Spectral Domain Methods in Electromagnetics EE4620

Lecture # 1

Introduction and Spectral Green's Functions for Stratified Media

Andrea Neto

About This Course

Several teachers from the THz Sensing Group:
A. Neto, D. Cavallo, N. LLombart

Preparation for Master Thesis topics

All about spectral techniques for analyzing antennas

Lots of Theory and Programming:

Structure
Per week: 1h theory / 1h Matlab instruction
Every week there are Matlab assignments (focus on smart programing)
Final Mark: Assignments + Final individual presentation (last day of the course)

Schedule EE4620 - Spectral Domain Methods in Electromagnetics 2023

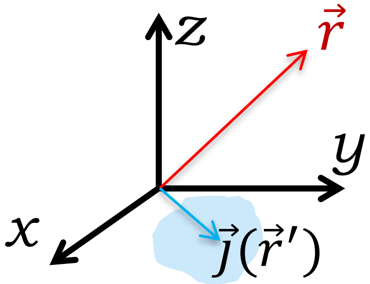
	Shahab	Huashen	Alejandr	Nick	Sander	Caspar	Riccardo
Tu Apr 25 10:45-12:45 Lecture Hall J **Stratified GF (Theory)**	**Tu May 2 10:45-12:45** Lecture Hall J **Stratified GF (Matlab)**	**Tu May 9 10:45-12:45** Lecture Hall J **Far field (Matlab)**	**Tu May 16 10:45-12:45** Lecture Hall J **Surface waves (Matlab)**	**Tu May 23 10:45-12:45** Lecture Hall J **Leaky waves (Matlab)**	**Tu May 30 10:45-12:45** Lecture Hall J **Artificial dielectrics (Matlab)**	**Tu Jun 6 10:45-12:45** Lecture Hall J **Connected array (Matlab)**	**Tu Jun 13 10:45-12:45** Lecture Hall J **Connected array (Matlab)**
We Apr 26 8:45-10:45 Lecture Hall H **Stratified GF (Theory)**	**We May 3 8:45-10:45** Lecture Hall H **Dominant Contrib. (Theory)**	**We May 10 8:45-10:45** Lecture Hall H **Surface waves (Theory)**	**We May 17 8:45-10:45** Lecture Hall H **Leaky waves (Matlab)**	**We May 24 8:45-10:45** Lecture Hall H **Artificial dielectrics (Theory)**	**We May 31 8:45-10:45** Lecture Hall H **Connected array (Theory)**	**We Jun 7 8:45-10:45** Lecture Hall H **Array equiv. circuits (Theory)**	**We Jun 14 8:45-10:45** Lecture Hall H **Antenna Array Design**

Andrea Nuria Daniele Matlab

Green's functions

$\tilde{G}(\vec{r},\vec{r}')$

A Green's function is
The Field, $\tilde{G}$, in $\vec{r}$
radiated by an elementary $\delta(\vec{r}-\vec{r}')\hat{p}'$
equivalent current source,
in $\vec{r}'$



In the case that one investigates
electric field radiated by **electric currents**

$$\boxed{\vec{e}(\vec{r},\vec{r}') = \iiint_V \tilde{G}(\vec{r},\vec{r}') * \vec{J}(\vec{r}')d\vec{r}'}$$

Convolution integral between Green's Functions and Equivalent currents descries the Fields

These representation is virtually the only one that allows you to solve a non trivial problem

Green's function of electric current in Free Space

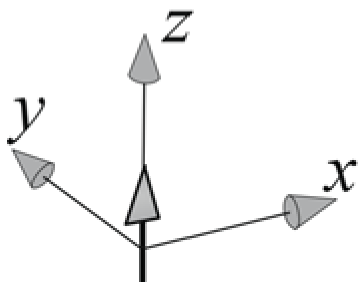
Potential are the tool invented to render easier the derivations of Green's functions

From the magnetic potential A

$$\vec{e}(\vec{r}) = -j\omega\vec{A} - \frac{j}{\omega\epsilon\mu}\nabla\left[\nabla\cdot\vec{A}\right]$$

$$\vec{h}(\vec{r}) = \frac{1}{\mu}\nabla\times\vec{A}$$

$$\vec{j}(\vec{r}) = \delta(x)\delta(y)\delta(z)\hat{z}$$



$$\begin{cases} \lim\limits_{r\to\infty}\vec{h}(\vec{r}) = jk\dfrac{sin\theta e^{-jkr}}{4\pi r}\hat{\phi} \\ \lim\limits_{r\to\infty}\vec{e}(\vec{r}) = jk\zeta\dfrac{sin\theta e^{-jkr}}{4\pi r}\hat{\theta} \end{cases}$$

$$\vec{A}(\vec{r},\vec{j}) = \frac{\mu e^{-jk|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}\hat{z}$$

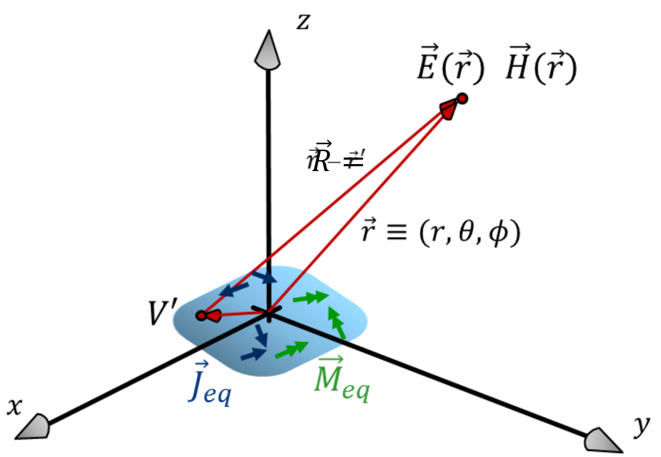
Free Space Green's function

for unbounded, isotropic, homogeneous, medium

Reminder: Quasi Optics

$$\vec{E}(\vec{r}) = -j\omega\mu \iiint_v \tilde{G}^e(\vec{r},\vec{r}')\,\vec{J}(\vec{r}')d\vec{r}' - \iiint_v \tilde{G}^m(\vec{r},\vec{r}')\vec{M}(\vec{r}')d\vec{r}'$$

$$\vec{H}(\vec{r}) = -j\omega\varepsilon \iiint_v \tilde{G}^e(\vec{r},\vec{r}')\,\vec{M}(\vec{r}')d\vec{r}' + \iiint_v \tilde{G}^m(\vec{r},\vec{r}')\vec{J}(\vec{r}')d\vec{r}'$$



where
$$\begin{cases} \tilde{G}^e(\vec{r},\vec{r}') = \left(\tilde{I} + \dfrac{\vec{\nabla}\vec{\nabla}}{k^2}\right) g(\vec{r},\vec{r}') \\ \tilde{G}^m(\vec{r},\vec{r}') = \vec{\nabla} g(\vec{r},\vec{r}') \times \tilde{I} \end{cases}$$

$$g(\vec{r},\vec{r}') = \frac{e^{-jk|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|} = \frac{e^{-jkR}}{4\pi R} \qquad \vec{R} = \vec{r} - \vec{r}'$$

$$\vec{\nabla} g(\vec{r},\vec{r}') = -\left(\frac{1}{R} + jk\right)\frac{e^{-jkR}}{4\pi R}\hat{R}$$

Spectral GF

Reminder: Quasi Optics

For ***free space***, we know the Fourier Transform of scalar Green's function

$$\frac{e^{-jk|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|} = -\frac{1}{(2\pi)^3}\iiint_{-\infty}^{\infty}\frac{e^{-jk_x(x-x')}e^{-jk_y(y-y')}e^{-jk_z(z-z')}}{k^2-k_x^2-k_y^2-k_z^2}dk_x dk_y dk_z$$

Dyadic Free Space Green's functions in spectral domain

$$\bar{\bar{G}}_{fs}^{ej}(\vec{r}-\vec{r}') =$$

$$\frac{-\zeta}{k8\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{bmatrix} k^2-k_x^2 & -k_xk_y & -k_x(\pm k_z) \\ -k_xk_y & k^2-k_y^2 & -k_y(\pm k_z) \\ -k_x(\pm k_z) & -k_y(\pm k_z) & -2j\delta(z-z')+(k^2-k_z^2)\end{bmatrix}\frac{e^{-jk_x(x-x')}e^{-jk_y(y-y')}e^{-jk_z|z-z'|}}{\sqrt{k^2-k_x^2-k_y^2}}dk_x dk_y$$

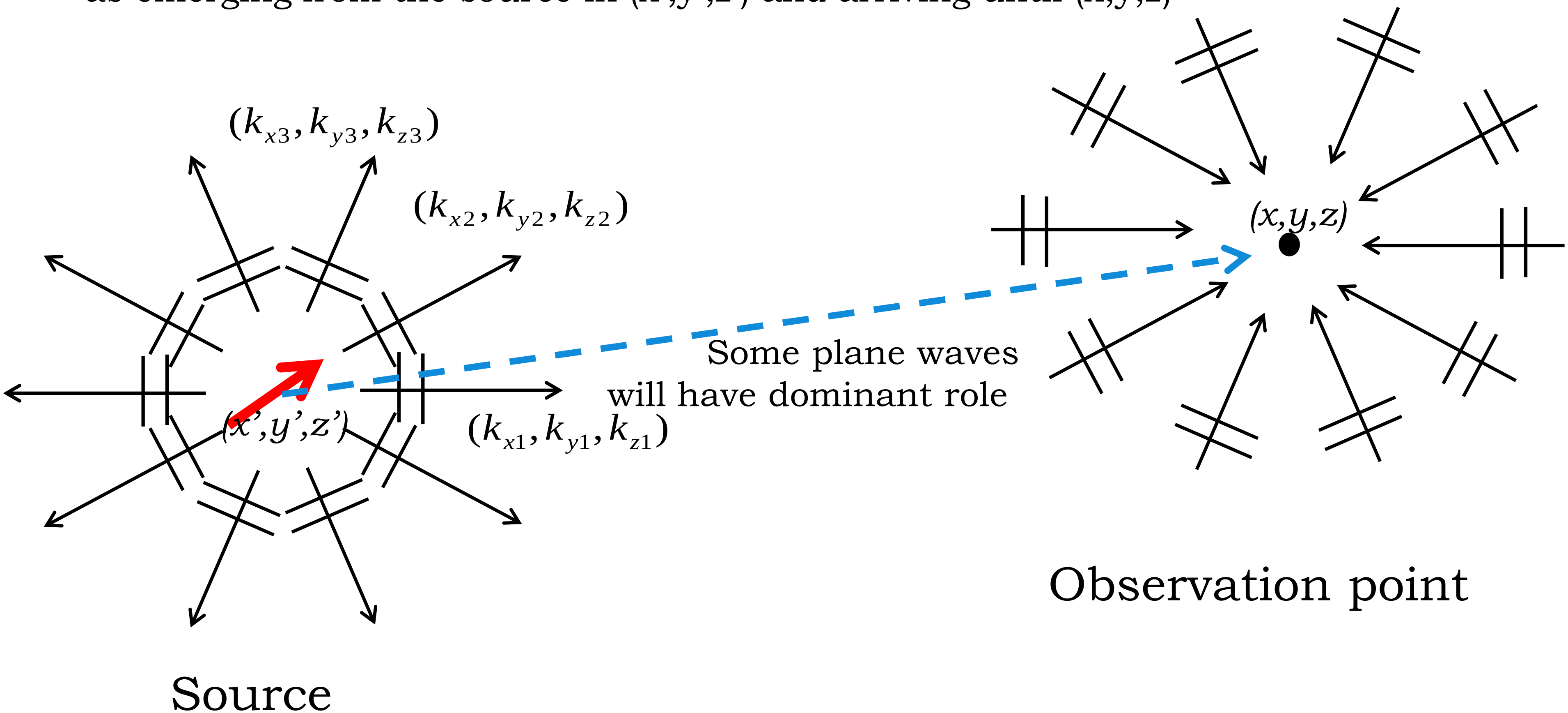
$$\bar{\bar{G}}_{fs}^{hm}(\vec{r}-\vec{r}') =$$

$$\frac{-1}{k\zeta8\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{bmatrix} k^2-k_x^2 & -k_xk_y & -k_x(\pm k_z) \\ -k_xk_y & k^2-k_y^2 & -k_y(\pm k_z) \\ -k_x(\pm k_z) & -k_y(\pm k_z) & -2j\delta(z-z')+(k^2-k_z^2)\end{bmatrix}\frac{e^{-jk_x(x-x')}e^{-jk_y(y-y')}e^{-jk_z|z-z'|}}{\sqrt{k^2-k_x^2-k_y^2}}dk_x dk_y$$

$$(z-z')\,{}^{>}\!\!/_{<}\,0 \Rightarrow \pm \quad k_z = \sqrt{k^2-k_x^2-k_y^2}$$

Plane Wave expansion

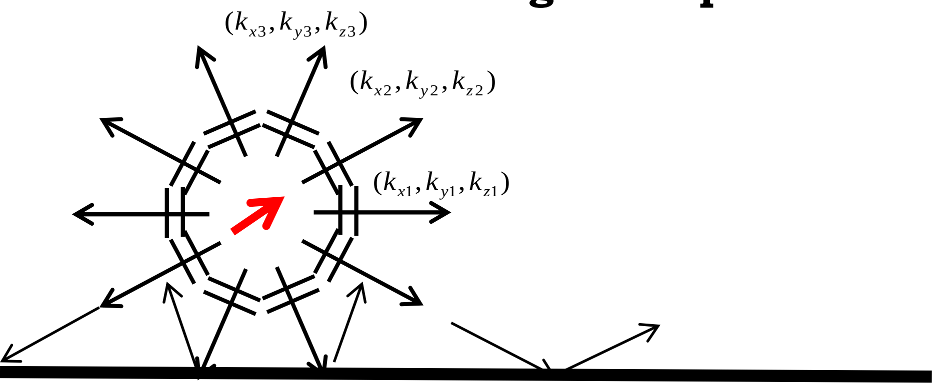
Plane waves are defined over the entire space.
Plane wave expansion represents the total radiated fields
as emerging from the source in (x’,y’,z’) and arriving until (x,y,z)

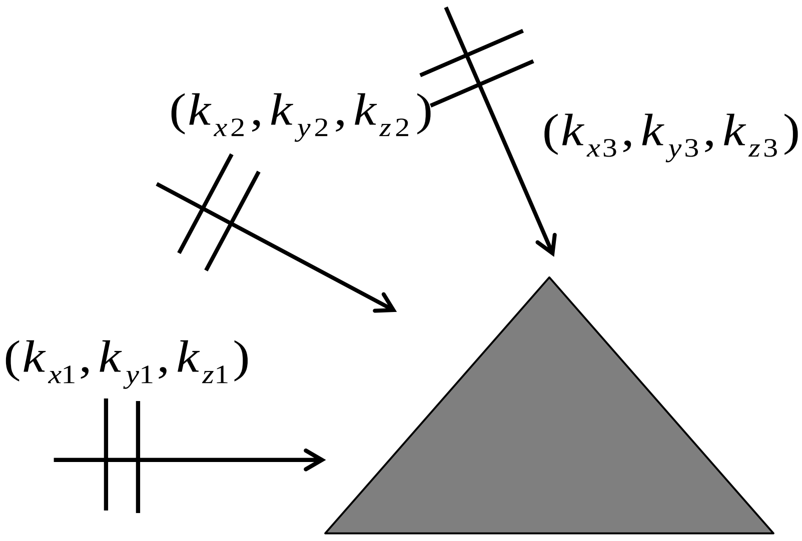


Generic Obstacles

- Power of spectral GF comes from possibility to build other Green's functions:
  if you know response to a single plane wave,
  than you can sum up all contributions to obtain response to elementary current

- **Infinite ground plane**





Far Field Radiation

$$\vec{\boldsymbol{f}}(x,y,z) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \widetilde{\boldsymbol{G}}^{fc}(k_x,k_y,z,z')\, \vec{\boldsymbol{C}}(k_x,k_y) e^{-jk_x x} e^{-jk_y y} dk_x dk_y$$

We know the solution of this integral

$$\vec{\boldsymbol{f}}(x,y,z) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \underbrace{\left(\widetilde{\boldsymbol{G}}^{fc}(k_x,k_y,z,z')\vec{\boldsymbol{C}}(k_x,k_y) k_{z0} e^{jk_{z0}|z-z'|}\right)}_{\text{slow varying function in the surrounding of the } \textbf{stationary phase point } k_{xs},k_{ys},k_{zs}} \frac{e^{-jk_{z0}|z-z'|}}{k_{z0}} e^{-jk_x x} e^{-jk_y y} dk_x dk_y$$

$$\vec{\boldsymbol{f}}^{far}(x,y,z) = \frac{1}{4\pi^2} \widetilde{\boldsymbol{G}}^{fc}(k_{xs},k_{ys},z,z')\vec{\boldsymbol{C}}(k_{xs},k_{ys}) k_{zs} e^{jk_{zs}|z-z'|} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-jk_{z0}|z-z'|}}{k_{z0}} e^{-jk_x x} e^{-jk_y y} dk_x dk_y$$

$$\vec{f}^{far}(\vec{\boldsymbol{r}}) = jk_{zs} \widetilde{\boldsymbol{G}}^{fc}(k_{xs},k_{ys},z,z')\vec{\boldsymbol{C}}(k_{xs},k_{ys}) e^{jk_{zs}|z-z'|} \frac{e^{-jkr}}{2\pi r}$$

Spectral Domain Methods in Electromagnetics EE4630

Topic 1

Transversalization of Maxwell's Equations

Dyadic Spectral Green's function for planar stratified media

Learning Objectives

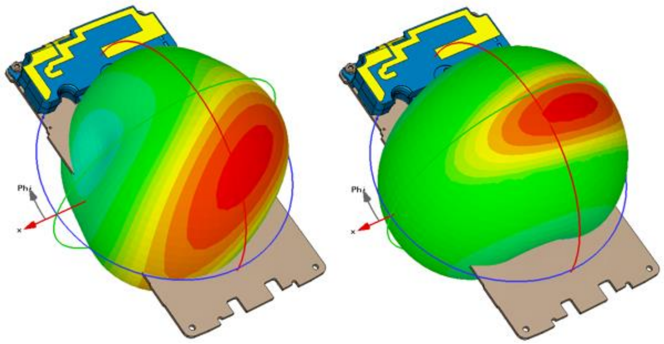
How to represent the Total fields in terms of TE and TM Waves

How to represent these waves in terms of Voltages and Currents in equivalent T.L.

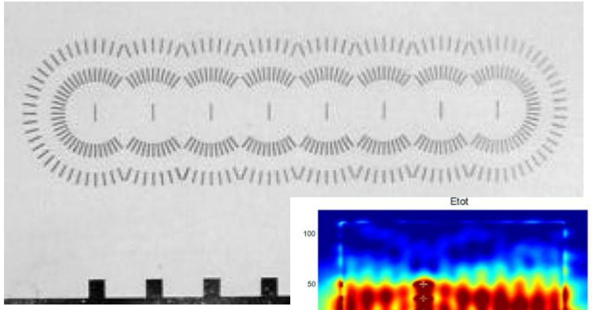
How to introduce the sources

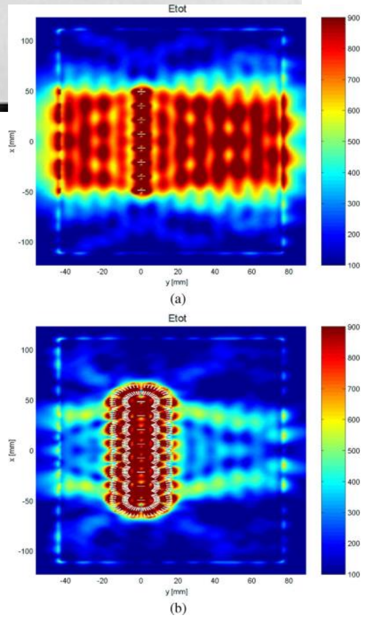
Learn how to construct Green's functions for layered media

Integrated Antennas



At low frequencies, the dielectrics are electrically very thin… you can design antennas as they were radiating in free space





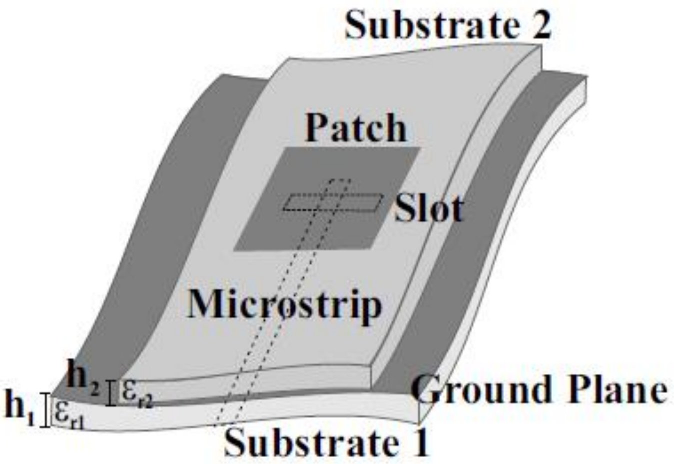
At mm-wave frequencies, the dielectric thickness is electrically significant… significant power remains trapped in the substrates (surface waves)

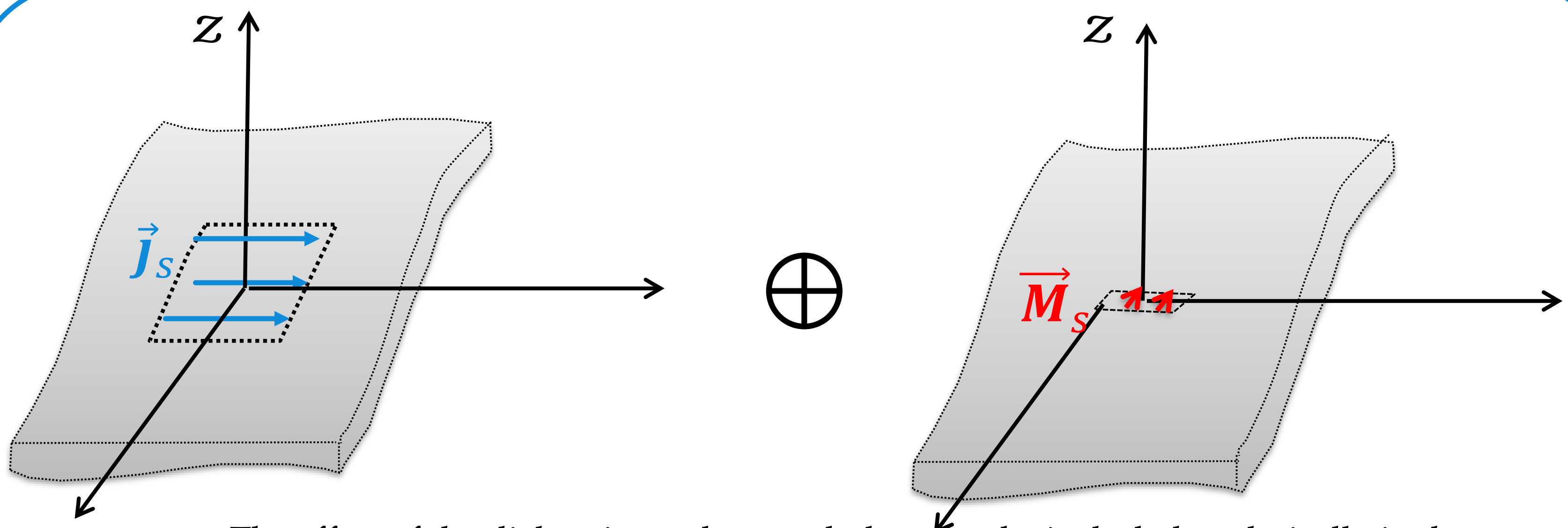
At THz frequencies, it is often chosen to use lenses (quasi-optical antennas)

SRON
Netherlands Institute for Space Research

Integrated Antennas

Integrated antennas can be modelled with *magnetic and electric equivalent surface currents*

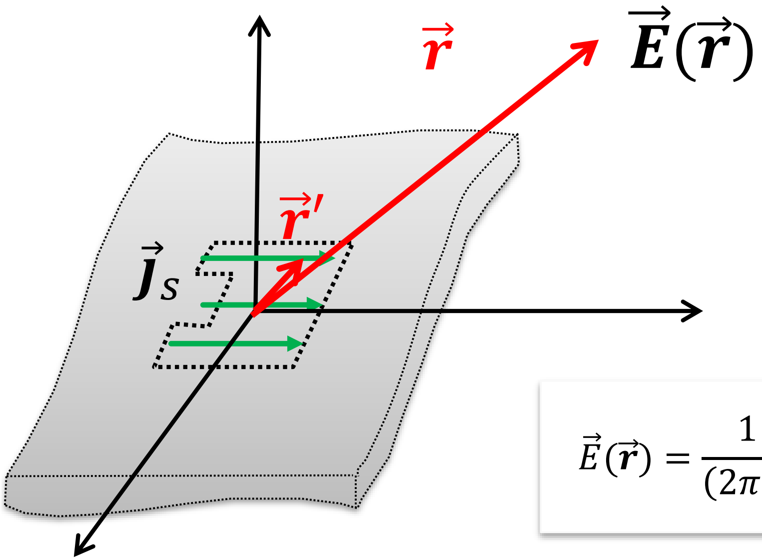




The effect of the dielectrics and ground plane can be included analytically in the Green's functions if we assume them to be infinite in x and y

Objective

We want to derive a general expression for the spectral GF for surface (xy) currents in planar stratified media



Surface Equivalent current spatial distribution

$$\vec{c}(\vec{r}') = \vec{c}_s(\vec{\rho}')\delta(z - z') \qquad c = j \; or \; m$$

Surface Equivalent current spectral distribution

$$\vec{\tilde{C}}_s(\vec{k}_\rho) = \iint \vec{c}_s(x', y')\, e^{jk_x x'} e^{jk_y y'} dx' dy'$$

$$\vec{E}(\vec{r}) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \boxed{\tilde{\bar{\bar{G}}}^{ec}(k_x, k_y, z, z')} \boxed{\vec{\tilde{C}}_s(k_x, k_y)} e^{-jk_x x} e^{-jk_y y} dk_x dk_y$$

Green's function of stratified dielectric media

$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \begin{bmatrix} G_{xx}(k_x, k_y, z, z') & G_{xy}(k_x, k_y, z, z') \\ G_{yx}(k_x, k_y, z, z') & G_{yy}(k_x, k_y, z, z') \\ G_{zx}(k_x, k_y, z, z') & G_{zy}(k_x, k_y, z, z') \end{bmatrix} \begin{bmatrix} C_x(k_x, k_y) \\ C_y(k_x, k_y) \end{bmatrix} e^{-jk_x x} e^{-jk_y y} dk_x dk_y$$

Spectral Vector $\vec{k}_\rho = k_x\hat{x} + k_y\hat{y}$

Spatial Vector $\vec{\rho}' = x'\hat{x} + y'\hat{y}$

Green's Functions in stratified media

f="e" or "h"
c="j" or "m"

$$\bar{\bar{G}}^{fc}(\vec{r},\vec{r}\,') = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{bmatrix} G_{xx}(k_x,k_y,z,z') & G_{xy}(k_x,k_y,z,z') & G_{xz}(k_x,k_y,z,z') \\ G_{yx}(k_x,k_y,z,z') & G_{yy}(k_x,k_y,z,z') & G_{yz}(k_x,k_y,z,z') \\ G_{zx}(k_x,k_y,z,z') & G_{zy}(k_x,k_y,z,z') & G_{zz}(k_x,k_y,z,z') \end{bmatrix} e^{-jk_x(x-x')}e^{-jk_y(y-y')}dk_x dk_y$$

$$\cancel{\boxed{\text{For } fs \quad e^{-jk_z|z-z'|}}}$$

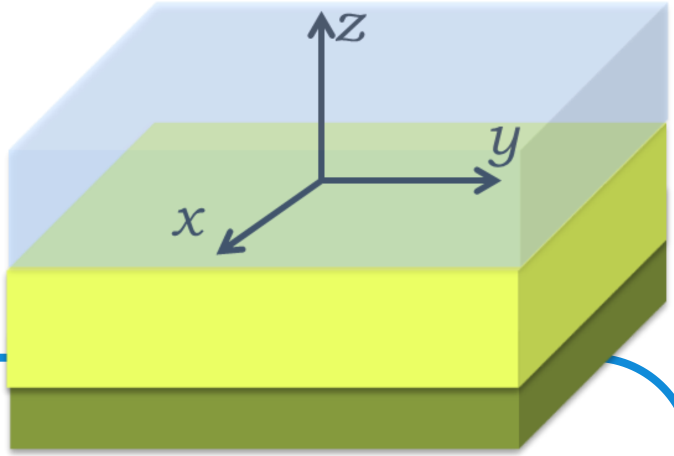
General expression for the spectral GF

that provides the "f" field component in $\vec{r} = (x,y,z)$

starting from the "c" current component in $\vec{r}\,' = (x',y',z')$

How do we do that?

Strategy to derive GF's in stratified media



Transversalization of Maxwell Equation

Introduce A_z (TM) and F_z (TE) potentials

Divide electric and magnetic fields in TE and TM fields with respect to normal

Introduce spectral equations for TE and TM potentials

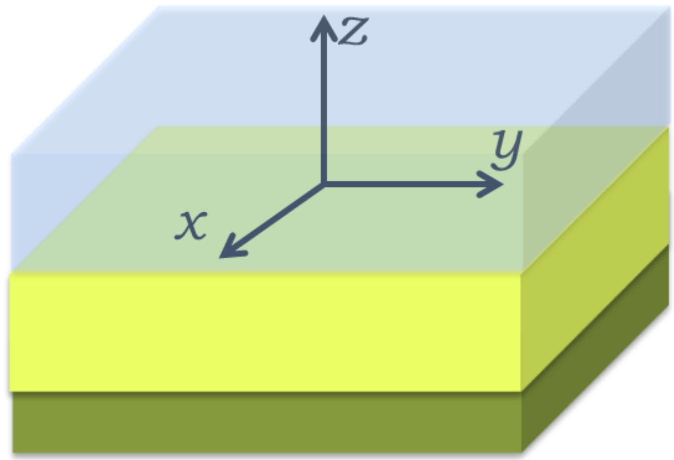
Solve separately spectral equations for TE and TM potentials

Evaluate TE and TM Spectral Fields as function of potentials

Nothing in this procedure is standard

Introduce the sources in the potential equations

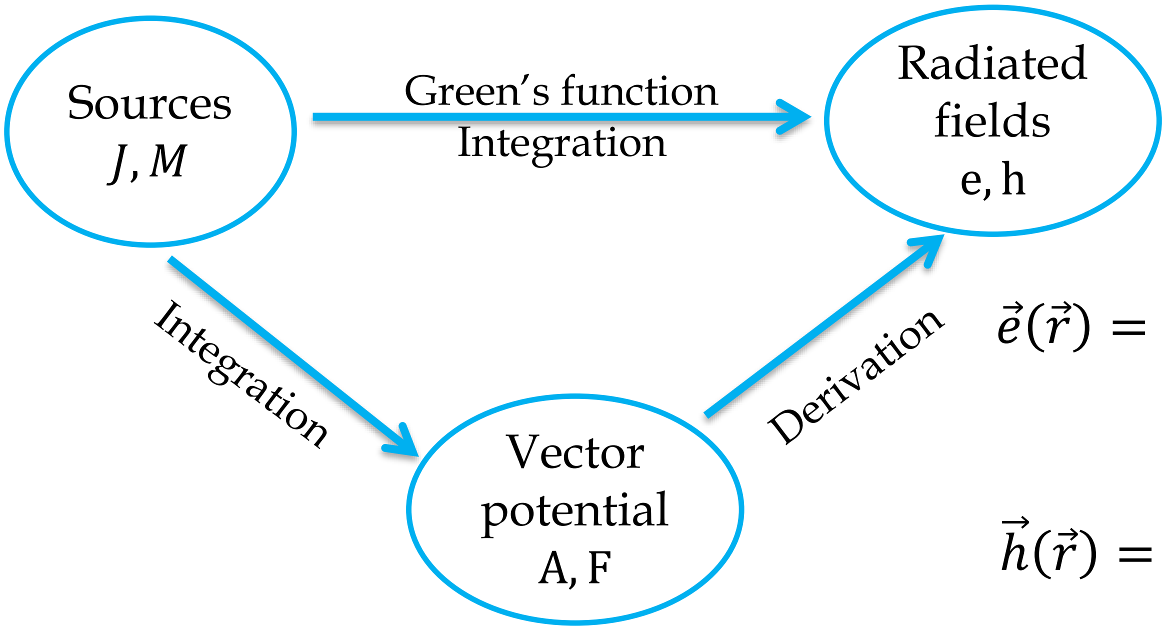
Introduce A_z (TM) and F_z (TE) potentials



$$\bar{\bar{G}}^{fc}(\vec{r},\vec{r}') = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}
\begin{bmatrix} G_{xx}(k_x,k_y,z,z') & G_{xy}(k_x,k_y,z,z') & G_{xz}(k_x,k_y,z,z') \\ G_{yx}(k_x,k_y,z,z') & G_{yy}(k_x,k_y,z,z') & G_{yz}(k_x,k_y,z,z') \\ G_{zx}(k_x,k_y,z,z') & G_{zy}(k_x,k_y,z,z') & G_{zz}(k_x,k_y,z,z') \end{bmatrix} e^{-jk_x(x-x')}e^{-jk_y(y-y')}dk_x dk_y$$

Definition of vector potentials

The electric and magnetic field in any problem can be expressed as functions of auxiliary vector electric (***F***) and magnetic (***A***) functions



$$\vec{e}(\vec{r}) = -j\omega\vec{A} - \frac{j}{\omega\epsilon\mu}\nabla[\nabla\cdot\vec{A}] - \frac{1}{\epsilon}\nabla\times\vec{F}$$

$$\vec{h}(\vec{r}) = -j\omega\vec{F} - \frac{j}{\omega\epsilon\mu}\nabla[\nabla\cdot\vec{F}] + \frac{1}{\mu}\nabla\times\vec{A}$$

$$\nabla^2\vec{A}(\vec{r}) + k^2\vec{A}(\vec{r}\,) = -\mu\vec{J}$$

$$\nabla^2\vec{F}(\vec{r}) + k^2\vec{F}(\vec{r}\,) = -\epsilon\vec{M}$$

in free space

$$\vec{A}(\vec{r},\vec{J}) = \frac{\mu e^{-jk|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}\hat{z}$$

$$\vec{F}(\vec{r},\vec{M}) = \frac{\epsilon e^{-jk|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}\hat{z}$$

In Stratified Media.... Transversalization

$$\nabla^2 \vec{A}(\vec{r}) + k^2 \vec{A}(\vec{r}\,) = -\mu \vec{J}$$

$$\nabla^2 \vec{F}(\vec{r}) + k^2 \vec{F}(\vec{r}\,) = -\epsilon \vec{M}$$

$$\vec{e}(\vec{r}) = -j\omega \vec{A} - \frac{j}{\omega\epsilon\mu} \nabla\left[\nabla \cdot \vec{A}\right] - \frac{1}{\epsilon} \nabla \times \vec{F}$$

$$\vec{h}(\vec{r}) = -j\omega \vec{F} - \frac{j}{\omega\epsilon\mu} \nabla\left[\nabla \cdot \vec{F}\right] + \frac{1}{\mu} \nabla \times \vec{A}$$

$$\vec{A}' = \vec{A}/\mu$$

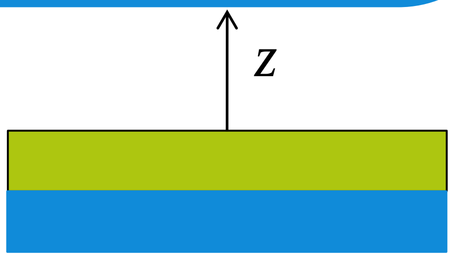
$$\vec{F}' = \vec{F}/\epsilon$$

$$\nabla^2 \vec{A}'(\vec{r}) + k^2 \vec{A}'(\vec{r}\,) = -\vec{J}$$

$$\nabla^2 \vec{F}'(\vec{r}) + k^2 \vec{F}'(\vec{r}\,) = -\vec{M}$$

$$\vec{e}(\vec{r}) = -j\omega\mu \vec{A}' - \frac{j}{\omega\epsilon} \nabla\left[\nabla \cdot \vec{A}'\right] - \nabla \times \vec{F}'$$

$$\vec{h}(\vec{r}) = -j\omega\epsilon \vec{F}' - \frac{j}{\omega\mu} \nabla\left[\nabla \cdot \vec{F}'\right] + \nabla \times \vec{A}'$$



Since the selection of the potentials is essentially arbitrary, for stratifications orthogonal to z,
.... we choose potentials along z:

$$\vec{A}' = A'_z \hat{z}$$

$$\vec{F}' = F'_z \hat{z}$$

And forget about the prime notation

Transverse Gradient Operator

$$\vec{A} = A_z \hat{z}$$

$$\vec{F} = F_z \hat{z}$$

$$\vec{e}(\vec{r}) = -j\omega\mu\vec{A} - \frac{j}{\omega\epsilon}\nabla\left[\nabla\cdot\vec{A}\right] - \nabla\times\vec{F} \qquad \vec{h}(\vec{r}) = -j\omega\epsilon\vec{F} - \frac{j}{\omega\mu}\nabla\left[\nabla\cdot\vec{F}\right] + \nabla\times\vec{A}$$

We introduce the transverse Nabla Operator

$$\nabla = \nabla_t + \frac{\partial}{\partial z}\hat{z} \qquad \nabla_t = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y}$$

$$\nabla\times(F\hat{z}) = \nabla_t\times(F\hat{z})$$

$$\nabla\cdot(F\hat{z}) = \left(\frac{\partial}{\partial z}\hat{z}\right)\cdot(F\hat{z})$$

$$\omega\varepsilon = k/\varsigma$$

$$\omega\mu = k\varsigma$$

We divide explicitly the field components deriving from A and F

$$\boxed{\vec{e}_A(\vec{r}) = -jk\zeta\left[A_z\hat{z} + \frac{1}{k^2}\frac{\partial^2}{\partial z^2}A_z\hat{z} + \frac{1}{k^2}\nabla_t\frac{\partial}{\partial z}A_z\right]}$$

$$\boxed{\vec{e}_F = -\nabla_t\times F_z\hat{z}}$$

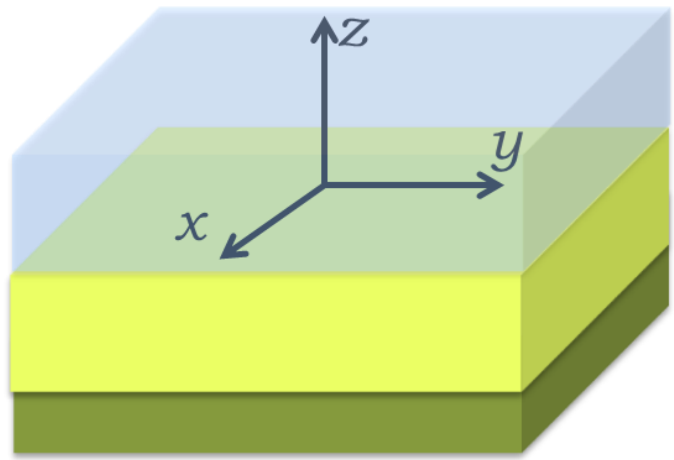
F

A

$$\boxed{\vec{h}_A = \nabla_t\times A_z\hat{z}}$$

$$\boxed{\vec{h}_F(\vec{r}) = -\frac{jk}{\zeta}\left[F_z\hat{z} + \frac{1}{k^2}\frac{\partial^2}{\partial z^2}F_z\hat{z} + \frac{1}{k^2}\nabla_t\frac{\partial}{\partial z}F_z\right]}$$

Divide electric and magnetic fields into TE and TM components with respect to z



$$\bar{\bar{G}}^{fc}(\vec{r},\vec{r}') = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \begin{bmatrix} G_{xx}(k_x,k_y,z,z') & G_{xy}(k_x,k_y,z,z') & G_{xz}(k_x,k_y,z,z') \\ G_{yx}(k_x,k_y,z,z') & G_{yy}(k_x,k_y,z,z') & G_{yz}(k_x,k_y,z,z') \\ G_{zx}(k_x,k_y,z,z') & G_{zy}(k_x,k_y,z,z') & G_{zz}(k_x,k_y,z,z') \end{bmatrix} e^{-jk_x(x-x')}e^{-jk_y(y-y')}dk_x dk_y$$

TE and TM Solutions

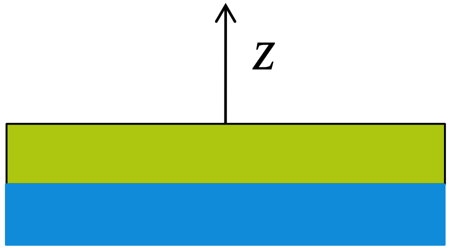
$$\vec{e}_F = -\nabla_t \times F_z \hat{z}$$

Electric field associated to Fz is entirely transverse to z ($E_z^{TE} = 0$), **Transverse Electric, TE**

$$\vec{h}_A = \nabla_t \times A_z \hat{z}$$

Magnetic field associated to Az is entirely transverse to z ($H_z^{TE} = 0$), **Transverse Magnetic, TM**

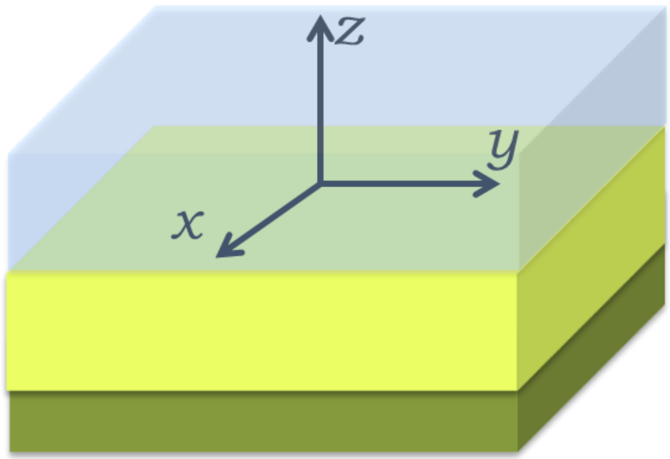
Accordingly we call TE all fields due to F, also the magnetic ones.
we call TM all fields due to A, also the electric ones.



$$\vec{h}_F(\vec{r}) = -\frac{jk}{\zeta}\left[F_z\hat{z} + \frac{1}{k^2}\frac{\partial^2}{\partial z^2}F_z\hat{z} + \frac{1}{k^2}\nabla_t\frac{\partial}{\partial z}F_z\right]$$

$$\vec{e}_A(\vec{r}) = -jk\zeta\left[A_z\hat{z} + \frac{1}{k^2}\frac{\partial^2}{\partial z^2}A_z\hat{z} + \frac{1}{k^2}\nabla_t\frac{\partial}{\partial z}A_z\right]$$

Introduce spectral equations for TE-TM potentials



$$\bar{\bar{G}}^{fc}(\vec{r},\vec{r}') = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \begin{bmatrix} G_{xx}(k_x,k_y,z,z') & G_{xy}(k_x,k_y,z,z') & G_{xz}(k_x,k_y,z,z') \\ G_{yx}(k_x,k_y,z,z') & G_{yy}(k_x,k_y,z,z') & G_{yz}(k_x,k_y,z,z') \\ G_{zx}(k_x,k_y,z,z') & G_{zy}(k_x,k_y,z,z') & G_{zz}(k_x,k_y,z,z') \end{bmatrix} e^{-jk_x(x-x')}e^{-jk_y(y-y')}dk_x dk_y$$

TM/TE Potentials, scalar wave equation

$$\nabla^2 \vec{A}(\vec{r}) + k^2 \vec{A}(\vec{r}\,) = -\vec{J}$$

$$\nabla^2 \vec{F}(\vec{r}) + k^2 \vec{F}(\vec{r}\,) = -\vec{M}$$

$$\nabla^2 A_z + k^2 A_z = -J_z$$

$$\nabla^2 F_z + k^2 F_z = -M_z$$

$$\vec{A} = A_z \hat{z} \qquad \vec{F} = F_z \hat{z}$$

$$\nabla^2 \vec{A} = \nabla^2 A_x \hat{x} + \nabla^2 A_y \hat{y} + \nabla^2 A_z \hat{z}$$

We can only deal naturally with currents oriented along z

The choice of using potentials along z only, was completely arbitrary. If the real currents are not only along z, for instance they are along x and y, we will have to consider some **different but equivalent currents,** rather than the original currents

Let us first see how to solve this equation in the spectral domain

Spectral Potentials

$$A_z(\vec{r}') \equiv \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{A}_z(k_x, k_y, z, z')\, e^{-jk_x x} e^{-jk_y y} dk_x dk_y$$

$$F_z(\vec{r}') \equiv \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{F}_z(k_x, k_y, z, z')\, e^{-jk_x x} e^{-jk_y y} dk_x dk_y$$

$$\nabla^2 A_z + k^2 A_z = -J_z$$

$$\nabla^2 F_z + k^2 F_z = -M_z$$

$$\Rightarrow$$

$$\boxed{\begin{aligned} k_z^2 \tilde{A}_z + \frac{\partial^2}{\partial z^2}\tilde{A}_z &= -\tilde{J}_z \\ k_z^2 \tilde{F}_z + \frac{\partial^2}{\partial z^2}\tilde{F}_z &= -\tilde{M}_z \end{aligned}}$$

Demonstration in the next page

Demonstration

$$\nabla^2 A_z + k^2 A_z = -J_z \Rightarrow k_z^2 \tilde{A}_z + \frac{\partial^2}{\partial z^2} \tilde{A}_z = -\tilde{J}_z$$

$$\nabla = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z} \qquad \nabla_t \equiv \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} \rightarrow \nabla = \nabla_t + \frac{\partial}{\partial z}\hat{z}$$

$$\boldsymbol{\nabla^2} = \left(\nabla_t + \frac{\partial}{\partial z}\hat{z}\right) \cdot \left(\nabla_t + \frac{\partial}{\partial z}\hat{z}\right) = \boldsymbol{\nabla_t^2} + \boldsymbol{\frac{\partial^2}{\partial z^2}}$$

$$\nabla_t^2 = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y}\right) \cdot \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y}\right) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

In the spectrum

$$\frac{\partial}{\partial x} e^{-jk_x x} = -jk_x e^{-jk_x x}$$

$$\boldsymbol{\nabla_t^2} e^{-jk_x x} e^{-jk_y y} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) e^{-jk_x x} e^{-jk_y y} = -\left(k_x^2 + k_y^2\right) e^{-jk_x x} e^{-jk_y y} = \boldsymbol{-k_\rho^2} e^{-jk_x x} e^{-jk_y y}$$

$$-k_\rho^2 \tilde{A}_z + \frac{\partial^2}{\partial z^2} \tilde{A}_z + k^2 \tilde{A}_z = -\tilde{J}_z$$

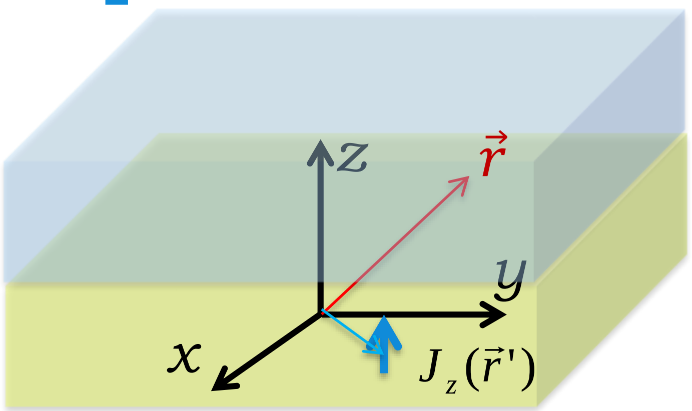
$$k_x^2 + k_y^2 + k_z^2 = k^2 \rightarrow k^2 - k_\rho^2 = k_z^2$$

$$k_z^2 \tilde{A}_z + \frac{\partial^2}{\partial z^2} \tilde{A}_z = -\tilde{J}_z$$

Spectral/Space Domain Equations

$$k_z^2 \tilde{A}_z + \frac{\partial^2}{\partial z^2} \tilde{A}_z = -\tilde{J}_z$$ ***TM***

$$J_z(\vec{r}\,')$$ ***TE***



In the space domain

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(k_z^2+\frac{\partial^2}{\partial z^2}\right)\tilde{A}_z(\vec{k}_\rho,z)\,e^{-j\vec{k}_\rho\cdot\vec{\rho}}d\vec{k}_\rho = -\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{J}_z(\vec{k}_\rho,z')\,e^{-j\vec{k}_\rho\cdot\vec{\rho}}d\vec{k}_\rho$$

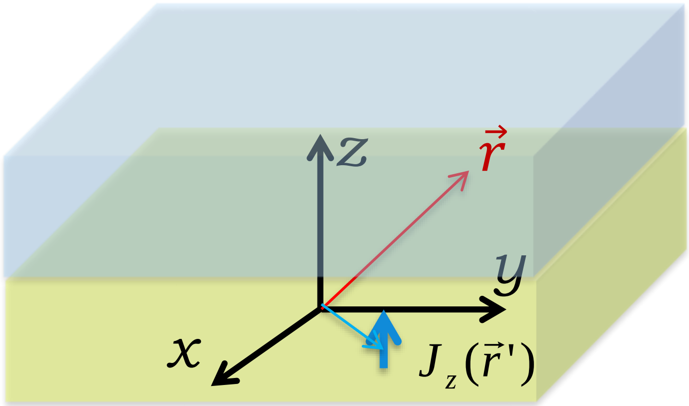
$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(k_z^2+\frac{\partial^2}{\partial z^2}\right)\tilde{F}_z(\vec{k}_\rho,z)\,e^{-j\vec{k}_\rho\cdot\vec{\rho}}d\vec{k}_\rho = -\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{M}_z(\vec{k}_\rho,z')\,e^{-j\vec{k}_\rho\cdot\vec{\rho}}d\vec{k}_\rho$$

Introducing Impulsive Sources

$$J_z(\vec{r}') = \delta(\vec{\rho} - \vec{\rho}')\,\delta(z - z')$$

$$M_z(\vec{r}') = \delta(\vec{\rho} - \vec{\rho}')\,\delta(z - z')$$

In the space domain



$$\tilde{J}_z(\vec{k}_\rho) = e^{j\vec{k}_\rho\cdot\vec{\rho}'}\delta(z - z')$$

$$\tilde{M}_z(\vec{k}_\rho) = e^{j\vec{k}_\rho\cdot\vec{\rho}'}\delta(z - z')$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(k_z^2 + \frac{\partial^2}{\partial z^2}\right)\tilde{A}_z(\vec{k}_\rho, z)\,e^{-j\vec{k}_\rho\cdot\vec{\rho}}dk_\rho = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j\vec{k}_\rho\cdot\vec{\rho}'}\delta(z - z')\,e^{-j\vec{k}_\rho\cdot\vec{\rho}}d\vec{k}_\rho$$

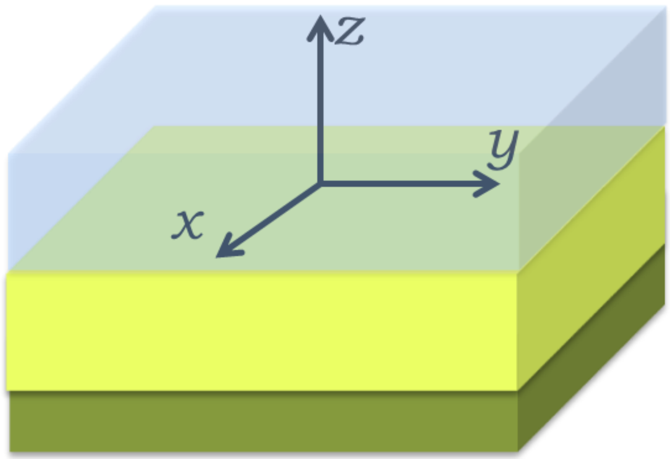
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(k_z^2 + \frac{\partial^2}{\partial z^2}\right)\tilde{F}_z(\vec{k}_\rho, z)\,e^{-j\vec{k}_\rho\cdot\vec{\rho}}dk_\rho = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j\vec{k}_\rho\cdot\vec{\rho}'}\delta(z - z')\,e^{-j\vec{k}_\rho\cdot\vec{\rho}}d\vec{k}_\rho$$

Equating the spectra

$$\left(k_z^2 + \frac{\partial^2}{\partial z^2}\right)\tilde{A}_z(\vec{k}_\rho, z) = -e^{j\vec{k}_\rho\cdot\vec{\rho}'}\delta(z - z')$$

$$\left(k_z^2 + \frac{\partial^2}{\partial z^2}\right)\tilde{F}_z(\vec{k}_\rho, z) = -e^{j\vec{k}_\rho\cdot\vec{\rho}'}\delta(z - z')$$

Solve spectral equations for TE and TM potentials in Free Space



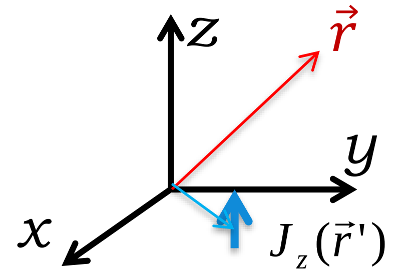
$$\bar{\bar{G}}^{fc}(\vec{r},\vec{r}') = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \begin{bmatrix} G_{xx}(k_x,k_y,z,z') & G_{xy}(k_x,k_y,z,z') & G_{xz}(k_x,k_y,z,z') \\ G_{yx}(k_x,k_y,z,z') & G_{yy}(k_x,k_y,z,z') & G_{yz}(k_x,k_y,z,z') \\ G_{zx}(k_x,k_y,z,z') & G_{zy}(k_x,k_y,z,z') & G_{zz}(k_x,k_y,z,z') \end{bmatrix} e^{-jk_x(x-x')}e^{-jk_y(y-y')}dk_x dk_y$$

Spectral Solution in Free Space

$$\left(k_z^2 + \frac{\partial^2}{\partial z^2}\right)\tilde{A}_z(\vec{k}_\rho, z) = -e^{j\vec{k}_\rho\cdot\vec{\rho}\prime}\delta(z-z')$$

$$\left(k_z^2 + \frac{\partial^2}{\partial z^2}\right)\tilde{F}_z(\vec{k}_\rho, z) = -e^{j\vec{k}_\rho\cdot\vec{\rho}\prime}\delta(z-z')$$

Need some boundary conditions: *potentials at infinity go to zero*



Free Space Potential in the Space Domain

in free space

$$A_z(\vec{r}\,) = F_z(\vec{r}) = \frac{e^{-jk|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}\hat{z}$$

Identity demonstrated in Advanced EM

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-jk_x(x-x\prime)}e^{-jk_y(y-y\prime)}}{2j\sqrt{k^2-k_x^2-k_y^2}}e^{-j\sqrt{k^2-k_x^2-k_y^2}|z-z\prime|}dk_x dk_y$$

Spectral Solution in Free Space

$$\left(k_z^2 + \frac{\partial^2}{\partial z^2}\right)\tilde{A}_z(\vec{k}_\rho, z) = -e^{j\vec{k}_\rho\cdot\vec{\rho}\prime}\delta(z - z')$$

$$\left(k_z^2 + \frac{\partial^2}{\partial z^2}\right)\tilde{F}_z(\vec{k}_\rho, z) = -e^{j\vec{k}_\rho\cdot\vec{\rho}\prime}\delta(z - z')$$

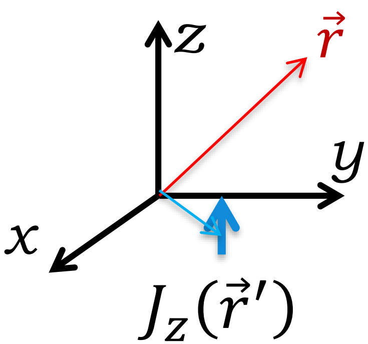
$$\tilde{A}_z(\vec{k}_\rho, z) = e^{j\vec{k}_\rho\cdot\vec{\rho}\prime}\frac{e^{-jk_z|z-z\prime|}}{2jk_z} \quad \textbf{TM}$$

$$\tilde{F}_z(\vec{k}_\rho, z) = e^{j\vec{k}_\rho\cdot\vec{\rho}\prime}\frac{e^{-jk_z|z-z\prime|}}{2jk_z} \quad \textbf{TE}$$

The spectral potential depends on the product of two functions:

$$k_z = \sqrt{k^2 - k_x^2 - k_y^2}$$

$$\tilde{A}_z(\vec{k}_\rho, z, z') = e^{j\vec{k}_\rho\cdot\vec{\rho}\prime}I_{TM}(\vec{k}_\rho, z, z')$$



$$\tilde{F}_z(\vec{k}_\rho, z, z') = e^{j\vec{k}_\rho\cdot\vec{\rho}\prime}V_{TE}(\vec{k}_\rho, z, z')$$

An exponential depending on the transverse location of the source

A function that depends on (z,z')

The power of transversalization

The **transverse** location of the origin of the reference system does not affect the spectrum in z.

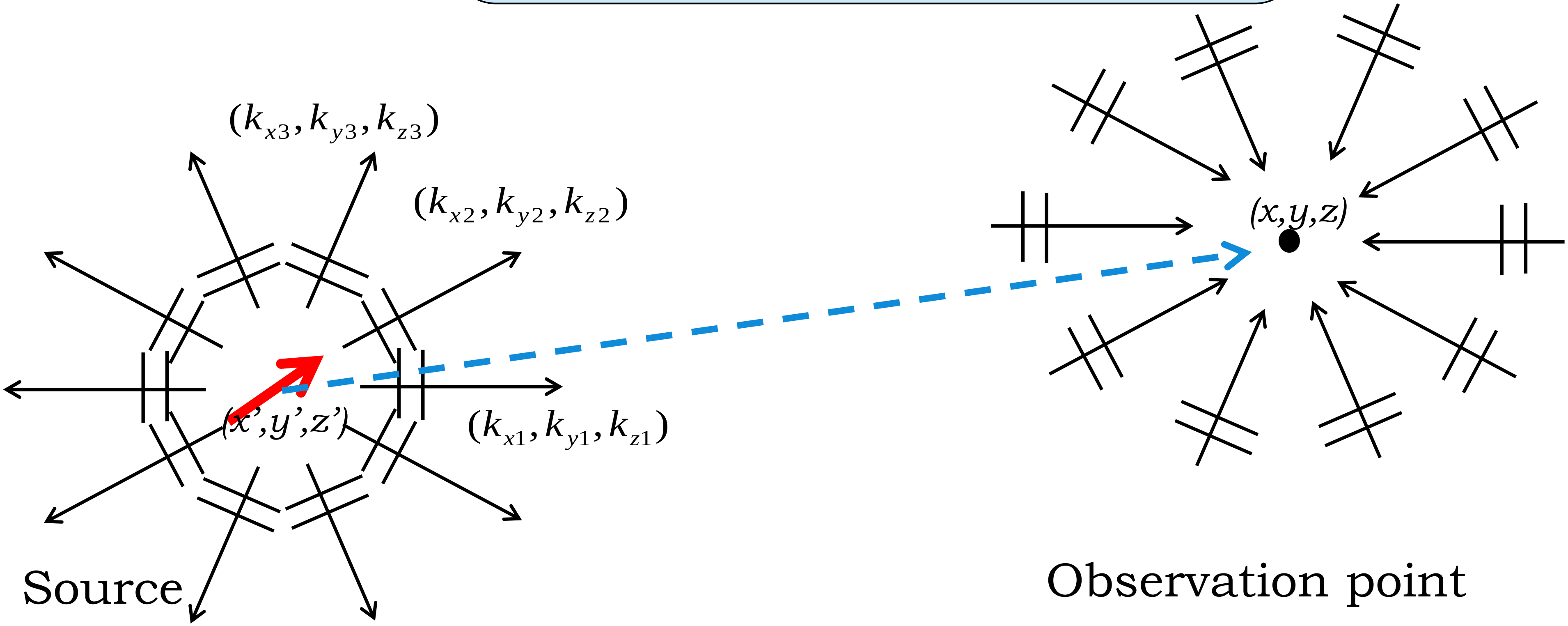
Similarly the spectrum in z does not affect the radial spectral dependence

Most Important Aspect of Free Space Solution

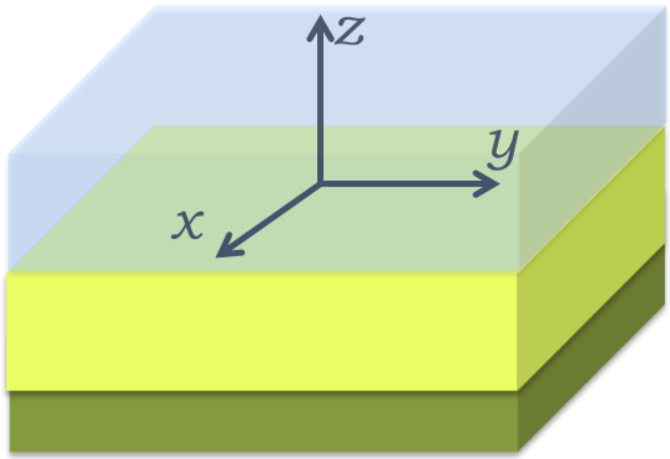
The potentials are expressed in terms of plane waves

$$\tilde{A}_z(\vec{k}_\rho, z) = e^{-j\vec{k}_\rho\cdot(\vec{\rho}-\vec{\rho}')}\frac{e^{-jk_z|z-z'|}}{2jk_z} \quad \textbf{TM}$$

$$\tilde{F}_z(\vec{k}_\rho, z) = e^{-j\vec{k}_\rho\cdot(\vec{\rho}-\vec{\rho}')}\frac{e^{-jk_z|z-z'|}}{2jk_z} \quad \textbf{TE}$$



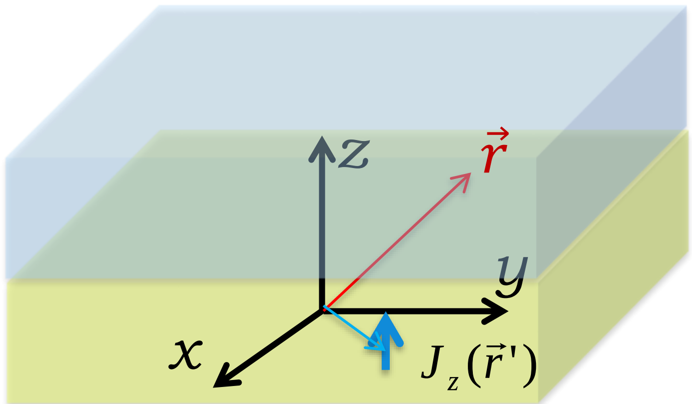
Solution of spectral equations for TE and TM potentials in Stratified Media



$$\bar{\bar{G}}^{fc}(\vec{r}-\vec{r}') = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}$$

$$\begin{bmatrix} G_{xx}(k_x,k_y,z,z') & G_{xy}(k_x,k_y,z,z') & G_{xz}(k_x,k_y,z,z') \\ G_{yx}(k_x,k_y,z,z') & G_{yy}(k_x,k_y,z,z') & G_{yz}(k_x,k_y,z,z') \\ G_{zx}(k_x,k_y,z,z') & G_{zy}(k_x,k_y,z,z') & G_{zz}(k_x,k_y,z,z') \end{bmatrix} e^{-jk_x(x-x\prime)}e^{-jk_y(y-y\prime)}dk_x dk_y$$

Stratified Media



Spectral Potential equations in each medium ***i*** :

$$\boxed{\begin{aligned}\left(k_{zi}^2+\frac{\partial^2}{\partial z^2}\right)\tilde{A}_z(\vec{k}_\rho,z) &= -e^{j\vec{k}_\rho\cdot\vec{\rho}'}\delta(z-z')\\ \left(k_{zi}^2+\frac{\partial^2}{\partial z^2}\right)\tilde{F}_z(\vec{k}_\rho,z) &= -e^{j\vec{k}_\rho\cdot\vec{\rho}'}\delta(z-z')\end{aligned}}$$

$$k_{zi}=\sqrt{k_i^2-k_x^2-k_y^2}$$

*Let us assume that we **know** already the solution, also in terms of plane waves. We cannot know until we introduce the sources… next lecture. So now we look for properties*

Let us assume that the source is at $\rho'=0$

$$\tilde{A}_z(\vec{k}_\rho,z,z')=e^{j\vec{k}_\rho\cdot\vec{\rho}'}I_{TM}(\vec{k}_\rho,z,z')$$

$$\tilde{F}_z(\vec{k}_\rho,z,z')=e^{j\vec{k}_\rho\cdot\vec{\rho}'}V_{TE}(\vec{k}_\rho,z,z')$$

$$\Rightarrow$$

$$\tilde{A}_z(\vec{k}_\rho,z,z')=I_{TM}(\vec{k}_\rho,z,z')$$

$$\tilde{F}_z(\vec{k}_\rho,z,z')=V_{TE}(\vec{k}_\rho,z,z')$$

TE Electric Field

Space Domain:

$$\vec{e}_F = -\nabla_t \times F_z \hat{z}$$

$$\boxed{\tilde{F}_z(\vec{k}_\rho, z, z') = V_{TE}(\vec{k}_\rho, z, z')}$$

$$\bar{\bar{g}}_{TE}^{ec}(\vec{r}, \vec{r}') = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \vec{\tilde{E}}_{TE}(k_x, k_y, z, z') e^{-jk_x(x-x')} e^{-jk_y(y-y')} dk_x dk_y$$

$$\nabla_t = -j\vec{k}_\rho$$

$\vec{x} \times \vec{y} = |x||y| \sin\vartheta \, \hat{z}$

Spectral Domain:

$$\vec{\tilde{E}}_{TE}(\vec{k}_\rho, z, z') = j\vec{k}_\rho \times V_{TE}(\vec{k}_\rho, z, z')\hat{z}$$
$$= jk_\rho \hat{k}_\rho \times \hat{z} V_{TE}(\vec{k}_\rho, z, z')$$

Introduced a third vector

$$\hat{k}_\rho \times \hat{z} = -\hat{\alpha}$$

$$\hat{k}_\rho \times \hat{\alpha} = \hat{z}$$

$$\boxed{\vec{\tilde{E}}_{TE}(\vec{k}_\rho, z, z') = -jk_\rho V_{TE}(\vec{k}_\rho, z, z')\hat{\alpha}}$$

TE Magnetic Field $\forall\, z \neq z'$

Space Domain:

$$\vec{h}_F(\vec{r}) = -\frac{jk}{\zeta}\left[F_z\hat{z} + \frac{1}{k^2}\frac{\partial^2}{\partial z^2}F_z\hat{z} + \frac{1}{k^2}\nabla_t\frac{\partial}{\partial z}F_z\right]$$

$$\boxed{\tilde{F}_z(\vec{k}_\rho, z, z') = V_{TE}(\vec{k}_\rho, z, z')}$$

$$\nabla_t = -j\vec{k}_\rho$$

Spectral Domain:

$$\vec{\tilde{H}}_{TE}(\vec{k}_\rho, z, z') = -\frac{jk}{\zeta}\left[\frac{1}{k^2}[k^2 - k_z^2]V_{TE}\hat{z} - jk_\rho\hat{k}_\rho\frac{1}{k^2}\frac{\partial}{\partial z}V_{TE}\right]$$

$$\vec{\tilde{H}}_{TE}(\vec{k}_\rho, z, z') = -\frac{j}{k\zeta}k_\rho^2V_{TE}\hat{z} + jk_\rho\hat{k}_\rho\underbrace{\left(j\frac{1}{k\zeta}\frac{\partial}{\partial z}V_{TE}\right)}$$

Since

$$\left(k_z^2 + \frac{\partial^2}{\partial z^2}\right)V_{TE}(k_z; z) = -\delta(z - z')$$

$$\frac{\partial^2}{\partial z^2}V_{TE} = -k_z^2V_{TE}; \quad \forall z \neq z'$$

Defined $\quad I_{TE} = j\frac{1}{k\zeta}\frac{\partial}{\partial z}V_{TE}$

$$\boxed{\vec{\tilde{H}}_{TE}(\vec{k}_\rho, z, z') = -\frac{j}{k\zeta}k_\rho^2V_{TE}\hat{z} + jk_\rho\hat{k}_\rho I_{TE}}$$

TE Magnetic Field $\forall z$

Space Domain:

$$\vec{h}_F(\vec{r}) = -\frac{jk}{\zeta}\left[F_z\hat{z} + \frac{1}{k^2}\frac{\partial^2}{\partial z^2}F_z\hat{z} + \frac{1}{k^2}\nabla_t\frac{\partial}{\partial z}F_z\right]$$

$$\nabla_t = -j\vec{k}_\rho$$

$$\boxed{\tilde{F}_z(\vec{k}_\rho, z, z') = V_{TE}(\vec{k}_\rho, z, z')}$$

Since

$$\left(k_z^2 + \frac{\partial^2}{\partial z^2}\right)V_{TE}(k_z; z) = -\delta(z - z')$$

$$\frac{\partial^2}{\partial z^2}V_{TE} = -k_z^2 V_{TE} - \delta(z - z')$$

Spectral Domain:

$$\vec{\tilde{H}}_{TE}(\vec{k}_\rho, z, z') = -\frac{jk}{\zeta}\left[\frac{1}{k^2}[k^2 - k_z^2]V_{TE}\hat{z} - \frac{1}{k^2}\delta(z - z')\hat{z} - jk_\rho\hat{k}_\rho\frac{1}{k^2}\frac{\partial}{\partial z}V_{TE}\right]$$

$$\vec{\tilde{H}}_{TE}(\vec{k}_\rho, z, z') = -\frac{j}{k\zeta}k_\rho^2 V_{TE}\hat{z} - \frac{j}{k\zeta}\delta(z - z')\,\hat{z} + jk_\rho\hat{k}_\rho\underbrace{\left(j\frac{1}{k\zeta}\frac{\partial}{\partial z}V_{TE}\right)}$$

Defined $I_{TE} = j\dfrac{1}{k\zeta}\dfrac{\partial}{\partial z}V_{TE}$

$$\boxed{\vec{\tilde{H}}_{TE}(\vec{k}_\rho, z, z') = -\frac{j}{k\zeta}k_\rho^2 V_{TE}\hat{z} - \frac{j}{k\zeta}\delta(z - z')\,\hat{z} + jk_\rho\hat{k}_\rho I_{TE}}$$

Equivalent TE Transmission Line

$$\vec{\vec{E}}_{TE} = -jk_\rho V_{TE}\hat{\alpha}$$

$$\vec{\vec{H}}_{TE} = -\frac{j}{k\zeta}k_\rho^2 V_{TE}\hat{z} - \frac{j}{k\zeta}\delta(z-z')\,\hat{z} + jk_\rho\hat{k}_\rho I_{TE}$$

Electric and magnetic fields are written as function of these two quantities V_{TE} I_{TE}

V_{TE} is the solution of the following differential equation + the pertinent boundary conditions

$$\left(k_z^2 + \frac{\partial^2}{\partial_z z^2}\right)V_{TE}(k_z;z) = -\delta(z-z')$$

In Free Space

$$V_{TE}(\vec{k}_\rho, z) = \frac{e^{-jk_z|z-z'|}}{2jk_z} \Rightarrow V_{TE} = V_{TE}^+ e^{-jk_z|z-z'|};\ V_{TE}^+ = \frac{1}{2jk_z}$$

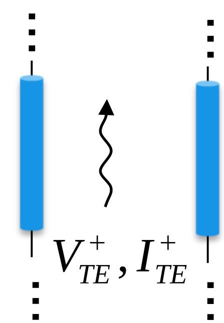
Since we defined $I_{TE}(z) = j\frac{1}{k\varsigma}\frac{\partial}{\partial z}V_{TE}(z)$

$$I_{TE}(z) = j\frac{1}{k\varsigma}\frac{\partial}{\partial z}V_{TE}^+ e^{-jk_z|z-z'|} = \frac{1}{\varsigma}\frac{k_z}{k}V_{TE}^+ e^{-jk_z|z-z'|}$$

$$I_{TE}(z) = I_{TE}^+ e^{-jk_z|z-z'|}$$

Therefore $I_{TE}^+ = V_{TE}^+\frac{1}{\varsigma}\frac{k_z}{k}$

*So in **free space** V_{TE} and I_{TE} are related as the solutions of a general **transmission line** characterized by*



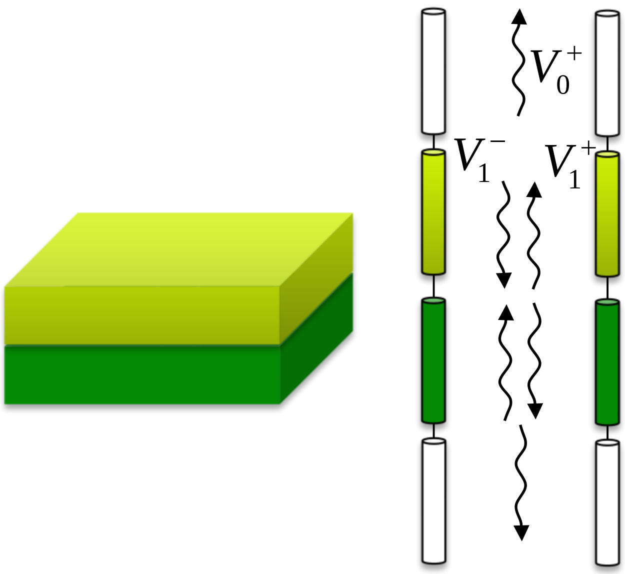
$$k_z = \sqrt{k^2 - k_x^2 - k_y^2} \qquad Z_{TE} = \frac{k\varsigma}{k_z}$$

Equivalent TE Transmission Line

V_{TE} is the solution of the following differential equation in each medium
with *boundary conditions* at infinity
and continuity at a number of stratifications
(modelled via an equivalent transmission line)

$$\left(k_z^2 + \frac{\partial^2}{\partial_z z^2}\right) V_{TE}(k_z; z) = -\delta(z - z')$$

For every homogeneous stratification: *i*



$$V_{TE,i}(z) = V_{TE,i}^{+} e^{-jk_z(z-z')} + V_{TE,i}^{-} e^{jk_z(z-z')}$$

$$I_{TE,i}(z) = j\frac{1}{\varsigma_i k_i}\frac{\partial}{\partial z} V_{TE,i}$$

$$= V_{TE,i}^{+}\frac{k_{z,i}}{\varsigma_i k_i} e^{-jk_{z,i}(z-z')} - V_{TE,i}^{-}\frac{k_{z,i}}{\varsigma_i k_i} e^{jk_{z,i}(z-z')}$$

$$= I_{TE,i}^{+} Z_{TE,i} e^{-jk_{z,i}(z-z')} + I_{TE,i}^{-} Z_{TE,i} e^{jk_{z,i}(z-z')}$$

$$I_{TE,i}^{+} = V_{TE,i}^{+} / Z_{TE,i} \qquad I_{TE,i}^{-} = -V_{TE,i}^{-} / Z_{TE,i}$$

$$Z_{TE,i} = \frac{k_i \varsigma_i}{k_{zi}}$$

TM Fields $\forall\, z \neq z'$

Space Domain:

$$\tilde{A}_z(\vec{k}_\rho, z, z') = I_{TM}(\vec{k}_\rho, z, z')$$

$$\vec{h}_{TM} = \nabla_t \times A_z\hat{z}$$

$$\vec{e}_{TM}(\vec{r}) = -jk\zeta\left[A_z\hat{z} + \frac{1}{k^2}\frac{\partial^2}{\partial z^2}A_z\hat{z} + \frac{1}{k^2}\nabla_t\,\frac{\partial}{\partial z}A_z\right]$$

Spectral Domain:

$$\vec{\tilde{H}}_{TM} = jk_\rho I_{TM}\hat{\alpha}$$

$$\vec{\tilde{E}}_{TM} = -\frac{j\zeta}{k}k_\rho^2 I_{TM}(z)\hat{z} + jk_\rho\hat{k}_\rho V_{TM}(z)$$

Defined

$$V_{TM}(z) = j\frac{\varsigma}{k}\frac{\partial}{\partial z}I_{TM}(z)$$

TM Fields $\forall z$

Space Domain:

$$\tilde{A}_z(\vec{k}_\rho, z, z') = I_{TM}(\vec{k}_\rho, z, z')$$

$$\vec{h}_{TM} = \nabla_t \times A_z \hat{z}$$

$$\vec{e}_{TM}(\vec{r}) = -jk\zeta\left[A_z\hat{z} + \frac{1}{k^2}\frac{\partial^2}{\partial z^2}A_z\hat{z} + \frac{1}{k^2}\nabla_t\,\frac{\partial}{\partial z}A_z\right]$$

Spectral Domain:

$$\vec{\tilde{H}}_{TM} = jk_\rho I_{TM}\hat{\alpha}$$

$$\vec{\tilde{E}}_{TME} = -\frac{j\zeta}{k}k_\rho^2 I_{TM}(z)\hat{z} - \frac{j\zeta}{k}\delta(z-z')\hat{z} + jk_\rho\hat{k}_\rho V_{TM}(z)$$

Defined

$$V_{TM}(z) = j\frac{\varsigma}{k}\frac{\partial}{\partial z}I_{TM}(z)$$

Equivalent TM Transmission Line

I_{TM} is the solution of the following differential equation in each medium with boundary conditions are infinity and continuity at a number of stratifications (modelled via an equivalent transmission line)

$$\left(k_z^2 + \frac{\partial^2}{\partial_z z^2}\right) I_{TM}(k_z; z) = -\delta(z - z')$$

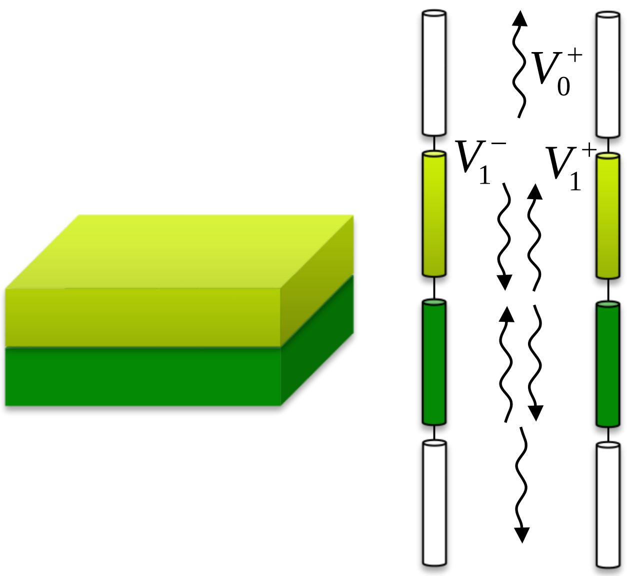
For every homogeneous stratification: *i*

$$I_{TM} = I_{TM}^+ e^{-jk_z(z-z')} + I_{TM}^- e^{jk_z(z-z')}$$

$$V_{TM}(z) = j\frac{\varsigma}{k}\frac{\partial}{\partial z} I_{TM} = j\frac{\varsigma}{k}\frac{\partial}{\partial z}\left(I_{TM}^+ e^{-jk_z(z-z')} + I_{TM}^- e^{jk_z(z-z')}\right)$$

$$= I_{TM}^+ \frac{k_z\varsigma}{k} e^{-jk_z(z-z')} - I_{TM}^- \frac{k_z\varsigma}{k} e^{jk_z(z-z')}$$

$$= V_{TM}^+ e^{-jk_z(z-z')} + V_{TM}^- e^{jk_z(z-z')}$$



$$Z_{TM} = \frac{k_z\varsigma}{k}$$

$$V_{TM}^+ = I_{TM}^+ Z_{TM}$$

$$V_{TM}^- = -I_{TM}^- Z_{TM}$$