Fall of Objects with Air Resistance

Cagatay Kavas
cagataykavas2001@gmail.com

***Abstract*—An engineer's job is to find an analytical solution to a mathematical model. Comparison of two objects with different masses with and without air resistance is also a problem that needs an analytical solution to its mathematical model. We used Newton's principles and made a simultaneous model of an object free falling under constant gravity which an user can enter the falling height of the objects and the masses of the objects while code is plotting displacement/time and velocity/time graphs of each objects with and without the air resistance.**

Keywords—Terminal velocity, air resistance, distance, height, velocity, acceleration, gravity, free fall.

I. Introduction

In this project I simulated the free fall of two objects with different mass with and without air resistance according to the Newton principles on Matlab in a m-file while not calculating the change in gravity at height User can enter a desired height value and masses of two objects than the fall of the object will be simulated according Newton's principles. Velocities of the object and distance traveled by the object are displayed simultaneously.

Velocity is the displacement over time and acceleration is the change in velocity over time so using Newton's principles we can derive the mathematical relation of displacement, velocity and acceleration to build a mathematical model to our problem and find its analytical solution. However, if we include air resistance objects will reach a maximum speed over time which leads to a linear displacement graph and unvarying speed graph after time.

If we want to compare falling of objects with and without air resistance in real life in order for an area to have no air resistance all the air in the room needs to be pomped out which takes too much time, space and money. So experiment with the actual system is quite hard to do and experiment with the actual system is out of question if we want to make many tests on the mode. Instead we make the mathematical model about the problem and bring an analytical solution to it and simulate this problem in computer environment. If we get the desired outcome than we experiment with the real life system.

II. Theoritical Background

The Earth has tendency to pull objects towards its center because of its mass. This effect is called gravity. Since Earth is constantly pulling objects towards its center so objects are under force. If an object does not initiate force in opposite direction force upon the object will be unbalanced (there will be a net force different than 0 N) and acceleration occurs. Free falling is one of the occurances of an object under unbalanced forces.

Acceleration is the change in velocity over time. Gravity changes as the object's distance from Earth's center changes but we will ignore this phenomena as well as the resistance of air. Constant gravity (g) is widely accepted as 9.81 m/s^2 and it is denoted as g.

Fall of an unmoving object from the air is our system to find a mathematical solution to. So starting speed of the object is considered as $V_s = 0$ m/s. Starting point x_s is the height of fall and x_f final height (when an object hits the ground) so its height is 0 m. So Newton's principles are:

Equation for final speed: (1)

$$V_f = V_s + g\ x\ t = g\ x\ t$$

Equation for total displacement: (2)

$$x_f = x_s - V_s\ x\ t - \frac{1}{2} g\ x\ t^2 = \frac{1}{2} g\ x\ t^2$$

$$0 = x_s - \frac{1}{2} g\ x\ t^2 \rightarrow x_s = \frac{1}{2} g\ x\ t^2$$

Since x_s is the height of the fall and g is considered as a constant we can find the fall time from this equation. (3)

$$x_s = \frac{1}{2} g\ x\ t^2 \rightarrow t^2 = \frac{2x_s}{g} \rightarrow t = \sqrt{\frac{2x_s}{g}}$$

Since we derived mathematical equations for falling of an object without air resistance now we need to derive the equations for falling of an object with air resistance. Force applied to an object causes acceleration. Acceleration is inversely proportional to the mass of the object. Air resistance and mass pulling force are the two applied forces to a falling object. Air resistance constant multiplied by velocity's square give the force applied by the air resistance. Gravity multiplied by the mass of the object is the other force.

Forces applied to a free falling object including air resistance

$$m \; x \; a = m \; x \; g - k \; x \; V^2 (4)$$

k is the air resistance constant, a is object's acceleration, g is the gravity, V is object's speed and m is the mass of the object. Since a is in m/s^2 and m is in kg and V is in m/s we can say that k is defined as kg/m.

$$kg \; x \frac{m}{s^2} = k \; x \frac{m^2}{s^2} \to k = \frac{kg}{m}$$

When object reaches its max velocity is when acceleration is 0.

$$g = k \; x \frac{V^2}{m} \to V_f = \sqrt{\frac{mg}{k}} \; (5)$$

If we derivate *(1):*

$$m\frac{dV}{dt} = mg - kV^2$$

$$\frac{dV}{dt} = g - \frac{kV^2}{m} = -\frac{k}{m}(V^2 - \frac{gm}{k})$$

$$\frac{\boldsymbol{dV}}{(\boldsymbol{V^2} - \frac{\boldsymbol{gm}}{\boldsymbol{k}})} = -\frac{\boldsymbol{k}}{\boldsymbol{m}}\boldsymbol{dt} \; (6)$$

Integrating both sides of (6):

$$\int_0^V \frac{\boldsymbol{dV}}{(\boldsymbol{V^2} - \frac{\boldsymbol{gm}}{\boldsymbol{k}})} = \int_0^t -\frac{\boldsymbol{k}}{\boldsymbol{m}}\boldsymbol{dt} \; (7)$$

By applying integral table conversions we get:

$$\sqrt{\frac{k}{mg}} atanh\left(\frac{V}{\sqrt{\frac{gm}{k}}}\right) = -\frac{k}{m}t \; (8)$$

By leaving V alone at (8) we get the velocity for time formula

$$V(t) = \sqrt{\frac{gm}{k}} tanh\left(t\sqrt{\frac{gk}{m}}\right) \; (9)$$

By integrating V(t) we can find displacement over time X(t) function.

$$\boldsymbol{x(t)} = \frac{\boldsymbol{m}}{\boldsymbol{k}}\boldsymbol{ln}\left[\boldsymbol{cosh}\left(\boldsymbol{t}\sqrt{\frac{\boldsymbol{gk}}{\boldsymbol{m}}}\right)\right] \; (10)$$

When object falls the displacement will be equal to height. If we put x(t)=h and leave t alone we can find fall time.

$$t = \sqrt{\frac{m}{gk}} \; x \; acosh\left(e^{\frac{hk}{m}}\right) (11)$$

Than I compared these results to [4] at the references and I verified that these formulas are true.

III. Description of the Simulation

First of all I set gravity as 9.81 m/s^2 and air resistance constant as 0.24 kg/m than I defined a newtonFalling function which simulates the falling of an object according to Newton's principals. Visualization of the two objects of different masses falling simulated with and without air resistance is as smooth as possible also velocity/time and displacement/time graphs are displayed simultaneously. Code works according to height and masses of the two objects entered by the user.

Since the masses of the objects doesn't effect anything in falling of the objects without air resistance both objects's fall without air resistance is displayed in color green at graphs (height/time, velocity/time) and the falling simulation. However masses of the objects effect the falling of the objects with air resistance so I used color red for the first object and the color blue for the second object in the falling simulation and in the graphs.

First I defined inputs in the function. Falling time of the objects without air resistance (fallTimeGreen is calculated according to (3) at theoretical background. My code calculates the fallTime which is greatest fall time so I defined it as 0 first. fallTime and fallTimeGreen values are used in limits of graphs used in graphs and breaking out of the newtonFalling function. Than code updates fallTime according to the output from the fallCalcul function which is defined below newtonFalling function. fallCalcul function uses gravity, air resistance constant, mass of object and height inputs and calculates the time when object falls according to (11) formula at the theoretical background.

Since my code from last week's assignment worked and I tested the accuracy of that code I modified my homework to display three screens by using subplot function.

newtonFalling function requires height and gravity inputs to print the fall times of objects, falling of the objects in both regions (with and without air resistance) and two graphs (height/time and velocity/time).

Since I simulated the fall of object on a graph and I needed 2 graphs as well so I divided the figure to three parts by using subplot function. ax3 is for visualizing the falling of the objects, ax1 is for velocity/time graphs of objects and ax2 is for height/time graphs.

The moment object starts to fall is defined as t=0, t increases in the while loop after each iteration.

While loop breaks when t (current time) > fallTime condition is reached.

Locations of the falling objects without air resistances are defined by the (2) and objects with air resistances are defined by (10) at the theoretical background and I used dots on graphs to visualize the falling of balls from [1] at the references. Sizes of the objects are proportional to their masses. Air resistance constant k consists volume too but in order for better visualization I just multiplied mass with a constant value to be the size of ball. Which is not an accurate representation but in sake of visualization the masses of the objects that is a solution I came up with.

I used if functions since the falling time for objects with and without air resistance is different and masses of the objects with different mass is different in air resistance condition. So when an individual object's falling time value is reached they stay still at ground.

Title of the graph, x and y labels, line between two areas, name of the areas (with/without air resistance) are entered according to mathlab user's guide from [2] at the references. In order to visualize the falling of the object I set limits to x and y axises so while ground and starting point stay still and object keeps moving towards the ground.

Instead of t I used a value (a) to plot displacement/time and velocity/time. It is because a is a vector starting from 0 and each element increases with 0.1 seconds until it reaches the current time (t) so that when iterations keep happening both graphs will continue to plot graphs simultaneously.

I displayed velocity/time graphs without air resistance according to (1) and with the air resistances according to (9) from the theoretical background.

I displayed displacement/time graphs without air resistance according to (2) and with the air resistances according to (10) from the theoretical background.

I followed same steps (title of graph, x and y labels) for displacement/time and velocity/time graphs.

I added a pause so that we can observe the code's outputs. If we didn't put a pause we wouldn't be able to observe the code's outputs since computer runs the code very fast.

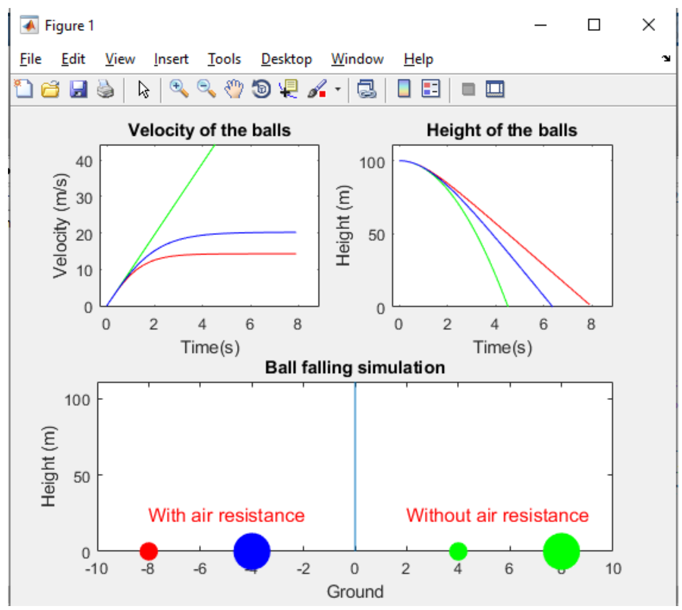
After while loop ends the fall times of the objects are printed and all objects are at the ground.

IV. Simulational Results

I modified my working code from last week. I verified my last week's code's results by changing gravity value to 10 and placed pausing my simulation in every second. For example if our object were to fall from 80 meters in explained conditions: $80\text{-}g*t^2/2 = 80\text{-}10*1^2/2 = 75$ would be the position of our object after a second. After verifying that my code works I set gravity to its widely accepted value (9.81 m/s^2).

For the falling with the air resistance part I compared my code's outputs to the calculator [4] at references and I verified that my code works.

I ran my code with 100 meters height red object mass 5 kg and blue object mass 10 kg here is the output.



We can see that velocity of the green objects kept increasing linearly and reached the ground in a curve before blue and red objects.

Blue object, the object with the greater mass, however reached the ground later than green object and its velocity reached a limit value (terminal velocity) and stayed there after increasing linearly for a while. Height of the blue object reduced in a curve for a while after object's velocity reached a limit value. After reaching that limit value blue object's height started to decrease linearly until it reached the ground.

Red object, the object with the lesser mass, reached the ground later than blue object. Red object reached its limit velocity before the blue object. Red object's limit velocity is smaller than blue object's.

We deduced that from this experiment:

-Objects at the area without air resistance and constant gravity reach the ground at the same time. Their velocity kept increasing linearly and their height decreased in a curve until they hit the ground.

-Objects in area without the air resistance in constant gravity reached the ground before the objects in area with air resistance.

-The velocities of the objects in area with air resistance and constant gravity kept increasing linearly until they reached their terminal velocities. Their heights kept decreasing in a curve until they reached their terminal velocities. When objects reached their terminal velocity their height decreased linearly until they hit the ground.

-Object with greater mass in area with air resistance and constant gravity reached a higher terminal velocity than object with less mass. Object with lesser mass reached its terminal velocity before object with greater mass.

-Object with greater mass in area with air resistance reached the ground before object with lesser mass.

There is also a different formula at [5] in which the forces applied on a falling object are defined as:

$$m \, x \, a = m \, x \, g - k \, x \, V$$

Final velocity of the object is:

$$V_f = g \, x \frac{m}{k}$$

Compared to the final velocity at (5) from theoretical background this value is greater so the objects with that formula fall faster, with reaching higher terminal velocity even before compared to the results we calculated in theoretical background.

Graphs are simulated correctly however since our pause time couldn't match the iteration speed of our while segment there may be differences in higher height values.

Conclusions

When studying a system instead of experimenting with the actual system we experiment with the model of a system because experimenting on the actual system is harder. An engineer must know the mathematics behind an event and built its mathematical model and bring an analytical solution to this problem. If the solution to this problem is as its expected to be than we can experiment with the actual system.

We used Newton's principles to bring an analytical solution to comparison of fall of an object with and without the air resistance with constant gravity.

There are two known formulas for falling of an object with the air resistance. Compared to the formula at the [5] at the references we got different results. Formula at the [5] uses V instead of V^2 so it reaches a higher terminal velocity at a time later when the object moving with the formula with V^2 (formulas from [3] and [4] at the references).

However because of the iteration speed of our code there may be some differences in the falling of the object simultaneously.

Code with the formula from [5] at references can be requested upon request.

References

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