Spectral Domain Methods in Electromagnetics EE4620

Topic # 3

Dominant Spectral Contributions

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EE4580 Quasi Optical Systems

Topic 2
Far Field
Branch and Polar Singularities
Surface Waves
Leaky Waves

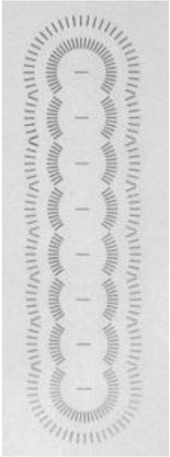
Learning Objectives

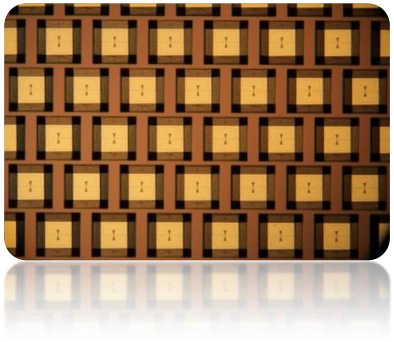
- Identify the spectral singularities and relate to field contributions
- Introduction to the Branch and polar singularities
- Far field asymptotic evolution

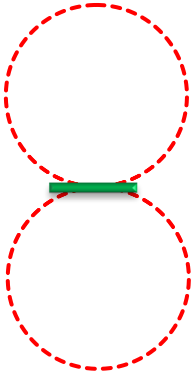
Integrated Antennas

Radar, Space, Sensing applications…
The objective is to radiate the power into a certain direction

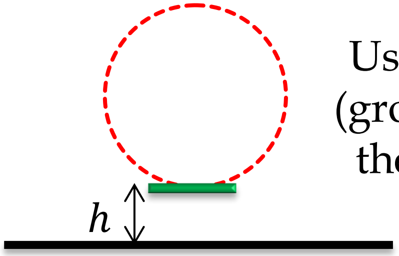
Dielectric allows using integrated technology (PCB, lithography)







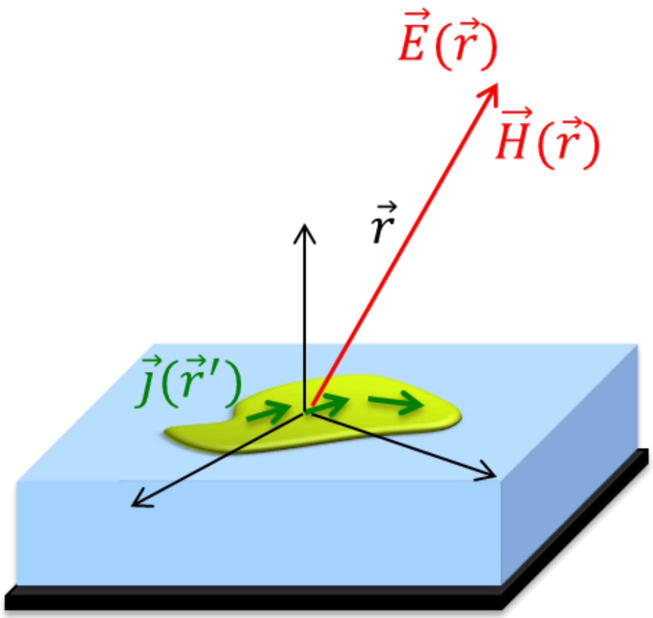
A dipole in free space radiates the same power in the upper and lower regions

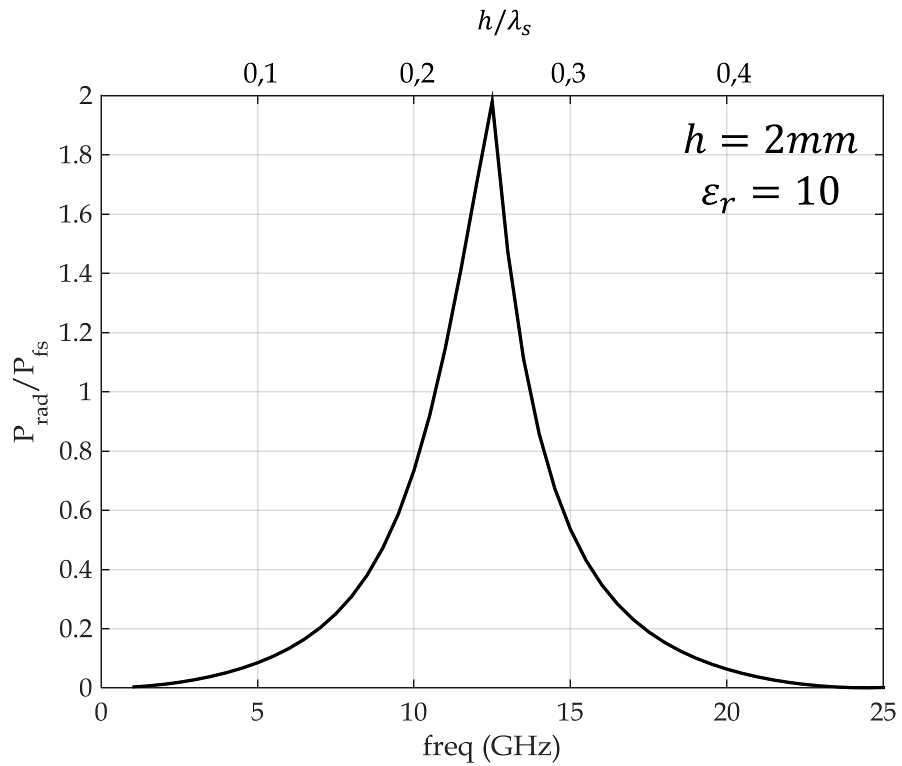


Using a back reflector (ground plane) reduces the radiation into the lower region

$$h_{opt} = \lambda/4$$

Integrated Antennas

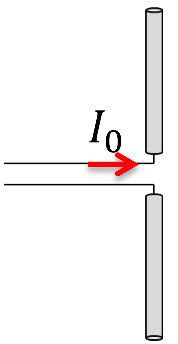


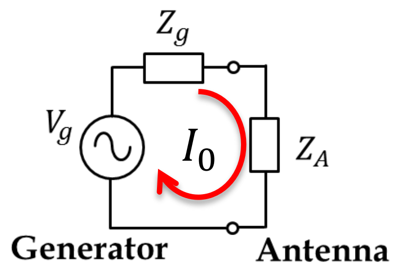


Optimum radiation for $h_{opt} = \lambda_s/4$

Power Budget

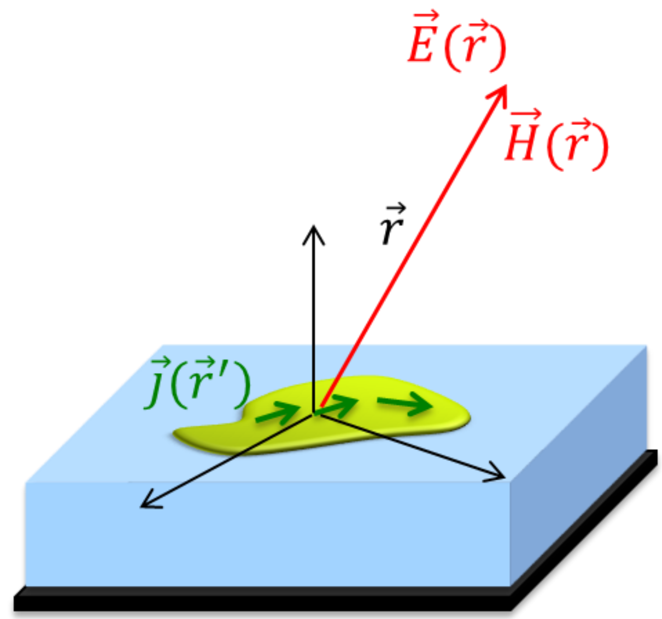
For antennas in free space
with no ohmic losses





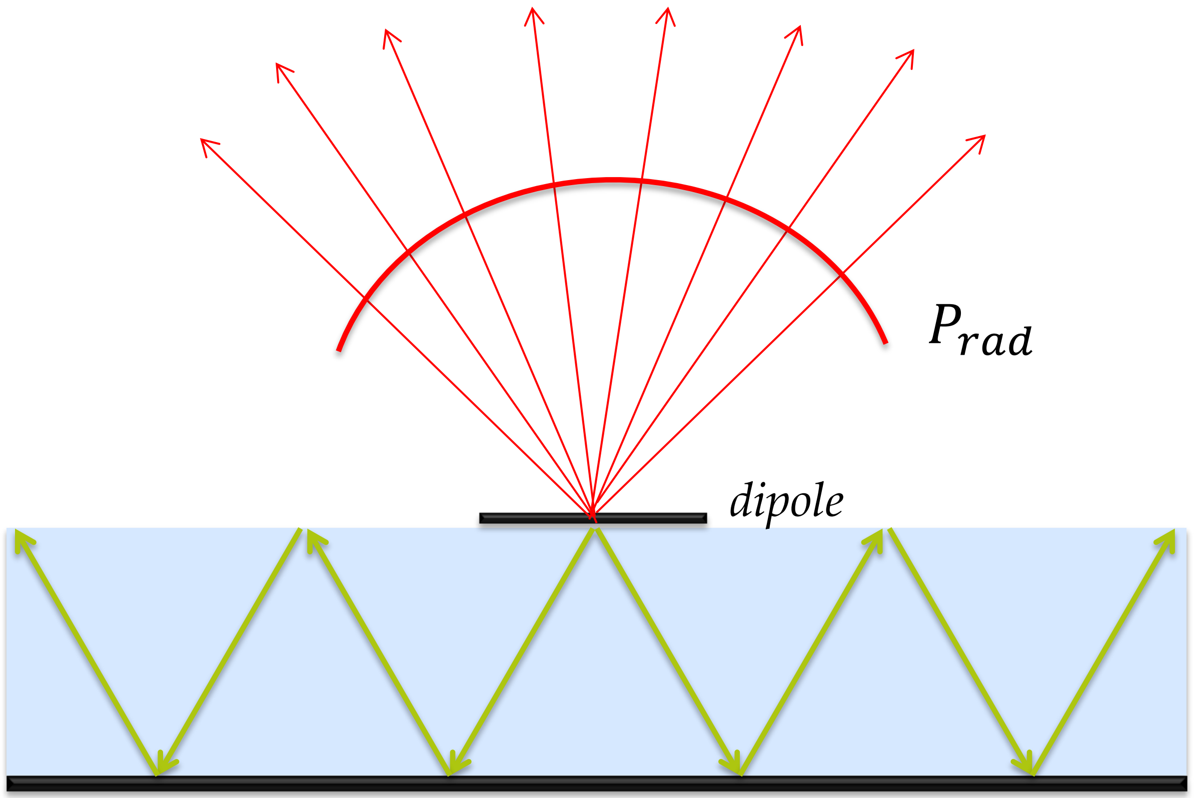
$$\boldsymbol{j}_{eq} = I_0 \boldsymbol{f}(x,y)$$

$$P_{rad} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} U(\theta,\phi) \sin\theta \, d\theta d\phi = P_A \quad = \frac{1}{2} |I_0|^2 Re\{Z_A\}$$



What happens in antennas radiating into infinitely extended dielectrics?

Surface Waves

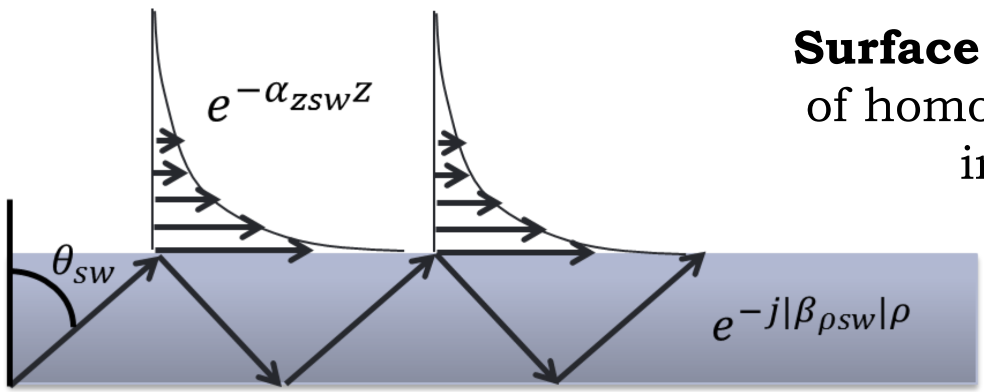


There is a part of the power delivered to the antenna that is radiated inside the dielectric!

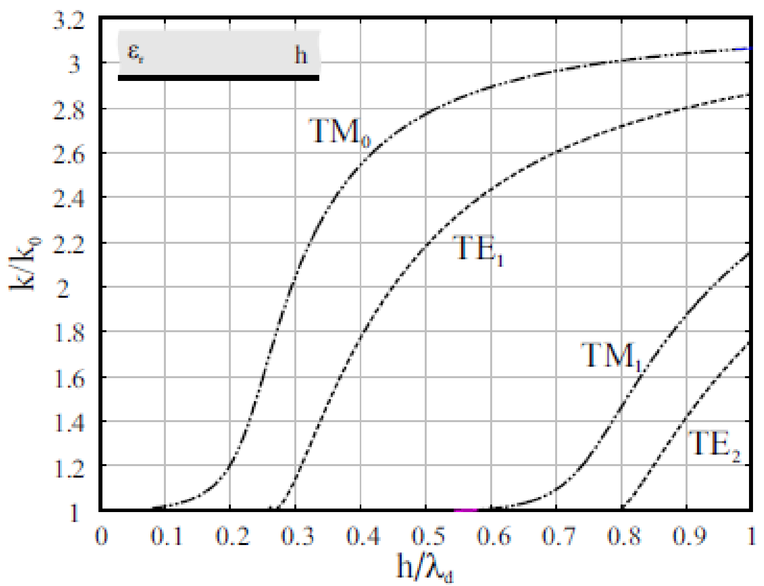
P_{sw}

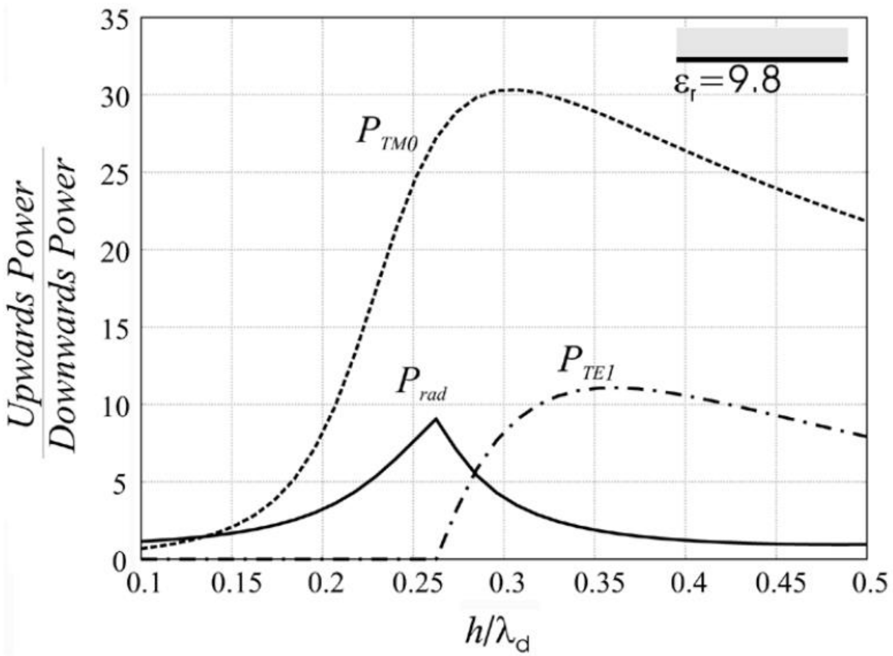
$$P_A = \frac{1}{2}|I_0|^2 Re\{Z_A\} = P_{rad} + P_{sw}$$

Surface Waves

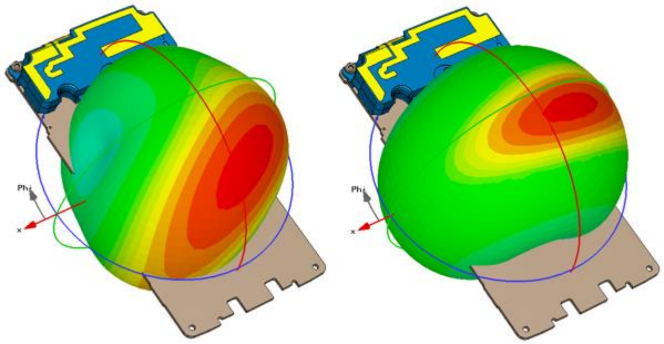


Surface Waves: they are a couple of homogenous PWs propagating inside the dielectric

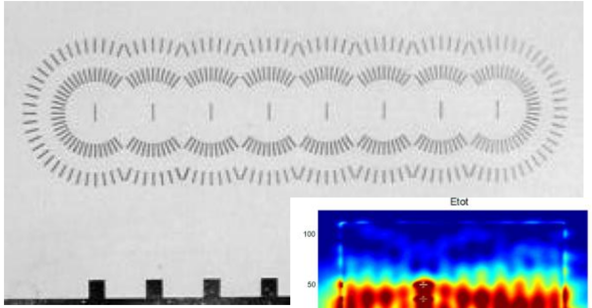


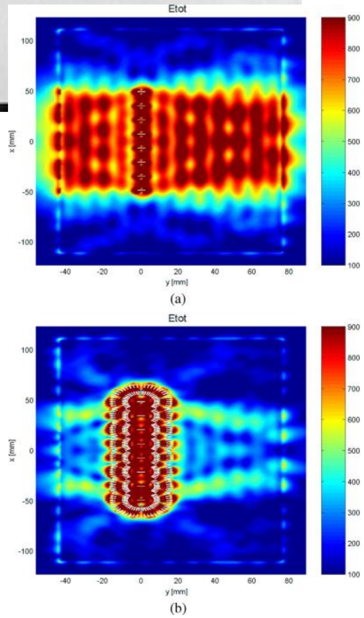


Integrated Antennas



At low frequencies, the dielectrics are electrically very thin… you can design antennas as they were radiating in free space





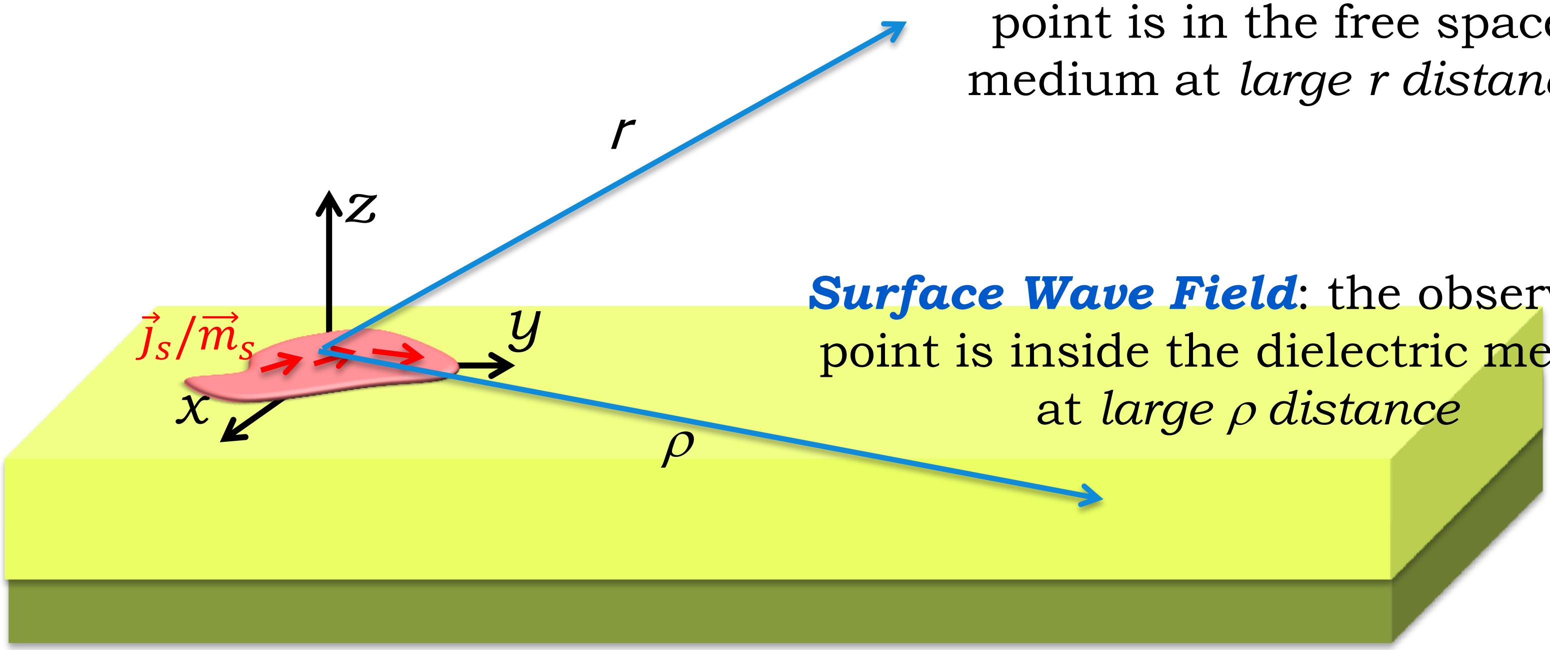
At mm-wave frequencies, the dielectric thickness is electrically significant… significant power remains trapped in the substrates (surface waves)

At THz frequencies, it is actually better to use lenses (quasi-optical antennas)

Fields Radiated by Printed Antennas

$$\vec{\boldsymbol{f}}(x,y,z) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{\boldsymbol{G}}^{fc}(k_x,k_y,z,z')\,\vec{\boldsymbol{C}}(k_x,k_y)e^{-jk_x x}e^{-jk_y y}dk_x dk_y$$

Far Field: the observation point is in the free space medium at *large r distance*



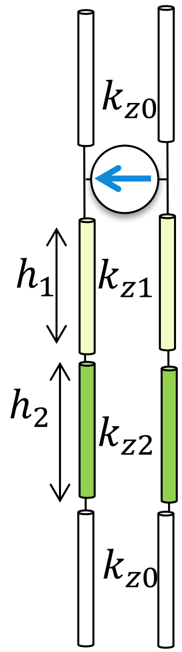
Surface Wave Field: the observation point is inside the dielectric medium at *large ρ distance*

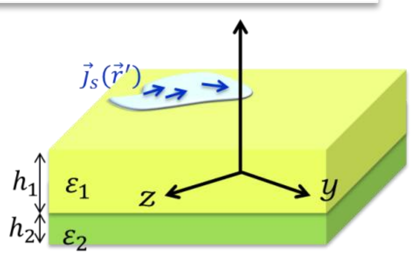
Dyadic Green's Function for Stratified Media

$$\boldsymbol{f}(\vec{r}) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{\boldsymbol{G}}^{fc}(k_x,k_y,z,z')\boldsymbol{C}_s(k_x,k_y)e^{-jk_x x}e^{-jk_y y}\,dk_x dk_y$$

$$\tilde{\boldsymbol{G}}^{ej} = \begin{bmatrix} -\dfrac{v_{TM}k_x^2 + v_{TE}k_y^2}{k_\rho^2} & \dfrac{(v_{TE}-v_{TM})k_xk_y}{k_\rho^2} \\ \dfrac{(v_{TE}-v_{TM})k_xk_y}{k_\rho^2} & -\dfrac{v_{TE}k_x^2 + v_{TM}k_y^2}{k_\rho^2} \\ \varsigma\dfrac{k_x}{k}i_{TM} & \varsigma\dfrac{k_y}{k}i_{TM} \end{bmatrix}$$

$$\tilde{\boldsymbol{G}}^{hj} = \begin{bmatrix} \dfrac{(i_{TM}-i_{TE})k_xk_y}{k_\rho^2} & \dfrac{i_{TE}k_x^2 + i_{TM}k_y^2}{k_\rho^2} \\ \dfrac{i_{TE}k_y^2 + i_{TM}k_x^2}{k_\rho^2} & \dfrac{(i_{TE}-i_{TM})k_xk_y}{k_\rho^2} \\ \dfrac{k_y}{\varsigma k}v_{TE} & \dfrac{k_x}{\varsigma k}v_{TE} \end{bmatrix}$$

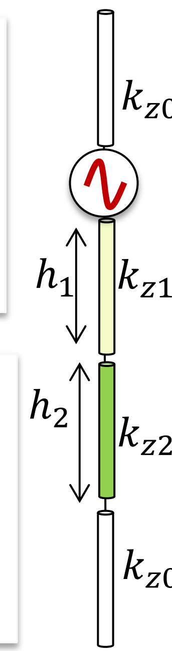


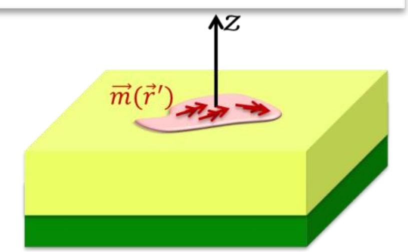


$$k_{zi} = \sqrt{k_i^2 - k_x^2 - k_y^2}$$

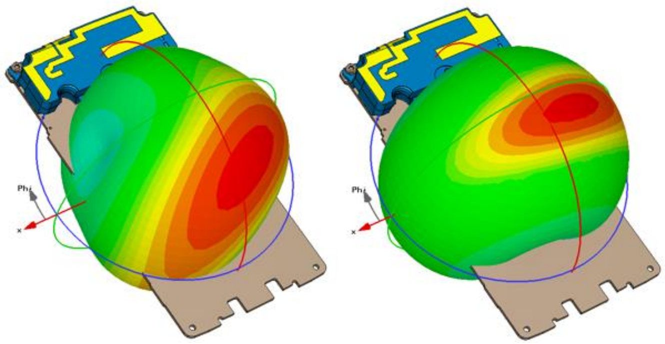
$$\tilde{\boldsymbol{G}}^{hm} = \begin{bmatrix} -\dfrac{i_{TE}k_x^2 + i_{TM}k_y^2}{k_\rho^2} & \dfrac{(i_{TM}-i_{TE})k_xk_y}{k_\rho^2} \\ \dfrac{(i_{TM}-i_{TE})k_xk_y}{k_\rho^2} & -\dfrac{i_{TM}k_x^2 + i_{TE}k_y^2}{k_\rho^2} \\ \dfrac{k_x}{\varsigma k}v_{TE} & \dfrac{k_y}{\varsigma k}v_{TE} \end{bmatrix}$$

$$\tilde{\boldsymbol{G}}^{em} = \begin{bmatrix} \dfrac{(v_{TM}-v_{TE})k_xk_y}{k_\rho^2} & -\dfrac{v_{TE}k_y^2 + v_{TM}k_x^2}{k_\rho^2} \\ \dfrac{v_{TE}k_x^2 + v_{TM}k_y^2}{k_\rho^2} & \dfrac{(v_{TE}-v_{TM})k_xk_y}{k_\rho^2} \\ -\varsigma\dfrac{k_y}{k}i_{TM} & \varsigma\dfrac{k_x}{k}i_{TM} \end{bmatrix}$$





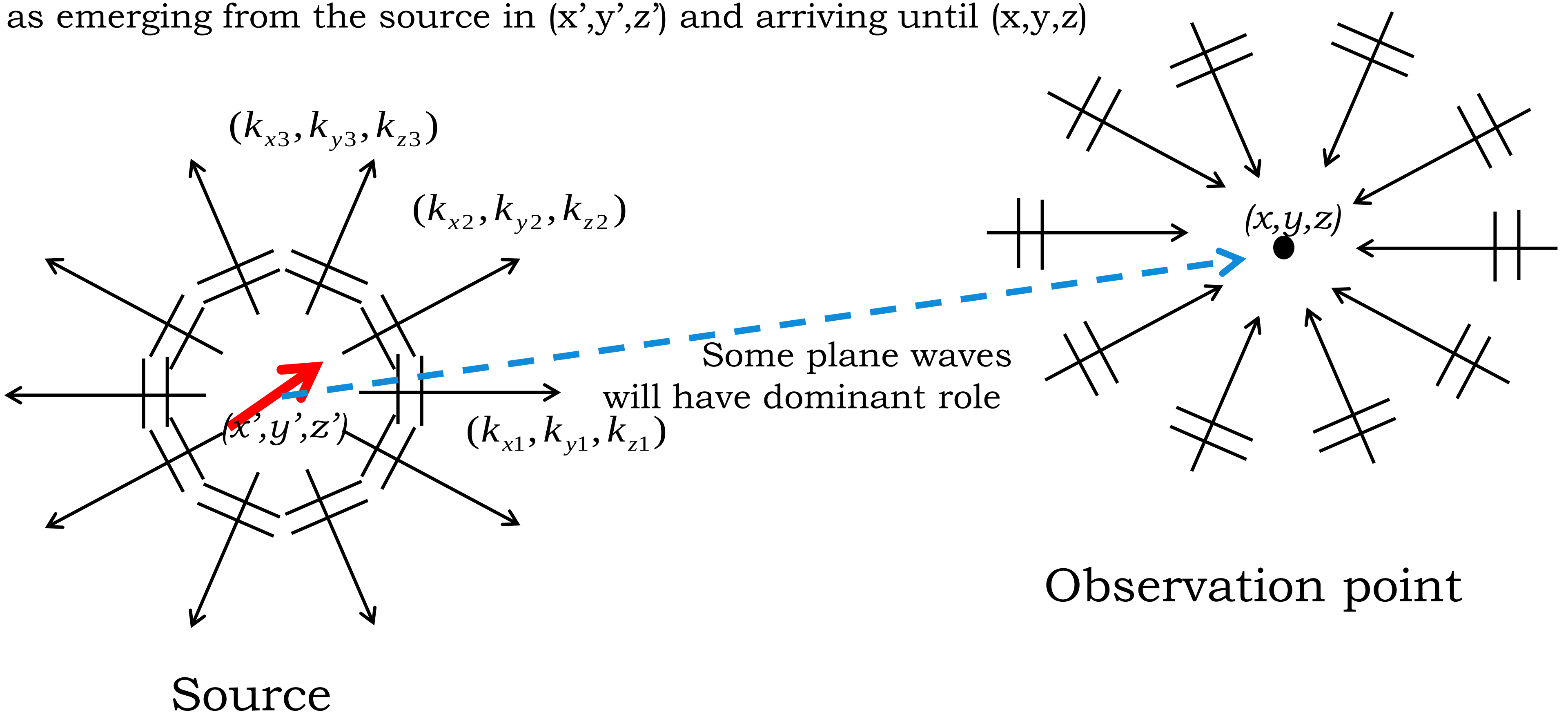
Fields radiated into free space (far field)



$$\bar{\bar{G}}^{fc}(\vec{r}-\vec{r}\,') = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \begin{bmatrix} G_{xx}(k_x,k_y,z,z') & G_{xy}(k_x,k_y,z,z') & G_{xz}(k_x,k_y,z,z') \\ G_{yx}(k_x,k_y,z,z') & G_{yy}(k_x,k_y,z,z') & G_{yz}(k_x,k_y,z,z') \\ G_{zx}(k_x,k_y,z,z') & G_{zy}(k_x,k_y,z,z') & G_{zz}(k_x,k_y,z,z') \end{bmatrix} e^{-jk_x(x-x')}e^{-jk_y(y-y')}\,dk_x\,dk_y$$

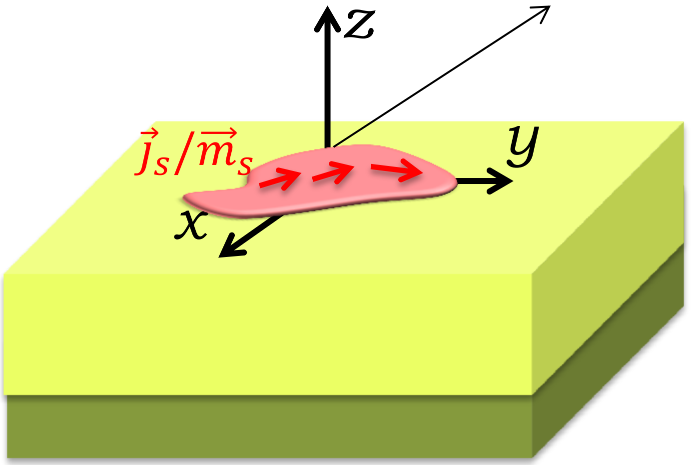
Plane Wave expansion

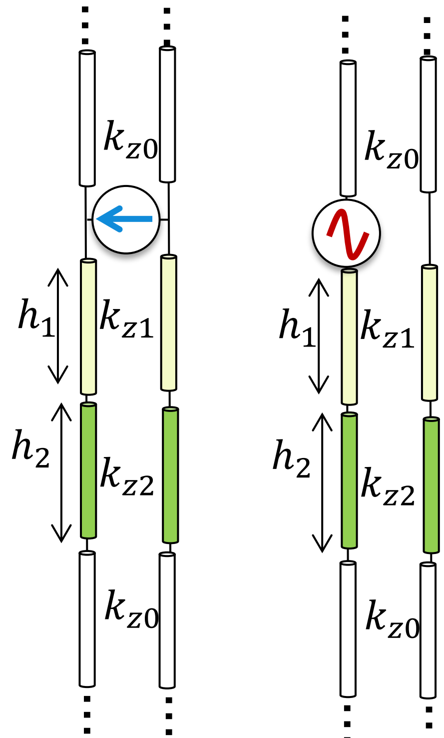
Plane waves are defined over the entire space.
Plane wave expansion represents the total radiated fields as emerging from the source in (x',y',z') and arriving until (x,y,z)



Printed Antennas

$$\vec{\boldsymbol{f}}(x,y,z)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{\boldsymbol{G}}^{fc}(k_x,k_y,z,z')\,\vec{\boldsymbol{C}}(k_x,k_y)e^{-jk_xx}e^{-jk_yy}dk_xdk_y$$





Far Field: the observation point is in the infinitely extended transmission lines. Therefore

$$\tilde{G}^{fc}(k_x,k_y,z,z')\propto e^{-jk_{z0}z}$$

Far Field Radiation

$$\vec{f}(x,y,z) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{G}^{fc}(k_x,k_y,z,z')\,\vec{C}(k_x,k_y)e^{-jk_xx}e^{-jk_yy}dk_xdk_y$$

We know the solution of this integral

$$\vec{f}(x,y,z) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\underbrace{\left(\tilde{G}^{fc}(k_x,k_y,z,z')\vec{C}(k_x,k_y)k_{z0}e^{jk_{z0}|z-z'|}\right)}\frac{e^{-jk_{z0}|z-z'|}}{k_{z0}}e^{-jk_xx}e^{-jk_yy}dk_xdk_y$$

slow varying function in the surrounding of the **stationary phase point** k_{xs}, k_{ys}, k_{zs}

$$\vec{f}^{far}(x,y,z) = \frac{1}{4\pi^2}\tilde{G}^{fc}(k_{xs},k_{ys},z,z')\vec{C}(k_{xs},k_{ys})k_{zs}e^{jk_{zs}|z-z'|}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-jk_{z0}|z-z'|}}{k_{z0}}e^{-jk_xx}e^{-jk_yy}dk_xdk_y$$

$$\vec{f}^{far}(\vec{r}) = jk_{zs}\tilde{G}^{fc}(k_{xs},k_{ys},z,z')\vec{C}(k_{xs},k_{ys})e^{jk_{zs}|z-z'|}\frac{e^{-jkr}}{2\pi r}$$

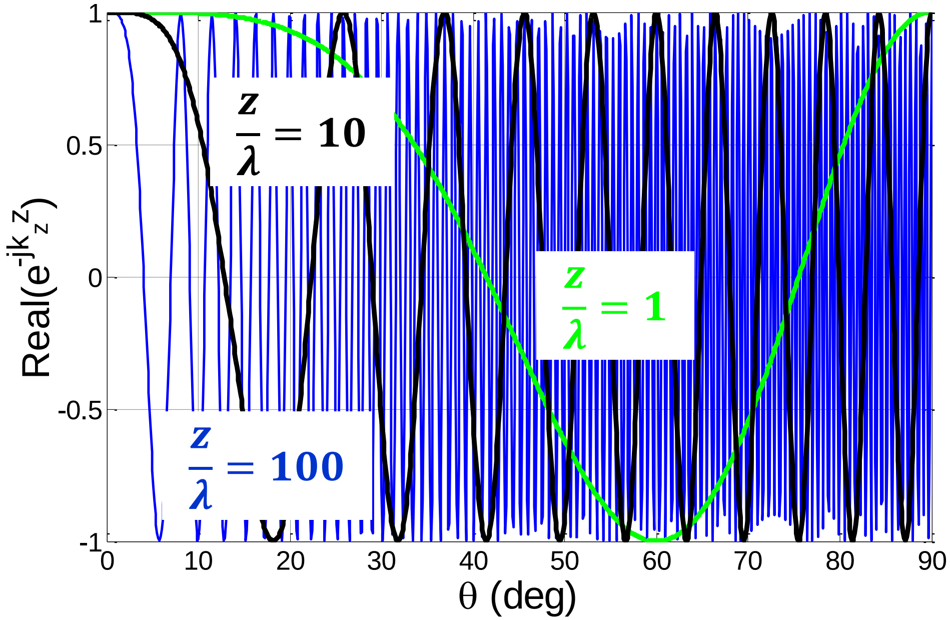
Stationary Phase Asymptotic Evaluation

- The original integration domain is infinite $\int_{-\infty}^{\infty}$
- In the far field, we can assume that the distance is much larger than the wavelength.

For instance if z is large:

$$e^{-jk_z z} = e^{-j\frac{2\pi}{\lambda}cos\theta z}$$

It is a highly oscillating function!



The integral can be calculated using an asymptotic evaluation

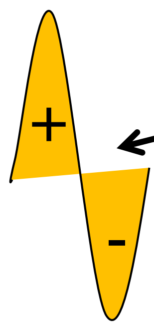
The stationary phase method

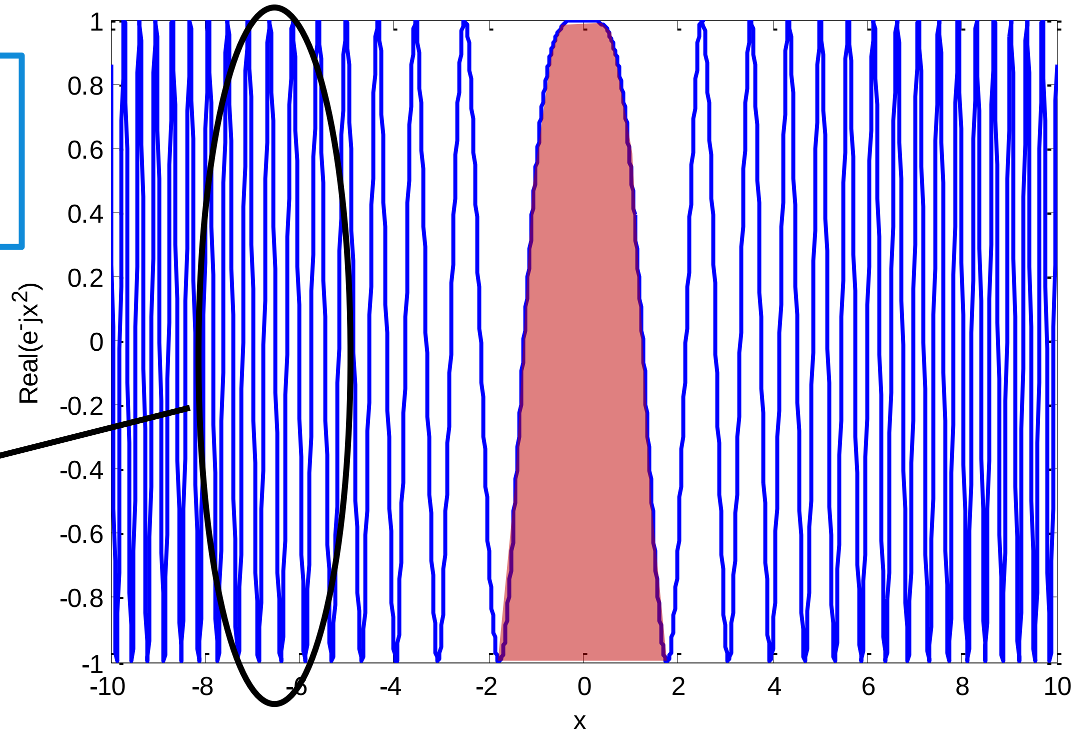
Oscillating function

Identity

$$\boxed{I = \int_{-\infty}^{\infty} e^{-j\Omega x^2} dx = \sqrt{\frac{\pi}{|\Omega|}} e^{\mp j\pi/4}}$$

$\mp : \Omega \,{}^{>}/_{<}\, 0$





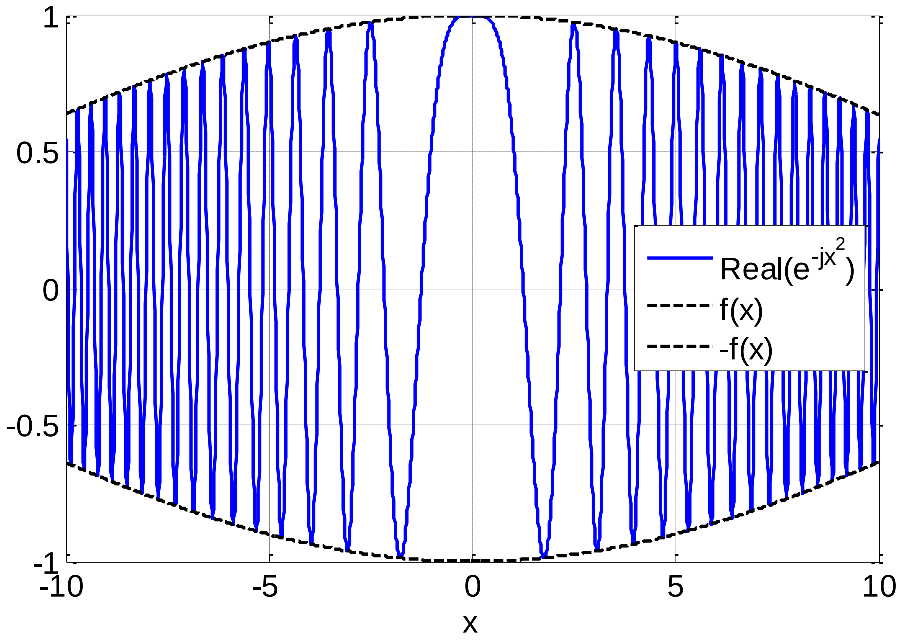
$$\int_{-\infty}^{\infty} e^{-|\Omega| x^2} dx = \sqrt{\frac{\pi}{|\Omega|}}$$

The integral of this function is dominated by the area of the portion of the function closed to the origin. Indeed, the portion far from the origin oscillates rapidly and leads to destructive contributions

Slow varying function

$$I = \int_{-\infty}^{\infty} f(x) e^{-j\Omega x^2} dx \approx \sqrt{\frac{\pi}{\Omega}} f(0) e^{\mp j\pi/4}$$

Where f is a function slow varying function that changes negligibly over a cycle of variation $\cos(\Omega x^2)$



Demonstration

1) Taylor approximation around the origin

$$f(x) \approx f(0) + x f'(0) + x^2 \frac{1}{2} f''(0) + \cdots$$

2) Place it into the integral. **The odd exponents are zero**.

$$I \approx f(0) \int_{-\infty}^{\infty} e^{-j\Omega x^2} dx + f''(0) \frac{1}{2} \int_{-\infty}^{\infty} x^2 e^{-j\Omega x^2} dx$$

$$+ f''''(0) \frac{1}{4!} \int_{-\infty}^{\infty} x^4 e^{-j\Omega x^2} dx + \cdots =$$

3) Using $\displaystyle \int_{-\infty}^{\infty} x^2 e^{-j\Omega x^2} dx = j \frac{\delta}{\delta\Omega} \int_{-\infty}^{\infty} e^{-j\Omega x^2} dx$

$$I \approx f(0) \sqrt{\frac{\pi}{j\Omega}} + \frac{1}{2} f''(0) \frac{1}{2} \frac{1}{j\Omega} \sqrt{\frac{\pi}{j\Omega}} + \frac{1}{4!} f''''(0) j \frac{3}{4} \frac{1}{-\Omega^2} \sqrt{\frac{\pi}{j\Omega}} + \cdots$$

For large Ω, only the first term is significant

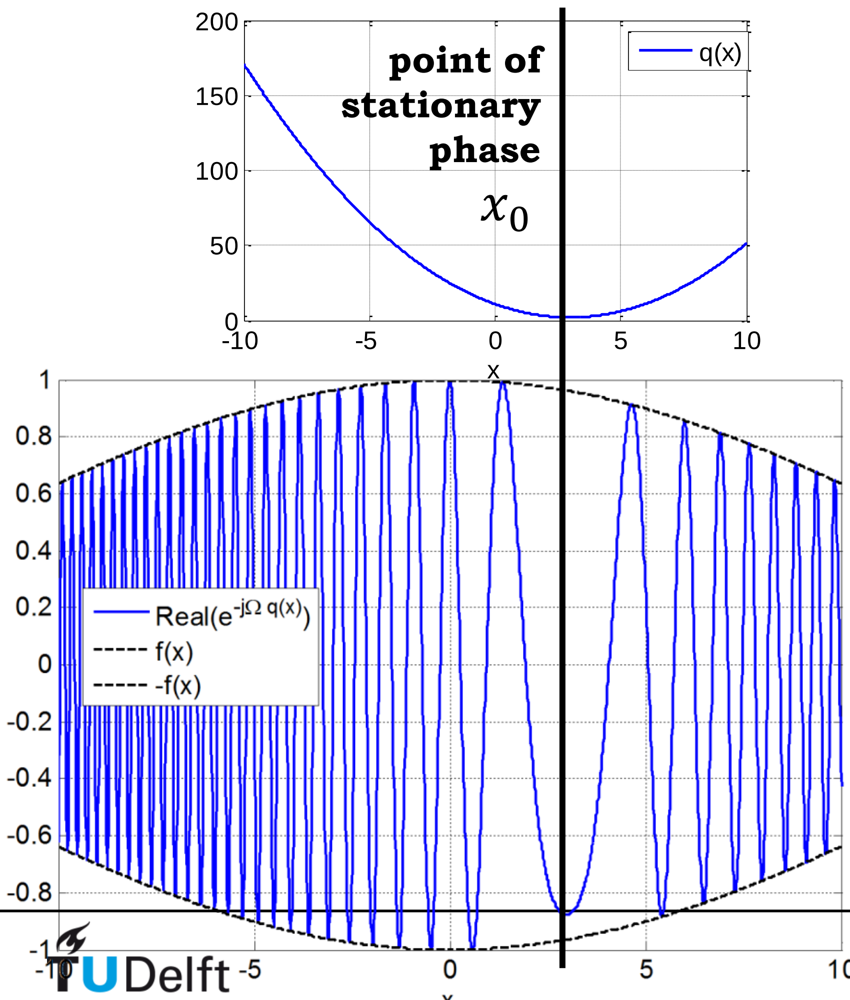
Stationary phase method

$$I = \int_{-\infty}^{\infty} f(x)e^{-j\Omega q(x)}dx \approx f(x_0)e^{-j\Omega q(x_0)} \sqrt{\frac{\pi}{\frac{1}{2}\Omega|q''(x_0)|}}\, e^{\mp j\pi/4}$$

$\Omega > 0 \qquad \mp q''(x_0) \,{}^{>}/_{<}\, 0$

Where $q(x)$ is a function with a minimum/ maximum in x_0

$$q'(x_0) = 0$$



Demonstration

1) Taylor approximation of the phase

$$q(x) \approx q(x_0) + \frac{1}{2}q''(x_0)(x - x_0)^2$$

2) The integral is now dominated by the area around x_0

$$I \approx \int_{-\infty}^{\infty} f(x)e^{-j\Omega q(x_0)}e^{-j\Omega\frac{1}{2}q''(x_0)(x-x_0)^2}dx$$

3) Asymptotic evaluation

$$I \approx f(x_0)e^{-j\Omega q(x_0)} \int_{-\infty}^{\infty} e^{-j\Omega\frac{1}{2}q''(x_0)(x-x_0)^2}dx$$

$$= f(x_0)e^{-j\Omega q(x_0)} \int_{-\infty}^{\infty} e^{-j\Omega\frac{1}{2}q''(x_0)x'^2}dx'$$

Far Field Calculation: Stationary Phase Point

$$\vec{f}(x,y,z) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} \tilde{\tilde{\vec{G}}}^{fc}(k_x,k_y,z,z')e^{jk_z|z-z'|}\vec{C}(k_x,k_y)e^{-jk_x x}e^{-jk_y x}e^{-jk_z|z-z'|}dk_x dk_y$$

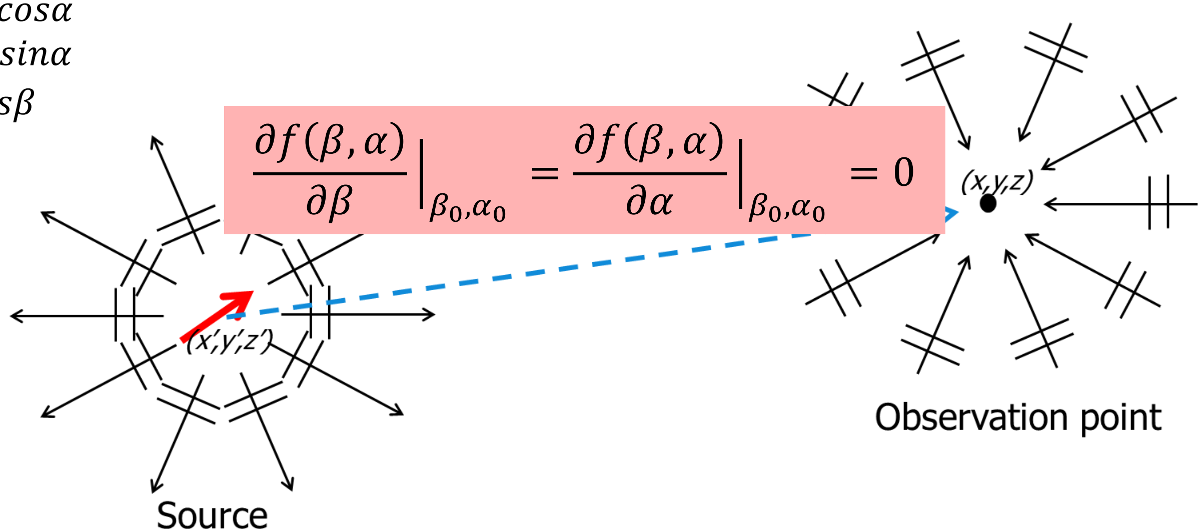
In order to find a stationary phase point, it is simpler to the spectral integral into cylindrical integration domain by applying the following change of variables

The observation points are defined by

$$x = r sin\theta cos\phi$$
$$y = r sin\theta sin\phi$$
$$|z-z'| = r cos\theta$$

$$k_x = k_0 sin\beta cos\alpha$$
$$k_y = k_0 sin\beta sin\alpha$$
$$k_z = k_0 cos\beta$$

$$\left.\frac{\partial f(\beta,\alpha)}{\partial \beta}\right|_{\beta_0,\alpha_0} = \left.\frac{\partial f(\beta,\alpha)}{\partial \alpha}\right|_{\beta_0,\alpha_0} = 0$$



Far Field Calculation: Stationary Phase Point

$$\vec{\boldsymbol{f}}(x,y,z) \approx \frac{\vec{F}\left(k_{xs},k_{ys},z,z'\right)}{(2\pi)^2}\iint\limits_{-\infty}^{\infty}\frac{e^{-jf(k_x,k_yk_z,x,y,z,z\prime)}}{k_z}dk_xdk_y$$

$$\vec{F}\left(k_{xs},k_{ys},z,z'\right) = k_{zs}\widetilde{\boldsymbol{G}}^{fc}\left(k_{xs},k_{ys},z,z'\right)\vec{\boldsymbol{C}}\left(k_{xs},k_{ys}\right)e^{jk_{zs}|z-z\prime|}$$

k_{xs},k_{ys}

β_s,α_s are the saddle points defined by the oscillating phase function f

Change integration domain $\quad k_x = ksin\beta cos\alpha \quad k_y = ksin\beta sin\alpha$

$$\vec{\boldsymbol{f}}(x,y,z) \approx \frac{\vec{F}(\beta_s,\alpha_s)}{(2\pi)^2}\iint\limits_{\beta,\alpha} e^{-jf(\beta,\alpha;\theta,\phi,r)}k^2 sin\beta d\beta d\alpha$$

Change observation points $\quad x = rsin\theta cos\phi \quad y = ksin\theta sin\phi$

$$f(\beta,\alpha) = k_0 r(sin\beta cos\alpha sin\theta cos\phi + sin\beta sin\alpha sin\theta sin\phi + cos\beta cos\theta) = k_0 r\,(sin\beta sin\theta cos(\alpha-\phi) + cos\beta cos\theta)$$

Stationary Phase Points

$$f(\beta, \alpha) = k_0 r \left(sin\beta sin\theta cos(\alpha - \phi) + cos\beta cos\theta\right)$$

There are two stationary phase points k_{xs1}, k_{ys1} k_{xs2}, k_{ys2}

$$\frac{\partial}{\partial\alpha} f(\beta, \alpha) = -k_0 r sin\beta sin\theta sin(\alpha - \phi) = 0 \longrightarrow \boxed{\alpha_1 = \phi,\ \alpha_2 = \phi + \pi}$$

$$f(\beta, \alpha_1) = k_0 r \left(sin\beta sin\theta + cos\beta cos\theta\right) \qquad f(\beta, \alpha_2) = k_0 r \left(-sin\beta sin\theta + cos\beta cos\theta\right)$$

$$\frac{\partial}{\partial\beta} f(\beta, \alpha_1) = k_0 r(cos\beta sin\theta - sin\beta cos\theta) = k_0 r sin(\beta - \theta)$$

$$\frac{\partial}{\partial\beta} f(\beta, \alpha_2) = k_0 r(-cos\beta sin\theta - sin\beta cos\theta) = k_0 r sin(\beta + \theta)$$

$$\frac{\partial}{\partial\beta} f(\beta, \alpha_1) = k_0 r sin(\beta - \theta) = 0 \longrightarrow \beta_1 = \theta$$

$$\frac{\partial}{\partial\beta} f(\beta, \alpha_2) = k_0 r sin(\beta + \theta) = 0 \longrightarrow \beta_2 = \pi - \theta$$

Stationary Phase Points

There are two stationary phase points

$\alpha_1 = \phi,\ \alpha_2 = \phi + \pi$

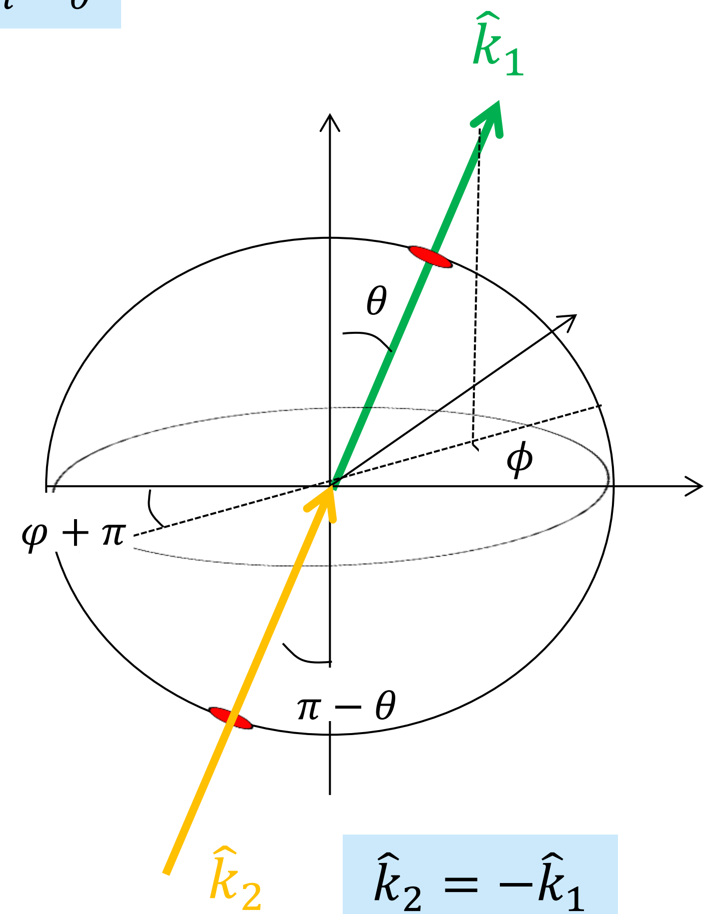
$\beta_1 = \theta \quad \beta_2 = \pi - \theta$

$$\hat{k}_1 = k_{xs1}\hat{x} + k_{ys1}\hat{y} + k_{zs1}\hat{z}$$

$$k_{xs1} = ksin\beta_1 cos\alpha_1 = ksin\theta cos\phi$$
$$k_{ys1} = ksin\beta_1 sin\alpha_1 = ksin\theta sin\phi$$
$$k_{zs1} = kcos\beta_1 = kcos\theta$$

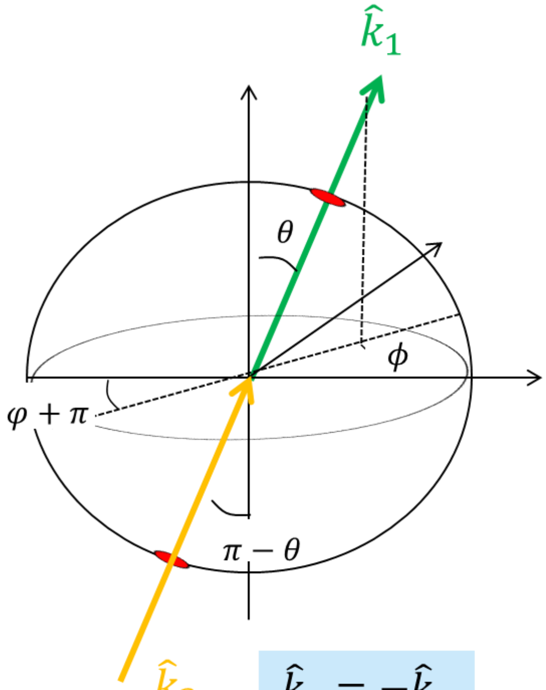
$$\hat{k}_2 = k_{xs2}\hat{x} + k_{ys2}\hat{y} + k_{zs2}\hat{z}$$

$$k_{xs2} = ksin\beta_2 cos\alpha_2 = -ksin\theta cos\phi$$
$$k_{ys2} = ksin\beta_1 sin\alpha_1 = -ksin\theta sin\phi$$
$$k_{zs2} = kcos\beta_1 = -kcos\theta$$



$\hat{k}_2 = -\hat{k}_1$

Stationary Phase Points



$\hat{k}_2 = -\hat{k}_1$

$$\vec{\boldsymbol{f}}(x, y, z) = \frac{\vec{F}(\beta_s, \alpha_s)}{(2\pi)^2} \iint_{\beta,\alpha} e^{-jf(\beta,\alpha;\theta,\phi,r)} k^2 sin\beta d\beta d\alpha$$

$$f(\beta_1 = \theta, \alpha_1 = \phi) = k_0 r\,(\sin^2\theta + \cos^2\theta) = k_0 r$$

$$f(\beta_2 = \pi - \theta, \alpha_2 = \phi + \pi) = -k_0 r$$

$$\vec{f}^{far}(\vec{\boldsymbol{r}}) = \vec{F}\left(\hat{k}_1\right)\frac{je^{-jkr}}{2\pi r}$$

Outward propagating field

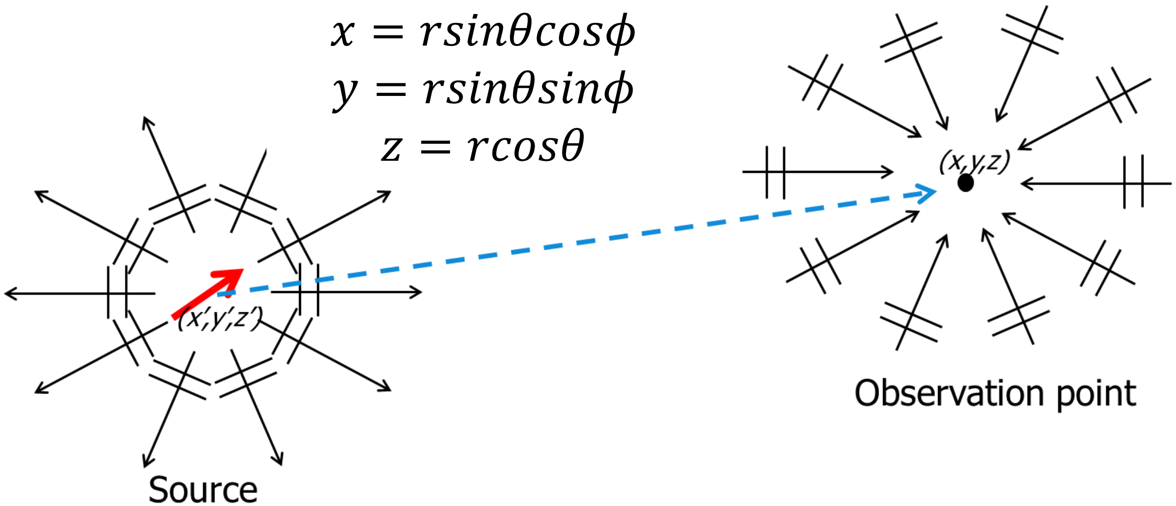
$$\vec{f}^{far}(\vec{\boldsymbol{r}}) = -\vec{F}\left(-\hat{k}_1\right)\frac{je^{jkr}}{2\pi r}$$

Inward propagating field,
we exclude it

Dominant PW in the Far Field

$$\vec{f}^{far}(\vec{\boldsymbol{r}}) = jk_{zs}\widetilde{\boldsymbol{G}}^{fc}(k_{xs}, k_{ys}, z, z')\vec{\boldsymbol{C}}(k_{xs}, k_{ys})e^{jk_{zs}|z-z\prime|}\frac{e^{-jkr}}{2\pi r}$$

$$x = r sin\theta cos\phi$$
$$y = r sin\theta sin\phi$$
$$z = r cos\theta$$



The plane wave with the dominant role is the one given by the direct ray

$$k_{xs} = k_0 sin\theta cos\phi$$

$$k_{ys} = k_0 sin\theta sin\phi$$

$$k_{zs} = k_0 cos\theta$$

It corresponds to a stationary phase point

$$\alpha_1 = \varphi$$

$$\beta_1 = \theta$$

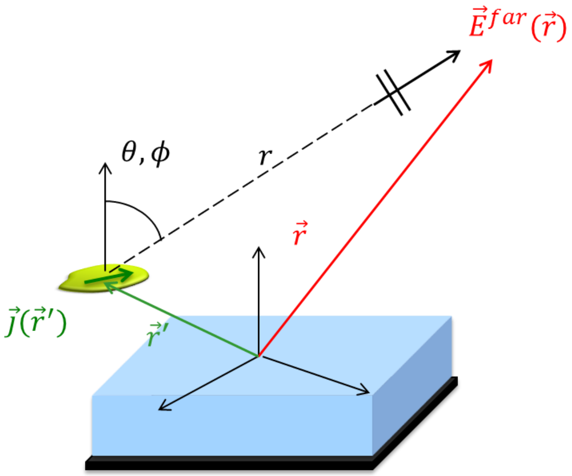
The far field of any source is proportional to $\dfrac{e^{-jkr}}{r}$ *Spherical Wave*

The region of the spectrum that impacts the far field is $\boldsymbol{k_{\rho s} < k_0}$

When this approximation is valid?

Direct PW

$$\vec{f}^{far}(\vec{\boldsymbol{r}}) = jk_{zs}\widetilde{\boldsymbol{G}}^{fc}(k_{xs},k_{ys},z,z_s)\vec{\boldsymbol{C}}_0(k_{xs},k_{ys})e^{jk_{zs}|z-z_s|}\frac{e^{-jkr}}{2\pi r}$$

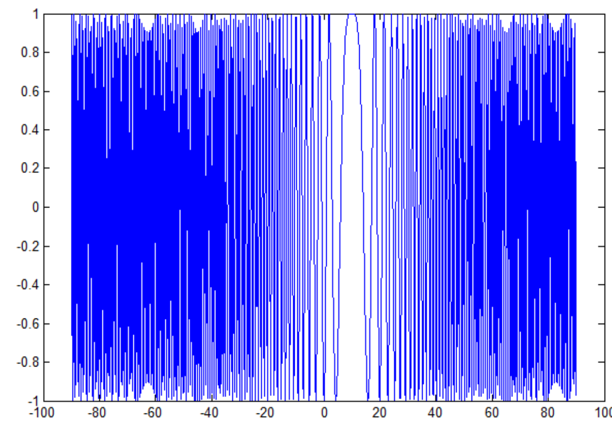


$$k_{xs} = k_0 sin\theta cos\phi \qquad k_{ys} = k_0 sin\theta sin\phi$$

$k_0 r \gg 1$
Highly oscilalting phase

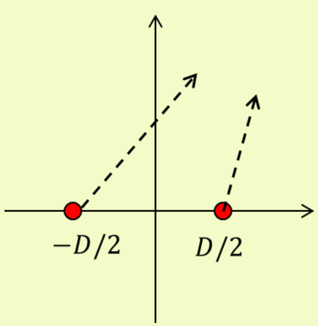
$\vec{C}_0$ has to be slowly varying

$$\frac{2D^2}{\lambda} < r$$

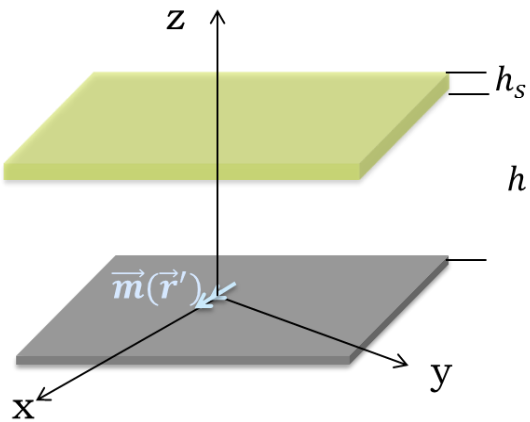


One could extend the region of applicability by *identifying the oscillating terms of the current and performing an asymptotic evaluation for each of them:*

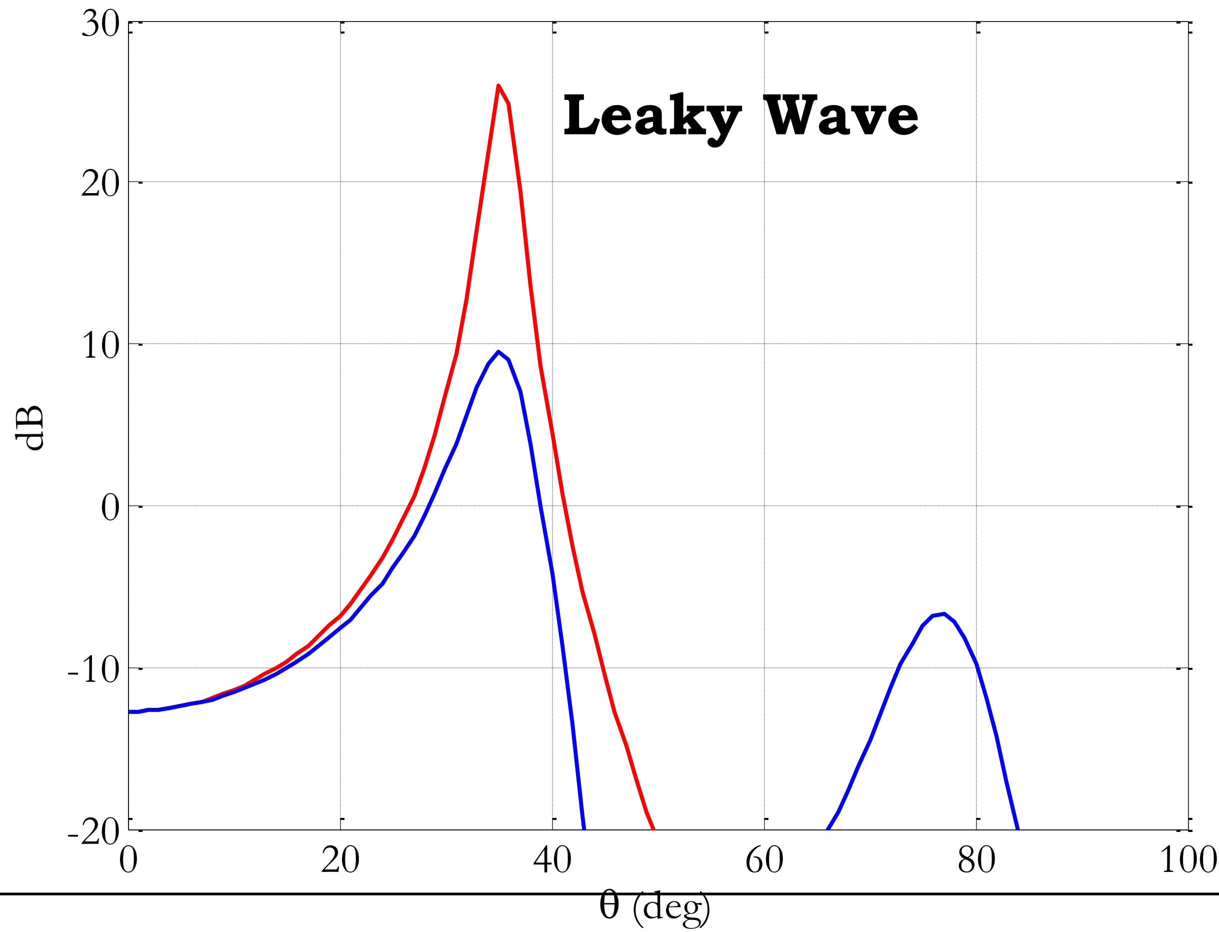
$$C(k_x) = D sinc\left(\frac{k_x D}{2}\right)$$
$$= \frac{e^{jk_xD/2} - e^{-jk_xD/2}}{jk_x}$$



Example 2: Slot in a dielectric stratification



$$\vec{E}^{far}(\vec{\boldsymbol{r}}) = jk_{zs}\tilde{\boldsymbol{G}}^{\boldsymbol{ej}}(k_{xs}, k_{ys}, z, 0)\vec{\boldsymbol{J}}_{eq}(k_{xs}, k_{ys})e^{jk_{zs}|z|}\frac{e^{-jkr}}{2\pi r}$$



The far field of the small source is proportional to the SGF region defined by

$$k_\rho \leq k_0$$

Properties of the spectral GF integral

$$\bar{\bar{G}}^{fc}(\vec{r}-\vec{r}') = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \begin{bmatrix} G_{xx}(k_x,k_y,z,z') & G_{xy}(k_x,k_y,z,z') & G_{xz}(k_x,k_y,z,z') \\ G_{yx}(k_x,k_y,z,z') & G_{yy}(k_x,k_y,z,z') & G_{yz}(k_x,k_y,z,z') \\ G_{zx}(k_x,k_y,z,z') & G_{zy}(k_x,k_y,z,z') & G_{zz}(k_x,k_y,z,z') \end{bmatrix} e^{-jk_x(x-x\prime)}e^{-jk_y(y-y\prime)}dk_x dk_y$$

Changing the Integration Domain

$$\boldsymbol{f}(\vec{r}) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\widetilde{\boldsymbol{G}}^{fc}(k_x,k_y,z,z')\boldsymbol{C}_s(k_x,k_y)e^{-jk_xx}e^{-jk_yy}dk_xdk_y$$

$$\widetilde{\boldsymbol{G}}^{hm} = \begin{bmatrix} -\frac{i_{TE}k_x^2 + i_{TM}k_y^2}{k_\rho^2} & \frac{(i_{TM} - i_{TE})k_xk_y}{k_\rho^2} \\ \frac{(i_{TM} - i_{TE})k_xk_y}{k_\rho^2} & -\frac{i_{TM}k_x^2 + i_{TE}k_y^2}{k_\rho^2} \\ \frac{k_x}{\varsigma k}v_{TE} & \frac{k_y}{\varsigma k}v_{TE} \end{bmatrix}$$

The solution of the transmission line does not depend on α

$$i_{TE} = i_{TE}(k_\rho),$$
$$i_{TM} = i_{TM}(k_\rho)$$

$$k_{zi} = \sqrt{k_i^2 - k_\rho^2}$$

Change integration domain into cylindrical coordinates:

$$k_x = k_\rho cos\alpha \quad k_y = k_\rho sin\alpha \quad \vec{k}_\rho = k_x\hat{x} + k_y\hat{y}$$

$$x = \rho cos\phi \quad y = \rho sin\phi \quad \vec{\rho} = x\hat{x} + y\hat{y}$$

$$\boldsymbol{f}(\vec{r}) = \frac{1}{4\pi^2}\int_{0}^{\pi}\int_{-\infty}^{\infty}\widetilde{\boldsymbol{G}}^{fc}(k_\rho,\alpha,z,z')\boldsymbol{C}_s(k_\rho,\alpha)e^{-j\vec{\boldsymbol{k}}_\rho\cdot\vec{\boldsymbol{\rho}}}k_\rho dk_\rho d\alpha$$

Can we simplify the integration?

$$\vec{f}(\vec{r}) = \frac{1}{4\pi^2}\int_0^{\pi}\int_{-\infty}^{\infty} \tilde{G}^{fc}(k_\rho, \alpha, z, z')\,\vec{C}(k_\rho, \alpha)\, e^{-j\vec{k}_\rho\cdot\vec{\rho}}\; k_\rho dk_\rho d\alpha$$

If we want to calculate the field at a *relatively large distance* from the source:

- Far field
- Surface wave contribution

The integral will be dominated by the oscillating terms (exponentials) or/and by the singularities (critical spectral points) of the SGF

The current can be considered a slow varying function at a critical point of the spectrum $(k_{\rho C}, \alpha_C)$, i.e. a stationary point or singularity, and therefore it can be extracted from the integral

$$\vec{f}_c(\vec{r}) \approx \frac{1}{4\pi^2}\int_0^{\pi}\int_{-\infty}^{\infty} \tilde{G}^{fc}(k_\rho, \alpha, z, z')\, e^{-j\vec{k}_\rho\cdot\vec{\rho}}\; k_\rho dk_\rho d\alpha\,\vec{C}(k_{\rho c}, \alpha_c)$$

Properties of the k_ρ-integral

$$\vec{\boldsymbol{f}}(\vec{r}) = \frac{1}{4\pi^2}\int_0^{\pi}\int_{-\infty}^{\infty} \tilde{\boldsymbol{G}}^{fc}(k_\rho, \alpha, z, z')e^{-j\vec{\boldsymbol{k}}_{\boldsymbol{\rho}}\cdot\vec{\boldsymbol{\rho}}}\; k_\rho dk_\rho d\alpha \vec{\boldsymbol{C}}(k_{\rho s}, \alpha_s)$$

While $\tilde{\boldsymbol{G}}^{fc}(k_\rho, \alpha, z, z')$ depends on the nature of the z-stratification, and position of source and observation in z, general **asymptotic properties of the field** can be inferred from the analytic behaviour of this integrand by looking into its critical points:

- Stationary points
- **Branch points**
- **Poles**

Let us begin by looking into the integrand of the free space case

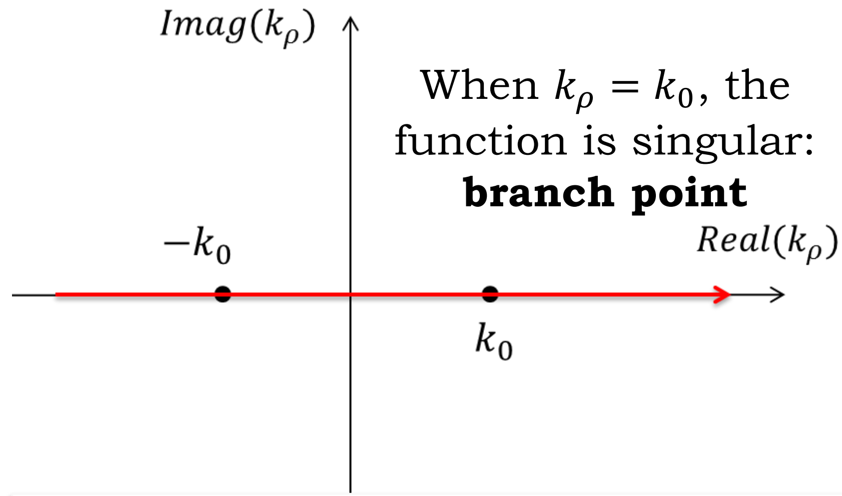
$$v_{TM}(k_\rho, z, z') = \frac{\varsigma_0 k_{z0}}{2k_0} e^{-jk_{z0}|z-z'|}$$

$$v_{TE}(k_\rho, z, z') = \frac{\varsigma_0 k_0}{2k_{z0}} e^{-jk_{z0}|z-z'|}$$

k_ρ- Integration Path

$$\int_{-\infty}^{\infty} \tilde{\boldsymbol{G}}^{fc}(k_\rho, \alpha, z, z') e^{-j\vec{k}_\rho \cdot \vec{\rho}} \; k_\rho dk_\rho$$

$$v_{TE}(k_\rho, z, z') = \frac{\varsigma_0 k_0}{2k_{z0}} e^{-jk_{z0}|z-z'|}$$



When $k_\rho = k_0$, the function is singular: **branch point**

Radiation condition: the integral should converge at infinity.
A propagating solution is required for |z-z'| from 0 to infinity

$$k_z = \beta + j\alpha$$

$$e^{-jk_z|z-z'|} = e^{-j\beta|z-z'|} e^{\alpha\,|z-z'|}$$

$$\beta > 0 \; for \; |k_\rho| < k_0$$

$$\alpha < 0 \; for \; |k_\rho| > k_0$$

One cannot integrate over the real axis.

Need to look for an *appropriate* integration path

This path will depend on how the square root in k_{z0} is calculated

Multivalued function in the Complex Plane

The square root is a multivalued function:

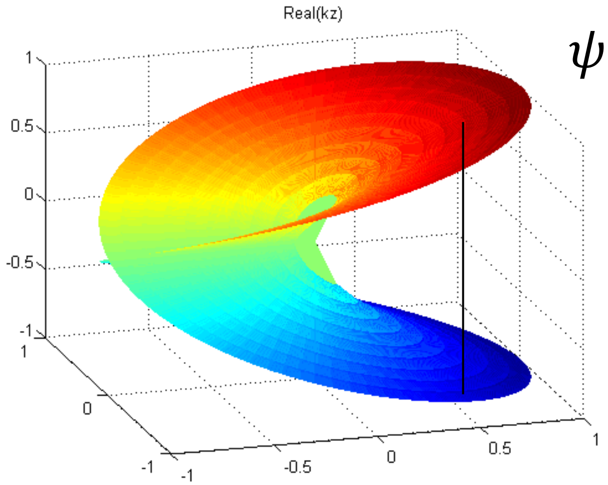
$$k_z = \pm\sqrt{k^2 - k_\rho^2}$$

We can still use a unique value (e.g. the positive sign)

However **this choice will not make the function continuous over the whole complex plane**

$$s = \rho e^{j\psi} \qquad k_z = \sqrt{s} = \sqrt{\rho}e^{j\psi/2}$$

$$s = \rho e^{j(\psi+2\pi)} \quad k_z = \sqrt{s} = \sqrt{\rho}e^{j\left(\frac{\psi}{2}+\pi\right)}$$



$$\psi = [-2\pi, 2\pi]$$

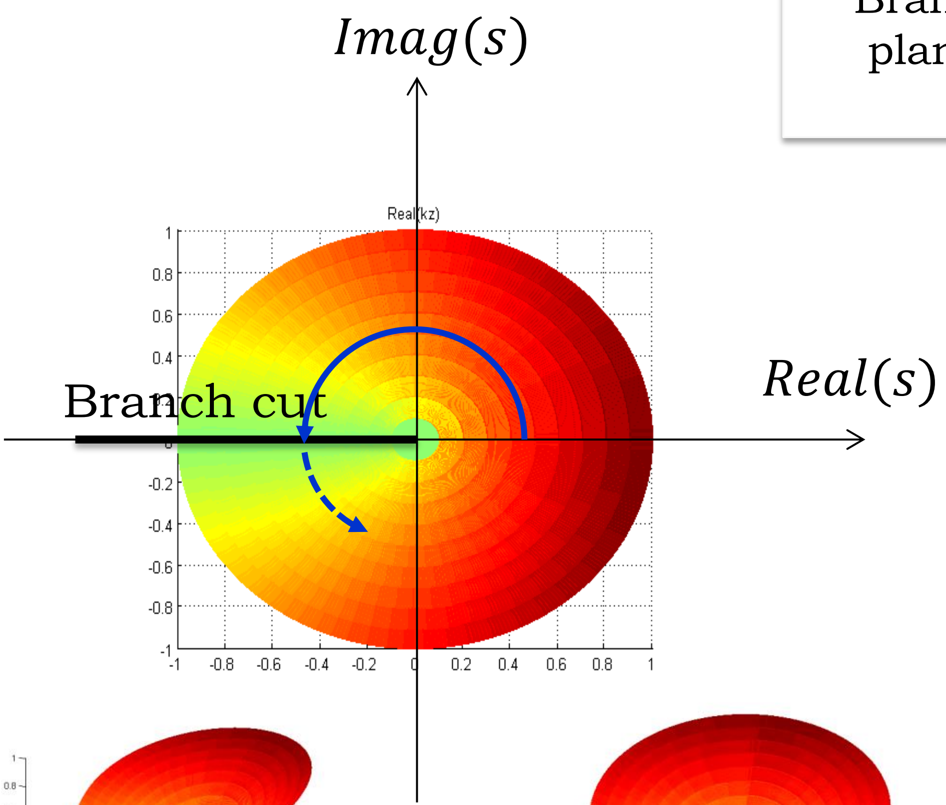
$$\rho = 0.5, \psi = \pi$$
$$k_z = j/\sqrt{2}$$

$$\rho = 0.5, \psi = -\pi$$
$$k_z = -j/\sqrt{2}$$

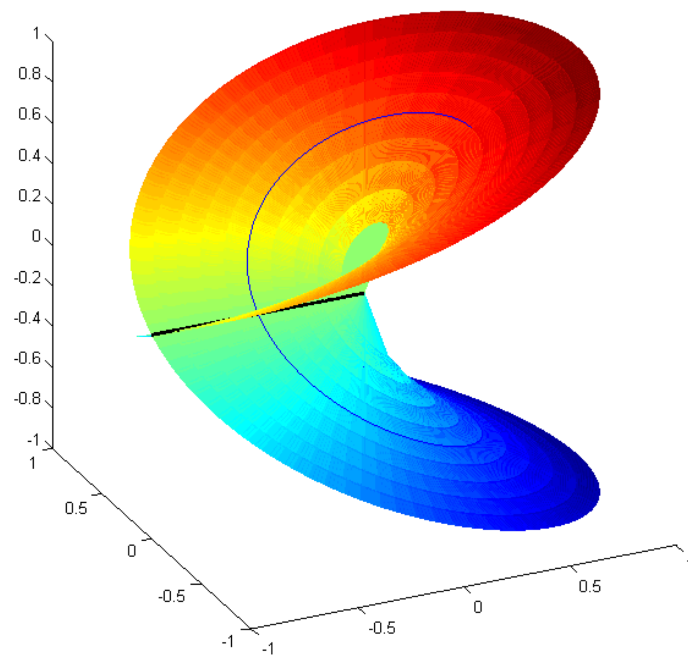
Branch Cuts in the Complex Plane

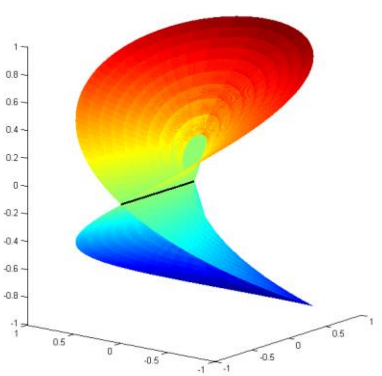
$$k_z = \sqrt{s} = \sqrt{\rho} e^{j\psi/2}$$

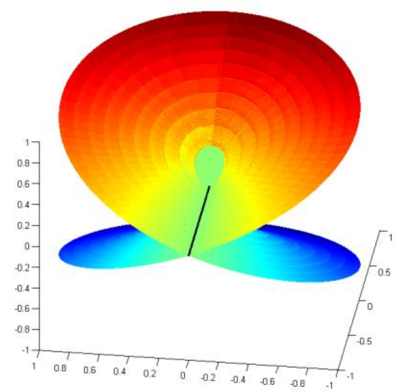
Branch cut is a curve in the complex plane across which the multivalued function is discontinuous



$\psi = [-2\pi, 2\pi]$

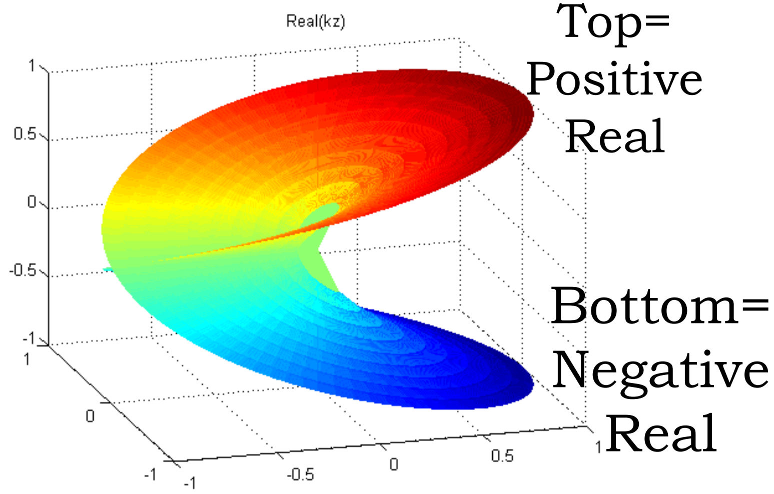


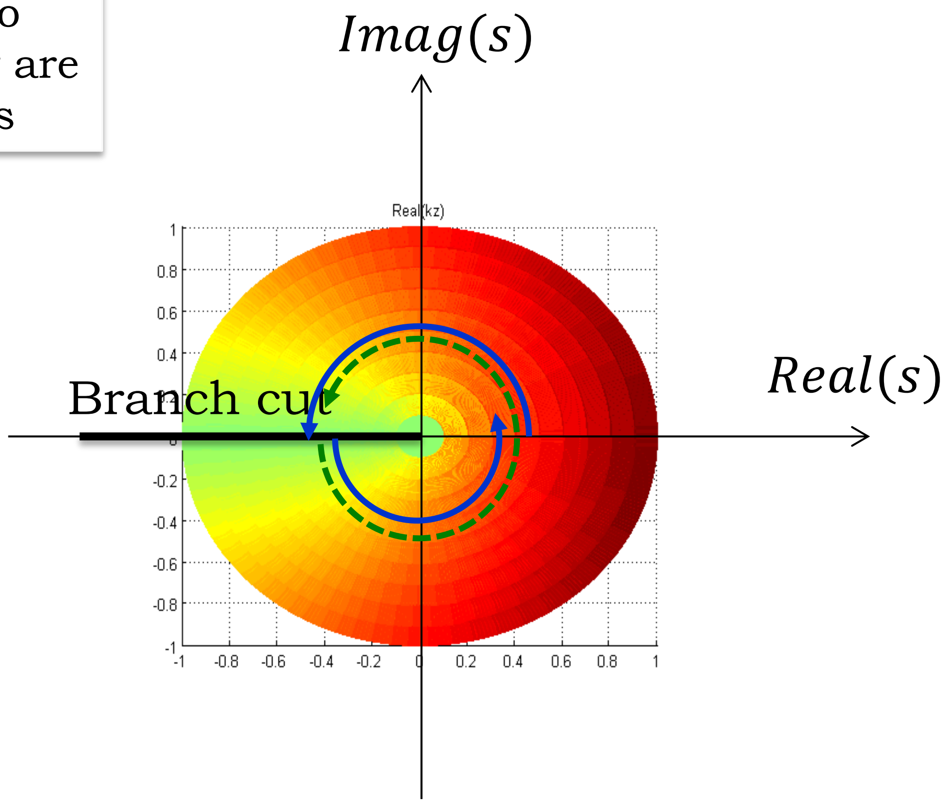




Riemann Surfaces

Riemann surfaces is another way to represent multi-value functions. They are connected through the branch cuts





$\rho = 0.5, \psi = 0$
$k_z = 1/\sqrt{2}$

$\rho = 0.5, \psi = \pi/2$
$k_z = (1 + j)/2$

$\rho = 0.5, \psi = \pi$
$k_z = j/\sqrt{2}$

$\rho = 0.5, \psi = 3\pi/2$
$k_z = (-1 + j)/2$

$\rho = 0.5, \psi = 2\pi$
$k_z = -1/\sqrt{2}$

$\rho = 0.5, \psi = 5\pi/2$
$k_z = -(1 + j)/2$

$\rho = 0.5, \psi = 3\pi$
$k_z = -j/\sqrt{2}$

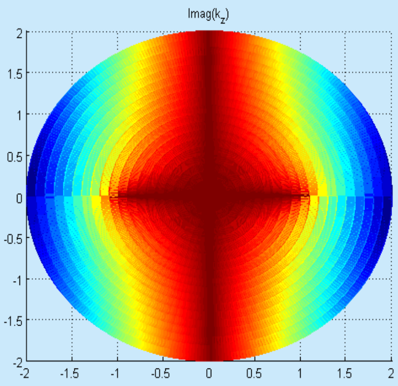
$\rho = 0.5, \psi = 7\pi/2$
$k_z = (1 - j)/2$

Preferred Convention

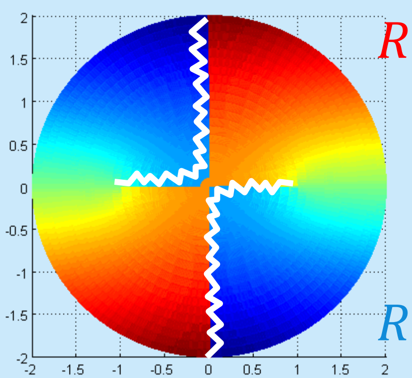
Selection of square root multivalues on the Riemann sheets to obtain a **unique specification of the integrand in the complex plane**

$$k_z = -j\sqrt{-\left(k^2 - k_\rho^2\right)}$$

Top Riemann Sheet:



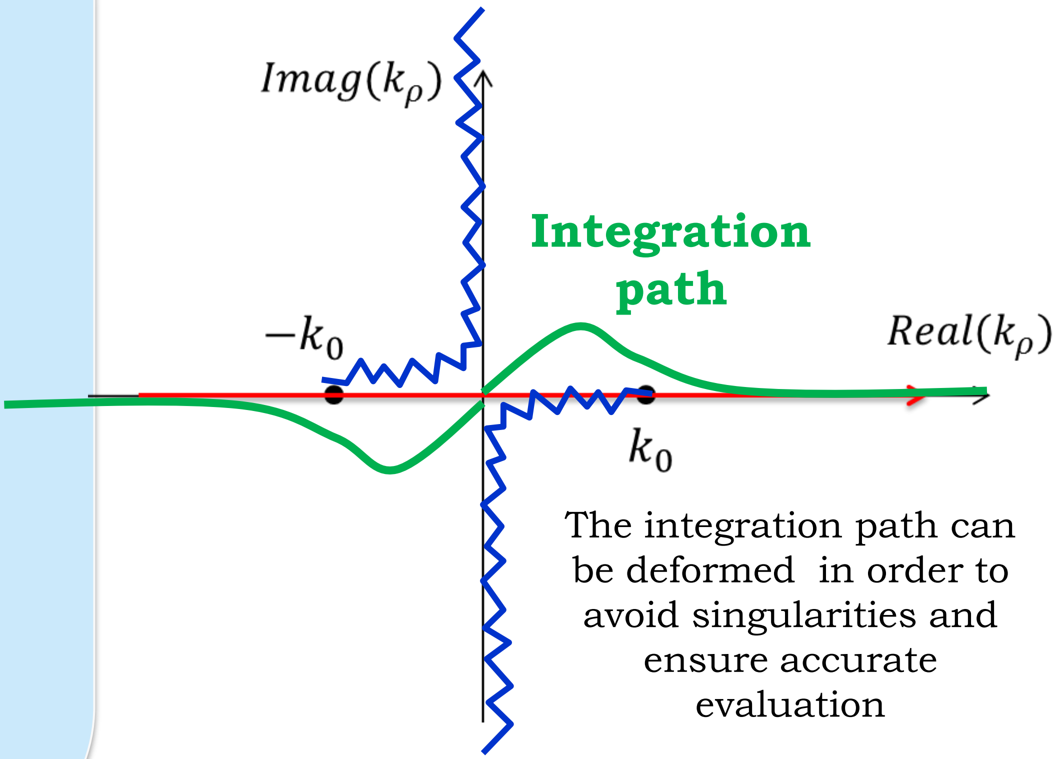
$Imag(k_z) < 0$



$Real(k_z) > 0$

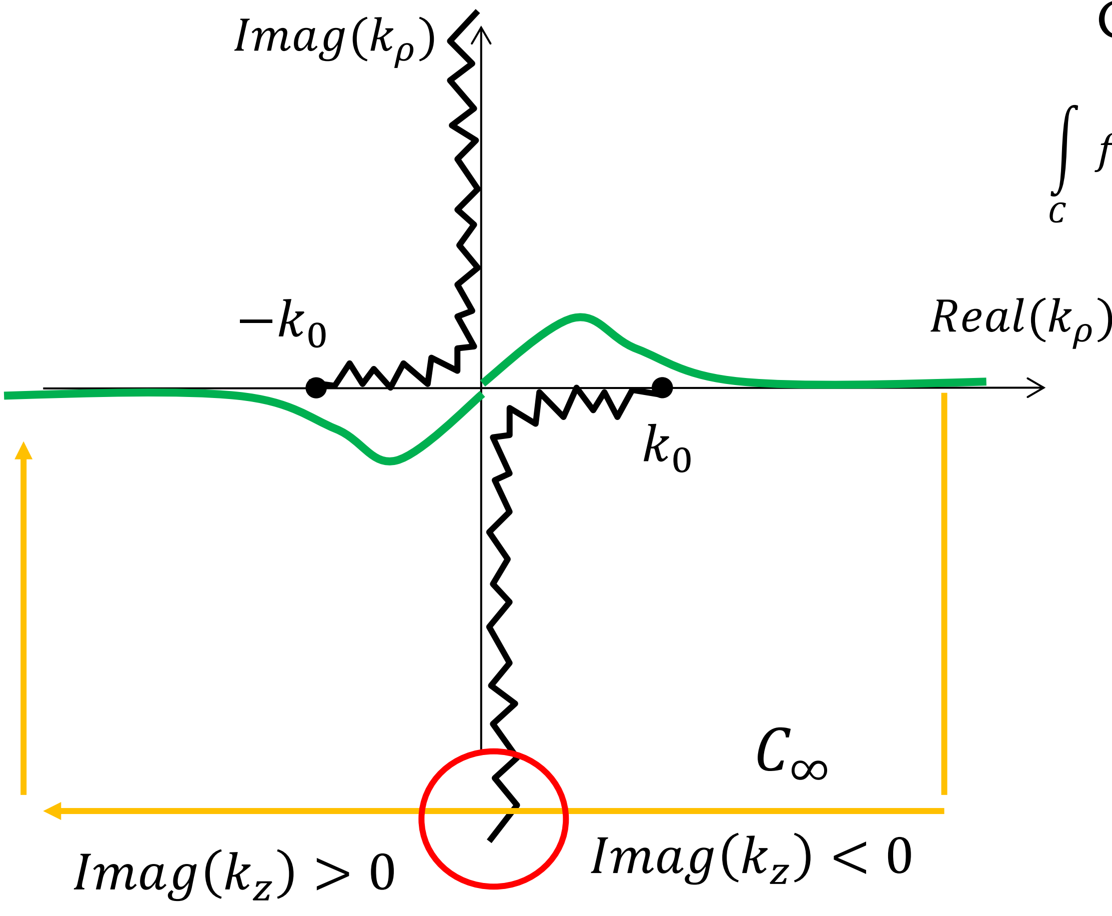
$Real(k_z) < 0$

Branch cuts



The integration path can be deformed in order to avoid singularities and ensure accurate evaluation

Changing integration domain



Crossing the branch

Cauchy's theorem:

$$\int_C f(k_\rho)dk_\rho = -2\pi j \sum_i Res[k_{\rho i}]$$

In free space, there is no pole singularities:

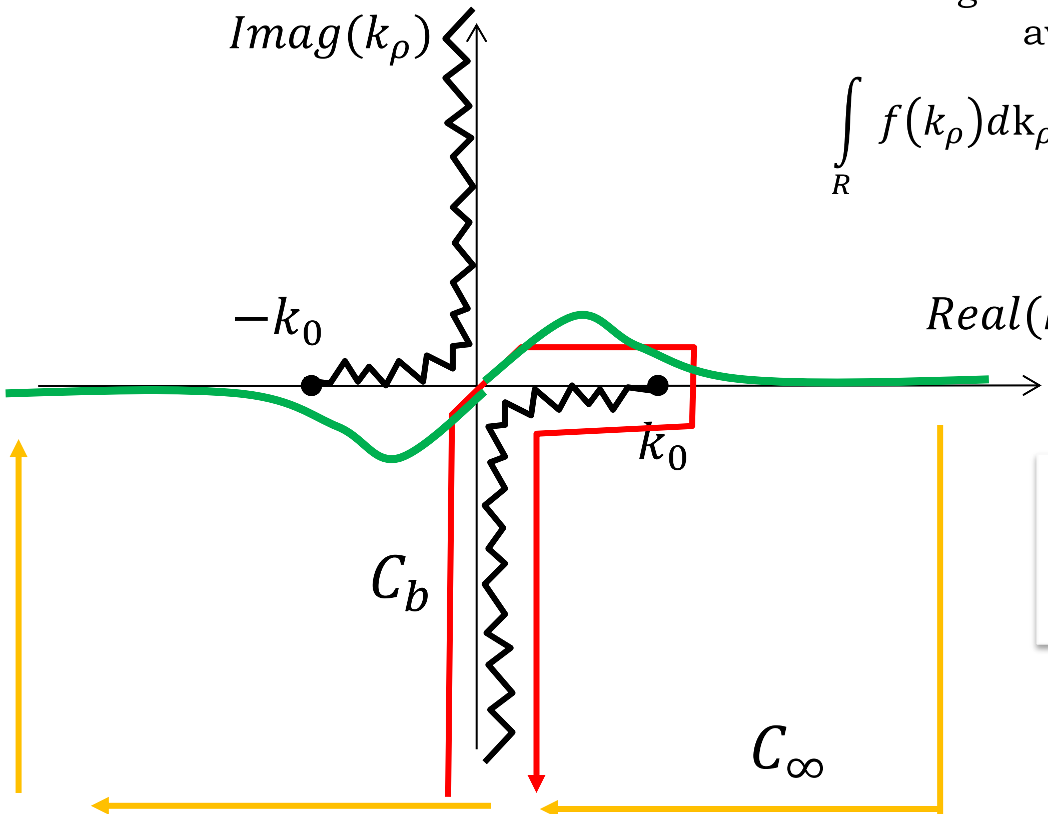
$$\int_C f(k_\rho)dk_\rho =$$

$$\int_R f(k_\rho)dk_\rho + \int_{C_\infty} f(k_\rho)dk_\rho = 0$$

Integration around the branch

The integration path can be deformed in order to avoid crossing the branch

$$\int_R f(k_\rho)d\mathrm{k}_\rho + \int_{C_\infty} f(k_\rho)d\mathrm{k}_\rho - \int_{C_b} f(k_\rho)d\mathrm{k}_\rho = 0$$



$$\int_R f(k_\rho)d\mathrm{k}_\rho = \int_{C_b} f(k_\rho)d\mathrm{k}_\rho$$

$$\int_{-\infty}^{\infty} \tilde{\boldsymbol{G}}^{fc}(k_\rho, \alpha, z, z')e^{-j\vec{\boldsymbol{k}}_\rho\cdot\vec{\boldsymbol{\rho}}} \; k_\rho dk_\rho$$

$$\int_{C_b} \tilde{\boldsymbol{G}}^{fc}(k_\rho, z, z')e^{-j\vec{\boldsymbol{k}}_\rho\cdot\vec{\boldsymbol{\rho}}} \; k_\rho dk_\rho$$

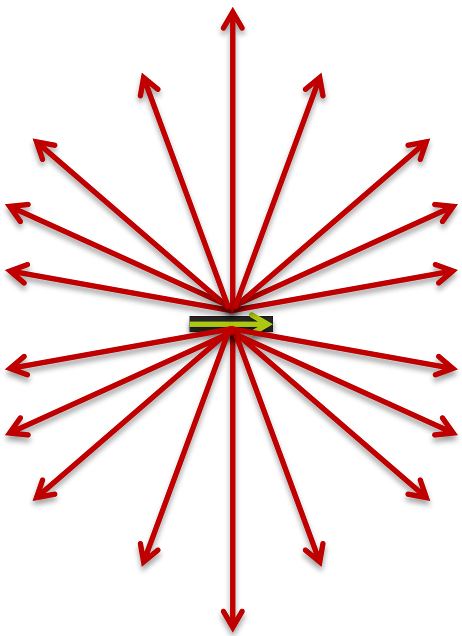
The integration along the branch cut gives the radiated field (**Space Wave)**

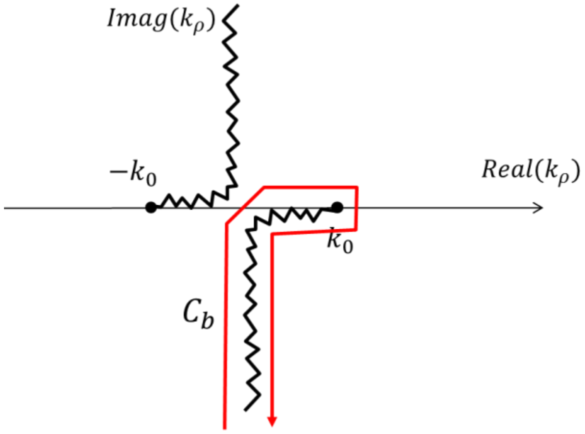
$$e^{-jk_z z} = e^{-\alpha z}$$

$$Imag(k_z) < 0$$

$$\int_{C_\infty} f(k_\rho)d\mathrm{k}_\rho = 0$$

Space Wave





The space wave integral is proportional to the free space scalar potential

The field associated has a decay of **1/r**

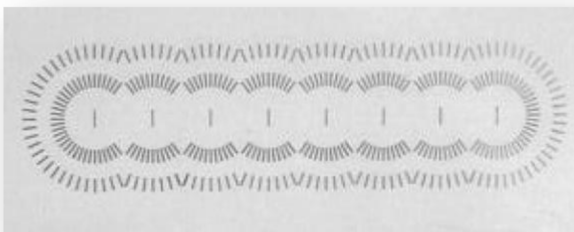
It can be demonstrated using the GF derived from a potential parallel to the source

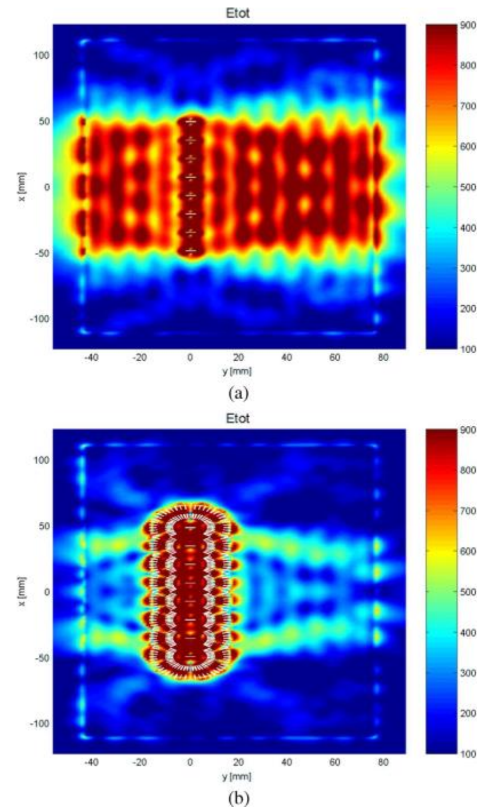
The far field is a space wave evaluated at large observation distances

In free space:

$$\int_{-\infty}^{\infty} \tilde{\boldsymbol{G}}^{fc}(k_\rho, \alpha, z, z') e^{-j\vec{\boldsymbol{k}}_\rho \cdot \vec{\boldsymbol{\rho}}} \; k_\rho dk_\rho = I_b$$

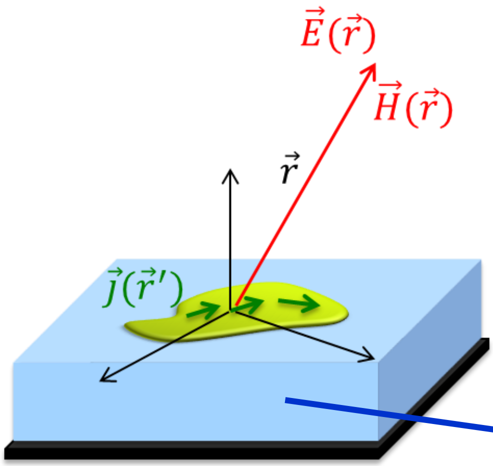
Fields radiated into the dielectric (surface wave)





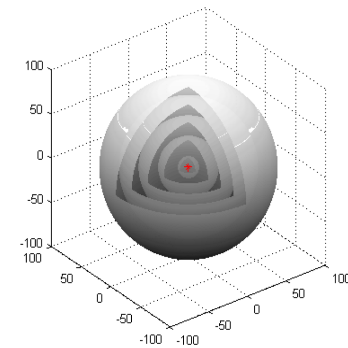
Fields Radiated by Printed Antennas

$$\boldsymbol{f}(\vec{r}) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\widetilde{\boldsymbol{G}}^{fc}(k_x,k_y,z,z')\boldsymbol{C}_s(k_x,k_y)e^{-jk_x x}e^{-jk_y y}dk_x dk_y$$



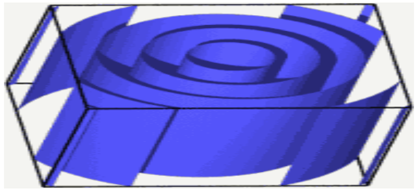
Far field: Spherical wave is emerging when the observation point is in the infinite medium

$$\frac{e^{-jk_0 r}}{r}$$



Surface wave: Cylindrical wave is emerging when the observation point is inside a dielectric substrate with finite thickness

$$\frac{e^{-jk_{sw}\rho}}{\sqrt{\rho}}$$



General Stratification

$$\int_{-\infty}^{\infty} \tilde{\boldsymbol{G}}^{fc}(k_\rho, \alpha, z, z') e^{-j\vec{\boldsymbol{k}}_\rho \cdot \vec{\boldsymbol{\rho}}} \; k_\rho dk_\rho$$

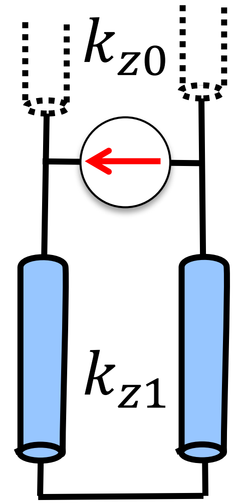
For n-dielectric layers, we have n different square roots

$$k_{zi} = \pm\sqrt{k_i^2 - k_\rho^2}$$

The dependency of $\tilde{G}^{fc}(k_\rho, \alpha, z, z')$ on k_{zi} is always even except for infinite mediums

There are not two different values associated to the $\pm$

$$Z_d = jZ_s \tan(k_{zs}h)$$



Branch cuts are only present in infinite open mediums. They give rise to the space wave field.

Pole singularities in k_ρ arise in dielectric stratifications.

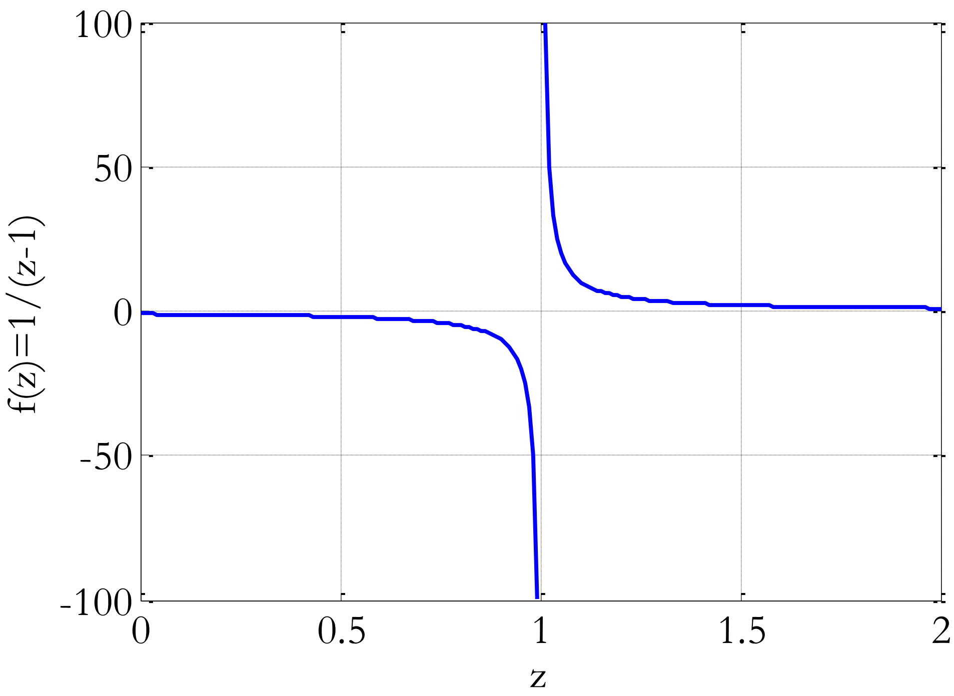
Pole Singularities

$$f(z) = \frac{g(z)}{(z-a)^n}$$

a is a pole of $f(z)$of order n provided that $g(a) \neq 0$

$$f(z) = \frac{1}{h(z)}$$

The zeros of $h(z)$ are the poles of $f(z)$

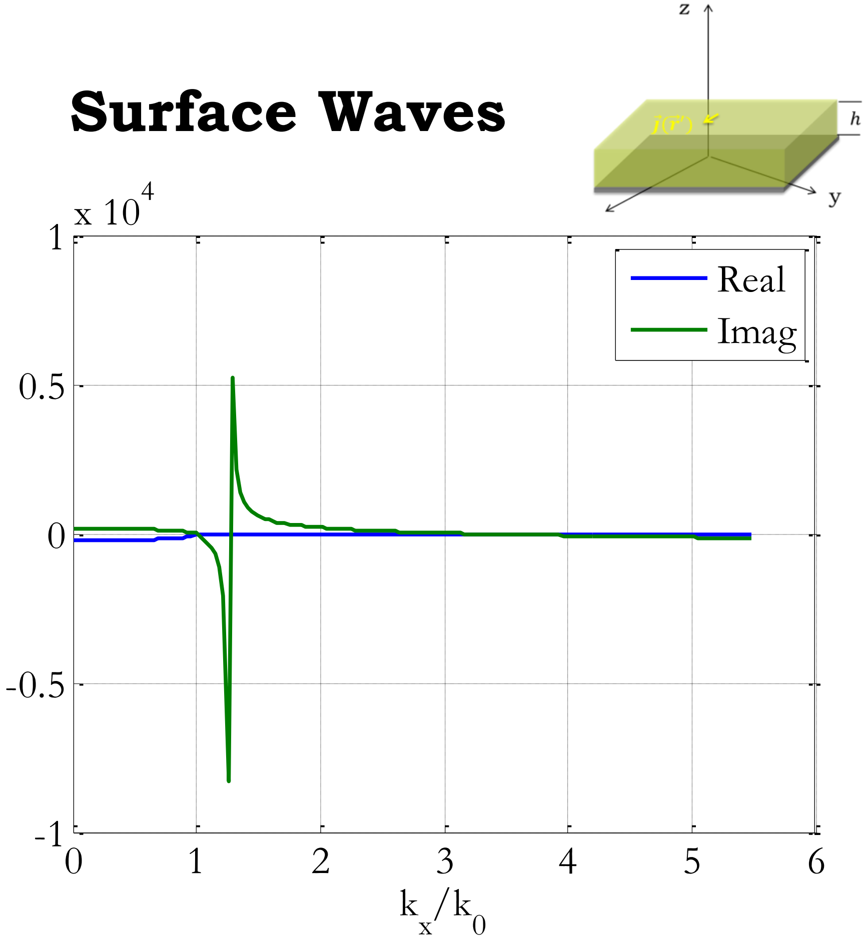


$$\boxed{h(z) = 0}$$

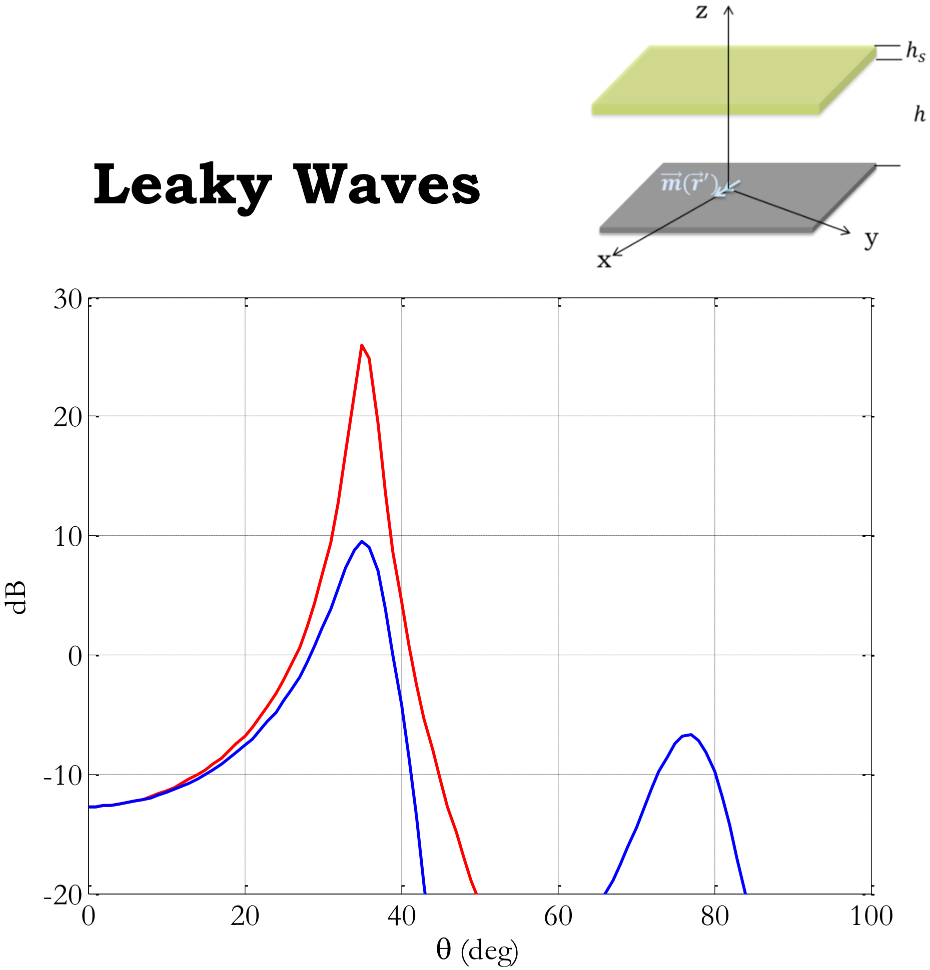
Dispersion equation to find the pole singularities

Examples

Surface Waves



Leaky Waves

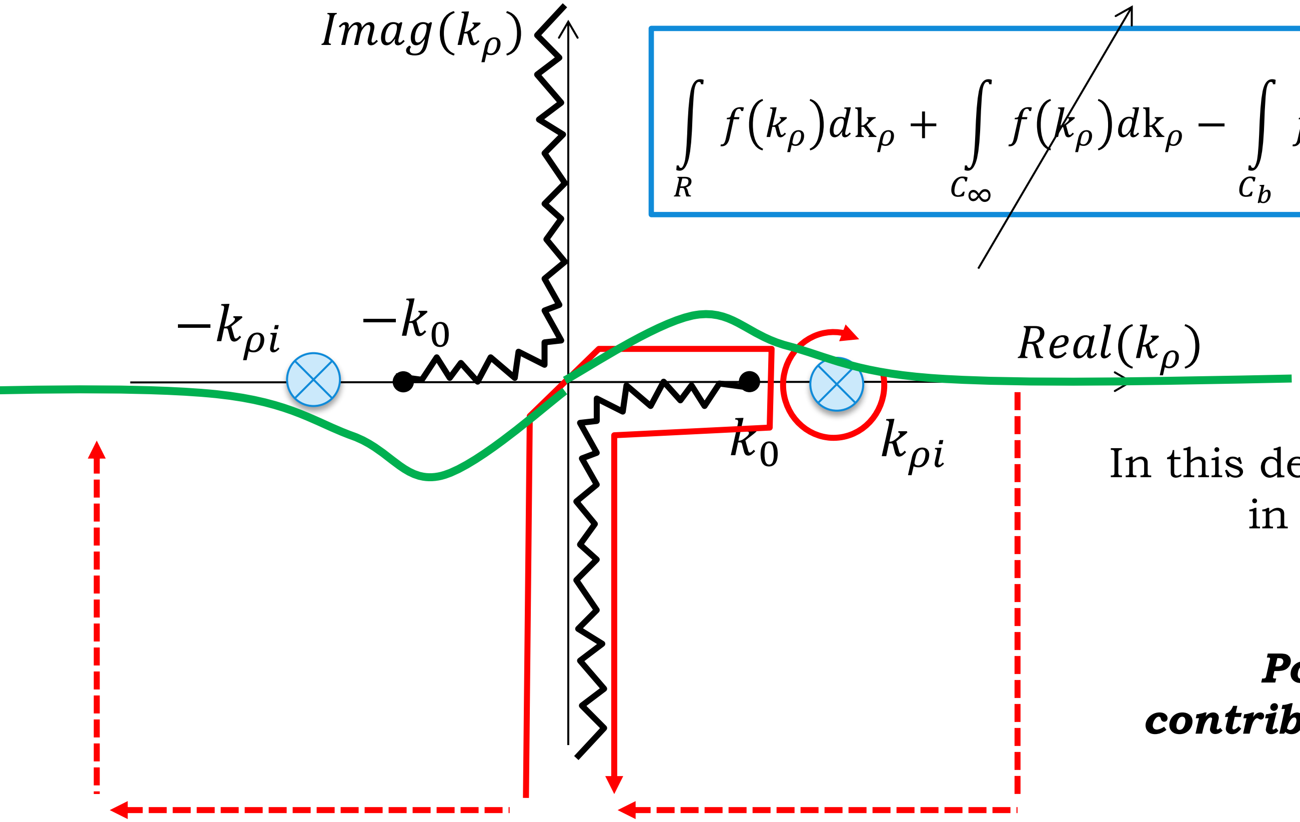


SW Field Contribution

$$\int_{-\infty}^{\infty} \tilde{\boldsymbol{G}}^{fc}(k_\rho, \alpha, z, z') e^{-j\vec{k}_\rho \cdot \vec{\rho}} \; k_\rho dk_\rho$$

In dielectric stratifications there is also pole singularities:

$$\int_R f(k_\rho) dk_\rho + \int_{C_\infty} f(k_\rho) dk_\rho - \int_{C_b} f(k_\rho) dk_\rho = -2\pi j \sum_i Res[k_{\rho i}]$$



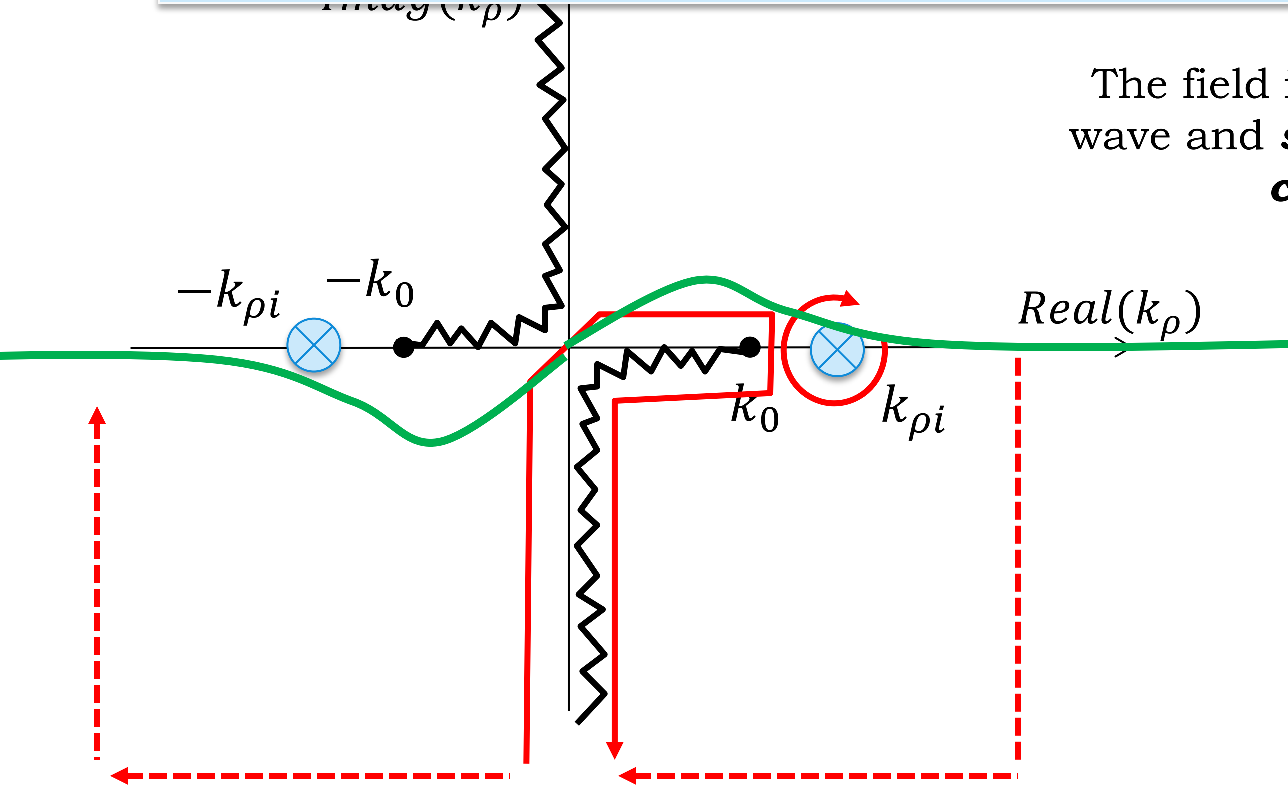
In this deformation path, only poles in the top Riemann sheet are captured

Poles gives rise to discrete contributions to the field in the form of waves

SW Field Contribution

$$\int_{-\infty}^{\infty} \tilde{\boldsymbol{G}}^{fc}(k_\rho, \alpha, z, z') e^{-j\vec{k}_\rho \cdot \vec{\rho}} \; k_\rho dk_\rho$$

$$= \int_{C_b} \tilde{\boldsymbol{G}}^{fc}(k_\rho, z, z') e^{-j\vec{k}_\rho \cdot \vec{\rho}} \; k_\rho dk_\rho - 2\pi j \sum_i Res\left[\tilde{\boldsymbol{G}}^{fc}(k_\rho, z, z') e^{-j\vec{k}_\rho \cdot \vec{\rho}} \; k_\rho\right]\Big|_{k_\rho = k_{\rho i}}$$



The field is divided into a space wave and ***surface wave discrete contributions***

Important Points

- The SGF has two **main singularities**: branch points and poles
- Depending on the observation point, the spectral Green's function has different critical points: stationary phase (far field), branch (space wave), poles (surface wave field)
- **Branch cuts** are defined depending on the *sign of the square root* in an infinite medium. This sign is related to the radiation condition.
- The **integration path** is deformed to avoid these singularities and achieve best convergence.
- The integration around the branch gives the **space wave**
- The **stationary phase points** give the field at large distances. For instance the far field as a single plane wave contribution in each observation point
- **Real pole singularities** give rise to surface wave *discrete* field contributions