EE 503
Final Exam
(Duration: 150 minutes)

1. (15 pts) Let $\mathbf{x}$ and $\mathbf{y}$ be the samples of a zero-mean Gaussian random process with variances σ_x^2 and σ_y^2 and the correlation coefficient ρ_{xy}. Suppose we have the following transformation:

$$\begin{aligned}\mathbf{u} &= \mathbf{x} + k\mathbf{y} \\ \mathbf{v} &= \mathbf{x} - k\mathbf{y}\end{aligned}$$

a) Find a value of k such that $\mathbf{u}$ and $\mathbf{v}$ are statistically independent. What is the dependence of k on ρ_{xy}?

b) Write the joint pdf of $\mathbf{u}$ and $\mathbf{v}$ for the k value determined in part (a).

2. (20 pts) In the following filtering scheme, $w[n]$ and $v[n]$ are independent WSS processes with zero-mean and auto-correlations $r_w[k] = \sigma_w^2 \delta[k]$ and $r_v[k] = \sigma_v^2 \delta[k]$, respectively.

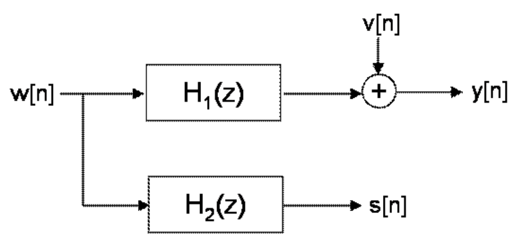


Figure 1: Filtering Scheme

a) Find the cross-correlation and the cross power spectral density of $s[n]$ and $y[n]$.

b) Find the non-causal IIR Wiener filter to estimate $s[n]$ given $y[n]$, $n \in (-\infty, \infty)$.

c) Show that the Wiener filter in part (b) can be decomposed as two filters, one of which is the non-causal IIR Wiener filter for the estimation of $w[n]$ and the other one is $h_2[n]$. Comment on this decomposition.

3. (35 pts) A random variable c is observed under noisy conditions. The signal model for the observations of x_1 and x_2 is as follows:

$$x_1 = c + v_1$$
$$x_2 = c + v_2$$

It is known that $E\{c\} = 1$, $E\{c^2\} = 2$ and the measurement noise samples, v_1 and v_2, are zero mean and uncorrelated with c. The correlation matrix for $[v_1 \ v_2]^T$ is given below:

$$\mathbf{R}_v = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}.$$

a) Find α_1, α_2 such that $\widehat{c} = \alpha_1 x_1 + \alpha_2 x_2$ has the minimum mean square estimation error. Find the minimum mean square error achieved by the estimator. Is the estimator biased?

b) Find the optimal β minimizing the mean square estimation error of $\widehat{c} = \beta x_1 + (1-\beta) x_2$. Is the estimator biased? Comment on the dependence of this estimator on the statistical charaterization of the desired signal, namely $E\{c\}$ and $E\{c^2\}$.

c) Find the optimal affine estimator in the form $\widehat{c} = \gamma_1 x_1 + \gamma_2 x_2 + \gamma_3$ that minimizes the estimation error. Find the minimum mean square error achieved by the estimator.

d) Compare your findings in parts (a), (b) and (c) in terms of estimator bias and MSE.

4. (15 points) Let **y** and **u** be independent random variables. Assume that **y** is normal with zero mean and unit variance; and **u** is binomial (takes two values which are $u = -1$ and $u = 1$ for this problem) with $P(u = 1) = P(u = -1) = \frac{1}{2}$. Random variable **z** is defined as:

$$\mathbf{z} = \mathbf{u}\mathbf{y}$$

a) Show that **z** is Gaussian with zero mean and unit variance.

b) Show that **y** and **z** are uncorrelated.

c) Show that **y** and **z** are not independent.

d) State whether the following is true or false: If two random variables are uncorrelated and normal distributed, but not jointly normal distributed; they are not necessarily independent. Justify your answer.

5. (20 pts) In this problem, we study the design of Wiener filters with the multi-band structure.

The desired process $d[n]$ is observed in the presence of white noise process $v[n]$. The observed process is

$$x[n] = d[n] + v[n]$$

where $d[n]$ and $v[n]$ are uncorrelated, zero-mean processes which are jointly WSS.

We would like to estimate $d[n]$ from the observations $x[n]$ with a filter having the frequency response of

$$H(e^{j\omega}) = \begin{cases} a, & 0 < \omega < 2\pi/3 \\ b, & 2\pi/3 < \omega < 4\pi/3 \\ c, & 4\pi/3 < \omega < 2\pi \end{cases},$$

as shown in Figure 2.

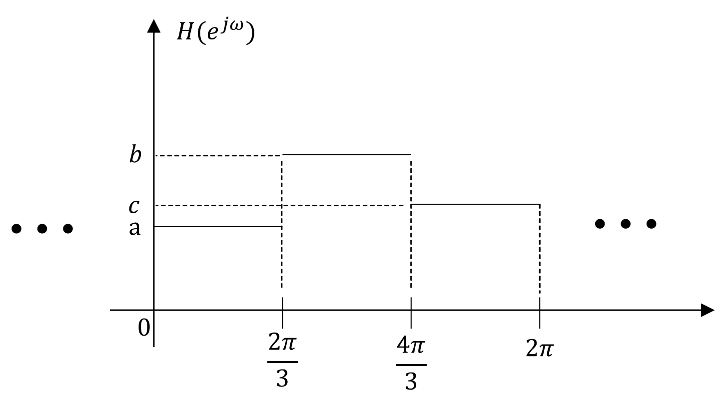


Figure 2: Frequency response of a 3-band filter

Here a, b and c are <u>real-valued</u> scalars representing the gain in the low, mid and high frequency bands, respectively.

a) Write the cost function of the problem and determine a, b and c values to minimize the mean-square estimation error.

b) If the filter designed in part (a) (having 3 degrees of freedom) is called a 3-band filter, extend your solution to a 5-band filter. Extend your solution to an infinite band filter. Explain your reasoning in these extensions clearly and make connections with other structured Wiener filters (IIR, FIR filters).

EE 503

Final

1. $u = x + ky$, $v = x - ky$, $Var(x) = \sigma_x^2$, $Var(y) = \sigma_y^2$, $Cov(x,y) = \rho_{xy}\sigma_x\sigma_y$, $E\{x\} = E\{y\} = 0$

a) $E\{uv\} = 0$ (for the independence of jointly Gaussian $\underset{\sim}{u}, \underset{\sim}{v}$)

1

$$E\{uv\} = E\{(x+ky)(x-ky)\}$$
$$= E\{x^2\} - k^2 E\{y^2\}$$
$$= \sigma_x^2 - k^2\sigma_y^2$$

x7

Then $k = \mp\sqrt{\frac{\sigma_x^2}{\sigma_y^2}} = \mp\frac{\sigma_x}{\sigma_y}$ for independence of u, v Gaussian $\underset{\sim}{u}, \underset{\sim}{v}$

b) $f_{U,V}(u,v) = f_U(u) \cdot f_V(v) = \frac{1}{\sqrt{2\pi\sigma_u^2}} e^{-\frac{u^2}{2\sigma_u^2}} \cdot \frac{1}{\sqrt{2\pi\sigma_v^2}} e^{-\frac{v^2}{2\sigma_v^2}}$

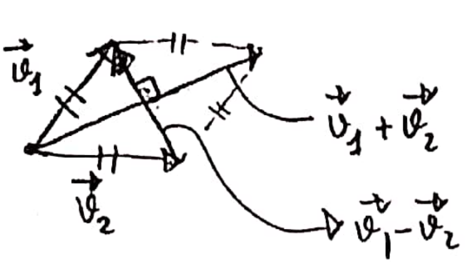
x8

$$\sigma_u^2 = E\{(x+ky)^2\} = \sigma_x^2 + 2k\rho_{xy}\sigma_x\sigma_y + k^2\sigma_y^2 = 2\sigma_x^2(1 \mp \rho_{xy})$$

Th. $\sigma_v^2 = E\{(x-ky)^2\} = \sigma_x^2 - 2k\rho_{xy}\sigma_x\sigma_y + k^2\sigma_y^2 = 2\sigma_x^2(1 \pm \rho_{xy})$

In this problem, it is interesting to see that the decorrelation operation does not depend on ρ_{xy}; but only depends on σ_x^2 and σ_y^2. This result can be interpretted in terms of Euclidean vectors as follows:

Remember that an equilateral parallelogram (eşkenar dörtgen) has orthogonal diagonals.



$\rightarrow$ $\vec{v}_1 + \vec{v}_2 \perp \vec{v}_1 - \vec{v}_2$

if

$\|\vec{v}_1\| = \|\vec{v}_2\|$.

Note that $\|\vec{x}\|^2$ corresponds to $E\{x^2\} = Var(x)$ in the space of r.v.'s. By multiplying $\underset{\sim}{y}$ r.v. with $k = \mp \frac{\sigma_x}{\sigma_y}$, we get $E\{(ky)^2\} = E\{x^2\}$ and then adding and subtracting two r.v.'s with equal norm ($\underset{\sim}{x}$ and $k\underset{\sim}{y}$) results is an orthogonal r.v.'s $\underset{\sim}{u}, \underset{\sim}{v}$.

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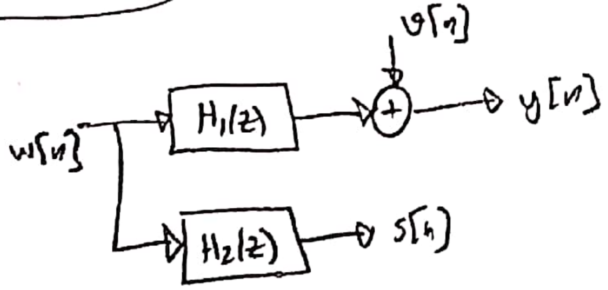
(2)



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a) $s[n] = \sum_{l=-\infty}^{\infty} h_2[l]\, w[n-l]$

$y[n] = \sum_{l'=-\infty}^{\infty} h_1[l]\, w[n-l] + v[n]$

$$r_{sy}[k] = E\{ s[n]\, y^*[n-k] \} \qquad y[n]\big|_{(n-k)\leftarrow n}$$

$$= E\Big\{\Big(\sum_{\ell} h_2[\ell]\, w[n-\ell]\Big)\Big(\sum_{\ell'} h_1[\ell']\, w[n-k-\ell'] + v[n]\Big)^*\Big\}$$

$$= \sum_{\ell}\sum_{\ell'} h_2[\ell]\, h_1^*[\ell']\, \underbrace{r_W[k+\ell'-\ell]}_{\sigma_W^2\,\delta[k+\ell'-\ell]} + \cancel{\sum h_2^*[\ell]\, E\{w[n-\ell]\, v^*[n]\}}^{\,0}$$

$$= \sigma_W^2 \sum_{\ell'=-\infty}^{\infty} h_1^*[\ell']\, h_2[k+\ell'] \qquad (3)$$

$$= \sigma_W^2 \sum_{\ell''=-\infty}^{\infty} h_1^*[-\ell'']\, h_2[k-\ell'']$$

$$= \sigma_W^2\, h_1^*(-k) * h_2(k). \qquad (*)$$

$$S_{sy}(e^{j\omega}) = \mathcal{F}\{r_{sy}[k]\} = \sigma_W^2\, H_2(e^{j\omega})\, H_1^*(e^{j\omega}).$$

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b) $$H^{IIR\text{-}NC}(e^{j\omega}) = \frac{S_{sy}(e^{j\omega})}{S_y(e^{j\omega})} = \frac{\sigma_W^2\, H_2(e^{j\omega})\, H_1^*(e^{j\omega})}{\sigma_W^2\,|H_1(e^{j\omega})|^2 + \sigma_V^2}.$$

[redacted] (+5)

$$\boxed{S_y(e^{j\omega}) = \sigma_W^2\,|H_1(e^{j\omega})|^2 + \sigma_V^2}$$

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c) $$H^{IIR\text{-}NC}(e^{j\omega}) = \underbrace{\frac{\sigma_V^2\, H_1^*(e^{j\omega})}{\sigma_W^2\,|H_1(e^{j\omega})|^2 + \sigma_V^2}}\cdot H_2(e^{j\omega})$$

[redacted] (+5)

$$r_{wy}[k] = \sigma_W^2\, h_1^*[-k]$$

[The easiest way to see this is set $h_2[n] = \delta[n]$ in (*) or check your lecture notes]

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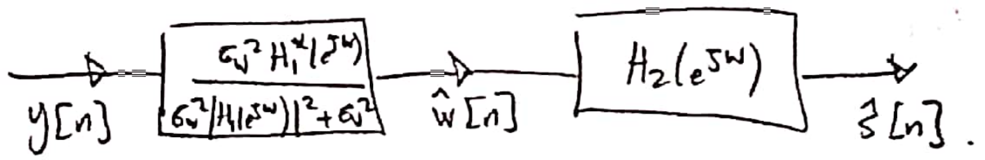
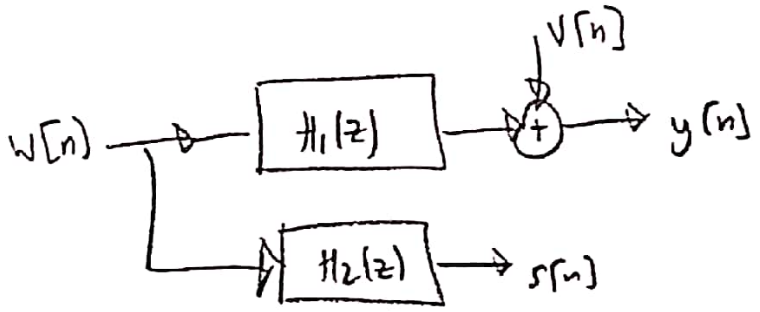
Then

$$H^{IIR\text{-}NC}(e^{j\omega}) = \underbrace{\frac{\overbrace{\sigma_w^2 H_1^*(e^{j\omega})}^{S_{wy}(e^{j\omega})}}{\underbrace{\sigma_v^2 |H_1(e^{j\omega})|^2 + \sigma_v^2}_{S_y(e^{j\omega})}}}_{\frac{S_{wy}(e^{j\omega})}{S_y(e^{j\omega})}} \cdot H_2(e^{j\omega})$$

and $\dfrac{S_{wy}(e^{j\omega})}{S_y(e^{j\omega})}$ is the IIR-NC Wiener filter for the estimation of $w[n]$. Hence, the optimal filter for $s[n]$ estimate, be represented as



This is not surprising, since from the block diagram



it is obvious that $s[n]$ is a linear processing result of $w[n]$. Hence, the optimal LMMSE estimator of $w[n]$ can be used to form the optimal LMMSE estimates of $s[n] = \sum_{l=-\infty}^{\infty} h_2[n-l]\, w[l]$.

Problem 3:

$\underset{\sim}{x_1} = \underset{\sim}{c} + \underset{\sim}{v_1}$, $E\{c\} = 1$, $E\{c^2\} = 2$, $E\{v_k\} = 0$, $R_v = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

$\underset{\sim}{x_2} = \underset{\sim}{c} + \underset{\sim}{v_2}$

$\underset{\sim}{v_k}$ and $\underset{\sim}{c}$ are uncorrelated $k = \{1, 2\}$

a) $\hat{c} = [\alpha_1 \ \alpha_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow R_x \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = r_{cx}$, $E\left\{c \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right\} = \underbrace{E\{c^2\}}_{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

+10

$$E\left\{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} [x_1 \ x_2]\right\} = \begin{bmatrix} E\{x_1^2\} & E\{x_1 x_2\} \\ E\{x_2 x_1\} & E\{x_2^2\} \end{bmatrix} = \ldots$$

5

$$= \begin{bmatrix} \underbrace{E\{c^2\}}_{2} + \underbrace{E\{v_1^2\}}_{1} \ (3) & E\{c^2\} \ (2) \\ E\{c^2\} \ (2) & \underbrace{E\{c^2\}}_{2} + \underbrace{E\{v_2^2\}}_{4} \ (6) \end{bmatrix}$$

$$R_x \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = r_{cx}$$

$$\rightarrow \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} \alpha_1^{opt} \\ \alpha_2^{opt} \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/7 \\ 1/7 \end{bmatrix}$$

$$\text{min. MSE} = E\Big\{\underbrace{\Big(c - [\alpha_1^{opt} \ \alpha_2^{opt}] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\Big)^2}_{e^2}\Big\} = E\{ec\} = E\{c^2\} - [\alpha_1^{opt} \ \alpha_2^{opt}] r_{cx}$$

$$= 2 - \left[\tfrac{4}{7} \ \ \tfrac{1}{7}\right] \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\boxed{\hat{c} = \frac{4}{7} x_1 + \frac{1}{7} x_2}$$

min MSE $\rightarrow = 4/7$

Bias: $E\{\hat{c}\} \stackrel{?}{=} E\{c\}$

$$E\{\hat{c}\} = E\left\{[\alpha_1^{opt} \ \alpha_2^{opt}] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right\} = \left[\tfrac{4}{7} \ \ \tfrac{1}{7}\right] \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{5}{7}$$

$$E\{c\} = 1.$$

Since $E\{\hat{c}\} \neq E\{c\}$, the estimator is biased.

6

b) $J(\beta) = E\{\underbrace{[c-(\beta x_1 + (1-\beta)x_2)]}_{e}^2\} = E\{e^2\}$

+10

$\frac{d}{d\beta}J(\beta) = 2E\{e\frac{d}{d\beta}e\} = -2E\{e(x_1 - x_2)\} = 0 \rightarrow$

$e = c - \beta x_1 - (1-\beta)x_2$

$\rightarrow E\{(c - \beta x_1 - (1-\beta)x_2)\underbrace{(x_1 - x_2)}_{v_1 - v_2}\} = -\beta \underbrace{E\{x_1 v_1\}}_{E\{v_1^2\}=1} + (1-\beta)\underbrace{E\{x v_2\}}_{E\{v_2^2\}=4} = 0$

$c \perp v_1$
$c \perp v_2$

Equation does not depend on signal characteristics $E\{c\}$ or $E\{c^2\}$

$\rightarrow -\beta + (1-\beta)4 = 0 \rightarrow \beta = 4/5$

Then, $\hat{c} = \frac{4}{5}x_1 + \frac{1}{5}x_2$

minMSE:

$E\{(c-\hat{c})^2\} = E\{[c - \underbrace{(\frac{4}{5}x_1 + \frac{1}{5}x_2)}_{c + \frac{4}{5}v_1 + \frac{1}{5}v_2}]^2\} = E\{(\frac{4}{5}v_1 + \frac{1}{5}v_2)^2\}$

$= \frac{16.1 + 1.4}{25} = \frac{20}{25} = 4/5$

Bias:

$E\{\hat{c}\} = \frac{4}{5}\overbrace{E\{x_1\}}^{1} + \frac{1}{5}\overbrace{E\{x_2\}}^{1} = 1 = E\{c\} \rightarrow$ Estimator is unbiased!

c) $\hat{c} = r_1 x_1 + r_2 x_2 + r_3 \cdot 1 = [r_1 \ \ r_2 \ \ r_3]\begin{bmatrix}x_1\\x_2\\1\end{bmatrix}$

[redacted]

+10

$$\underline{R}_x \begin{bmatrix}r_1\\r_2\\r_3\end{bmatrix} = r_{c\underline{x}} \leftarrow E\left\{c\begin{bmatrix}x_1\\x_2\\1\end{bmatrix}\right\} = E\left\{\begin{bmatrix}c^2\\c^2\\c\end{bmatrix}\right\} = \begin{bmatrix}2\\2\\1\end{bmatrix}$$

$$\begin{array}{c} -3 \\ -2 \end{array}\begin{bmatrix}3 & 2 & 1\\2 & 6 & 1\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}r_1\\r_2\\r_3\end{bmatrix} = \begin{bmatrix}2\\2\\1\end{bmatrix}$$

$E\left\{\begin{bmatrix}x_1\\x_2\\1\end{bmatrix}[x_1 \ \ x_2 \ \ 1]\right\}$, Applying row operations to simplify the operations

$$\times 4 \begin{bmatrix}0 & -1 & -2\\0 & 4 & -1\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}r_1\\r_2\\r_3\end{bmatrix} = \begin{bmatrix}-1\\0\\1\end{bmatrix}$$

$$\begin{bmatrix}0 & -1 & -2\\0 & 0 & -9\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}r_1\\r_2\\r_3\end{bmatrix} = \begin{bmatrix}-1\\-4\\1\end{bmatrix} \rightarrow \text{the operations}$$

$\rightarrow r_3 = 4/9$, $r_2 = 1/9$ (1st row), $r_1 = 4/9$ (3rd row) $\rightarrow \begin{bmatrix}r_1\\r_2\\r_3\end{bmatrix} = \frac{1}{9}\begin{bmatrix}4\\1\\4\end{bmatrix}$

$$\hat{c} = \frac{4}{9}x_1 + \frac{1}{9}x_2 + \frac{4}{9}$$

(8)

min MSE

$$E\{\underbrace{(c-\hat{c})}_{e}{}^2\} = E\{ec\} = E\{c^2\} - \frac{1}{9}[4\ 1\ 4]\begin{bmatrix}2\\2\\1\end{bmatrix}$$

$$e = c - [\gamma_1^{opt}\ \gamma_2^{opt}\ \gamma_3^{opt}]\begin{bmatrix}x_1\\x_2\\1\end{bmatrix}$$

$$= 2 - 14/9 = \underline{4/9} \leftarrow \text{min MSE.}$$

Bias:

$$E\{\hat{c}\} = E\{\tfrac{4}{9}x_1 + \tfrac{1}{9}x_2 + \tfrac{4}{9}\} = 1 \rightarrow \text{Estimator is unbiased.}$$

(8)

d) (+5)

	Deg. of freedom	Bias/Unbiased	min MSE	Type of Estimator
a) $\hat{c} = \frac{4}{7}x_1 + \frac{1}{7}x_2$	2	Biased	$\frac{4}{7} \approx 0.57$	LMMSE (no bias correction)
b) $\hat{c} = \frac{4}{5}x_1 + \frac{1}{5}x_2$	1	Unbiased	$\frac{4}{5} = 0.8$	BLUE
c) $\hat{c} = \frac{4}{9}x_1 + \frac{1}{9}x_2 + \frac{4}{9}$	3	Unbiased	$\frac{4}{9} = 0.\overline{4}$	LMMSE (with bias correction)

Note that BLUE estimator in part b has only one degree of freedom left; since BLUE is a linear estimator, hence it is a combiner of two samples in this problem and due to unbiasness condition, one degree of freedom is utilized (weights should sum to 1 for unbiased estimator in this problem) and therefore, the remaining degree of freedom for BLUE estimator is just 1.

(4) $y \sim N(0,1)$
$$U = \begin{cases} 1, & \text{w.p. } 1/2 \\ -1, & \text{w.p. } 1/2 \end{cases} \qquad z = uy$$

a) $F_Z(z) = P\{Z \leq z\} = P\{UY \leq z\}$

$= P\{UY \leq z \mid U = 1\} P(U=1)$ + $P\{UY \leq z \mid U = -1\} P(U=1)$ $\frac{1}{2}$

+6

$= P\{Y \leq z \mid \cancel{U=1}\}\frac{1}{2} + P\{\underbrace{-Y \leq z}_{Y \geq -z} \mid \cancel{U=-1}\}\frac{1}{2}$

$= F_Y(z)\frac{1}{2} + (1 - F_Y(-z))\frac{1}{2}$

$$f_Z(z) = \frac{d}{dz}F_Z(z) = f_Y(z)\frac{1}{2} + \frac{1}{2}\underbrace{f_Y(-z)}_{f_Y(z) = f_Y(-z)} = f_Y(z)$$

since $y \sim N(0,1)$

Hence, $Z \sim N(0,1)$.

+3 b) $Cov(Y,Z) = E\{YZ\} = E\{Y^2 U\} = E\{Y^2\}E\{\overset{0}{\cancel{U}}\} = 0.$

+3 c) Assume z is independent of y. Then $f_{Z|Y}(z|y)$ should be identical $f_Z(z)$, i.e. $f_{Z|Y}(z|y) = f_Z(z)$.

(10)

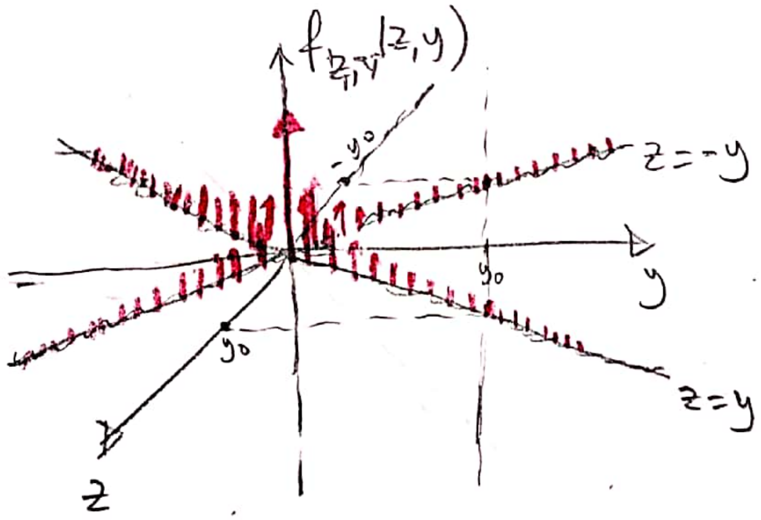
But $f_{\mathfrak{z}}(z|y) = \begin{cases} +y & \text{with prob. } 1/2 \\ -y & \text{with prob } 1/2 \end{cases} = \frac{1}{2}\left[\delta(z-y) + \delta(z+y)\right]$

$\mathfrak{z} = \upsilon y$

Hence $\underline{f_{\mathfrak{z}}(z|y) \text{ is not } f_{\mathfrak{z}}(z)}$ which is $N(0,1)$

d) In this problem $\begin{matrix} \mathfrak{z} \sim N(0,1) \\ y \sim N(0,1) \end{matrix}$, but z and y are not <u>jointly</u> Gaussian distributed. The joint distribution is $f_{\mathfrak{z},y}(z,y) = \underbrace{f_y(y)}_{N(0,1)} \cdot \underbrace{f_{\mathfrak{z}}(z|y)}_{\begin{cases} y & \text{w.p. } 1/2 \\ -y & \text{w.p. } 1/2 \end{cases}}$

(+3)



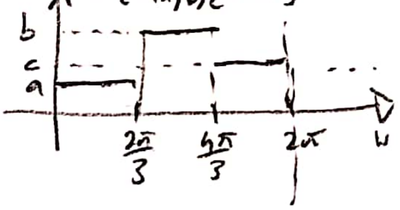
Q5. This is a structured estimator (filter) design problem. Different from previous structures, filter is specified much easily in freq. domain.

a) $x[n] = \underset{\sim}{d}[n] + \underset{\sim}{v}[n]$ $\quad$ $\left.\begin{array}{l} d[n]: \text{desired r.p.} \\ v[n]: \text{noise} \end{array}\right\}$ uncorrelated and zero mean.

$$\hat{d}[n] = \sum_{\ell=-\infty}^{\infty} h_{a,b,c}[\ell]\, x[n-\ell]$$

IIR filter with 3-degrees of freedom

$\mathcal{F}\{h_{a,b,c}[n]\} = H(e^{jw})$



$$x[n] \rightarrow \boxed{h_{abc}[n]} \rightarrow \hat{d}[n]$$

$$\underset{a,b,c}{MSE} = E\{(d(n) - \hat{d}(n))^2\} = E\{d^2(n) - 2\hat{d}(n)d(n) + \hat{d}^2(n)\}$$

$$= r_d(0) - 2r_{\hat{d}d}(0) + r_{\hat{d}}(0)$$

$$\boxed{r_{\hat{d}}[k] = r_x[k] * h_{abc}[k] * h^*_{abc}[-k]}$$

$$r_{\hat{d}d}(k) = E\Big\{\underbrace{\sum_\ell h_{abc}[\ell]\, x[n-\ell]}_{\hat{d}[n]}\, d^*[n-k]\Big\}$$

$$= h_{abc}[k] * \underbrace{r_{xd}[k]}_{r_d[k]}$$

+15

$$= \frac{1}{2\pi}\int_0^{2\pi} \Big[S_d(e^{jw}) - 2\underbrace{S_{\hat{d}d}(e^{jw})}_{\boxed{H_{abc}(e^{jw})S_d(e^{jw})}} + \underbrace{S_{\hat{d}}(e^{jw})}_{\boxed{|H_{abc}(e^{jw})|^2 S_x(e^{jw})}}\Big]dw$$

$$MSE(a,b,c) = \frac{1}{2\pi}\cdot\left[\begin{array}{l} \displaystyle\int_0^{2\pi/3}\{S_d(e^{jw}) - 2a\,S_d(e^{jw}) + a^2 S_x(e^{jw})\}\,dw \\ \quad + \\ \displaystyle\int_{2\pi/3}^{4\pi/3}\{S_d(e^{jw}) - 2b\,S_d(e^{jw}) + b^2 S_x(e^{jw})\}\,dw \\ \quad + \\ \displaystyle\int_{4\pi/3}^{2\pi}\{S_d(e^{jw}) - 2c\,S_d(e^{jw}) + c^2 S_x(e^{jw})\}\,dw \end{array}\right]$$

$$\frac{\partial MSE(a,b,c)}{\partial c} = 0 \rightarrow \quad -\cancel{2}\int_0^{\frac{2\pi}{3}} S_d(e^{jw})\,dw + \cancel{2}a\int_0^{2\pi/3} S_x(e^{jw})\,dw = 0$$

$$\Downarrow$$

$$a = \frac{\int_0^{2\pi/3} S_d(e^{jw})\,dw}{\int_0^{2\pi/3} \underbrace{S_x(e^{jw})}_{S_d(e^{jw}) + S_v(e^{jw})}\,dw}$$

We can interpret the gain "a" as

$$a = \frac{\frac{1}{2\pi/3}\int_0^{2\pi/3} S_d(e^{jw})\,dw}{\frac{1}{2\pi/3}\int_0^{2\pi/3} S_d(e^{jw})\,dw + \frac{1}{\frac{2\pi}{3}}\int_0^{2\pi/3} S_v(e^{jw})\,dw}$$

$\overline{S_{d|a}}$: Avg. signal power in Band-a

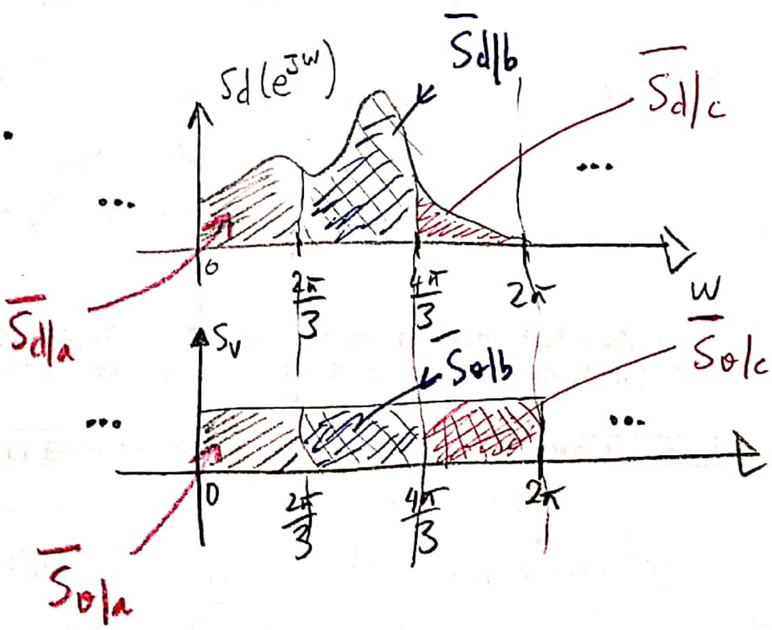
$\overline{S_{v|a}}$: Avg. noise power in Band-a

$$a = \frac{\overline{S_{d|a}}}{\overline{S_{d|a}} + \overline{S_{v|a}}} = \frac{1}{1 + \frac{1}{SNR_a}}$$

$$\boxed{SNR_a \triangleq \frac{\overline{S_{d|a}}}{\overline{S_{v|a}}}}$$

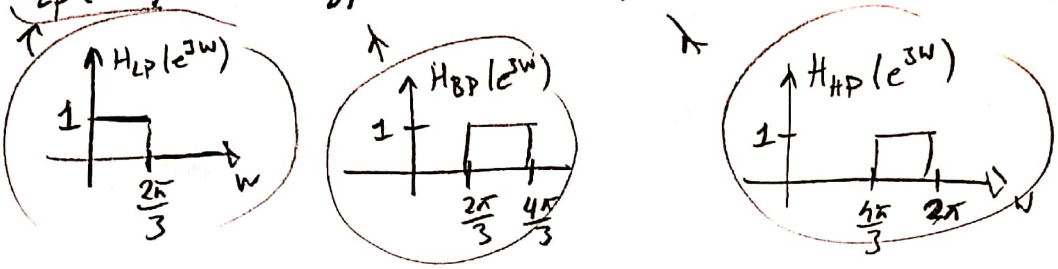
Hence,

$$b = \frac{1}{1+\frac{1}{SNR_b}}, \quad c = \frac{1}{1+\frac{1}{SNR_c}}.$$

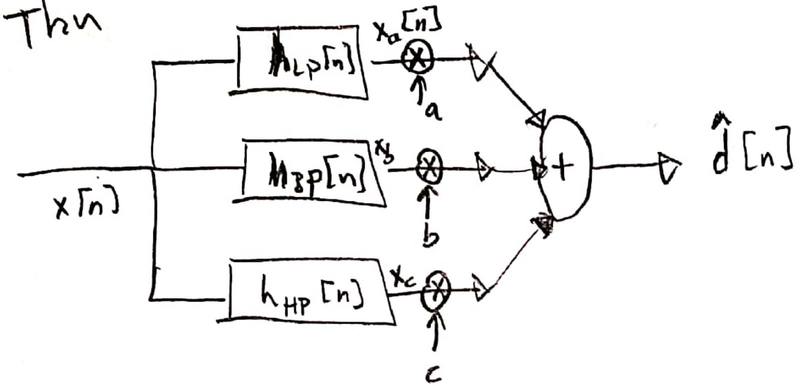


Another way of looking at the same problem is

$$H_{a,b,c}(e^{jw}) = aH_{LP}(e^{jw}) + b\,H_{BP}(e^{jw}) + c\,H_{HP}(e^{jw})$$



Then



Filtering structure to be optimized

$$\text{Then, } MSE(a,b,c) = E\Big\{\big(d(n) - \overbrace{(a\,x_a[n] + b\,x_b[n] + c\,x_c[n])}^{\hat{d}[n]}\big)^2\Big\}$$

$$\frac{\partial MSE(a,b,c)}{\partial a} = 0 \rightarrow E\{e[n]\,x_a[n]\} = 0$$

$$\Downarrow$$

$$r_{dx_a}[0] - a\,r_{x_a}[0] - b\,\cancel{r_{x_a x_b}[0]}^{\,0} - c\,\cancel{r_{x_a x_c}[0]}^{\,0} = 0$$

$$\boxed{\begin{array}{l} r_{x_a x_b}[k] = 0 \quad \forall k \\ \text{Since} \\ S_{x_a x_b}(e^{jw}) = S_x(e^{jw})H_{LP}(e^{jw}) \cdots H^*_{BP}(e^{jw}) \\ = 0, \ \forall w \end{array}}$$

Then

$$a = \frac{r_{dx_a}[0]}{r_{x_a}[0]} = \frac{\frac{1}{2\pi}\int_0^{2\pi/3} S_d(e^{jw})\,dw}{\frac{1}{2\pi}\int_0^{2\pi/3}\left[S_x(e^{jw}) + S_d(e^{jw})\right]dw}$$

b) For 5-band filter, the coefficients/gains are selected as $\dfrac{1}{1+\dfrac{1}{SNR_{band}}}$.

x5

For infinite-band filter, we get IIR-Non-causal Wiener filter.