Spectral Domain Methods in Electromagnetics EE4620

Lecture # 2

Truly Important Points

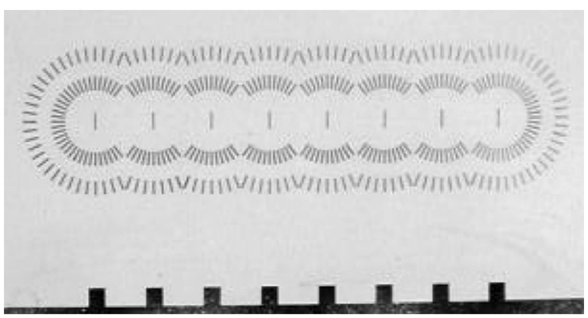
$$\bar{\bar{G}}^{fc}(\vec{r},\vec{r}') = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{\bar{G}}(k_x,k_y,z,z')\,e^{-jk_x(x-x')}e^{-jk_y(y-y')}dk_x dk_y$$

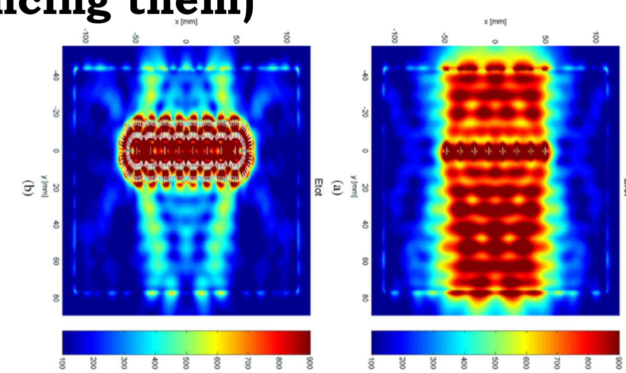
1) How to construct this GF is an entrance price to pay.

2) You can survive without understanding it, but you need to be able to use it.

3) No one cares about the analysis in itself. It’s for the **design of structures**
   a) The GF has polar singularities, each pole corresponding to TE or TM wave
   b) Capturing the residue of these poles will provide the dominant contributions to the Analysis without full integrations.

   c) These waves are the ones for which one has to design!!
   (either killing them or enhancing them)

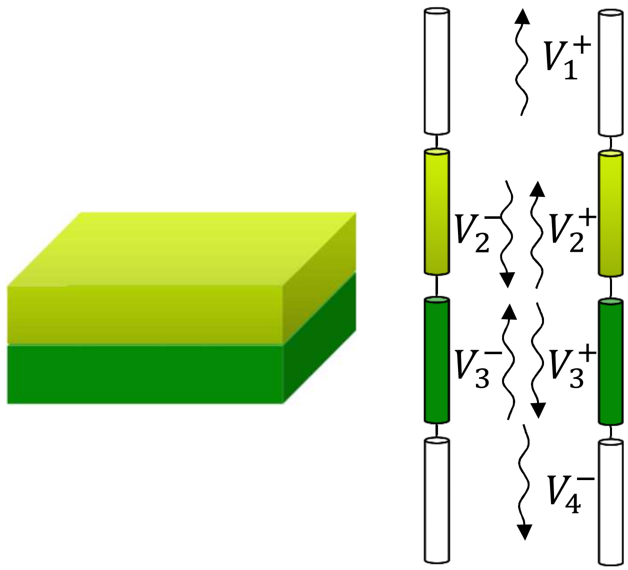




Equivalent Transmission Lines

$$\begin{cases} \vec{\tilde{E}}_{TE} = -jk_\rho V_{TE}\hat{\alpha} \\ \vec{\tilde{H}}_{TE} = -\dfrac{j}{k\zeta}k_\rho^2 V_{TE}\hat{z} + jk_\rho \hat{k}_\rho I_{TE} \end{cases} \qquad \begin{cases} \vec{\tilde{H}}_{TM} = jk_\rho I_{TM}\hat{\alpha} \\ \vec{\tilde{E}}_{TM} = -\dfrac{j\zeta}{k}k_\rho^2 I_{TM}(z)\hat{z} + jk_\rho \hat{k}_\rho V_{TM}(z) \end{cases}$$

The fields



For every homogeneous stratification: *i*

$$V^i(z) = V_i^+ e^{-jk_{zi}z} + V_i^- e^{jk_{zi}z}$$

$$I^i(z) = I_i^+ e^{-jk_z z} + I_i^- e^{jk_{zi}z}$$

$$I_i^+ Z_{ci} = V_i^+ \qquad I_i^- Z_{ci} = -V_i^-$$

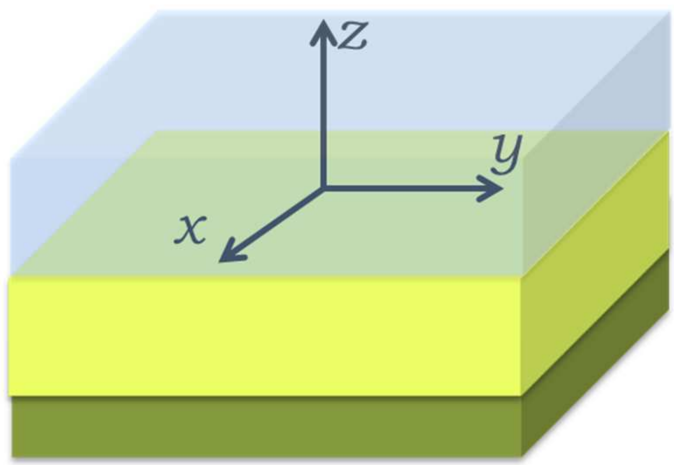
TE and TM differ for the characteristic impedance

$$Z_{ci} = \begin{cases} Z_{TE} = \dfrac{\zeta k}{k_z} \\ Z_{TM} = \dfrac{\zeta k_z}{k} \end{cases}$$

TE and TM have the same propagation constant

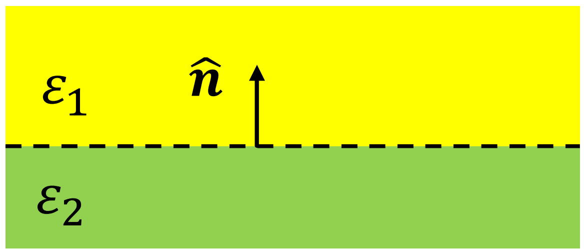
$$k_{zi} = \sqrt{k_i^2 - k_x^2 - k_y^2}$$

Introducing the sources



$$\bar{\bar{G}}^{fc}(\vec{r},\vec{r}') = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{bmatrix} G_{xx}(k_x,k_y,z,z') & G_{xy}(k_x,k_y,z,z') & G_{xz}(k_x,k_y,z,z') \\ G_{yx}(k_x,k_y,z,z') & G_{yy}(k_x,k_y,z,z') & G_{yz}(k_x,k_y,z,z') \\ G_{zx}(k_x,k_y,z,z') & G_{zy}(k_x,k_y,z,z') & G_{zz}(k_x,k_y,z,z') \end{bmatrix} e^{-jk_x(x-x')}e^{-jk_y(y-y')}dk_x dk_y$$

A step back: Boundary conditions

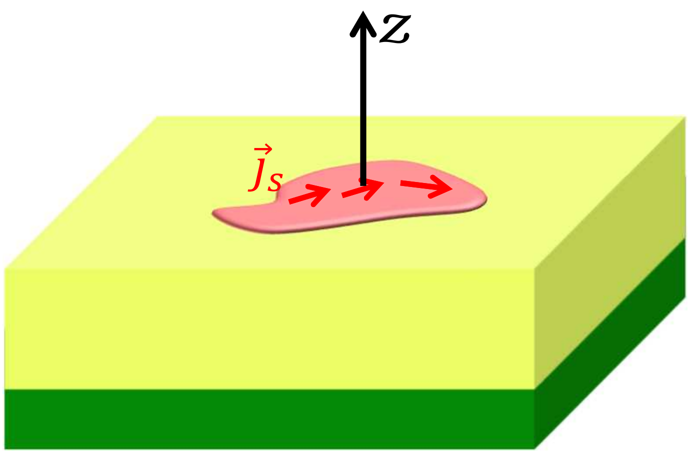


Boundary conditions at
any interface

$$\hat{\boldsymbol{n}} \times (\boldsymbol{H}_1 - \boldsymbol{H}_2) = \boldsymbol{J}_s$$

Surface current densities

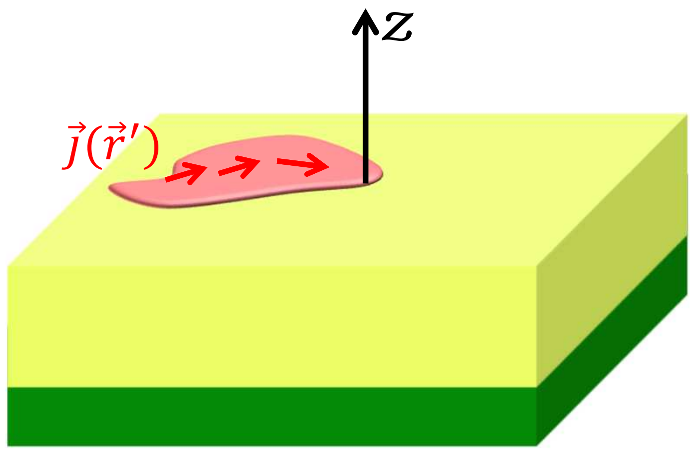
$$\vec{J}_t = \vec{J}_s(\vec{\rho})\delta(z)$$



The fact that there are surface **electric current sources** means that there is a **magnetic field discontinuity** at the source location

Tangent Electric Sources

Let us assume there is an electric source *tangent* to the x,y plane



Tangent **electric sources correspond** to **magnetic field discontinuities**

Spatial current distribution

$$\vec{j}_t(\vec{r}') = \delta(\vec{\rho} - \vec{\rho}')\, \delta(z - z')\hat{p}_j$$

Spectral current distribution

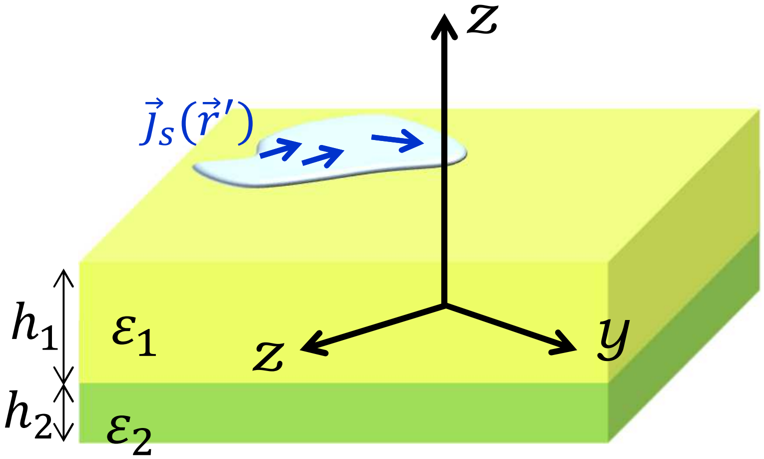
$$\vec{J}_t(\vec{k}_\rho) = e^{j\vec{k}_\rho \cdot \vec{\rho}'}\delta(z - z')\hat{p}_j$$

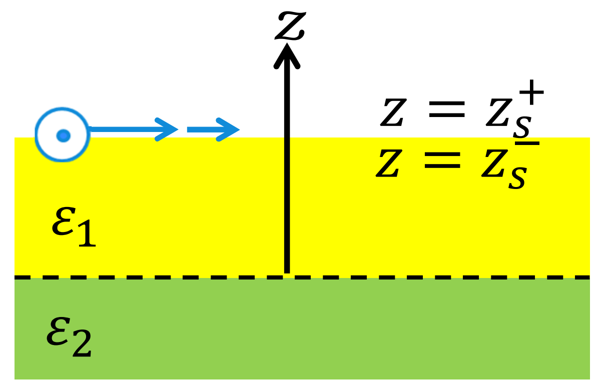
We would like introduce an equivalent source that can be represented in the transmission line with a generator

How?

$$\vec{J}_t(\vec{k}_\rho) \Rightarrow I_{TM}^{eq}(k_\rho)\, I_{TE}^{eq}(k_\rho)$$

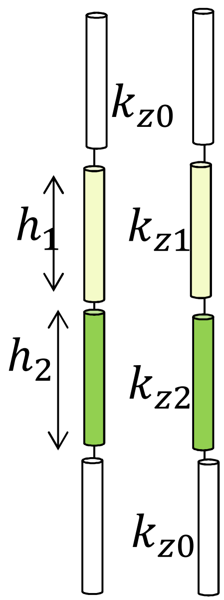
Introducing tangent electric sources





We interpret Surface **electric current sources** as **discontinuity** in the **tangent magnetic field** at z_s

$$\hat{\boldsymbol{z}} \times \left(\boldsymbol{H}(z_s^+) - \boldsymbol{H}(z_s^-)\right) = \boldsymbol{J}_s$$



How we can relate this boundary condition to the fields obtained from the transmission line representation??

$$\vec{\tilde{H}}_{TM} = jk_\rho I_{TM}\hat{\alpha}$$

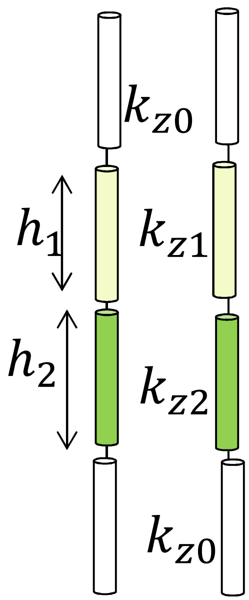
$$\vec{\tilde{H}}_{TE} = -\frac{j}{k\zeta}k_\rho^2 V_{TE}\hat{z} + jk_\rho \hat{k}_\rho I_{TE}$$

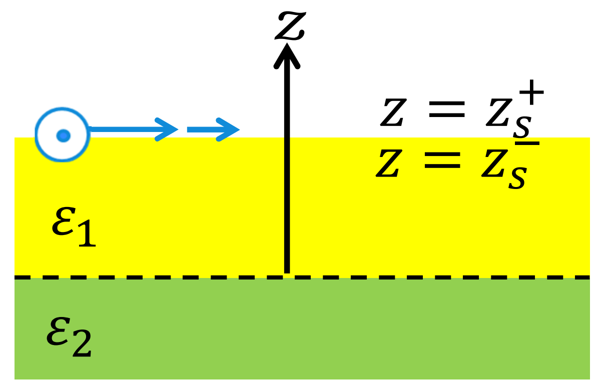
The **tangent** magnetic field in the transmission line representation is

$$\boldsymbol{H_t} = jk_\rho I_{TM}\widehat{\boldsymbol{\alpha}} + jk_\rho I_{TE}\widehat{\boldsymbol{k}}_{\boldsymbol{\rho}}$$

We drop the tilde, $\tilde{H}$, since evrything is now in the spectrum

Introducing tangent electric sources





$$\hat{\boldsymbol{z}} \times \left(\boldsymbol{H}_t(z_s^+) - \boldsymbol{H}_t(z_s^-)\right) = \boldsymbol{J}_s$$

Projection $\vec{H}_t$ into the spectral unit vectors (TE/TM)

$$\boldsymbol{H}_t = jk_\rho I_{TM}\hat{\boldsymbol{\alpha}} + jk_\rho I_{TE}\hat{\boldsymbol{k}}_{\boldsymbol{\rho}} \Longleftrightarrow \boldsymbol{H}^{dis}(z_s) = \boldsymbol{H}_t(z_s^+) - \boldsymbol{H}_t(z_s^-) = H_{TE}^{dis}(z_s)\hat{\boldsymbol{k}}_{\boldsymbol{\rho}} + H_{TM}^{dis}(z_s)\hat{\boldsymbol{\alpha}}$$

Relate to a ***current discontinuity*** in the transmission line

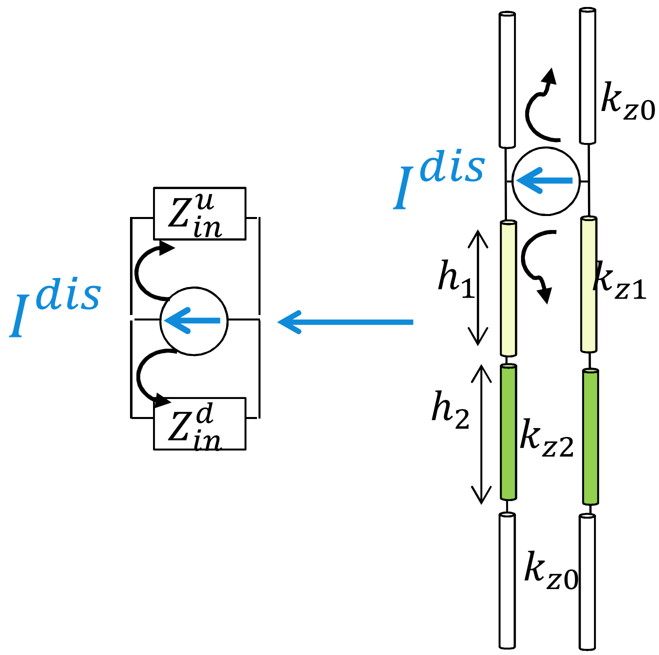
$$H_{TE}^{dis}(z_s) = jk_\rho I_{TE}^{dis} \quad H_{TM}^{dis}(z_s) = jk_\rho I_{TM}^{dis}$$

Equivalent current generator

$$I_{TE}^{dis} = \frac{H_{TE}^{dis}(z_s)}{jk_\rho} \qquad I_{TM}^{dis} = \frac{H_{TM}^{dis}(z_s)}{jk_\rho}$$

The electric source can be represented with a current discontinuity (i.e. current generator in the transmission line)

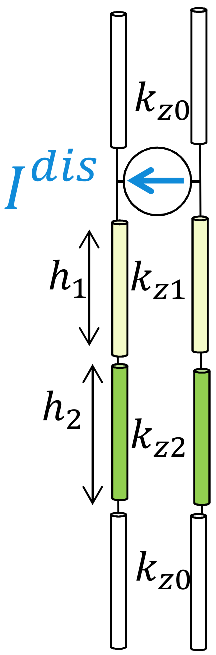
In parallel



Can remember that only with current generator in parallel it provides a discontinuity in the tangent magnetic field

$$\boldsymbol{H_t} = jk_\rho I_{TM}\widehat{\boldsymbol{\alpha}} + jk_\rho I_{TE}\widehat{\boldsymbol{k}}_{\boldsymbol{\rho}}$$

Equivalent current generator



$$I_{TE}^{dis} = \frac{H_{TE}^{dis}(z_s)}{jk_\rho} \qquad I_{TM}^{dis} = \frac{H_{TM}^{dis}(z_s)}{jk_\rho}$$

Last step… relate the magnetic field discontinuity to the actual source distribution

$$\hat{\boldsymbol{z}} \times \left(\boldsymbol{H}_t(z_s^+) - \boldsymbol{H}_t(z_s^-)\right) = \hat{\boldsymbol{z}} \times \boldsymbol{H}_t^{dis} = \boldsymbol{J}_s$$

$$\left\{ \begin{array}{c} (\hat{\boldsymbol{z}} \times \boldsymbol{H}_t^{dis}) \times \hat{\boldsymbol{z}} = \boldsymbol{J}_s \times \hat{\boldsymbol{z}} \\ \Downarrow \\ \boldsymbol{H}_t^{dis} = \boldsymbol{J}_s \times \hat{\boldsymbol{z}} \end{array} \right.$$

1) Mechanic representation of tangent currents in terms of spectral unit vectors

$$\boldsymbol{J}_s = J_x \hat{\boldsymbol{x}} + J_y \hat{\boldsymbol{y}} \quad \Rightarrow \quad \boldsymbol{J}_s = (\boldsymbol{J}_s \cdot \hat{\boldsymbol{k}}_{\boldsymbol{\rho}})\hat{\boldsymbol{k}}_{\boldsymbol{\rho}} + (\boldsymbol{J}_s \cdot \hat{\boldsymbol{\alpha}})\hat{\boldsymbol{\alpha}}$$

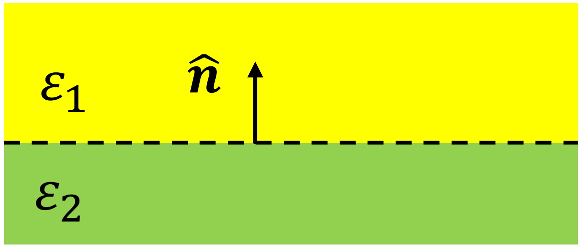
2) Relate the TE and TM magnetic field discontinuity to these electric currents

$$\begin{aligned} H_{TE}^{dis}(z_s)\hat{\boldsymbol{k}}_{\boldsymbol{\rho}} + H_{TM}^{dis}(z_s)\hat{\boldsymbol{\alpha}} &= \left[(\boldsymbol{J}_s \cdot \hat{\boldsymbol{k}}_{\boldsymbol{\rho}})\hat{\boldsymbol{k}}_{\boldsymbol{\rho}} + (\boldsymbol{J}_s \cdot \hat{\boldsymbol{\alpha}})\hat{\boldsymbol{\alpha}}\right] \times \hat{\boldsymbol{z}} \\ &= (\boldsymbol{J}_s \cdot \hat{\boldsymbol{\alpha}})\hat{\boldsymbol{k}}_{\boldsymbol{\rho}} - (\boldsymbol{J}_s \cdot \hat{\boldsymbol{k}}_{\boldsymbol{\rho}})\hat{\boldsymbol{\alpha}} \end{aligned}$$

$$H_{TE}^{dis}(z_s) = \boldsymbol{J}_s \cdot \hat{\boldsymbol{\alpha}} \qquad H_{TM}^{dis}(z_s) = -\boldsymbol{J}_s \cdot \hat{\boldsymbol{k}}_{\boldsymbol{\rho}}$$

$$I_{TE}^{\ dis}(z_s) = \frac{\boldsymbol{J}_s \cdot \hat{\boldsymbol{\alpha}}}{jk_\rho} \qquad I_{TM}^{\ dis}(z_s) = \frac{-\boldsymbol{J}_s \cdot \hat{\boldsymbol{k}}_{\boldsymbol{\rho}}}{jk_\rho}$$

Boundary conditions

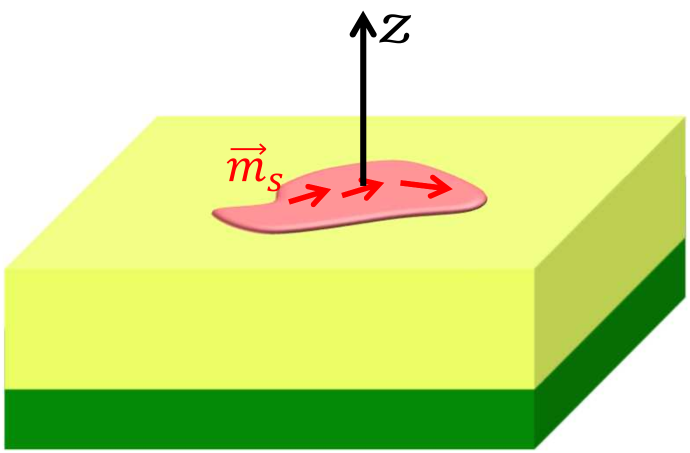


Boundary conditions at any interface

$$\hat{\boldsymbol{n}} \times (\boldsymbol{E}_1 - \boldsymbol{E}_2) = -\boldsymbol{M}_s$$

Surface current densities

$$\vec{m}_t = \vec{m}_s(\vec{\rho})\delta(z)$$

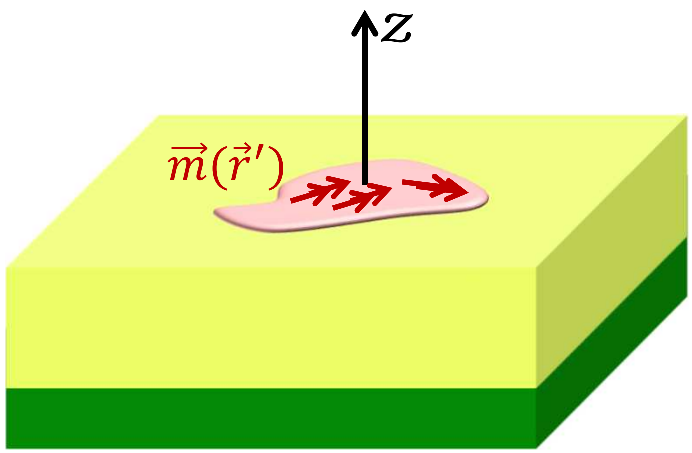


The fact that there are surface **magnetic current sources** means that there is a **electric field discontinuity** at the source location

Equivalent steps for magnetic sources

Tangent Magnetic Sources

Let us now assume there is a magnetic source *tangent* to the x,y plane



Spatial current distribution

$$\vec{m}_t(\vec{r}\,') = \delta(\vec{\rho} - \vec{\rho}\,')\,\delta(z - z')\hat{p}_m$$

Spectral current distribution

$$\vec{M}_t(\vec{k}_\rho) = e^{j\vec{k}_\rho \cdot \vec{\rho}\,'}\delta(z - z')\hat{p}_m$$

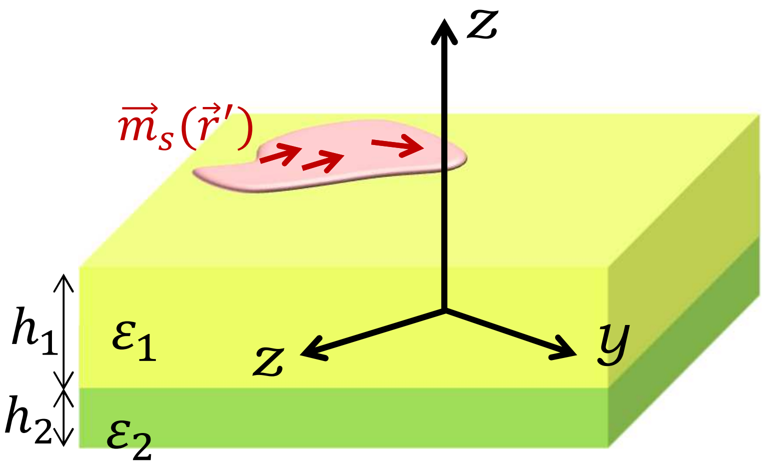
Real tangent **magnetic sources** correspond to **electric field discontinuities**

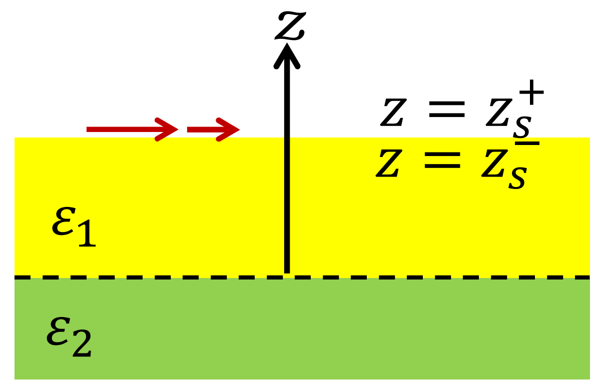
What would it be the equivalent generator in the transmission line??

$$\vec{M}_t \Rightarrow V_{TE}^{eq}, V_{TM}^{eq}$$

Equivalent steps for magnetic sources

Introducing tangent magnetic sources



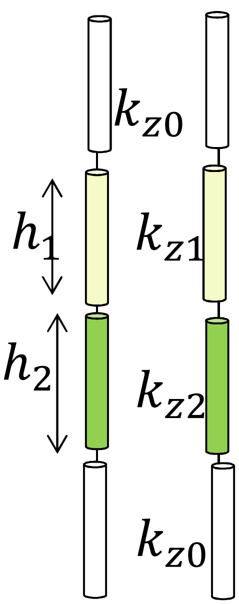


$$\boxed{\hat{\boldsymbol{z}} \times \left(\boldsymbol{E}(z_s^+) - \boldsymbol{E}(z_s^-)\right) = \boldsymbol{M}_s}$$

$$\boldsymbol{E}^{dis}(z_s) = E^{dis}_{TM}(z_s)\hat{\boldsymbol{k}}_{\boldsymbol{\rho}} + E^{dis}_{TE}(z_s)\hat{\boldsymbol{\alpha}}$$

Projection into the spectral unit vectors

$$E^{dis}_{TE}(z_s) = -jk_\rho V^{dis}_{TE} \quad E^{dis}_{TM}(z_s) = jk_\rho V^{dis}_{TM}$$



How we can relate this boundary condition to the fields obtained from the transmission line representation??

$$\vec{\tilde{E}}_{TE} = -jk_\rho V_{TE}\hat{\alpha}$$

$$\vec{\tilde{E}}_{TM} = -\frac{j\zeta}{k}k_\rho^2 I_{TM}(z)\hat{z} + jk_\rho\hat{k}_\rho V_{TM}(z)$$

The **tangent** electric field in the transmission line representation is

$$\boldsymbol{E_t} = -jk_\rho V_{TE}\hat{\boldsymbol{\alpha}} + jk_\rho V_{TM}\hat{\boldsymbol{k}}_{\boldsymbol{\rho}}$$

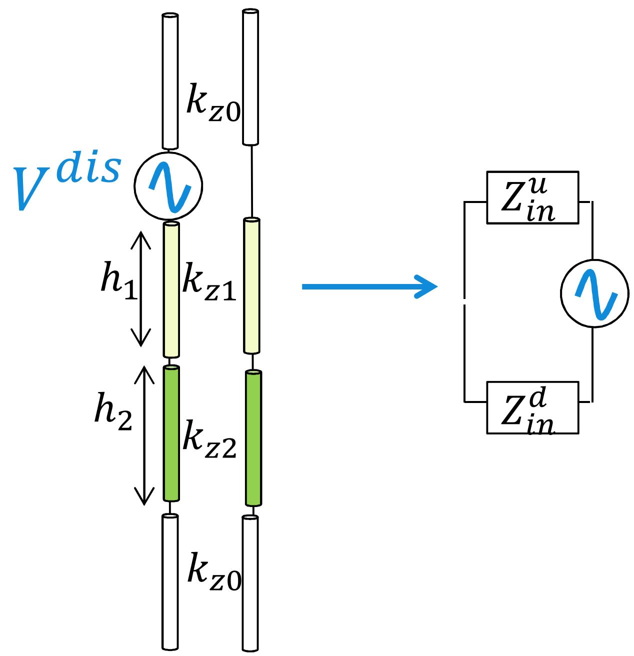
Equivalent steps for mag sou

Equivalent voltage generator

$$V_{TE}^{dis} = \frac{E_{TE}^{dis}(z_s)}{-jk_\rho} \qquad V_{TM}^{dis} = \frac{E_{TM}^{dis}(z_s)}{jk_\rho}$$

The magnetic source can be represented with a voltage discontinuity (i.e. voltage generator in the transmission line)

In parallel or in series??

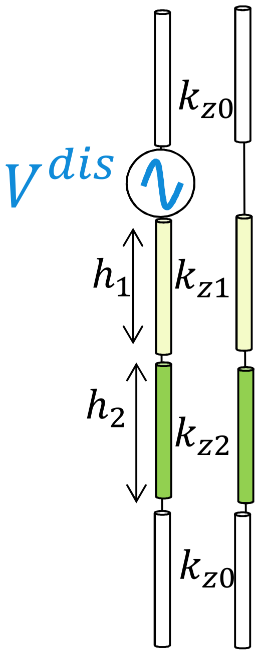


In series

Only in series it provides a discontinuity in the tangent electric field!

$$\boldsymbol{E_t} = -jk_\rho V_{TE}\boldsymbol{\hat{\alpha}} + jk_\rho V_{TM}\boldsymbol{\hat{k}_\rho}$$

Equivalent voltage generator



$$V_{TE}^{dis} = \frac{E_{TE}^{dis}(z_s)}{-jk_\rho} \qquad V_{TM}^{dis} = \frac{E_{TM}^{dis}(z_s)}{jk_\rho}$$

Last step… relate the magnetic field discontinuity to the actual source distribution

$$\hat{\boldsymbol{z}} \times \boldsymbol{E}_{\boldsymbol{t}}^{dis} = -\boldsymbol{M}_s$$

1) We need to represent the real current with spectral unit vector

$$\boldsymbol{M}_s = M_x\hat{\boldsymbol{x}} + M_y\hat{\boldsymbol{y}} \quad \Rightarrow \quad \boldsymbol{M}_s = (\boldsymbol{M}_s \cdot \hat{\boldsymbol{k}}_{\boldsymbol{\rho}})\hat{\boldsymbol{k}}_{\boldsymbol{\rho}} + (\boldsymbol{M}_s \cdot \hat{\boldsymbol{\alpha}})\hat{\boldsymbol{\alpha}}$$

2) Relate the TE and TM magnetic field discontinuity to these electric currents

$$E_{TM}^{dis}(z_s)\hat{\boldsymbol{k}}_{\boldsymbol{\rho}} + E_{TE}^{dis}(z_s)\hat{\boldsymbol{\alpha}} = -\left[(\boldsymbol{M}_s \cdot \hat{\boldsymbol{k}}_{\boldsymbol{\rho}})\hat{\boldsymbol{k}}_{\boldsymbol{\rho}} + (\boldsymbol{M}_s \cdot \hat{\boldsymbol{\alpha}})\hat{\boldsymbol{\alpha}}\right] \times \hat{\boldsymbol{z}}$$
$$= -(\boldsymbol{M}_s \cdot \hat{\boldsymbol{\alpha}})\hat{\boldsymbol{k}}_{\boldsymbol{\rho}} + (\boldsymbol{M}_s \cdot \hat{\boldsymbol{k}}_{\boldsymbol{\rho}})\hat{\boldsymbol{\alpha}}$$

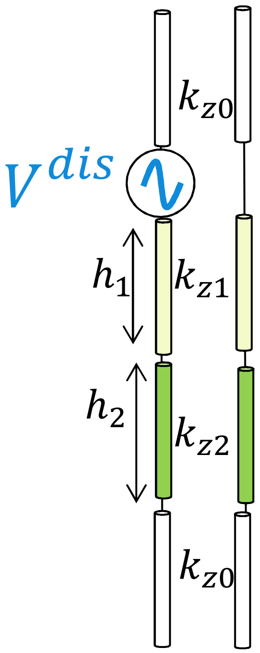
$$\boxed{E_{TM}^{dis}(z_s)} = -\boldsymbol{M}_s \cdot \hat{\boldsymbol{\alpha}} \qquad \boxed{E_{TE}^{dis}(z_s) = \boldsymbol{M}_s \cdot \hat{\boldsymbol{k}}_{\boldsymbol{\rho}}}$$

$$V_{TM}^{dis} = \frac{-\boldsymbol{M}_s \cdot \hat{\boldsymbol{\alpha}}}{jk_\rho} \qquad V_{TE}^{dis} = \frac{\boldsymbol{M}_s \cdot \hat{\boldsymbol{k}}_{\boldsymbol{\rho}}}{-jk_\rho}$$

Equivalent steps for magnetic sources

TU Delft

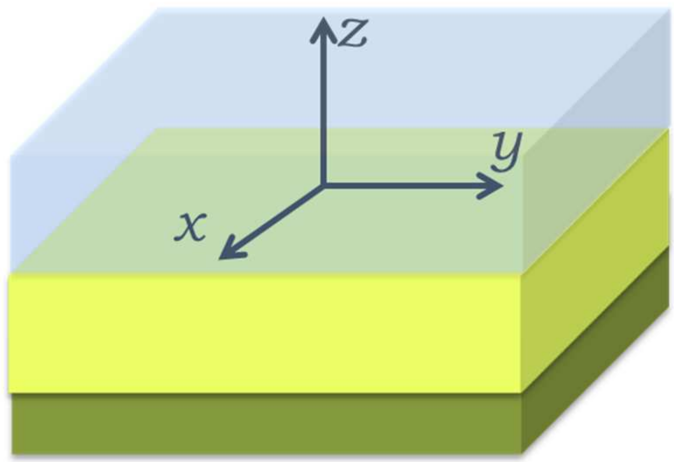
Equivalent voltage generator



$$V_{TE}^{dis} = \frac{E_{TE}^{dis}(z_s)}{-jk_\rho} \qquad V_{TM}^{dis} = \frac{E_{TM}^{dis}(z_s)}{jk_\rho}$$

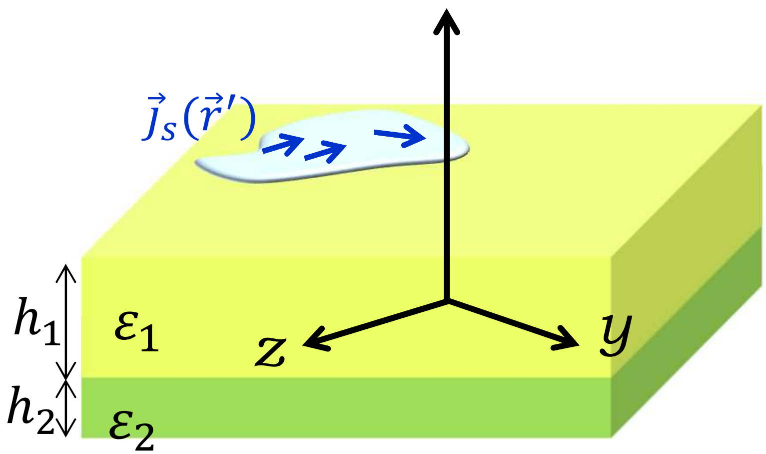
$$V_{TM}^{dis} = \frac{-\boldsymbol{M_s} \cdot \hat{\boldsymbol{\alpha}}}{jk_\rho} \qquad V_{TE}^{dis} = \frac{\boldsymbol{M_s} \cdot \hat{\boldsymbol{k}}_{\boldsymbol{\rho}}}{-jk_\rho}$$

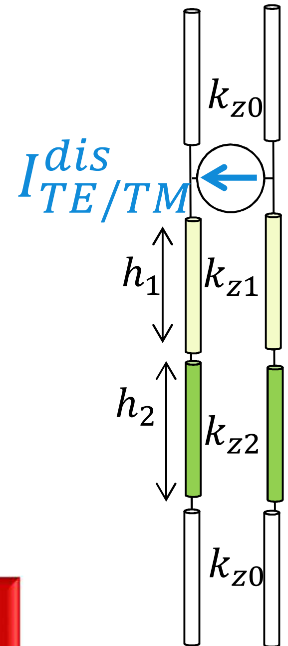
Introducing the TL with Unitary current sources



$$\bar{\bar{G}}^{fc}(\vec{r},\vec{r}') = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \begin{bmatrix} G_{xx}(k_x,k_y,z,z') & G_{xy}(k_x,k_y,z,z') & G_{xz}(k_x,k_y,z,z') \\ G_{yx}(k_x,k_y,z,z') & G_{yy}(k_x,k_y,z,z') & G_{yz}(k_x,k_y,z,z') \\ G_{zx}(k_x,k_y,z,z') & G_{zy}(k_x,k_y,z,z') & G_{zz}(k_x,k_y,z,z') \end{bmatrix} e^{-jk_x(x-x')}e^{-jk_y(y-y')}dk_x dk_y$$

E/H Field – Electric Current Dyadic Spectral GF





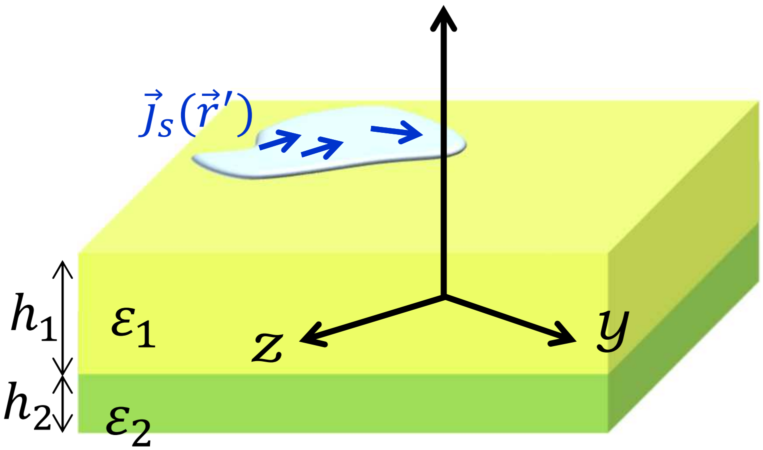
$$I^{dis}_{TE} = \frac{\boldsymbol{J}_s \cdot \hat{\boldsymbol{\alpha}}}{jk_\rho}$$

$$I^{dis}_{TM} = \frac{-\boldsymbol{J}_s \cdot \hat{\boldsymbol{k}}_{\boldsymbol{\rho}}}{jk_\rho}$$

The GF is the field generated by an impulsive source of unitary amplitude in the considered medium

$$\vec{E}(\vec{\boldsymbol{r}}) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} \tilde{\tilde{G}}^{ej}(k_x, k_y, z, z') \tilde{\vec{J}}_s(k_x, k_y) e^{-jk_x x} e^{-jk_y y} dk_x dk_y$$

Equivalent Current Generators



Spatial current distribution at z’

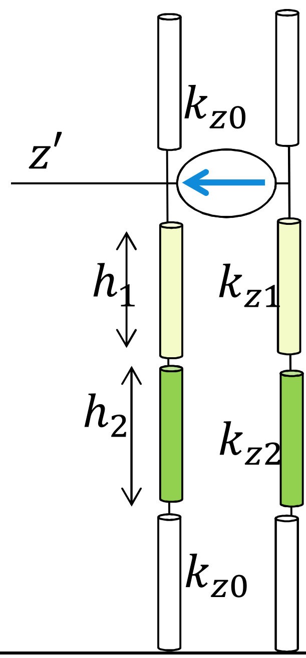
$$\vec{j}(\vec{r}') = \vec{j}_s(\vec{\rho}')\delta(z - z')$$

$$\vec{j}_s(\vec{\rho}') = \hat{p}\delta(\vec{\rho} - \vec{\rho}') \longrightarrow$$

Spectral distribution of the surface current

$$\vec{J}_s(\vec{k}_\rho) = \hat{p}_j e^{j\vec{k}_\rho \cdot \vec{\rho}'}$$

FT in x,y



These generators include both the orientation of the source and its amplitude

$$I_{TE}^{dis} = \frac{\boldsymbol{J}_s \cdot \hat{\boldsymbol{\alpha}}}{jk_\rho}$$

$$I_{TM}^{dis} = \frac{-\boldsymbol{J}_s \cdot \hat{\boldsymbol{k}}_{\boldsymbol{\rho}}}{jk_\rho}$$

The generators for the Green’s function only need to include the orientation of the source (info necessary to build the dyadic)

$$I_{TE}^{j} = \frac{\hat{\boldsymbol{p}}_{\boldsymbol{j}} \cdot \hat{\boldsymbol{\alpha}}}{jk_\rho}$$

$$I_{TM}^{j} = \frac{-\hat{\boldsymbol{p}}_{\boldsymbol{j}} \cdot \hat{\boldsymbol{k}}_{\boldsymbol{\rho}}}{jk_\rho}$$

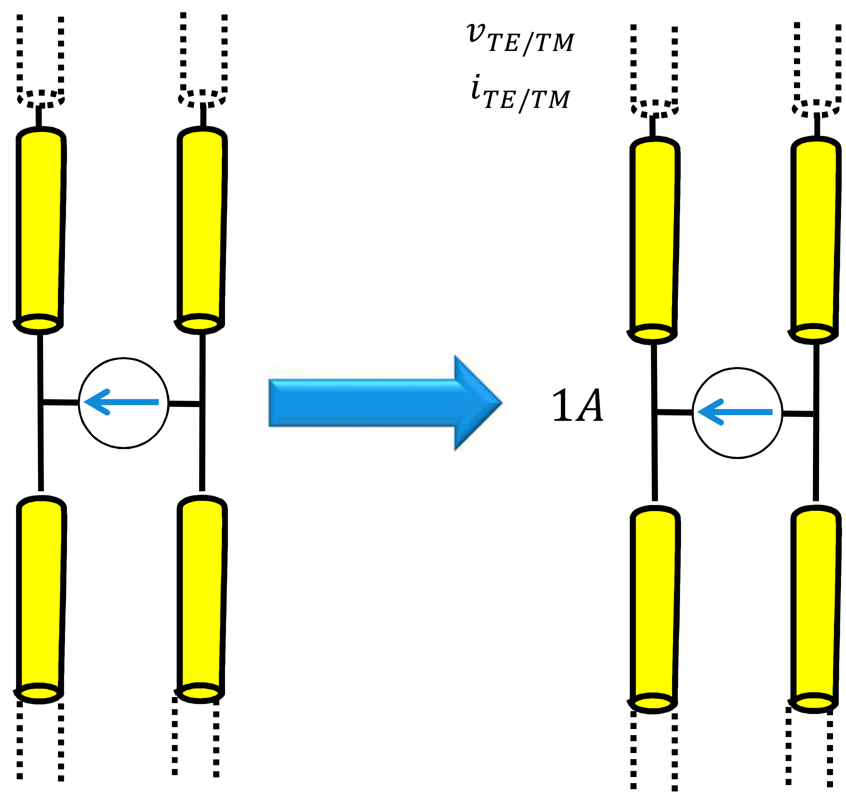
Equivalent "Unitary" Current Generators

The voltage and current solutions along the transmission line can be normalized to the amplitude of the $I^j_{TE/TM}$ current generators to obtain equivalent **transmission lines fed by unit generators**

$V_{TE/TM}$
$I_{TE/TM}$

$$I^j_{TE} = \frac{\hat{\boldsymbol{p}}_{\boldsymbol{j}} \cdot \hat{\boldsymbol{\alpha}}}{jk_\rho}$$

$$I^j_{TM} = \frac{-\hat{\boldsymbol{p}}_{\boldsymbol{j}} \cdot \hat{\boldsymbol{k}}_{\boldsymbol{\rho}}}{jk_\rho}$$



$$v_{TE} = \frac{V_{TE}}{I^j_{TE}}$$

$$i_{TE} = \frac{I_{TE}}{I^j_{TE}}$$

$$v_{TM} = \frac{V_{TM}}{I^j_{TM}}$$

$$i_{TM} = \frac{I_{TM}}{I^j_{TM}}$$

Normalized solutions of the transmission line to have a generator with unitary amplitude

Dyadic Spectral GF
For Electric Current Source

Electric/Magnetic Fields

$$\vec{\tilde{E}}_{TE} = -jk_\rho V_{TE}\hat{\alpha}$$

$$\vec{\tilde{E}}_{TM} = -\frac{j\zeta}{k}k_\rho^2 I_{TM}\hat{z} + jk_\rho \hat{k}_\rho V_{TM}$$

$$\vec{\tilde{H}}_{TM} = jk_\rho I_{TM}\hat{\alpha}$$

$$\vec{\tilde{H}}_{TE} = -\frac{j}{k\zeta}k_\rho^2 V_{TE}\hat{z} + jk_\rho \hat{k}_\rho I_{TE}$$

Non-normalized Voltage/current

$$V_{TE} = v_{TE} I_{TE}^j = v_{TE}\frac{\hat{\boldsymbol{p}}_{\boldsymbol{j}} \cdot \hat{\boldsymbol{\alpha}}}{jk_\rho}$$

$$I_{TE} = i_{TE} I_{TE}^j = i_{TE}\frac{\hat{\boldsymbol{p}}_{\boldsymbol{j}} \cdot \hat{\boldsymbol{\alpha}}}{jk_\rho}$$

$$V_{TM} = v_{TM} I_{TM}^j = -v_{TM}\frac{\hat{\boldsymbol{p}}_{\boldsymbol{j}} \cdot \hat{\boldsymbol{k}}_{\boldsymbol{\rho}}}{jk_\rho}$$

$$I_{TM} = i_{TM} I_{TM}^j = -i_{TM}\frac{\hat{\boldsymbol{p}}_{\boldsymbol{j}} \cdot \hat{\boldsymbol{k}}_{\boldsymbol{\rho}}}{jk_\rho}$$

These expression are explicit but we can build and equivalent notation much more compact

Next we build the ***dyadic Matrix notation:***
Information on the source and field orientation

TUDelft

Dyadic Spectral GF
For Magnetic Current Source

Electric/Magnetic Fields

$$\vec{\vec{E}}_{TE} = -jk_\rho V_{TE}\hat{\alpha}$$

$$\vec{\vec{E}}_{TM} = -\frac{j\zeta}{k}k_\rho^2 I_{TM}\hat{z} + jk_\rho\hat{k}_\rho V_{TM}$$

$$\vec{\vec{H}}_{TM} = jk_\rho I_{TM}\hat{\alpha}$$

$$\vec{\vec{H}}_{TE} = -\frac{j}{k\zeta}k_\rho^2 V_{TE}\hat{z} + jk_\rho\hat{k}_\rho I_{TE}$$

Also for magnetic currents the notation can be much more compact

Using a transmission line unit voltage generator

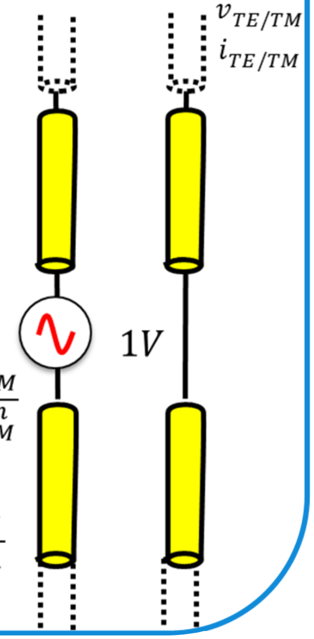
$$V_{TE} = v_{TE}\frac{\boldsymbol{p_m}\cdot\hat{\boldsymbol{k}}_{\boldsymbol{\rho}}}{-jk_\rho} \quad I_{TE} = i_{TE}\frac{\hat{\boldsymbol{p}}_{\boldsymbol{m}}\cdot\hat{\boldsymbol{k}}_{\boldsymbol{\rho}}}{-jk_\rho}$$

$$V_{TM} = -v_{TM}\frac{\hat{\boldsymbol{p}}_{\boldsymbol{m}}\cdot\hat{\boldsymbol{\alpha}}}{jk_\rho} \quad I_{TM} = -i_{TM}\frac{\hat{\boldsymbol{p}}_{\boldsymbol{m}}\cdot\hat{\boldsymbol{k}}_{\boldsymbol{\rho}}}{jk_\rho}$$

$$V_{TM}^m = \frac{-\boldsymbol{p_m}\cdot\hat{\boldsymbol{\alpha}}}{jk_\rho}$$

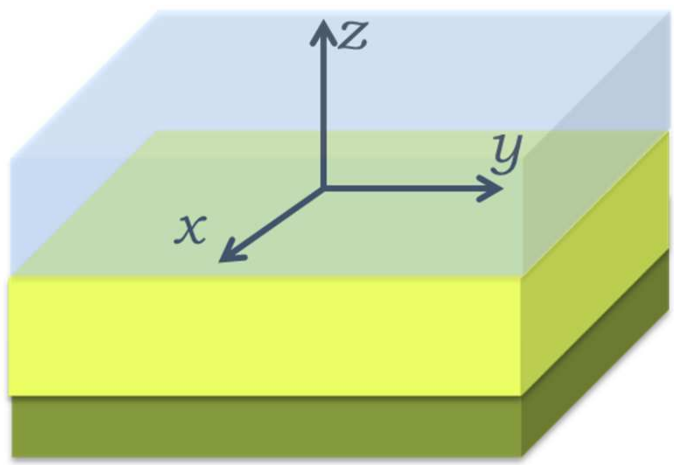
$$V_{TE}^m = \frac{\boldsymbol{p_m}\cdot\hat{\boldsymbol{k}}_{\boldsymbol{\rho}}}{-jk_\rho}$$

$$v_{TE} = \frac{V_{TE}}{V_{TE}^m} \qquad i_{TE} = \frac{I_{TE}}{V_{TE}^m} \qquad v_{TM} = \frac{V_{TM}}{V_{TM}^m} \qquad i_{TM} = \frac{I_{TM}}{V_{TM}^m}$$



Equivalent steps for magnetic sources

Building the Cartesian Dyadic



$$\bar{\bar{G}}^{fc}(\vec{r},\vec{r}') = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \begin{bmatrix} G_{xx}(k_x,k_y,z,z') & G_{xy}(k_x,k_y,z,z') & G_{xz}(k_x,k_y,z,z') \\ G_{yx}(k_x,k_y,z,z') & G_{yy}(k_x,k_y,z,z') & G_{yz}(k_x,k_y,z,z') \\ G_{zx}(k_x,k_y,z,z') & G_{zy}(k_x,k_y,z,z') & G_{zz}(k_x,k_y,z,z') \end{bmatrix} e^{-jk_x(x-x')}e^{-jk_y(y-y')}dk_x dk_y$$

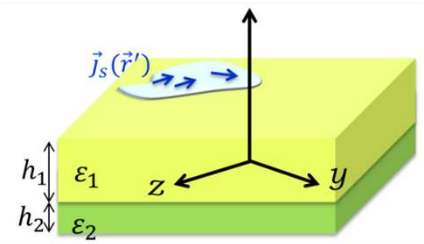
Goal:
Dyadic Green's Function for Stratified Media

Building

With $\vec{e}$ and $\vec{J}$ are in Cartesian Coordinates

$$\boldsymbol{e}(\vec{r}) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{\boldsymbol{G}}^{ej}(k_x,k_y,z,z')\boldsymbol{C}_s(k_x,k_y)e^{-jk_x x}e^{-jk_y y}dk_x dk_y$$

$$\tilde{\boldsymbol{G}}^{ej}(k_x,k_y,z,z') = \begin{bmatrix} -\dfrac{v_{TM}k_x^2+v_{TE}k_y^2}{k_\rho^2} & \dfrac{(v_{TE}-v_{TM})k_xk_y}{k_\rho^2} \\ \dfrac{(v_{TE}-v_{TM})k_xk_y}{k_\rho^2} & -\dfrac{v_{TE}k_x^2+v_{TM}k_y^2}{k_\rho^2} \\ \varsigma\dfrac{k_x}{k}i_{TM} & \varsigma\dfrac{k_y}{k}i_{TM} \end{bmatrix}$$



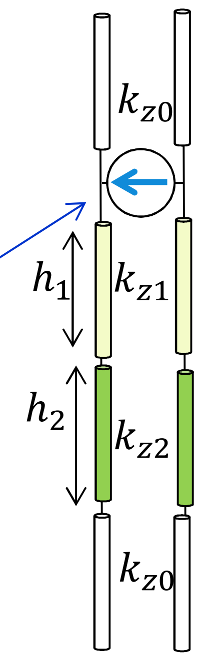
The step is only Book keeping!!!

Starting from

Knowledge that electric field is proportional to Voltages and currents in TE/TM T.L

$$\vec{\tilde{E}}_{TE} = -jk_\rho V_{TE}\hat{\alpha}$$

$$\vec{\tilde{E}}_{TM} = -\frac{j\zeta}{k}k_\rho^2 I_{TM}\hat{z} + jk_\rho\hat{k}_\rho V_{TM}$$



With generators in parallel

$$I_{TE}^{j} = \frac{\widehat{\boldsymbol{p}}_j\cdot\widehat{\boldsymbol{\alpha}}}{jk_\rho}$$

$$I_{TM}^{j} = \frac{-\widehat{\boldsymbol{p}}_j\cdot\widehat{\boldsymbol{k}}_\rho}{jk_\rho}$$

All fields in *cylindrical spectral unit vectors*

TE Electric Field – Electric Current

$$\vec{\tilde{E}}_{TE} = -jk_\rho V_{TE}\hat{\alpha}$$

$$V_{TE} = v_{TE}\frac{\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{\alpha}}}{jk_\rho}$$

$$\vec{\boldsymbol{E}}_{\boldsymbol{TE}} = -jk_\rho v_{TE}\frac{\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{\alpha}}}{jk_\rho}\hat{\boldsymbol{\alpha}}$$

$$\vec{\boldsymbol{E}}_{\boldsymbol{TE}} = -v_{TE}\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\alpha}}$$

$$\hat{\boldsymbol{\alpha}} = \frac{1}{k_\rho}(k_x\hat{\boldsymbol{y}} - k_y\hat{\boldsymbol{x}})$$

$$\hat{\boldsymbol{p}} = p_x\hat{\boldsymbol{x}} + p_y\hat{\boldsymbol{y}}$$

$$\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{\alpha}} = \frac{1}{k_\rho}(k_x p_y - k_y p_x)\text{: scalar}$$

$$\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\alpha}} = \frac{1}{k_\rho^2}(k_x^2 p_y - k_x k_y p_x)\,\hat{y} - \frac{1}{k_\rho^2}(k_x k_y p_y - k_y^2 p_x)\,\hat{x}\text{ : vector}$$

It can be expressed as a multiplication of a matrix and a vector

$$\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\alpha}} = \frac{1}{k_\rho^2}\begin{bmatrix} k_y^2 & -k_x k_y \\ -k_x k_y & k_x^2 \end{bmatrix}\begin{bmatrix} p_x \\ p_y \end{bmatrix} = \hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\alpha}}\cdot\hat{\boldsymbol{p}}$$

$\hat{\boldsymbol{p}}$

$$\frac{1}{k_\rho}\begin{bmatrix} -k_y \\ k_x \end{bmatrix}\frac{1}{k_\rho}\begin{bmatrix} -k_y & k_x \end{bmatrix} = \hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\alpha}}$$

$$\boxed{\vec{\boldsymbol{E}}_{\boldsymbol{TE}} = -v_{TE}\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\alpha}}\cdot\hat{\boldsymbol{p}}}$$

dyadic representation

Other Dyadic representations

Mathematical Steps

They relate the orientation of the source and radiated field

$$\hat{\boldsymbol{\alpha}} = \frac{1}{k_\rho}\left(k_x\hat{\boldsymbol{y}} - k_y\hat{\boldsymbol{x}}\right)$$

$$\hat{\boldsymbol{k}}_{\boldsymbol{\rho}} = \frac{1}{k_\rho}\left(k_x\hat{\boldsymbol{x}} + k_y\hat{\boldsymbol{y}}\right)$$

$$\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\alpha}} = \frac{1}{k_\rho}\begin{bmatrix}-k_y\\ k_x\\ 0\end{bmatrix}\frac{1}{k_\rho}\begin{bmatrix}-k_y & k_x & 0\end{bmatrix} = \frac{1}{k_\rho^2}\begin{bmatrix}k_y^2 & -k_xk_y & 0\\ -k_xk_y & k_x^2 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\hat{\boldsymbol{k}}_{\boldsymbol{\rho}}\hat{\boldsymbol{k}}_{\boldsymbol{\rho}} = \frac{1}{k_\rho}\begin{bmatrix}k_x\\ k_y\\ 0\end{bmatrix}\frac{1}{k_\rho}\begin{bmatrix}k_x & k_y & 0\end{bmatrix} = \frac{1}{k_\rho^2}\begin{bmatrix}k_x^2 & k_xk_y & 0\\ k_xk_y & k_y^2 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{k}}_{\boldsymbol{\rho}} = \frac{1}{k_\rho}\begin{bmatrix}-k_y\\ k_x\\ 0\end{bmatrix}\frac{1}{k_\rho}\begin{bmatrix}k_x & k_y & 0\end{bmatrix} = \frac{1}{k_\rho^2}\begin{bmatrix}-k_xk_y & -k_y^2 & 0\\ k_x^2 & k_xk_y & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\hat{\boldsymbol{k}}_{\boldsymbol{\rho}}\hat{\boldsymbol{\alpha}} = \frac{1}{k_\rho}\begin{bmatrix}k_x\\ k_y\\ 0\end{bmatrix}\frac{1}{k_\rho}\begin{bmatrix}-k_y & k_x & 0\end{bmatrix} = \frac{1}{k_\rho^2}\begin{bmatrix}-k_xk_y & k_x^2 & 0\\ -k_y^2 & k_xk_y & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\hat{\boldsymbol{z}}\hat{\boldsymbol{k}}_{\boldsymbol{\rho}} = \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}\frac{1}{k_\rho}\begin{bmatrix}k_x & k_y & 0\end{bmatrix} = \frac{1}{k_\rho}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ k_x & k_y & 0\end{bmatrix}$$

$$\hat{\boldsymbol{z}}\hat{\boldsymbol{\alpha}} = \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}\frac{1}{k_\rho}\begin{bmatrix}-k_y & k_x & 0\end{bmatrix} = \frac{1}{k_\rho}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ -k_y & k_x & 0\end{bmatrix}$$

Electric Field – Electric Current Dyadic Spectral GF

$$\vec{\tilde{E}}_{TE} = -jk_\rho V_{TE}\hat{\alpha}$$

$$\vec{\tilde{E}}_{TM} = -\frac{j\zeta}{k}k_\rho^2 I_{TM}\hat{z} + jk_\rho\hat{k}_\rho V_{TM}$$

$$V_{TE} = v_{TE}\frac{\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{\alpha}}}{jk_\rho}$$

$$I_{TM} = -i_{TM}\frac{\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{k}}_{\boldsymbol{\rho}}}{jk_\rho}$$

$$V_{TM} = -v_{TM}\frac{\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{k}}_{\boldsymbol{\rho}}}{jk_\rho}$$

$$\boxed{\vec{\boldsymbol{E}} = \left(-\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\alpha}}v_{TE} + \varsigma\frac{k_\rho}{k}i_{TM}\hat{\boldsymbol{z}}\hat{\boldsymbol{k}}_{\boldsymbol{\rho}} - v_{TM}\hat{\boldsymbol{k}}_{\boldsymbol{\rho}}\hat{\boldsymbol{k}}_{\boldsymbol{\rho}}\right)\cdot\hat{\boldsymbol{p}}}$$

$$\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\alpha}} = \frac{1}{k_\rho^2}\begin{bmatrix} k_y^2 & -k_xk_y \\ -k_xk_y & k_x^2 \\ 0 & 0 \end{bmatrix}$$

$$\hat{\boldsymbol{k}}_{\boldsymbol{\rho}}\hat{\boldsymbol{k}}_{\boldsymbol{\rho}} = \frac{1}{k_\rho^2}\begin{bmatrix} k_x^2 & k_xk_y \\ k_xk_y & k_y^2 \\ 0 & 0 \end{bmatrix}$$

$$\hat{\boldsymbol{z}}\hat{\boldsymbol{k}}_{\boldsymbol{\rho}} = \frac{1}{k_\rho}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ k_x & k_y \end{bmatrix}$$

$$\tilde{\boldsymbol{G}}^{ej} = \begin{bmatrix} -\dfrac{v_{TM}k_x^2 + v_{TE}k_y^2}{k_\rho^2} & \dfrac{(v_{TE} - v_{TM})k_xk_y}{k_\rho^2} \\ \dfrac{(v_{TE} - v_{TM})k_xk_y}{k_\rho^2} & -\dfrac{v_{TE}k_x^2 + v_{TM}k_y^2}{k_\rho^2} \\ \varsigma\dfrac{k_x}{k}i_{TM} & \varsigma\dfrac{k_y}{k}i_{TM} \end{bmatrix}$$

Similarly for all GF

$$\vec{\boldsymbol{f}}(\vec{r}) \approx \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{\infty} \widetilde{\boldsymbol{G}}^{fc}(k_\rho, \alpha, z, z')e^{-j\vec{\boldsymbol{k}}_{\boldsymbol{\rho}}\cdot\vec{\boldsymbol{\rho}}}k_\rho dk_\rho d\alpha$$

Basically all the SGF´s can be expressed in a similar format by separating the dependence on the **transmission line solution** and the **vectorial projections**

	EJ	HM
$G_{\rho\rho}$	$-v_{TM}$	$-i_{TE}$
$G_{\alpha\alpha}$	$-v_{TE}$	$-i_{TM}$
$G_{z\rho}$	$\varsigma\frac{k_\rho}{k}i_{TM}$	$\frac{k_\rho}{\varsigma k}v_{TE}$

	EM	HJ
$G_{\alpha\rho}$	v_{TE}	$-i_{TM}$
$G_{\rho\alpha}$	$-v_{TM}$	i_{TE}
$G_{z\alpha}$	$\varsigma\frac{k_\rho}{k}i_{TM}$	$-\frac{k_\rho}{\varsigma k}v_{TE}$

$$\widetilde{\boldsymbol{g}}^{fc}(\vec{r},\vec{r}') = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{\infty}(G_{\rho\rho}\widehat{\boldsymbol{k}}_{\boldsymbol{\rho}}\widehat{\boldsymbol{k}}_{\boldsymbol{\rho}} + G_{\alpha\alpha}\widehat{\boldsymbol{\alpha}}\widehat{\boldsymbol{\alpha}} + G_{z\rho}\widehat{\boldsymbol{z}}\widehat{\boldsymbol{k}}_{\boldsymbol{\rho}})e^{-j\vec{\boldsymbol{k}}_{\boldsymbol{\rho}}\cdot\vec{\boldsymbol{\rho}}}k_\rho dk_\rho d\alpha$$

vectorial projections

$$\widetilde{\boldsymbol{g}}^{fc}(\vec{\boldsymbol{r}},\vec{\boldsymbol{r}}') = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(G_{\alpha\rho}\widehat{\boldsymbol{\alpha}}\widehat{\boldsymbol{k}}_{\boldsymbol{\rho}} + G_{\rho\alpha}\widehat{\boldsymbol{k}}_{\boldsymbol{\rho}}\widehat{\boldsymbol{\alpha}} + G_{z\alpha}\widehat{\boldsymbol{z}}\widehat{\boldsymbol{\alpha}})e^{-j\vec{\boldsymbol{k}}_{\boldsymbol{\rho}}\cdot\vec{\boldsymbol{\rho}}}dk_x dk_y$$

Magnetic Field – Electric Current Dyadic Spectral GF

$$\vec{\tilde{H}}_{TM} = jk_\rho I_{TM}\hat{\alpha}$$

$$\vec{\tilde{H}}_{TE} = -\frac{j}{k\zeta}k_\rho^2 V_{TE}\hat{z} + jk_\rho \hat{k}_\rho I_{TE}$$

$$V_{TE} = v_{TE}\frac{\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{\alpha}}}{jk_\rho}$$

$$I_{TM} = -i_{TM}\frac{\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{k}}_{\boldsymbol{\rho}}}{jk_\rho}$$

$$I_{TE} = i_{TE}\frac{\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{\alpha}}}{jk_\rho}$$

$$\boxed{\vec{\boldsymbol{H}} = \left(\hat{\boldsymbol{k}}_{\boldsymbol{\rho}}\hat{\boldsymbol{\alpha}} i_{TE} - \frac{k_\rho}{\varsigma k} v_{TE}\hat{\boldsymbol{z}}\hat{\boldsymbol{\alpha}} - i_{TM}\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{k}}_{\boldsymbol{\rho}}\right)\cdot\hat{\boldsymbol{p}}}$$

$$\hat{\boldsymbol{k}}_{\boldsymbol{\rho}}\hat{\boldsymbol{\alpha}} = \frac{1}{k_\rho^2}\begin{bmatrix} -k_x k_y & k_x^2 \\ k_y^2 & k_x k_y \\ 0 & 0 \end{bmatrix}$$

$$\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{k}}_{\boldsymbol{\rho}} = \frac{1}{k_\rho^2}\begin{bmatrix} -k_x k_y & -k_y^2 \\ k_x^2 & k_x k_y \\ 0 & 0 \end{bmatrix}$$

$$\hat{\boldsymbol{z}}\hat{\boldsymbol{\alpha}} = \frac{1}{k_\rho}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -k_y & k_x \end{bmatrix}$$

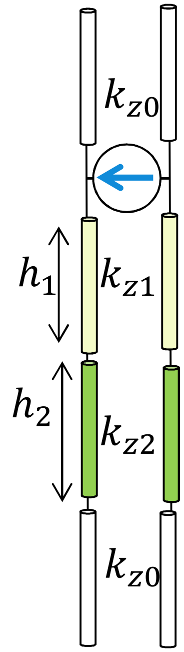
$$\tilde{\boldsymbol{G}}^{\boldsymbol{hj}} = \begin{bmatrix} \frac{(i_{TM} - i_{TE})k_x k_y}{k_\rho^2} & \frac{i_{TE}k_x^2 + i_{TM}k_y^2}{k_\rho^2} \\ \frac{i_{TE}k_y^2 + i_{TM}k_x^2}{k_\rho^2} & \frac{(i_{TE} - i_{TM})k_x k_y}{k_\rho^2} \\ \frac{k_y}{\varsigma k}v_{TE} & \frac{k_x}{\varsigma k}v_{TE} \end{bmatrix}$$

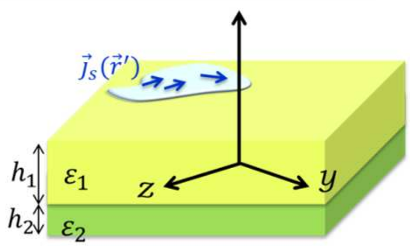
Dyadic Green's Function for Stratified Media

$$\boldsymbol{f}(\vec{r}) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{\boldsymbol{G}}^{fc}(k_x,k_y,z,z')\boldsymbol{C}_s(k_x,k_y)e^{-jk_x x}e^{-jk_y y}\,dk_x dk_y$$

$$\tilde{\boldsymbol{G}}^{ej} = \begin{bmatrix} -\dfrac{v_{TM}k_x^2 + v_{TE}k_y^2}{k_\rho^2} & \dfrac{(v_{TE}-v_{TM})k_xk_y}{k_\rho^2} \\ \dfrac{(v_{TE}-v_{TM})k_xk_y}{k_\rho^2} & -\dfrac{v_{TE}k_x^2 + v_{TM}k_y^2}{k_\rho^2} \\ \varsigma\dfrac{k_x}{k}i_{TM} & \varsigma\dfrac{k_y}{k}i_{TM} \end{bmatrix}$$

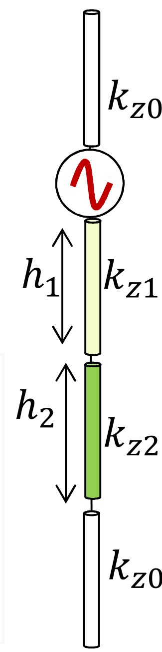
$$\tilde{\boldsymbol{G}}^{hj} = \begin{bmatrix} \dfrac{(i_{TM}-i_{TE})k_xk_y}{k_\rho^2} & \dfrac{i_{TE}k_x^2 + i_{TM}k_y^2}{k_\rho^2} \\ \dfrac{i_{TE}k_y^2 + i_{TM}k_x^2}{k_\rho^2} & \dfrac{(i_{TE}-i_{TM})k_xk_y}{k_\rho^2} \\ \dfrac{k_y}{\varsigma k}v_{TE} & \dfrac{k_x}{\varsigma k}v_{TE} \end{bmatrix}$$

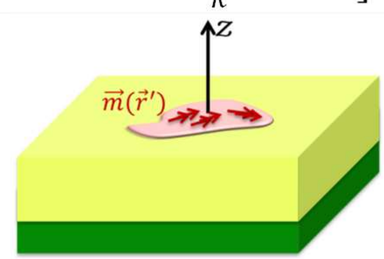




$$\tilde{\boldsymbol{G}}^{hm} = \begin{bmatrix} -\dfrac{i_{TE}k_x^2 + i_{TM}k_y^2}{k_\rho^2} & \dfrac{(i_{TM}-i_{TE})k_xk_y}{k_\rho^2} \\ \dfrac{(i_{TM}-i_{TE})k_xk_y}{k_\rho^2} & -\dfrac{i_{TM}k_x^2 + i_{TE}k_y^2}{k_\rho^2} \\ \dfrac{k_x}{\varsigma k}v_{TE} & \dfrac{k_y}{\varsigma k}v_{TE} \end{bmatrix}$$

$$\tilde{\boldsymbol{G}}^{em} = \begin{bmatrix} \dfrac{(v_{TM}-v_{TE})k_xk_y}{k_\rho^2} & -\dfrac{v_{TE}k_y^2 + v_{TM}k_x^2}{k_\rho^2} \\ \dfrac{v_{TE}k_x^2 + v_{TM}k_y^2}{k_\rho^2} & \dfrac{(v_{TE}-v_{TM})k_xk_y}{k_\rho^2} \\ -\varsigma\dfrac{k_y}{k}i_{TM} & \varsigma\dfrac{k_x}{k}i_{TM} \end{bmatrix}$$

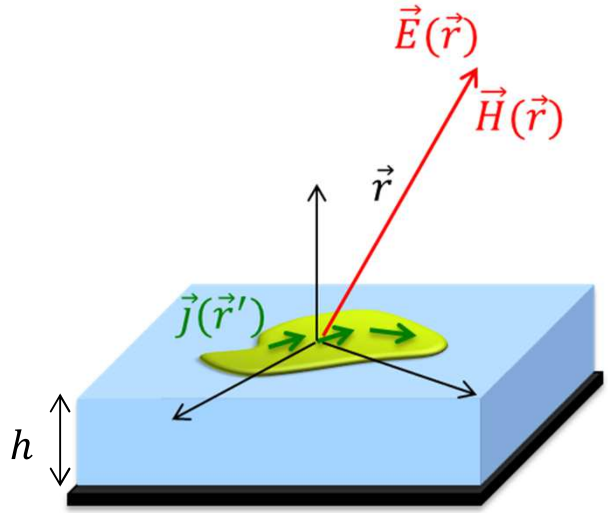




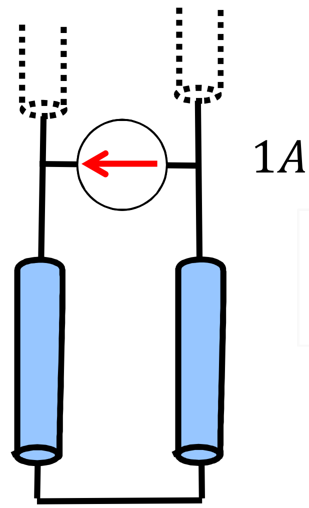
Example 1: Elementary electric source on a Grounded Slab

Electric current distribution

$\vec{j}(\vec{\boldsymbol{\rho}}') = rect(x,l)rect(y,w)\delta(z-h)\hat{\boldsymbol{x}}$



Equivalent transmission lines fed by unit generators



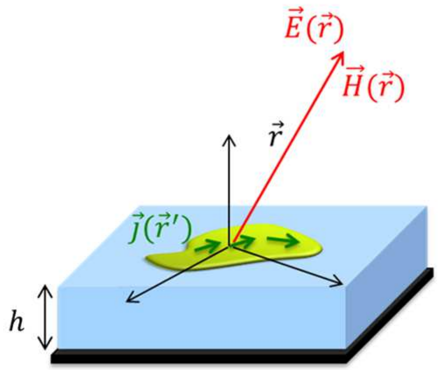
$v_{TE/TM}(k_\rho, z, z')$

$i_{TE/TM}(k_\rho, z, z')$

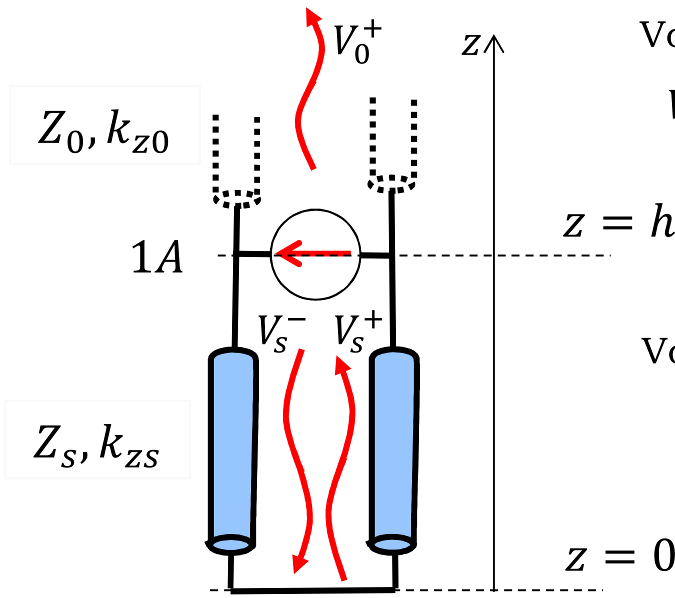
$$\vec{E}(\vec{\boldsymbol{r}}) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} \tilde{\bar{G}}^{ej}(k_x, k_y, z, z')\vec{\tilde{J}}_s(k_x, k_y\,)e^{-jk_x x}e^{-jk_y y}dk_x dk_y$$

$$\tilde{\boldsymbol{G}}^{ej} = \begin{bmatrix} -\frac{v_{TM}k_x^2 + v_{TE}k_y^2}{k_\rho^2} & \frac{(v_{TE} - v_{TM})k_x k_y}{k_\rho^2} \\ \frac{(v_{TE} - v_{TM})k_x k_y}{k_\rho^2} & -\frac{v_{TE}k_x^2 + v_{TM}k_y^2}{k_\rho^2} \\ \varsigma\frac{k_x}{k}i_{TM} & \varsigma\frac{k_y}{k}i_{TM} \end{bmatrix}$$

Example 1: Transmission Line solution



$v_{TM}(z),\ v_{TE}(z)$?



Voltage in the air:

$$V_0(z) = V_0^+ \quad e^{-jk_{z0}z}$$

Voltage in the slab:

$$V_s(z) = V_s^+ \quad e^{-jk_{zs}z} + V_s^- e^{jk_{zs}z}$$

TM Solution

$$Z_0 = \frac{\varsigma_0 k_{z0}}{k_0} \qquad Z_s = \frac{\varsigma_s k_{zs}}{k_s}$$

$$k_{z0} = \sqrt{k_0^2 - k_x^2 - k_y^2}$$

$$k_{zs} = \sqrt{k_s^2 - k_x^2 - k_y^2}$$

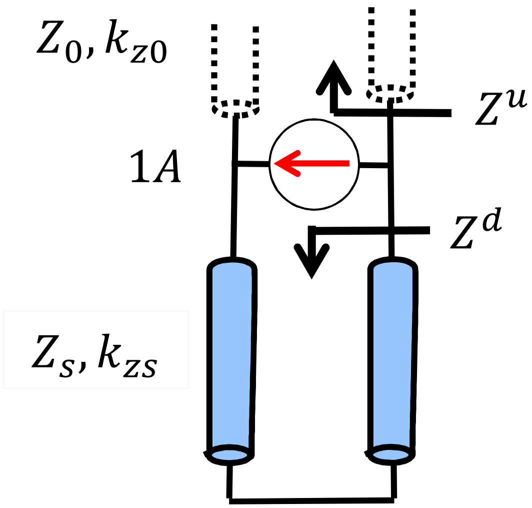
TE Solution

$$Z_0 = \frac{\varsigma_0 k_0}{k_{z0}} \qquad Z_s = \frac{\varsigma_s k_s}{k_{zs}}$$

$$k_{z0} = \sqrt{k_0^2 - k_x^2 - k_y^2}$$

$$k_{zs} = \sqrt{k_s^2 - k_x^2 - k_y^2}$$

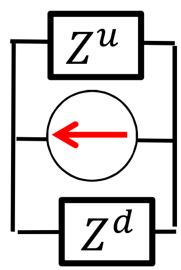
Example 1: Transmission Line solution



Voltage at z=0 $\quad V_s = V_s^+ + V_s^- = 0 \quad \Gamma = -1$

$$V_s(z) = V_s^+\left(e^{-jk_{zs}z} - e^{jk_{zs}z}\right) = -2jV_s^+ sin(k_{zs}z)$$

Voltage at z=h



$Z_u = Z_0$

$Z_d = jZ_s\tan(k_{zs}h)$

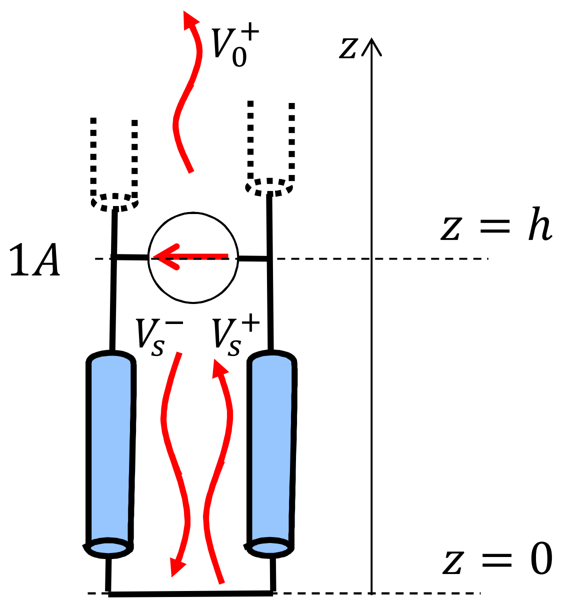
$$v(h) = \frac{Z_u Z_d}{Z_u + Z_d} I_g; \quad (I_g = 1)$$

$$v(h) = V_s(h) = -2jV_s^+ sin(k_{zs}h) = \frac{Z_u Z_d}{Z_u + Z_d} = V_0^+ \; e^{-jk_{z0}h} = V_0(h)$$

$$V_s^+ = \frac{Z_u Z_d}{Z_u + Z_d}\frac{j}{2sin(k_{zs}h)}$$

$$V_0^+ = \frac{Z_u Z_d}{Z_u + Z_d} e^{jk_{z0}h}$$

Example 1: Elementary electric source on a Grounded Slab



Voltage solution in the slab:

$$V_s(z) = \frac{Z_u Z_d}{Z_u + Z_d} \frac{sin(k_{zs} z)}{sin(k_{zs} h)}$$

Current solution in the slab:

$$I_s(z) = \frac{V_s^+}{Z_s}\left(e^{-jk_{zs}z} + e^{jk_{zs}z}\right) = \frac{1}{Z_s} \frac{Z_u Z_d}{Z_u + Z_d} \frac{j\cos(k_{zs} z)}{sin(k_{zs} h)}$$

Voltage solution in the air:

$$V_0(z) = \frac{Z_u Z_d}{Z_u + Z_d} e^{jk_{z0}h} e^{-jk_{z0}z}$$

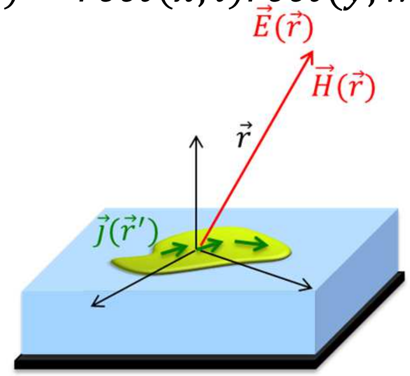
Current solution in the air:

$$I_0(z) = \frac{1}{Z_0} \frac{Z_u Z_d}{Z_u + Z_d} e^{jk_{z0}h} e^{-jk_{z0}z}$$

Example 1: Elementary electric source on a Grounded Slab

Electric current distribution

$\vec{\boldsymbol{j}}(\vec{\boldsymbol{\rho}}') = rect(x,l)rect(y,w)\delta(z-h)\hat{\boldsymbol{x}}$



How to evaluate the
<u>x-component field</u> ***in the air****?*

$$\vec{E}(\vec{\boldsymbol{r}}) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} \tilde{\vec{G}}^{ej}(k_x,k_y,z,z'=h)\vec{\tilde{J}}_s(k_x,k_y)e^{-jk_x x}e^{-jk_y y}dk_x dk_y$$

$$\vec{\tilde{J}}_s(k_x,k_y) = wsinc\left(\frac{k_x l}{2}\right) lsinc\left(\frac{k_y w}{2}\right)\hat{x}$$

Voltage solution in the air line

$$V_0(z) = \frac{Z_u Z_d}{Z_u+Z_d}\underbrace{e^{jk_{z0}h}}_{\text{Source quote}}\underbrace{e^{-jk_{z0}z}}_{\text{Obs. quote}}$$

$$\tilde{\boldsymbol{G}}^{ej} = \begin{bmatrix} -\frac{v_{TM}k_x^2+v_{TE}k_y^2}{k_\rho^2} & \frac{(v_{TE}-v_{TM})k_x k_y}{k_\rho^2} \\ \frac{(v_{TE}-v_{TM})k_x k_y}{k_\rho^2} & -\frac{v_{TE}k_x^2+v_{TM}k_y^2}{k_\rho^2} \\ \varsigma\frac{k_x}{k}i_{TM} & \varsigma\frac{k_y}{k}i_{TM} \end{bmatrix}$$

$$E_x(\vec{\boldsymbol{r}}) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}\left(-\frac{v_{TM}(z)k_x^2+v_{TE}(z)k_y^2}{k_\rho^2}\right) wsinc\left(\frac{k_x l}{2}\right) wsinc\left(\frac{k_y l}{2}\right)e^{-jk_x x}e^{-jk_y y}dk_x dk_y$$

Important Points

1) The spectral potentials in planar stratified dielectrics can be evaluated via two equivalent transmission lines (TE/TM)

2) The solution of the transmission lines depend on the spectral variable $\boldsymbol{k_\rho}(\, since\ k_z = \sqrt{k_i^2 - k_\rho^2})$

3) The tangent **sources** (j_x, j_y, m_x, m_y) are introduced as **discontinuities** in the tangent e/h fields

4) Electric currents are represented by **parallel current generators**

5) Magnetic currents are presented by **series voltage generators**

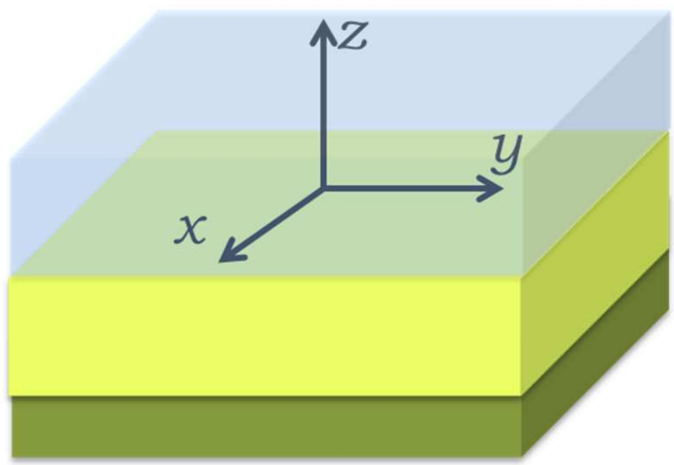
6) The Spectral Green's functions (fields) are calculated using the **voltage and current solutions** of the equivalent transmission lines

7) The actual dyadic depends on the orientation of the source and field. Today we have calculated the dyadic for **Cartesian coordinates.**

Truly Important Points

1) How to construct this GF is an entrance price to pay.
2) You do it because the GF has polar singularities, each pole corresponding to TE or TM wave
3) Capturing the residue of these poles will provide the dominant contributions to the Analysis without full integrations.
4) **These waves are the ones for which one has to design!!**
   (either killing them or enhancing them)

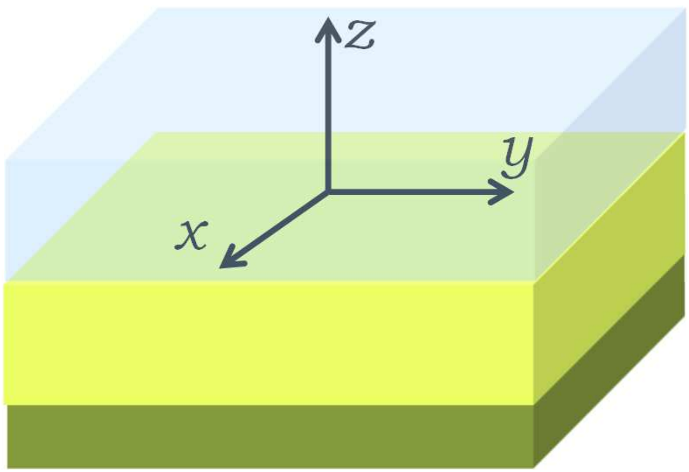
Reminder of the derivation of the Spectral GF for stratified media



$$G^{fc}(\vec{r},\vec{r}')=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{bmatrix}\tilde{G}_{xx}(k_x,k_y;z,z') & \tilde{G}_{xy}(k_x,k_y;z,z') & \tilde{G}_{xz}(k_x,k_y;z,z')\\ \tilde{G}_{yx}(k_x,k_y;z,z') & \tilde{G}_{yy}(k_x,k_y;z,z') & \tilde{G}_{yz}(k_x,k_y;z,z')\\ \tilde{G}_{zy}(k_x,k_y;z,z') & \tilde{G}_{zy}(k_x,k_y;z,z') & \tilde{G}_{zz}(k_x,k_y;z,z')\end{bmatrix}e^{-jk_x(x-x')}e^{-jk_y(y-y')}dk_x dk_y$$

Step 1

The total E and H fields can be calculated divided into TE and TM field components: ***Transversalization of Maxwell Equation***



Since stratified dielectric media are invariant in xy plane, we choose the potentials to be only along z:

Electric potential $\vec{F} = \varepsilon F_z \hat{z}$

Magnetic potential $\vec{A} = \mu A_z \hat{z}$

TE: transverse electric with respect to z ($E_z^{TE} = 0$)

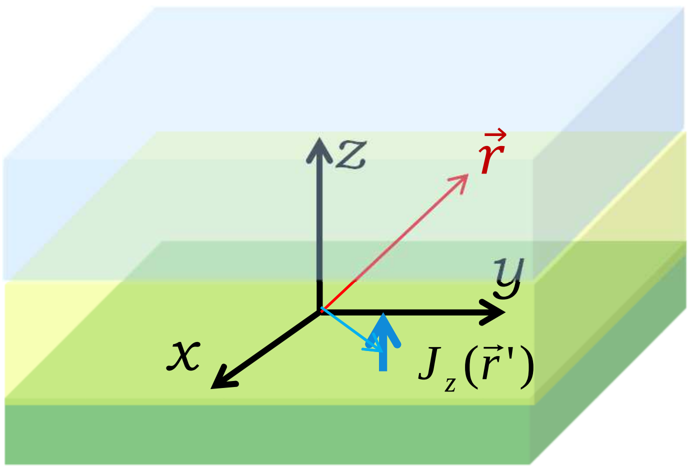
$$\vec{E}_{Fz} = -\nabla_t \times F_z \hat{z} \equiv \vec{E}_{TE} \qquad \vec{H}_{Fz} = -j\frac{k}{\varsigma}\left(F_z \hat{z} + \frac{1}{k^2}\nabla_t \frac{\partial}{\partial z} F_z + \frac{1}{k^2}\frac{\partial^2}{\partial z^2} F_z \hat{z} \right) \equiv \vec{H}_{TE}$$

TM: transverse magnetic with respect to z *c*

$$\vec{H}_{Az} = \nabla_t \times A_z \hat{z} \equiv \vec{H}_{TM} \qquad \vec{E}_{Az} = -jk\varsigma\left(A_z \hat{z} + \frac{1}{k^2}\frac{\partial^2}{\partial z^2} A_z \hat{z} + \frac{1}{k^2}\nabla_t \frac{\partial}{\partial z} A_z \right) \equiv \vec{E}_{TM}$$

Step 2

Find the TE and TM potential solutions in the ***spectral domain***



Introducing the *spectral potentials*…

$$A_z(\vec{r}) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{A}_z(k_x, k_y; z)\, e^{-jk_x x} e^{-jk_y y}\, dk_x dk_y$$

$$F_z(\vec{r}) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{F}_z(k_x, k_y; z)\, e^{-jk_x x} e^{-jk_y y}\, dk_x dk_y$$

$\vec{r}$: observation point
The FT is introduced only in xy
The spectral potential depends on z

We can only deal naturally with currents oriented along z

$$J_z(\vec{r}\,') = \delta(\vec{\rho} - \vec{\rho}\,')\delta(z - z')$$

$$M_z(\vec{r}\,') = \delta(\vec{\rho} - \vec{\rho})\delta(z - z')$$

$\vec{r}\,' = \vec{\rho}\,' + z'\hat{z}$: *source point*

TM $\nabla^2 A_z + k^2 A_z = -J_z$

TE $\nabla^2 F_z + k^2 F_z = -M_z$

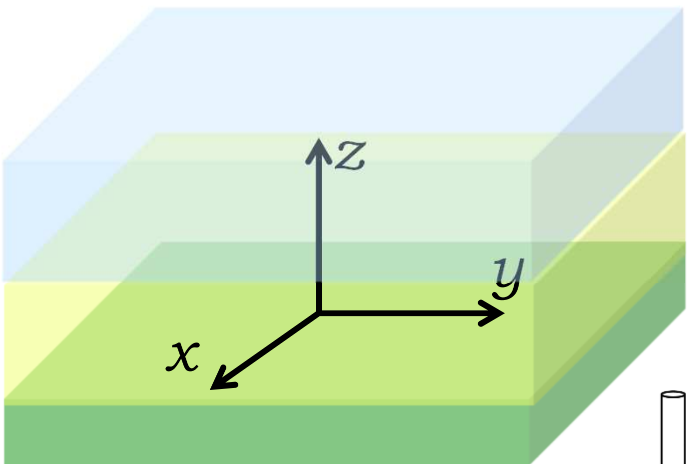
$$\left(k_z^2 + \frac{\partial^2}{\partial z^2}\right)\tilde{A}_z(\vec{k}_\rho; z) = -e^{j\vec{k}_\rho \cdot \vec{\rho}\,'}\delta(z - z')$$

$$\left(k_z^2 + \frac{\partial^2}{\partial z^2}\right)\tilde{F}_z(\vec{k}_\rho; z) = -e^{j\vec{k}_\rho \cdot \vec{\rho}\,'}\delta(z - z')$$

$\vec{k}_\rho = k_x\hat{x} + k_y\hat{y}$

Step 3

The spectral differential equations of the potentials can be solved using a ***transmission line representation***



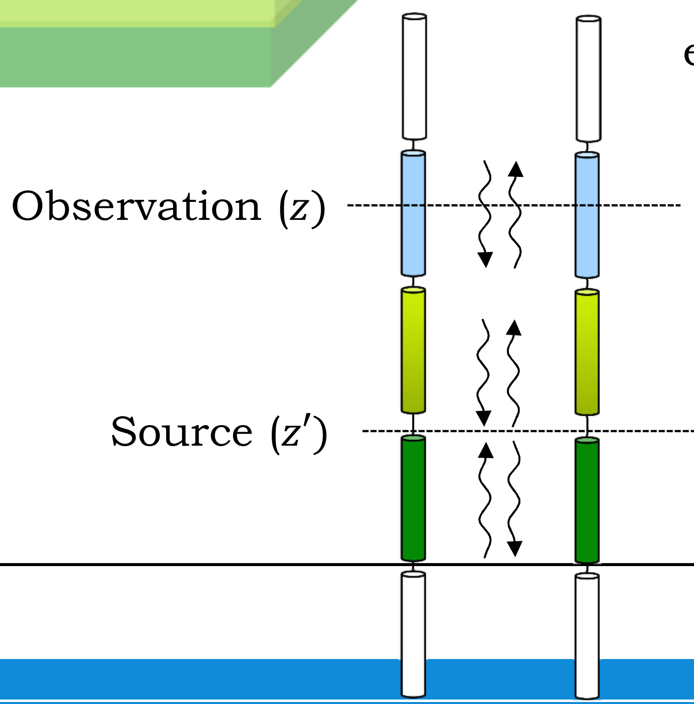
$$\left(k_z^2+\frac{\partial^2}{\partial z^2}\right)\tilde{A}_z(\vec{k}_\rho;z)=-e^{j\vec{k}_\rho\cdot\vec{\rho}'}\,\delta(z-z')$$

$$\left(k_z^2+\frac{\partial^2}{\partial z^2}\right)\tilde{F}_z(\vec{k}_\rho;z)=-e^{j\vec{k}_\rho\cdot\vec{\rho}'}\,\delta(z-z')$$

These equations are equivalent to those of a z-transmission line where

$$I_{TM}=e^{-j\vec{k}_\rho\cdot\vec{\rho}'}\tilde{A}_z(\vec{k}_\rho;z)$$

$$V_{TE}=e^{-j\vec{k}_\rho\cdot\vec{\rho}'}\tilde{F}_z(\vec{k}_\rho;z)$$



Step 4

The spectral differential equations of the potentials can be solved using a ***transmission line representation***

TM $I_{TM}(\vec{k}_\rho;z)=e^{-j\vec{k}_\rho\cdot\vec{\rho}'}\tilde{A}_z(\vec{k}_\rho;z)$

$$V_{TM}=V_{TM}^{+}e^{-jk_{zi}(z-z')}+V_{TM}^{-}e^{jk_{zi}(z-z')}$$

$$I_{TM}=I_{TM}^{+}e^{-jk_{zi}(z-z')}+I_{TM}^{-}e^{jk_{zi}(z-z')}$$

$$Z_{TM}=\frac{k_{zi}\varsigma_i}{k_i}=\frac{V_{TM}^{+}}{I_{TM}^{+}}=-\frac{V_{TM}^{-}}{I_{TM}^{-}}$$

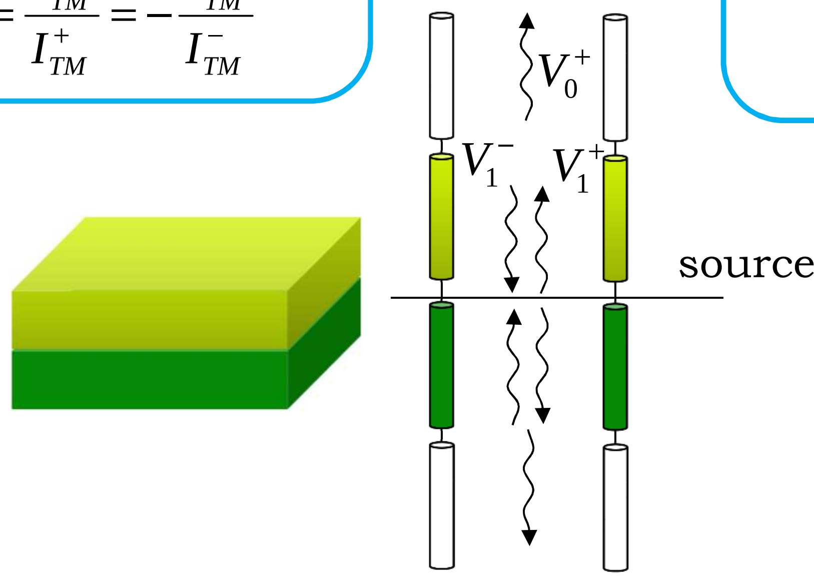
TE $V_{TE}(\vec{k}_\rho;z)=e^{-j\vec{k}_\rho\cdot\vec{\rho}'}\tilde{F}_z(\vec{k}_\rho;z)$

$$V_{TE}=V_{TE}^{+}e^{-jk_{zi}(z-z')}+V_{TE}^{-}e^{jk_{zi}(z-z')}$$

$$I_{TE}=I_{TE}^{+}e^{-jk_{zi}(z-z')}+I_{TE}^{-}e^{jk_{zi}(z-z')}$$

$$Z_{TE}=\frac{k_i\varsigma_i}{k_{zi}}=\frac{V_{TE}^{+}}{I_{TE}^{+}}=-\frac{V_{TE}^{-}}{I_{TE}^{-}}$$

$$\boxed{k_{zi}^2=k_i^2-k_\rho^2}$$



k_z: *changes in dielectric*
$Z_{TE/TM}$: *changes in dielectric*

Step 5

The ***spectral Electric and Magnetic fields*** can be calculated using the *Voltage and Current solutions* of the equivalent transmission line:

$$\vec{E}_{TE} = -\nabla_t \times F_z \hat{z} \longrightarrow \boxed{\vec{\mathbf{E}}_{\mathbf{TE}} = j\vec{k}_\rho \times \hat{z} V_{TE} = -jk_\rho V_{TE} \hat{\boldsymbol{\alpha}}}$$

$$\tilde{F}_z(\vec{k}_\rho; z) = V_{TE}(\vec{k}_\rho; z)$$

$$\boxed{\hat{k}_\rho \times \hat{\alpha} = \hat{z}}$$

$$\nabla_t = -j\vec{k}_\rho$$

Electric field

$$\vec{\boldsymbol{E}}_{\boldsymbol{TM}} = -j\varsigma \frac{k_\rho^2}{k} I_{TM} \hat{\boldsymbol{z}} + jk_\rho V_{TM} \hat{\boldsymbol{k}}_{\boldsymbol{\rho}}$$

$$\vec{\boldsymbol{E}}_{\boldsymbol{TE}} = -jk_\rho V_{TE} \hat{\boldsymbol{\alpha}}$$

Magnetic field

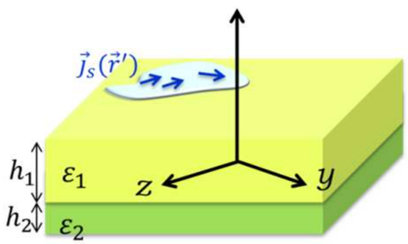
$$\vec{\boldsymbol{H}}_{\boldsymbol{TM}} = jk_\rho I_{TM} \hat{\boldsymbol{\alpha}}$$

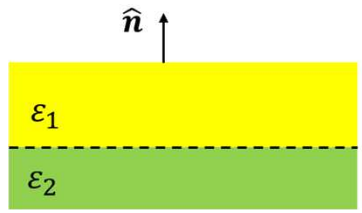
$$\vec{\boldsymbol{H}}_{\boldsymbol{TE}} = -j \frac{k_\rho^2}{\varsigma k} V_{TE} \hat{\boldsymbol{z}} + jk_\rho \hat{\boldsymbol{k}}_{\boldsymbol{\rho}} I_{TE}$$

$$\hat{\boldsymbol{k}}_{\boldsymbol{\rho}} = \frac{1}{k_\rho}\left(k_x \hat{\boldsymbol{x}} + k_y \hat{\boldsymbol{y}}\right) \qquad \hat{\boldsymbol{\alpha}} = \frac{1}{k_\rho}\left(k_x \hat{\boldsymbol{y}} - k_y \hat{\boldsymbol{x}}\right)$$

Step 6

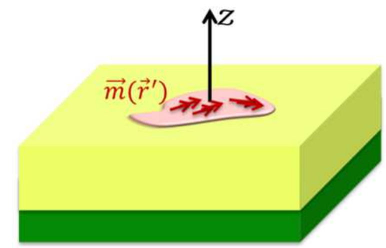
The ***tangent electric and magnetic sources*** are introduced as **discontinuities** in the tangent fields





$$\hat{\boldsymbol{n}} \times (\boldsymbol{E}_1 - \boldsymbol{E}_2) = -\boldsymbol{M}_s$$
$$\hat{\boldsymbol{n}} \times (\boldsymbol{H}_1 - \boldsymbol{H}_2) = \boldsymbol{J}_s$$

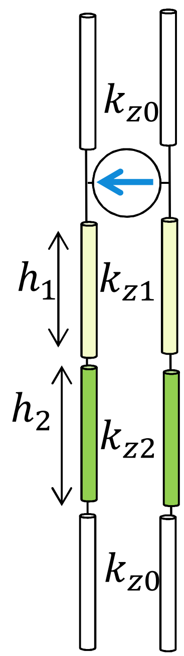
Surface current densities



$$I^{dis}_{TE} = \frac{H^{dis}_{TE}(z_s)}{jk_\rho} = \frac{\boldsymbol{J}_s \cdot \hat{\boldsymbol{\alpha}}}{jk_\rho}$$

$$I^{dis}_{TM} = \frac{H^{dis}_{TM}(z_s)}{jk_\rho} = \frac{-\boldsymbol{J}_s \cdot \hat{\boldsymbol{k}}_{\boldsymbol{\rho}}}{jk_\rho}$$

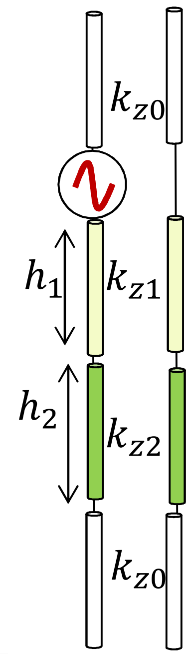
Electric currents are represented by **parallel current unit amplitude generators**



$$V^{dis}_{TM} = \frac{E^{dis}_{TM}(z_s)}{jk_\rho} = \frac{-\boldsymbol{M}_s \cdot \hat{\boldsymbol{\alpha}}}{jk_\rho}$$

$$V^{dis}_{TE} = \frac{E^{dis}_{TE}(z_s)}{-jk_\rho} = \frac{\boldsymbol{M}_s \cdot \hat{\boldsymbol{k}}_{\boldsymbol{\rho}}}{-jk_\rho}$$

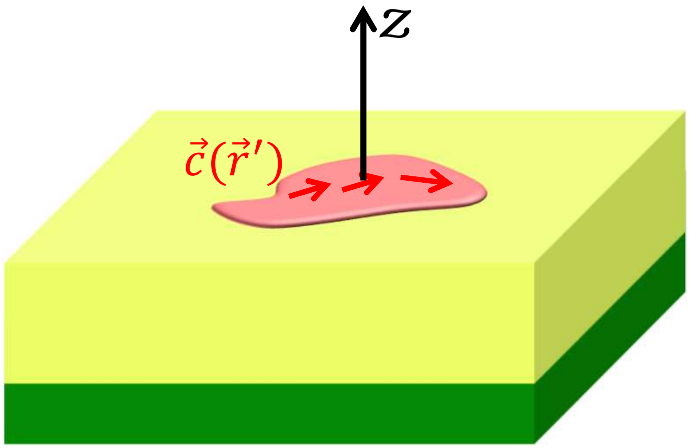
Magnetic currents are represented by **series voltage unit amplitude generators**



The solution of the transmission line problem depends only on $\boldsymbol{k}_{\boldsymbol{\rho}} (i.e.\, k_z)$

TU Delft

Introduction of Sources



$$\left(k_z^2 + \frac{\partial^2}{\partial z^2}\right) I_{TM}(k_z; z) = 0$$

Homogenous differential equation

For happens for sources oriented along x or y?

For sources oriented along z

Spatial current distribution

$$J_z(\vec{r}') = \delta(\vec{\rho} - \vec{\rho}')\,\delta(z - z')$$

$$M_z(\vec{r}') = \delta(\vec{\rho} - \vec{\rho}')\,\delta(z - z')$$

Spectral current distribution

$$\tilde{J}_z(\vec{k}_\rho) = e^{j\vec{k}_\rho \cdot \vec{\rho}'}\delta(z - z')$$

$$\tilde{M}_z(\vec{k}_\rho) = e^{j\vec{k}_\rho \cdot \vec{\rho}'}\delta(z - z')$$

$$\left(k_{zi}^2 + \frac{\partial^2}{\partial z^2}\right) I_{TM}(k_z, z) = -e^{j\vec{k}_\rho \cdot \vec{\rho}'}\delta(z - z')$$

$$\left(k_{zi}^2 + \frac{\partial^2}{\partial z^2}\right) V_{TE}(k_z, z) = -e^{j\vec{k}_\rho \cdot \vec{\rho}'}\delta(z - z')$$