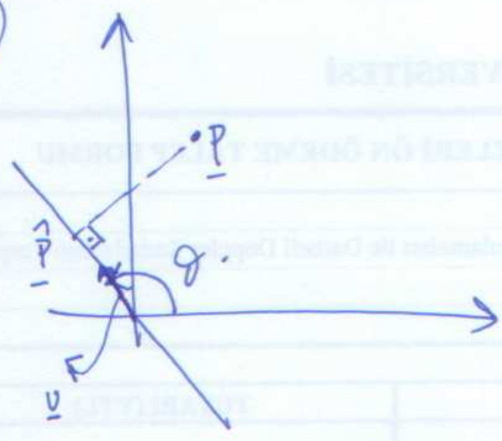
— EE 503, HW #1 Solutions —

Fall 2009-10

(1)



a) $\underline{p} = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$; $\hat{\underline{p}} = \alpha_{opt} \underbrace{\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}}_{\underline{u}}$

$\underline{e} = (\underline{p} - \hat{\underline{p}})$ error vector.

We know that $\underline{e} \perp \underline{u}$ at the closest point so $\longrightarrow$.

For optimal α, (α_{opt}) $\quad \underline{e}^T \cdot \underline{u} = 0$.

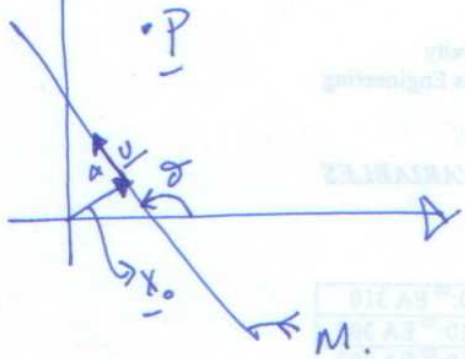
$$\underline{e}^T \cdot \underline{u} = 0 \longrightarrow \underline{p}^T \cdot \underline{u} - \alpha_{opt} \underbrace{\underline{u}^T \cdot \underline{u}}_{1} = 0$$

$$\boxed{\alpha_{opt} = \underline{p}^T \cdot \underline{u} = p_x \cos\theta + p_y \sin\theta}$$

b)
$$J(\alpha) = \|\underline{e}\|^2 = (\underline{p} - \hat{\underline{p}})^T (\underline{p} - \hat{\underline{p}})$$
$$= \underline{p}^T \cdot \underline{p} - 2 \hat{\underline{p}}^T \cdot \underline{p} + \hat{\underline{p}}^T \cdot \hat{\underline{p}}$$
$$= \underline{p}^T \cdot \underline{p} - 2\alpha \underline{u}^T \cdot \underline{p} + \alpha^2 \underline{u}^T \cdot \underline{u}$$
$$= \|\underline{p}\|^2 - 2\alpha (\underline{u}^T \cdot \underline{p}) + \alpha^2$$

$$\left. \frac{\partial J(\alpha)}{d\alpha} \right|_{\alpha = \alpha_{opt}} = 0 \longrightarrow -2(\underline{u}^T \cdot \underline{p}) + 2\alpha_{opt} = 0 \longrightarrow \boxed{\alpha_{opt} = \underline{u}^T \cdot \underline{p}}$$

(2)



$$\underline{\hat{P}} = \underline{x}_0 + \alpha \underline{u} \qquad \underline{u} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

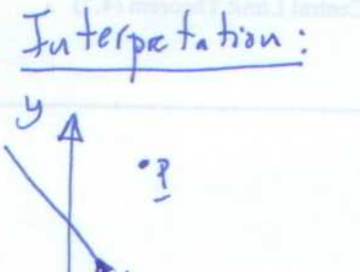
a) Points on line M do not form a sub-space of (x, y) plane. Since if m_1 and m_2 are two points in the space M; then $\underline{m}_1 + \underline{m}_2$ should also be in the space M which is not the case. (or $\underline{0}$ (zero vector) should be an element of the sub-space.) → (Think about it!)

b) $J^{(\alpha)} = \| \underline{P} - \underline{\hat{P}} \|^2 = \| \underline{P} - \underline{x}_0 - \alpha \underline{u} \|^2 = \| \underline{P}_2 - \alpha \underline{u} \|^2 \qquad \underline{P}_2 = \underline{P} - \underline{x}_0$

then $\alpha_{opt} = \underline{u}^T \cdot \underline{P}_2$ (from Problem 1 b).

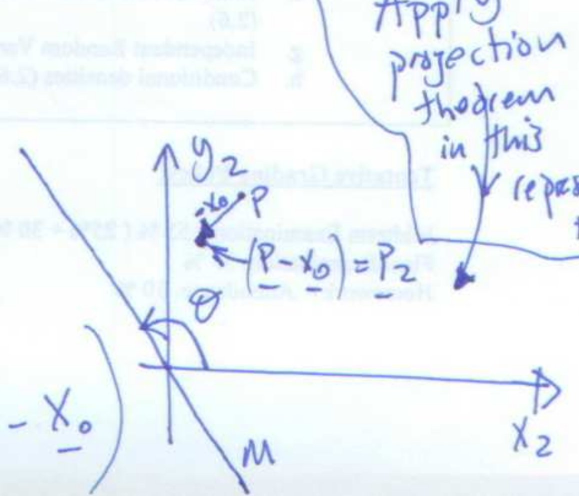
$$\alpha_{opt} = [\cos\theta \quad \sin\theta] \begin{bmatrix} P_x - x_{0_x} \\ P_y - x_{0_y} \end{bmatrix}$$

c) Interpretation:



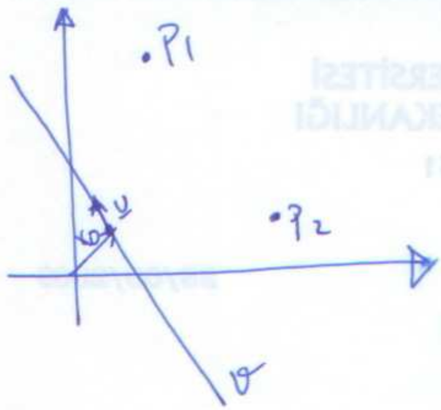
Move point $\underline{x}_0$ to origin.

$$\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} - \underline{x}_0 \right)$$



Apply projection theorem in this representation

(3)



$$\hat{\underline{p}} = \underline{x}_0 + \alpha \underline{u}$$

$$J(\alpha) = \|\hat{\underline{p}} - \underline{p}_1\|^2 + \|\hat{\underline{p}} - \underline{p}_2\|^2$$

$$= \|\alpha \underline{u} - \underbrace{(\underline{p}_1 - \underline{x}_0)}_{\underline{q}_1}\|^2 + \|\alpha \underline{u} - \underbrace{(\underline{p}_2 - \underline{x}_0)}_{\underline{q}_2}\|^2$$

$$J(\alpha) = \|\alpha \underline{u} - \underline{q}_1\|^2 + \|\alpha \underline{u} - \underline{q}_2\|^2$$

$$J(\alpha) = (\alpha \underline{u} - \underline{q}_1)^T(\alpha \underline{u} - \underline{q}_1) + (\alpha \underline{u} - \underline{q}_2)^T(\alpha \underline{u} - \underline{q}_2)$$

$$\frac{d}{d\alpha} J(\alpha) = \underline{u}^T \cdot (\alpha \underline{u} - \underline{q}_1) + (\alpha \underline{u} - \underline{q}_1)^T \cdot \underline{u} + \underline{u}^T \cdot (\alpha \underline{u} - \underline{q}_2) + (\alpha \underline{u} - \underline{q}_2)^T \cdot \underline{u}$$

$$= 2\underline{u}^T \cdot (\alpha \underline{u} - \underline{q}_1) + 2\underline{u}^T \cdot (\alpha \underline{u} - \underline{q}_2)$$

$$= 2\underline{u}^T \left(\alpha \underline{u} - (\underline{q}_1 + \underline{q}_2)\right)$$

$$\left.\frac{d}{d\alpha} J(\alpha)\right|_{\alpha = \alpha_{opt}} = 0 \longrightarrow \boxed{\alpha_{opt} = \underline{u}^T \cdot (\underline{q}_1 + \underline{q}_2)}$$

Comments: If we have more than two points, i.e. $p_1, p_2, p_3, \ldots, p_N$

than $\alpha_{opt} = \underline{u}^T \cdot \left(\sum_{k=1}^{N} \underline{q}_k\right)$