for $\alpha < z < 1/\alpha$.

2. (15 pts) In the following scheme, $w[n]$ and $v[n]$ are independent WSS processes with zero-mean and auto-correlations $r_w[k] = \sigma_w^2\delta[k]$ and $r_v[k] = \sigma_v^2\delta[k]$, respectively.

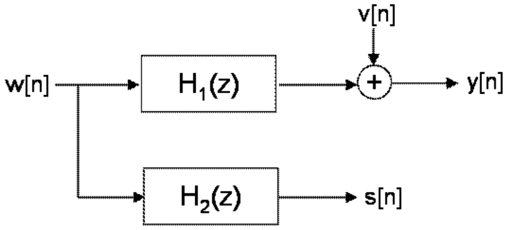


Figure 1: Filtering Scheme for Problem 2

Find the cross-correlation and the cross power spectral density of $s[n]$ and $y[n]$.

3. (15 pts.) The auto-correlation lags for $k = \{0, 1, 2\}$ of an AR(2) process is given as follows:

$$\mathbf{r}_x = [10,\ -5,\ 0]^T$$

a) Using Yule-Walker equations, find an LTI system, $H(z) = B(z)/A(z)$, generating the process.

b) Extend the given auto-correlation sequence to the lags $k = \{3, 4\}$.

4. (15 pts.) For a real valued WSS process $x(t)$, show that for $\mathbf{s} = \frac{1}{N}\sum_{k=1}^{N} x(kT)$, we have

$$\mathrm{E}\{\mathbf{s}^2\} = \frac{1}{2\pi N^2}\int_{-\infty}^{\infty} S_x(\omega)\frac{\sin^2(N\omega T/2)}{\sin^2(\omega T/2)}\,d\omega.$$

5. (20 pts.) In this problem, we examine the optimal smoothing operation. The process $x[n]$ is the superposition of the desired process $d[n]$ and noise process $v[n]$, as shown

$$x[n] = d[n] + v[n].$$

We assume that $d[n]$ and $v[n]$ are *real valued*, zero mean processes which are jointly WSS and uncorrelated from each other.

(a) Write the Wiener-Hopf equations for the coefficients w_k $k = \{-p, \ldots, p\}$ of the filter

$$\widehat{d}[n] = \sum_{k=-p}^{p} w_k x[n-k]$$

that minimizes the $E\{|d[n] - \widehat{d}[n]|^2\}$.

(b) Comment on the impulse response of the filter found in part (a). Is it an odd sequence, even sequence or neither even nor odd?

(c) Take $p = 3$ and $d[n]$ be an MA(1) process, i.e. a process synthesized by a system with one zero and no poles, and $v[n]$ be white noise. Comment on whether any one of filter coefficients, w_k for $k = \{-p, \ldots, p\}$, are equal to zero or not. Fully explain your reasoning. Briefly extend your answer for non-white noise.

(d) Take $p = 3$ and let $d[n]$ be an AR(1) process, i.e. a process synthesized by a system with one pole and no zeros and $v[n]$ be white noise. Comment on whether any one of filter coefficients, w_k $k = \{-p, \ldots, p\}$, are equal to zero or not. Fully explain your reasoning. Briefly extend your answer for non-white noise.

6. (15 pts) Consider the system given in Figure 2 for estimating the process $d[n]$ from $x[n]$.

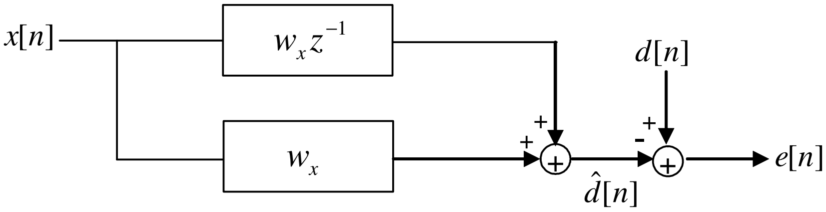


Figure 2: Filtering Scheme for Problem

If $\sigma_d^2 = 4$ and $r_x[0] = 1$, $r_x[1] = 0.5$, $r_x[2] = 0.25$ and $r_{dx}[0] = 1$ and $r_{dx}[1] = 0$; find w_x such that $E\{|e[n]|^2\}$ is minimized and find the minimum mean square error.

3.13 Suppose we are given a zero-mean process $x(n)$ with autocorrelation

$$r_x(k) = 10\left(\tfrac{1}{2}\right)^{|k|} + 3\left(\tfrac{1}{2}\right)^{|k-1|} + 3\left(\tfrac{1}{2}\right)^{|k+1|}$$

(a) Find a filter which, when driven by unit variance white noise, will yield a random process with this autocorrelation.

(b) Find a stable and causal filter which, when excited by $x(n)$, will produce zero mean, unit variance, white noise.

Solution

(a) The power spectrum of $x(n)$ is

$$P_x(z) = \frac{3/4}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z)}\left[10 + 3z^{-1} + 3z\right] = \frac{3}{4}\frac{(1+3z^{-1})(1+3z)}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z)}$$

Therefore, if

$$H(z) = \frac{\sqrt{3}}{2}\frac{1+3z^{-1}}{1-\frac{1}{2}z^{-1}} \quad ; \quad |z| > \tfrac{1}{2}$$

then the response of this filter to unit variance white noise will be a random process with the given autocorrelations.

(b) Consider the filter having a system function

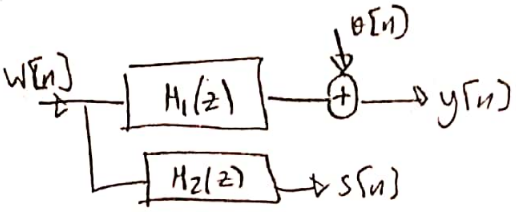
$$G(z) = \frac{2}{3\sqrt{3}}\frac{1-\frac{1}{2}z^{-1}}{1+\frac{1}{3}z^{-1}} \quad ; \quad |z| > \tfrac{1}{3}$$

Clearly this filter is stable and causal. Furthermore, if we filter $x(n)$ with $g(n)$ then the power spectrum of the filtered signal will be

$$P_y(z) = G(z)G(z^{-1})P_x(z) = 1$$

Therefore, $g(n)$ is the *whitening filter* that will produce unit variance white noise from $x(n)$.

Q2



$$y[n] = w[n] * h_1[n] + \theta[n] = \sum_{l_1=-\infty}^{\infty} h_1[l_1]\, w[n-l_1] + \theta[n]$$

$$s[n] = w[n] * h_2[n] = \sum_{l_2=-\infty}^{\infty} h_2[l_2]\, w[n-l_2]$$

$$r_{sy}[k] = E\{s[n]\, y^*[n-k]\} = E\Big\{\sum_{l_1}\sum_{l_2} h_1^*[l_1]\, h_2[l_2]\, w^*[n-k-l_1]\, w[n-l_2]\Big\}$$

$$= \sum_{l_1}\sum_{l_2} h_1^*[l_1]\, h_2[l_2]\, r_w[k+l_1-l_2]$$

$$= \sum_{l_1} h_1^*[l_1] \underbrace{\sum_{l_2} h_2[l_2]\, r_w[k+l_1-l_2]}_{r_w(k) * h_2[k] \triangleq g(k)}$$

$$= \sum_{l_1=-\infty}^{\infty} h_1^*[l_1]\, \overbrace{g(k+l_1)}$$

$$= \sum_{l_1'=-\infty}^{\infty} h_1^*[-l_1']\, g[k-l_1']$$

$$= g[k] * h_1^*[-k]$$

$$= r_w[k] * h_2[k] * h_1^*[-k]$$

and

$$S_{sy}(z) = S_w(z) \cdot H_2(z) \cdot H_1^*(1/z^*)$$

$$\left(\begin{aligned} r_w[k] &= \sigma_w^2\, \delta[k] \\ S_w(z) &= \sigma_w^2 \end{aligned} \right).$$

Note: When $H_1(z) = H_2(z)$, we get the known results.

causal recursion

unit-variance white noise

(Q3) **AR(2):**

a)

$$x[n] + a_1 x[n-1] + a_2 x[n-2] = b_0 w[n]$$

$$H(z) = \frac{b_0}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

$\downarrow \; x^*[n-k] \quad k \geq 0$

$\downarrow \; E\{\cdot\}$

Yule-Walker Eqn.

$$\rightarrow r_x[k] + a_1 r_x[k-1] + a_2 r_x[k-2] = |b_0|^2 \delta[k], \quad k \geq 0$$

$$\begin{array}{c} k=0 \rightarrow \\ k=1 \rightarrow \\ k=2 \rightarrow \end{array} \begin{bmatrix} \underset{(10)}{r_x[0]} & \underset{(-5)}{r_x[-1]} & \underset{(0)}{r_x[-2]} \\ \underset{(-5)}{r_x[1]} & \underset{(10)}{r_x[0]} & \underset{(-5)}{r_x[-1]} \\ \underset{(0)}{r_x[2]} & \underset{(-5)}{r_x[1]} & \underset{(10)}{r_x[0]} \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} |b_0|^2 \\ 0 \\ 0 \end{bmatrix}$$

From last 2eqn's:

$$\begin{bmatrix} 10 & -5 \\ -5 & 10 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} +5 \\ 0 \end{bmatrix} \rightarrow \begin{array}{l} a_1 = \frac{2}{3} \\ a_2 = 1/3 \end{array}$$

From 1^{st} eqn:

$$[10 \;\; -5 \;\; 0] \begin{bmatrix} 1 \\ 2/3 \\ 1/3 \end{bmatrix} = |b_0|^2 \rightarrow b_0 = \mp\sqrt{\frac{20}{3}}$$

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b) $k=3$:

$$r_x[3] + a_1 \underset{0}{r_x[2]} + \underset{1/3}{a_2} \underset{(-5)}{r_x[1]} = 0 \rightarrow r_x[3] = 5/3$$

$k=4$:

$$r_x[4] + \underset{2/3}{a_1} \underset{5/3}{r_x[3]} + a_2 \underset{0}{r_x[2]} = 0 \rightarrow r_x[4] = \frac{-10}{9}$$

④ $E\{s^2\} = \frac{1}{N^2} E\left\{ \sum_{k_1=1}^{N} \sum_{k_2=1}^{N} x(k_1T)\,x(k_2T) \right\}$

$$= \frac{1}{N^2}\sum_{k_1}\sum_{k_2} r_x\big((k_1-k_2)T\big)$$

$$\boxed{r_x(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(w)\, e^{+jw\tau}\, dw}$$

$$= \frac{1}{2\pi N^2}\sum_{k_1}\sum_{k_2}\int_{-\infty}^{\infty} S_x(w)\, e^{jw(k_1-k_2)T}\, dw$$

$$= \frac{1}{2\pi N^2}\int S_x(w)\left(\sum_{k_1=1}^{N} e^{jwk_1T}\right)\left(\sum_{k_2=1}^{N} e^{-jwk_2T}\right) dw.$$

$$\boxed{\sum_{k_1=1}^{N} r^{k_1} = r\,\frac{1-r^N}{1-r}}$$

$$= \frac{1}{2\pi N^2}\int S_x(w)\left(\underset{r=e^{jwT}}{e^{jwT}}\cdot\frac{1-e^{jwTN}}{1-e^{jwT}}\right)\left(\;-\;\right)^{*} dw$$

$$= \frac{1}{2\pi N^2}\int_{-\infty}^{\infty} S_x(w)\left|\frac{1-e^{jwTN}}{1-e^{jwT}}\right|^2 dw.$$

$$=$$

$$\frac{1-e^{jwTN}}{1-e^{jwT}} = \frac{e^{+jw\frac{TN}{2}}\left(e^{-jw\frac{TN}{2}} - e^{jw\frac{TN}{2}}\right)}{e^{jw\frac{T}{2}}\left(e^{-jw\frac{T}{2}} - e^{jw\frac{T}{2}}\right)}$$

$$= e^{jw\frac{T}{2}(N-1)}\,\frac{\sin(wNT/2)}{\sin(wT/2)}$$

$$= \frac{1}{2\pi N^2}\int_{-\infty}^{\infty} S_x(w)\left(\frac{\sin\left(w\frac{NT}{2}\right)}{\sin\left(\frac{wT}{2}\right)}\right)^2 dw$$

Q5 a) (x5)

$$\underline{x}[n] = \begin{bmatrix} x[n+p] \\ x[n+p-1] \\ \vdots \\ x[n] \\ \vdots \\ x[n-(p-1)] \\ x[n-p] \end{bmatrix}_{(2P+1)\times 1}$$

$$\hat{d}[n] = [w_p \; w_{p-1} \cdots w_0 \cdots w_{-p}]\underline{x}[n]$$

$$(x[n] = d[n] + v[n])$$

Then, $\underline{\underline{R}}_x \underline{w} = \underline{r}_{dx}$

$$\underline{\underline{R}}_x = \begin{bmatrix} r_x[0] & r_x[1] & \cdots & r_x[2P] \\ r_x[-1] & r_x[0] & r_x[1] & \\ r_x[-2] & \ddots & \ddots & \\ \vdots & & \ddots & r_x[1] \\ r_x[-2P] & \cdots & r_x[-1] & r_x[0] \end{bmatrix}$$

$E\{\underline{x}[n]\underline{x}^T[n]\}$

$$\underline{r}_{dx} = E\{d[n]\underline{x}[n]\} = \begin{bmatrix} r_d[-P] \\ r_d[-P+1] \\ \vdots \\ r_d[0] \\ \vdots \\ r_d[P-1] \\ r_d[P] \end{bmatrix}$$

(x3) b) $\underline{\underline{R}}_x$: Toeplitz symmetric matrix, $\underline{r}_{dx}$: an "even" vector w.r.t. to its center sample

$$\underline{w}^{up-down} = [w_{-p} \; w_{-p+1} \cdots w_0 \cdots w_{p-1} \; w_p]$$

last coln. first coln

$$\begin{bmatrix} r_x[0] & r_x[1] & \cdots & r_x[2P] \\ r_x[-1] & r_x[0] & \cdots & r_x[2P-1] \\ \vdots & & & \\ r_x[-2P] & r_x[-2P+1] & \cdots & r_x[0] \end{bmatrix} \underbrace{\begin{bmatrix} w_p \\ w_{p-1} \\ \vdots \\ w_0 \\ \vdots \\ w_{-p} \end{bmatrix}}_{\underline{w}} = \underline{r}_{dx} \rightarrow \begin{array}{l} \text{first row} \\ \\ \\ \text{last row} \end{array} \begin{bmatrix} r_x[2P] & r_x[2P-1] & \cdots & r_x[0] \\ r_x[2P-1] & r_x[2P-2] & & r_x[-1] \\ \vdots & \vdots & & \vdots \\ r_x[0] & r_x[-1] & \cdots & r_x[-2P] \end{bmatrix} \underbrace{\begin{bmatrix} w_{-p} \\ w_{-p+1} \\ \vdots \\ w_0 \\ \vdots \\ w_p \end{bmatrix}}_{\underline{w}^{up-down}} = \underline{r}_{dx}$$

$$\rightarrow \begin{array}{l} \text{first row} \\ \\ \\ \text{last row} \end{array} \begin{bmatrix} r_x[0] & r_x[-1] & \cdots & r_x[-2P] \\ & & & \\ & & & \\ r_x[2P] & \cdots & & r_x[0] \end{bmatrix} \underline{w}^{up-down} = \begin{bmatrix} r_d[P] \\ r_d[P-1] \\ \vdots \\ r_d[0] \\ \vdots \\ r_d[-P] \end{bmatrix} \rightarrow$$

Since $\underline{\underline{R}}_x$: toeplitz symmetric and $\underline{r}_{dx}$: even vector.

$\underline{w}^{up-down}$ also satisfies →

also satisfies Normal equations found in part (a).

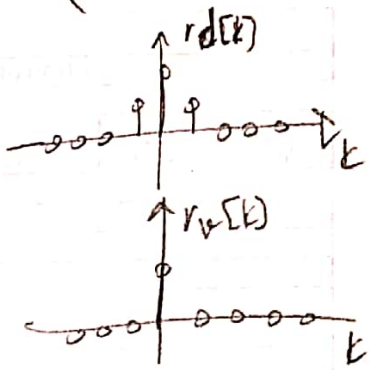
Hence, $\underline{w} = \underline{w}^{\text{up-down}}$, i.e. $\underline{w}$ is an even vector.

c) $d[n] : MA(1) \rightarrow d[n] = \alpha_0 w[n] + \alpha_1 w[n-1]$

+6

$$x[n] = d[n] + v[n] \quad \left(r_v[k] = \sigma_v^2 \delta[k] \text{ white noise} \right)$$

$$\begin{array}{r} \\ 1^{st}\text{eqn}\rightarrow \\ \\ 3^{rd}\text{eqn}\rightarrow \\ \\ \\ \\ \\ \end{array}
\begin{array}{ccccccc} w_3\downarrow & w_2\downarrow & w_1\downarrow & w_0\downarrow & & & \\ \sigma_x^2 & * & & & & 0 & \\ * & \sigma_x^2 & * & & & & \\ & * & \sigma_x^2 & * & & & \\ & & * & \sigma_x^2 & * & & \\ & 0 & & * & \sigma_x^2 & * & \\ & & & & * & \sigma_x^2 & * \\ & & & & & * & \sigma_x^2 \end{array}_{7\times7}
\begin{bmatrix} w_3 \\ w_2 \\ w_1 \\ w_0 \\ w_{-1} \\ w_{-2} \\ w_{-3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cancel{0} \\ \sigma_d^2 \\ \cancel{0} \\ 0 \\ 0 \end{bmatrix}_{7\times1}$$



Note that if $w_3 = 0 \rightarrow w_2 = 0 \rightarrow w_1 = 0 \rightarrow w_0 \neq 0$
(from 1st eqn) (from 2nd eqn)

but, that means $w_{-3} = w_{-2} = w_{-1} = 0$.

Hence, all zero $\underline{w}$ vector except for w_0 should be soln.

This is not possible. Hence, all coef. can be non-zero!!

Note that, $x[n]$ contains useful information $d[n]$ (Com bi $w[n], w[n-1]$)

$x[n+1]$ " " " $d[n]$

$x[n+2]$ contains useful information $x[n+1]$

$x[n+3]$ contain useful information $x[n+2]$

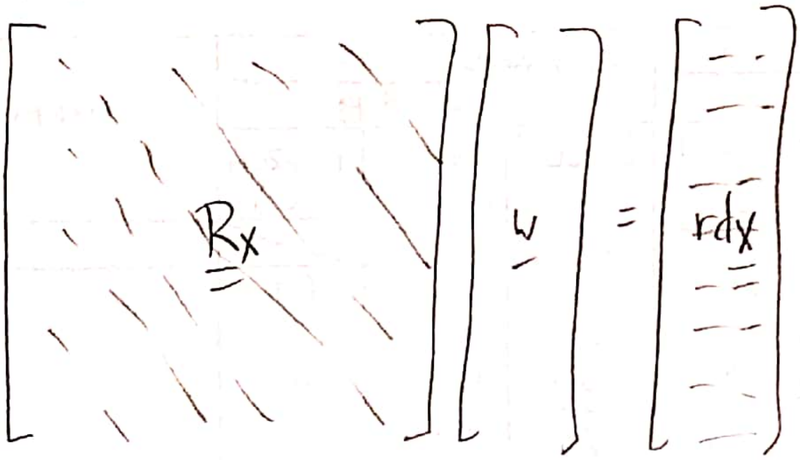
⋮

Hence, all observations contain useful information on $d[n]$ either directly or indirectly.

d) AR(1): $d[n] = \alpha d[n-1] + w[n]$ $\leftarrow \boxed{r_w[k] = \sigma_w^2 \delta[k]}$

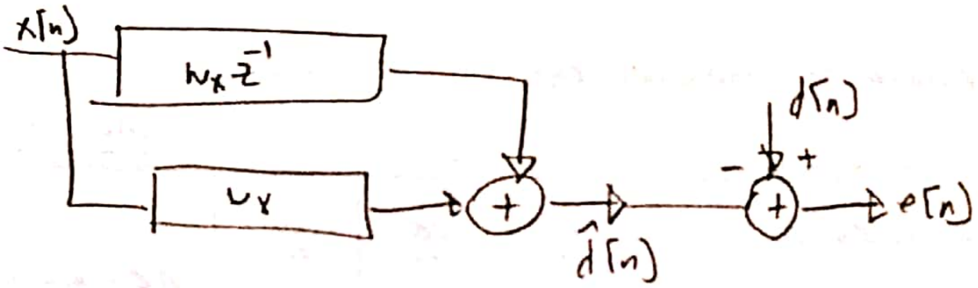
+6

$r_d[k] = r_d[0]\alpha^{|k|}$ | $r_x[k] = r_d[k] + \underbrace{r_v[k]}_{\uparrow \; \sigma_v^2\delta[k]}$



All entries of $\underline{\underline{R_x}}$ matrix and $r_{d\underline{x}}$ vector are non-zero. We also expect that all observations $x[n+k_0]$ $k_0 \in R$ contain useful information $d[n]$, since $d[n]$ and $d[n+k_0]$ are correlated. Hence, we do not expect to see any discarded information (i.e. having a zero weight in the estimator) for the optimal estimator.

Q6



$$e[n] = d[n] - \hat{d}[n] = d[n] - w_x(x[n] + \ast[n-1])$$

$$J(w_x) = E\{|e(n)|^2\} \rightarrow \frac{d}{dw_x} J(w_x) = -2E\{e[n](x[n]+x[n-1])\} = 0 \rightarrow$$

$$d[n) - w_x(x[n) + x[n-1]))$$

$$\rightarrow r_{dx}[0] + r_{dx}[1] = w_x(2r_x[0] + 2r_x[1])$$

$$w_x = \frac{r_{dx}[0) + r_{dx}[1]}{2(r_x[0] + r_x[1])} = \frac{1+0}{2(1+\frac{1}{2})} = \frac{1}{3}.$$

$$J_{min} = J\left(\frac{1}{3}\right) = E\left\{e[n]\left(d[n) - w_x(x[n) + x[n-1])\right)\Big|_{w_x=1/3}\right\}$$

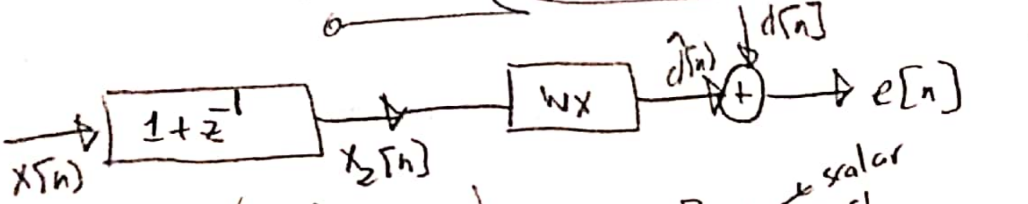
$$= E\{e[n]d[n]\}$$

$$= r_d[0] - \frac{1}{3}(r_{dx}[0] + r_{dx}[1])$$

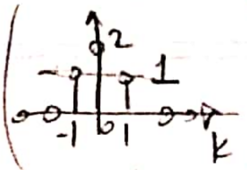
$$= 4 - \frac{1}{3}(1+0)$$

$$= 11/3.$$

Or



$$r_{dx_2}[0] = E\{d(n)(x(n) + x(n-1))\} = r_{dx}[0] + r_{dx}[1]$$

$r_{x_2}[k] = r_x[k] \ast ($  $)$ $\rightarrow$

$$= 2r_x[k] + r_x[k+1] + r_x[k-1]$$

$$\underline{\underline{R}}_{x_2} \underline{w}_x = \underline{r}_{dx_2} \quad \text{scalar}$$

$$\downarrow$$

$$r_{x_2}[0]\, w_x = r_{dx_2}(0) \rightarrow$$

$$w_x = \frac{r_{dx}[0] + r_{dx}[1]}{2r_x[0] + r_x[1] + r_x[-1]}$$