ECON 5322

Macroeconomic Theory for Applications

Topic 2

The Real Business Cycle model

Do Markets clear?

Is Money Neutral?	Y	N
Y	RBC	
N		

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Frictionless Case:

No market Failures

Most Important Benchmark in macro

*this material is a courtesy of Fabio Ghironi (UWashington) that was edited for this course

Introduction

- This course explores modern theories of macroeconomic fluctuations.

- Last time we saw there are multiple approaches to model the output fluctuations (aggregate supply)

- These changed according to the assumptions on the efficiency of the markets (frictions) and the role of policy

- Here we cover the simplest, yet most important benchmark: a frictionless case with no role for policy interventions

- This model is also known as the stochastic growth model, a setup in which fluctuations are the result of random shocks to technology and economic outcomes are efficient.

Introduction, Continued

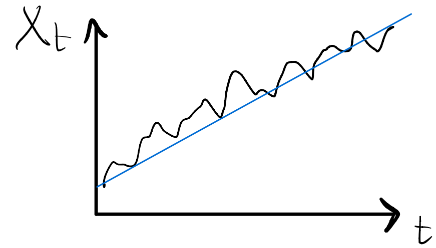
- In this course, the word *efficiency* has a very precise meaning:
  - The market economy is efficient when the outcome it generates is the same as the outcome chosen by a benevolent social planner in charge of allocating resources.
- The market economy is efficient in the RBC model: A benevolent planner who acts to maximize social welfare would not do better than the market.
  - In other words: the agents achieve a private outcome that no policy can improve on
- Crucial implication: No Role for Policy

Introduction, Continued

- **A useful benchmark:** We study the RBC model not because we believe that it is accurate, or realistic, but because it is a useful starting point to become familiar with concepts, tools, and techniques that we will use many times throughout the course.

- We will then introduce a number of more realistic features into our framework: monopoly power, nominal rigidity, financial market frictions, and more.

- These features, or frictions, will imply that the economy we model is no longer efficient: Policy can improve outcomes relative to the market.



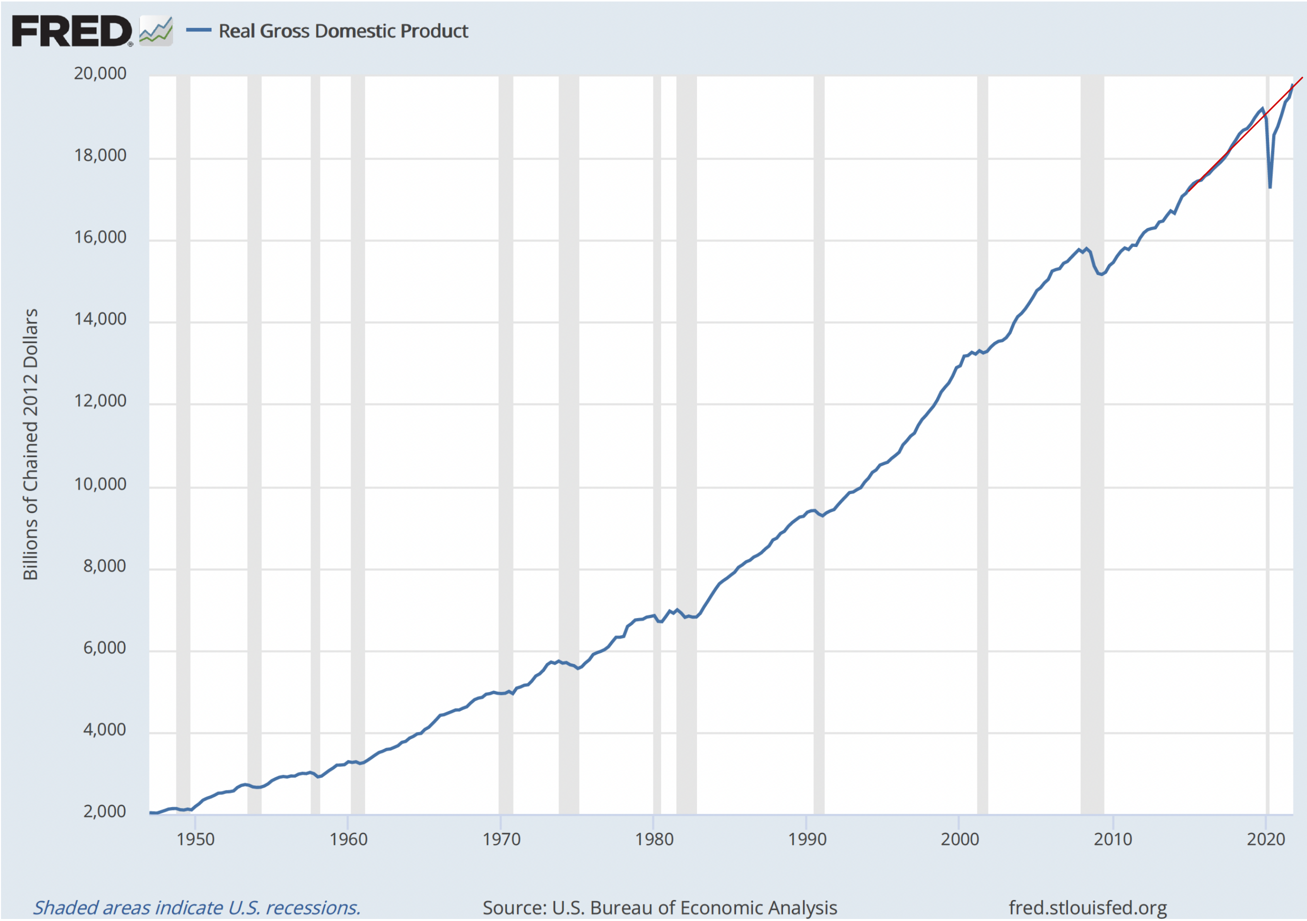
Introduction, Continued

- The tools used in the RBC model became the foundation of the mainstream framework for studying macroeconomic fluctuations in the 1980s, starting with seminal work by Finn Kydland and Edward Prescott published in *Econometrica* in 1982.

- The model studies fluctuations of the economy around its growth trend (business cycles) triggered by unexpected, random shocks, assuming that agents in the economy act to optimize intertemporal objective functions under rational expectations about the future.

- In its standard versions, the analysis assumes that shocks generate departures from trend that disappear over time: e.g., an unexpected improvement in technology causes the economy's GDP to rise above trend for temporarily

- The figure in the next slide shows the behavior of U.S. GDP since 1947. It gives you an illustration of situations in which the standard approach can work (much of the time) but also situations in which it will do very poorly (the aftermath of the Great Recession that followed the Global Financial Crisis of 2007-08).
  - The size and persistence of the shocks may be critical for determining when the usual approach is a good approximation. The COVID-19 crisis, for example, is an instance when although temporary, the size of the shock could create challenges for approximations with standard techniques


FRED
Real Gross Domestic Product
Billions of Chained 2012 Dollars
20,000
18,000
16,000
14,000
12,000
10,000
8,000
6,000
4,000
2,000
1950
1960
1970
1980
1990
2000
2010
2020
Shaded areas indicate U.S. recessions.
Source: U.S. Bureau of Economic Analysis
fred.stlouisfed.org

Introduction, Continued

- Market outcomes are efficient in the basic RBC model.
- This happens because there are no *distortions* (or *frictions*) that prevent the market's own devices from functioning well enough to deliver an outcome that is as good as the one a centralized and benevolent planner would choose.
- But the usefulness of the model *does not* depend the assumption that business cycles are triggered by technology shocks or that of an efficient model-economy.
- We will introduce distortions, obstacles to the smooth functioning of markets, realistic features that will allow us to tackle questions that the basic RBC setup cannot address—including issues that have taken center stage in discussions on macroeconomics.
- Studying the RBC model will prepare us to study those more realistic models, and it will help us understand exactly when and why market outcomes in those models are not efficient, and therefore when and why there is a role for policy in fixing frictions or distortions

Solving Models

- In studying the RBC model, we will pay special attention to the procedure for solving it.

- The difficulty in solving the model is a fundamental non-linearity that arises from the interaction between multiplicative elements, such as Cobb-Douglas production, and additive elements, such as capital accumulation and depreciation.

- This non-linearity makes it impossible, in general, to solve the model without resorting to some kind of approximation.
  - The only case in which this problem does not arise is when capital depreciates fully in one period and agents utility from consumption takes the logarithmic form.
  - This is a very special, unrealistic combination of assumptions.

Solving Models, Continued

- The solution method that we will study is called *log-linearization*.

- It starts from the model's optimality conditions and budget constraints and transforms them into a system of log-linear expectational difference equations in which all endogenous variables are function of the capital stock and of the exogenous shocks causing fluctuations.

Original Variable: X_t log-linear variable: $x_t = \frac{X_t - \bar{X}_t}{\bar{X}_t} \approx \log(X_t) - \log(\bar{X}_t)$

- Variables in the log-linearized model measure *percentage deviations of original variables from their trend (or steady-state) levels*.

  – We will use the words trend and steady state interchangeably, with the understanding that underlying variables are constant in steady state only if long-run growth is zero, otherwise they are moving at their trend-growth rate.

$100 \to 101$ (1% growth) $\to \log(101) - \log(100) = 0.01$
$(4.615) - (4.605)$

$100 \to 150$ (50% growth)

- The approximated model can then be solved using a method known as the *method of undetermined coefficients*.

$\log(150) - \log(100) = 0.4$
$(5.0106) - (4.605) = 0.4$

- An advantage of this solution method over alternatives is that it can be applied also to models in which the market outcome is not efficient.

Solving Models, Continued

- There are plenty of situations in which you would not want to log-linearize your model (and therefore assume that your variables always display a tendency to return to the steady state around which you approximated the original, non-linear model).
- Log-linearization limits the range of questions you can study, or it can yield very misleading conclusions.
- For example, log-linearization cannot handle phenomena like the Great Recession and the years that followed and situations in which accounting for nonlinearity (like the zero—or effective—lower bound on central bank policy interest rates) is necessary.
- But log-linearization is still used to work on many other interesting, important questions and understanding how it works also helps us understand its limitations and why alternative, more complicated techniques become necessary in other scenarios.
    - For this course we can work within the scope of this "local" approximation around trend

Households in the Basic RBC Model

- Consider an economy populated by a large number of identical, infinitely lived households, all subject to the same uncertainty.

- At time t, the representative household maximizes its expected intertemporal utility from t to the infinite future, discounting utility in future periods according to a discount factor β:

$$E_t\left[\sum_{s=t}^{\infty} \underbrace{\beta^{s-t}}_{\text{discount Factor}} \underbrace{u(C_s)}_{\text{Utility from Consumption}}\right] = E_t\left(\sum_{s=t}^{\infty} \beta^{s-t}\frac{C_s^{1-\gamma}}{1-\gamma}\right), \quad 0<\beta<1, \tag{1}$$

$u(c) = \frac{c^{1-\gamma}}{1-\gamma} \overset{\gamma=2}{\Rightarrow} \frac{c^{-1}}{-1}$

where E_t denotes the expectation based on the information available at time t, $\sum_{s=t}^{\infty}$ is the summation operator for time that goes from the current period (t) all the way to infinite, and C_s is consumption in period s ($s = t, ..., \infty$).

- We assume that this expectation is rational, *i.e.*, the household uses optimally all the information that is available to it.

Obj. Function:

$u(C_t) + \beta\, u(C_{t+1}) + \beta\beta\, u(C_{t+2}) + \beta\beta\beta\, u(C_{t+3}) + \ldots$

$= \beta^0 u(C_t) + \beta^1 u(C_{t+1}) + \beta^2 u(C_{t+2}) + \ldots$

$= \sum_{s=t}^{\infty} \beta^{s-t} u(C_s)$

\$100 $\quad X = 500$

(1)

$100 \approx \beta X$

The Intertemporal Utility Function

- The expression

$$E_t\left(\sum_{s=t}^{\infty}\beta^{s-t}\frac{C_s^{1-\gamma}}{1-\gamma}\right)$$

$0 < \beta < 1$

is a compact way of writing

$$E_t\left(\frac{C_t^{1-\gamma}}{1-\gamma}+\beta\frac{C_{t+1}^{1-\gamma}}{1-\gamma}+\beta^2\frac{C_{t+2}^{1-\gamma}}{1-\gamma}+...+\beta^{\infty}\frac{C_{t+\infty}^{1-\gamma}}{1-\gamma}\right).$$

- Discounting by β captures the idea that households care about utility from current consumption more than they care about utility from future consumption.

The Intertemporal Utility Function, Continued

- $\gamma > 0$ is the coefficient of relative risk aversion:
  - It measures the attitude of our representative household toward risk (uncertainty).
  - If γ were equal to zero—linear utility—the household would not care about risk. It would be perfectly indifferent between a certain level of consumption and an uncertain one.
  - If you are not familiar with the concept of risk aversion, you find more information in Appendix A.

$$u(c) = \begin{cases} \frac{c^{1-\gamma}}{1-\gamma} & \gamma < 1 \\ \log(c) & \text{if } \gamma = 1 \end{cases}$$

CRRA: Constant Relative Risk Aversion

The Intertemporal Utility Function, Continued

- Let us define the parameter $\boxed{\sigma \equiv \frac{1}{\gamma}.}$

- This is known as the elasticity of intertemporal substitution. σ

  – It measures the responsiveness of consumption to interest rate changes: the willingness of agents to postpone consumption across periods when interest rates change

- Tight connection between attitude toward uncertainty (γ) and toward intertemporal substitution (σ) is an undesirable feature of the model when studying important questions (for instance, related to asset pricing).

- Larry Epstein and Stanley Zin developed a framework that unties risk aversion from intertemporal substitution (*Journal of Political Economy*, 1989)

- We will stick to the basic framework, keeping in mind that it has significant limitations (see Appendix A for an example).

Capital Accumulation and Labor Supply

- Households can accumulate a single asset, homogeneous physical capital, K_t and

$$K_{t+1} = (1-\delta)K_t + I_t, \quad 0 < \delta \leq 1 \tag{2}$$

where K_t is the capital stock with which the household begins period t, I_t is investment in period t, δ is the rate of capital depreciation, and K_{t+1} is the stock of capital with which the household will begin period $t+1$.

- Each household supplies a fixed amount of labor ($N_t = 1$) in each period in a perfectly competitive labor market.

↳ Simplification: "Fixed Labor supply"

The Budget Constraint

$w.\cancel{N}$ 1

- The representative household's consumption is constrained by:

$$\underbrace{C_t + I_t}_{\text{Use of Resources}} = \underbrace{\widetilde{r}_t K_t + w_t}_{\text{Available Resources}}, \tag{3}$$

$\widetilde{r}_t$ is the rental rate the household receives from the firms that rent its capital in a perfectly competitive rental market, and w_t is the wage ($N_t = 1$ implies $w_t N_t = w_t =$ labor income).

  – This is the household's period budget constraint: A constraint like this applies to every period (to every s, for $s = 0, ..., \infty$).

- (2) and (3) imply that the household's budget constraint can be rewritten as:

take (2) solve for I_t, subs in (3):

$$C_t + K_{t+1} = (1 + \widetilde{r}_t - \delta)K_t + w_t. \tag{4}$$

- The problem of the household is to maximize (1) subject to (4).

- How do we solve such intertemporal optimization problem?

Present Value of the life utility stream

$$\max \sum_{s=t}^{\infty} \beta^{s-t} u(C_s)$$

s.t. (4)

Solving the Household's Problem: Intuitive Approach

- Let us start with in intuitive approach.
- Suppose we have a household that must decide what to do with 1 dollar in the current period (t). 1 it can use it to buy consumption in t (assume that 1 unit of consumption costs 1 dollar) or it can invest it in an asset that will generate the uncertain gross return R_{t+1} at time $t+1$.
- Consider the two possible choices of the household:
- Use the dollar to buy consumption: yields the benefit given by the increment in utility from consuming an extra unit of consumption today—the marginal utility of consumption: $u'(C_t)$.
- Invest the dollar: receives the return R_{t+1} at time $t+1$. In terms of the utility increment generated by the extra consumption this allows to do at $t+1$, this translates into $u'(C_{t+1})R_{t+1}$.

Option 1 $u'(C_t)$ marginal utility of consumption $\rightarrow$ Increase consumption by \$1

Option 2 $\beta R_{t+1} \cdot u'(C_{t+1})$ $\rightarrow$ Increase savings by \$1

In equilibrium 1 = 2

$$\Rightarrow u'(C_t) = \beta E_t[R_{t+1} u'(C_{t+1})]$$

$f(x)$ $f'(x)$ $f(x,y)$ $\cancel{f'(x,y)}$

$2x$ 2 $2xy$ $2x \to f_y(x,y)$

$2y \to f_x(x,y)$

Notation Digression

$U(C,1-N) \to U_C(C,1-N) = \dfrac{\partial U(C,1-N)}{\partial C}$

$U(C,1-N) \to U_{1-N}(C,1-N)$

- When we are dealing with functions of only one variable, we will denote the first derivative by using a superscript "$'$" and the second derivative by using a superscript "$''$."

- When we are dealing with functions of more than one variable, we will denote the first partial derivative with respect to a variable by having that variable indicated once as subscript and the second derivative by having the variable indicated twice as subscript.
  - Example: The first derivatives of the function $f(x, y)$ with respect to x and y are denoted $f_x(x, y)$ and $f_y(x, y)$, respectively, and some second derivatives are $f_{xx}(x, y)$ and $f_{yy}(x, y)$

- An alternative way of indicating partial first derivatives that may appear in the slides will be to use numerical subscripts referring to the variable with respect to which we take the derivative.
  - Example: The first derivatives of the function $f(x, y)$ with respect to x and to y are denoted $f_1(x, y)$ and $f_2(x, y)$, respectively, and some second derivatives are $f_{11}(x, y)$ and $f_{22}(x, y)$.

- Do *not* use "$'$" or "$''$" superscripts when denoting derivatives of functions of more than one variable.

- Why? Because $f'(x, y)$ does not tell anyone with respect to what variable the derivative is taken!

$$U(C,1-N) \to \begin{cases} U_C(\cdot) \to \dfrac{\partial U(C,1-N)}{\partial C} \\ U_{1-N}(\cdot) \end{cases}$$

Solving the Household's Problem: Intuitive Approach, Continued

- The household does not know $C_{t+1}$and R_{t+1} with certainty at time t, when taking its decision. Hence, it will compute its expectation of $R_{t+1}u'(C_{t+1})$ based on the information it has at time t: $E_t\left[u'(C_{t+1})R_{t+1}\right]$.
- Moreover, in comparing the benefit of consuming today to that of investing (and thus consuming the next period), the household discounts the future benefit with a discount factor β.
- Hence, the household will compare $u'(C_t)$ and $\beta E_t\left[u'(C_{t+1})R_{t+1}\right]$.
- When is the household happy with the allocation of its resources across periods?

Solving the Household's Problem: Intuitive Approach, Continued

- When it is indifferent between the two alternatives!
- In other words, for the household's behavior to be optimized, it must be the case that:

$$u'(C_t) = \beta E_t \left[u'(C_{t+1}) R_{t+1} \right].$$

- This optimality condition is known as Euler equation.
- In our model, the asset the household can invest in is capital, and the return that an extra unit of capital today generates at $t+1$ is $1 + \widetilde{r}_{t+1} - \delta$: the undepreciated portion of that unit of capital plus the rental rate that it generates.

MPK: $\widetilde{r}_{t+1}$

Undepreciated Capital: $1-\delta$

- Hence, the Euler equation for optimal capital accumulation in our model is:

$$u'(C_t) = \beta E_t \left[u'(C_{t+1})(1 + \widetilde{r}_{t+1} - \delta) \right],$$

$R_{t+1} = 1-\delta + \widetilde{r}_{t+1}$

or, given the assumed form of the period utility function,

$$C_t^{-\gamma} = \beta E_t \left[C_{t+1}^{-\gamma} (1 + \widetilde{r}_{t+1} - \delta) \right]. \quad (5)$$

$u(C_t) = \dfrac{C_t^{1-\gamma}}{1-\gamma}$

$u_c(C_t) = C_t^{-\gamma}$

$Cov(x,y) = E[xy] - E[x]E[y]$

Solving the Household's Problem: Doing the Math

- Now let us obtain this equation by doing math.
- The budget constraint (4) can be rearranged as:

Rearrange BC

$$C_t = -K_{t+1} + (1 + \widetilde{r}_t - \delta)K_t + w_t. \tag{6}$$

- Recall that the household faces a constraint like (4) in every period—put differently, it faces a sequence of constraints like (4) for time that goes from t to ∞.
- In the generic period s, it has to be:

$$C_s = -K_{s+1} + (1 + \widetilde{r}_s - \delta)K_s + w_s. \tag{7}$$

- We can substitute this constraint for C_s in the objective of the household, which will therefore be maximizing:

Substitute C_t

in $E_t\left\{\sum_{s=t}^{\infty} \beta^{s-t} u(C_s)\right\}$

$$E_t\left\{\sum_{s=t}^{\infty} \beta^{s-t} \frac{[-K_{s+1} + (1 + \widetilde{r}_s - \delta)K_s + w_s]^{1-\gamma}}{1-\gamma}\right\}. \tag{8}$$

Solving the Household's Problem: Doing the Math

- What does the household choose?
- The household takes the rental rate and the wage as given—as we mentioned above, they are determined in perfectly competitive markets in which all agents are price takers.
- Moreover, at any time s, K_s is *predetermined*: It is the capital stock with which the household *begins* the period. It was determined in the previous period.
- Having substituted investment and consumption out of the problem through our manipulation of constraints and substitutions (the substitution of (2) into (3), and the substitution of (7) into (1)) leaves K_{s+1} as the only variable that the household actually chooses at any time s.

$$\max_{\boxed{C_t},\, K_{t+1}} (8) \rightarrow \max_{K_{t+1}} (8) \qquad \frac{\partial (8)}{\partial K_{t+1}} = 0$$

Solving the Household's Problem: Doing the Math

- Without loss of generality, focus on $s = t$. The first-order condition for the household's optimal behavior follows from setting the derivative of (8) with respect to $K_{t+1:}$ equal to 0.

- To find this derivative most transparently, note what happens if we write the summation in (8) explicitly. The household maximizes:

$$\frac{[-K_{t+1} + (1 + \widetilde{r}_t - \delta)K_t + w_t]^{1-\gamma}}{1-\gamma} + \beta E_t \left\{ \frac{[-K_{t+2} + (1 + \widetilde{r}_{t+1} - \delta)K_{t+1} + w_{t+1}]^{1-\gamma}}{1-\gamma} \right\}$$
$$+\beta^2 E_t \left\{ \frac{[-K_{t+3} + (1 + \widetilde{r}_{t+2} - \delta)K_{t+2} + w_{t+2}]^{1-\gamma}}{1-\gamma} \right\} + \ldots$$

$$\beta E_t \frac{(1-\gamma)[\quad]^{1-\gamma-1}}{(1-\gamma)} (1+\widetilde{r}_{t+1}-\delta)$$

- Note also that everything in the first term is known at time t (K_{t+1} is chosen at t). Therefore, we can drop the expectation operator from that term.

- As you see, $K_{t+1:}$ appears in two consecutive terms of this expression. Hence, taking the derivative yields:

$$-(1-\gamma)\frac{\overbrace{[-K_{t+1} + (1 + \widetilde{r}_t - \delta)K_t + w_t]}^{C_t}{}^{-\gamma}}{1-\gamma}$$
$$+\beta E_t \left\{ \frac{(1-\gamma)\overbrace{[-K_{t+2} + (1 + \widetilde{r}_{t+1} - \delta)K_{t+1} + w_{t+1}]}^{C_{t+1}}{}^{-\gamma}}{1-\gamma} (1 + \widetilde{r}_{t+1} - \delta) \right\} = 0$$

Solving the Household's Problem: Doing the Math

- If you simplify the $1-\gamma$ terms, substitute (7) for $s=t$ and $s=t+1$, respectively, in the first and in the second term of this expression, and set it equal to 0, you immediately find:

$$-C_t^{-\gamma}+\beta E_t\left[C_{t+1}^{-\gamma}(1+\widetilde{r}_{t+1}-\delta)\right]=0,$$

  or

$$C_t^{-\gamma}=\beta E_t\left[C_{t+1}^{-\gamma}(1+\widetilde{r}_{t+1}-\delta)\right],$$

  i.e., the Euler equation (5).

- A sequence of such equations (one for every $s=t,...,\infty$) must be satisfied for the household to be optimizing its consumption and investment behavior over time.

The Transversality Condition

- The Euler equation is actually not the only optimality condition the household must satisfy:

- The Euler equation describes optimal behavior between any two consecutive periods (s and $s+1$, for $s = t, ..., \infty$), but the household is solving an infinite horizon problem that requires it to look beyond any pair of consecutive periods.

- The additional condition that must be satisfied is known as *transversality conditions*, and it has this form:

$$\lim_{T \longrightarrow \infty} E_t \left[\beta^T u'(c_{t+T})(1 + \widetilde{r}_{t+T} - \delta) K_{t+T} \right] = 0. \tag{9}$$

- We are not going to do the math to show why this condition must hold.

The Transversality Condition, Continued

- Intuitively, if the expression on the left-hand side were strictly positive, the household would be overaccumulating capital, so that a higher expected lifetime utility could be achieved by increasing consumption today.
- The counterpart to such non-optimality in a *finite* horizon model would be that the household dies with positively valued capital holdings: There is no bequest motive in our model for which anyone would want to die with positively valued assets!
- (9) cannot be violated on the negative side because the marginal utility of consumption is never negative, $0 < \delta \leq 1$, and capital (a factor of production) must be positive.

Euler Equations and Transversality Conditions

- One way to look at Euler equations and transversality conditions is to observe that Euler equations rule out arbitrage opportunities between consecutive periods (when the Euler equation holds, the household cannot increase its utility by changing consumption and capital holdings between two consecutive periods).

- Transversality conditions rule out permanent/infinite-horizon arbitrages (the household cannot increase its utility by increasing consumption permanently).

- Euler equations represent short-run optimality conditions, which all candidate paths for optimality of consumption and investment must satisfy, while the transversality condition gives an additional long-run optimality condition, which only one of the short-run optimal paths satisfies (given the assumptions we make on the utility function).
  - Concavity of the utility function ensures that we do not need to compute second-order conditions for the household's maximization problem.

The Rental Rate and Production

- Households rent capital to firms and, with competitive markets,

$$\tilde{r}_t = \frac{\partial Y_t}{\partial K_t} \quad \text{(marginal product of capital)},$$

  where Y_t is output.

- We assume that output in the economy is given by a Cobb-Douglas production function. In aggregate per capita terms,

$$Y_t = (A_t \cancelto{1}{N_t})^{\alpha} K_t^{1-\alpha} = \underline{A_t{}^{\alpha} K_t^{1-\alpha}} \tag{10}$$

  where $0 < \alpha < 1$ and A_t denotes exogenous technology (subject to random shocks)

$$\tilde{r}_t = \frac{\partial Y_t}{\partial K_t} = A_t^{\alpha} K_t^{1-\alpha-1}(1-\alpha)$$

$$= (1-\alpha)\frac{Y_t}{K_t}$$

- Therefore,

$$\underline{\tilde{r}_t = (1-\alpha)\left(\frac{A_t}{K_t}\right)^{\alpha}}$$

  and the Euler equation (5) becomes:

$$C_t^{-\gamma} = \beta E_t \left\{ C_{t+1}^{-\gamma} \left[\underbrace{(1-\alpha)\left(\frac{A_{t+1}}{K_{t+1}}\right)^{\alpha}}_{\tilde{r}_{t+1}} + 1 - \delta \right] \right\} \tag{11}$$

Efficiency and the Planner's Outcome

- There are no distortions in the model-economy we are considering (markets are perfectly competitive).

  ↳ No market distortions ⇒ Market outcome is efficient

- Then, the private market equilibrium coincides with the solution chosen by a social planner.

- Specifically, the planner would recognize that the following aggregate per capita resource constraint must be satisfied in each period:

Efficient: Allocation chosen by a benevolent planner

↳ distributes all resources in an economy

$$Y_t = C_t + I_t.$$

- Thus, from (3),

$$Y_t = \widetilde{r}_t K_t + w_t,$$

or

$$w_t = Y_t - \widetilde{r}_t K_t$$

(as implied by Euler's output exhaustion theorem).

Efficiency and the Planner's Outcome, Continued

- So, (4) becomes:

$(4): C_t + K_{t+1} = (1+\tilde{r}_t - \delta) K_t + w_t$

$$C_t + K_{t+1} = (1-\delta)K_t + Y_t,$$

or, taking (10) into account,

$$C_t + K_{t+1} = (1-\delta)K_t + A_t{}^{\alpha} K_t^{1-\alpha}. \qquad (12)$$

- A planner would recognize that the gross return at $t+1$ from investing one unit of the consumption good at t in capital would be:

$$R_{t+1} \equiv (1-\alpha)\left(\frac{A_{t+1}}{K_{t+1}}\right)^{\alpha} + 1 - \delta,$$

i.e., the marginal product of capital at $t+1$ plus undepreciated capital.

Problem of Planner:

Max $\sum_{s=t}^{\infty} \beta^s u(C_s)$

s.t. (12)

Efficiency and the Planner's Outcome, Continued

- Now, maximizing (1) subject to (12) yields:

$$C_t^{-\gamma} = \beta E_t \left\{ C_{t+1}^{-\gamma} \left[(1-\alpha) \left(\frac{A_{t+1}}{K_{t+1}} \right)^{\alpha} + 1 - \delta \right] \right\},$$

  or

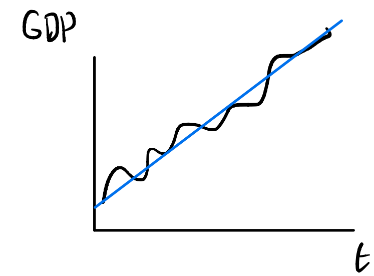
$$C_t^{-\gamma} = \beta E_t \left(C_{t+1}^{-\gamma} R_{t+1} \right), \tag{13}$$

  i.e., at an optimum, the cost of investing one unit of consumption today in capital accumulation (the marginal utility of one unit of consumption today) must be equal to the expected discounted marginal utility value of the gross return from investing one unit of consumption good in capital accumulation.

- As expected, (13) (the Euler equation from the solution of the planner's problem) and (11) (the Euler equation for the decentralized, market solution) are identical once the definition of R_{t+1} is taken into account.

Steady-State Growth

- Let us look for a steady-state, or balanced growth path of the model, in which technology, capital, and consumption all grow at a constant common growth rate.



- We denote this growth rate as:

$$G \equiv \frac{\overline{A}_{t+1}}{\overline{A}_t} \quad \frac{\bar{C}_{t+1}}{\bar{C}_t} = \frac{\bar{Y}_{t+1}}{\bar{Y}_t} = G$$

(overbars denote steady-state levels).

- In steady-state, the gross rate of return on capital, R_{t+1}, becomes a constant R, while the first-order condition (13) becomes:

$$\left(\frac{c_{t+1}}{c_t}\right)^{\gamma} = \beta R \longrightarrow \qquad G^{\gamma} = \beta R, \tag{14}$$

$$\frac{\bar{Y}_t}{\bar{K}_t} = \text{Constant}$$

or, in logs (letting $r \equiv \log R$ and $g \equiv \log G$):

$$\log(G^{\gamma}) = \log\beta + \log R$$

$$\gamma \log G = \log\beta + r \longrightarrow \qquad g = \frac{\log\beta + r}{\gamma} = \sigma\log\beta + \sigma r.$$

- This condition, relating the equilibrium growth rate of consumption to the intertemporal elasticity of substitution times the real interest rate, is a standard result of models with power utility.

$\left(\log(1+x) \simeq x, \ x \text{ small}\right)$

Steady-State Growth, Continued

The definition of R and equation (14) imply that, in steady state, the constant technology-capital ratio is:

$$\frac{\overline{A}_t}{\overline{\overline{K}}_t} = \left[\frac{G^\gamma/\beta - (1-\delta)}{1-\alpha}\right]^{1/\alpha}.$$

A higher rate of technology growth leads to a lower capital stock for a given level of technology

– The reason is that, in steady state, faster technology growth must be accompanied by faster consumption growth.

– Agents will accept a steeper consumption path only if the rate of return on capital is higher, which requires a lower capital stock.

$$\frac{G^r}{\beta} = R = 1+r$$

- Setting $G^\gamma/\beta = R \approx 1+r$ yields:

$$\frac{\overline{A}_t}{\overline{\overline{K}}_t} \approx \left(\frac{r+\delta}{1-\alpha}\right)^{1/\alpha} \leftarrow \left[\frac{\cancel{1}+r-\cancel{1}+\delta}{1-\alpha}\right]^{1/\alpha} \qquad (15)$$

Steady-State Growth, Continued

- It is possible to solve for various ratios of variables that are constant along a steady-state growth path.

- These ratios can be expressed in terms of four underlying parameters:

  - g, the log technology growth rate;

  - r, the log real return on capital;

    Strictly speaking, r is an endogenous variable of our model, but we treat is as a parameter as we recognize that it must satisfy $r = -\log\beta + \frac{g}{\sigma} = -\log\beta + \gamma g$.

  - α, the exponent on labor and technology in the production function, or equivalently, labor's share of output;

  - and δ, the rate of capital depreciation.

Steady-State Growth, Continued

- For purposes of “calibration,” interpreting periods as quarters, benchmark values of these parameters may be:

$$\begin{aligned} g &= .005 \quad (2\% \text{ at annual rate}), \\ r &= .015 \quad (6\% \text{ at annual rate}), \\ \alpha &= .667, \\ \delta &= .025 \quad (10\% \text{ at annual rate}). \end{aligned}$$

- These are all plausible numbers for the U.S. economy.

- Given $r = .015$ and $g = .005$, $r = -\log\beta + \gamma g$ defines the pairs of values for γ and β such that $r = .015$ and $g = .005$.

$$\frac{\bar{Y}_t}{\bar{K}_t} = \frac{\bar{A}_t^{\alpha}\,\bar{K}_t^{\cancel{1}-\alpha}}{\cancel{\bar{K}_t}} = \left(\frac{\bar{A}_t}{\bar{K}_t}\right)^{\alpha}$$

Steady-State Growth, Continued

- Using the production function and (15), we find the constant steady-state output capital ratio:

$$\frac{\overline{Y}_t}{\overline{K}_t} = \left(\frac{\overline{A}_t}{\overline{K}_t}\right)^{\alpha} \approx \left\{\left(\frac{r+\delta}{1-\alpha}\right)^{\frac{1}{\cancel{\alpha}}}\right\}^{\cancel{\alpha}}. \tag{16}$$

- Similarly, in steady state, the consumption-output ratio is constant at (see below for $\overline{C}_t/\overline{K}_t$):

$$\frac{C}{Y} = \frac{C/K}{Y/K}$$

$$\frac{\overline{C}_t}{\overline{Y}_t} = \frac{\overline{C}_t/\overline{K}_t}{\overline{Y}_t/\overline{K}_t} \approx 1 - \frac{(1-\alpha)(g+\delta)}{r+\delta}. \tag{17}$$

- At the benchmark parameter values above, it must be:

$$\frac{\overline{Y}_t}{\overline{K}_t} = .118 \;\; (.472 \text{ at annual rate}) \;\; \text{and} \;\; \frac{\overline{C}_t}{\overline{Y}_t} = .745,$$

fairly reasonable values.

- Getting steady state C_t/K_t: take the budget constraint (12), put bars on top of the variables, divide by K_t ; simplify and replace $G = 1 + g$

A Non-Linear Model of Fluctuations

- Outside the steady state ratios, the model we laid out is a system of non-linear equations for consumption, capital, output, and technology.
- Nonlinearities are caused by incomplete capital depreciation ($\delta < 1$ in (12) and in $R_{t+1} = (1-\alpha)\left(\frac{\overline{A}_{t+1}}{\overline{K}_{t+1}}\right)^{\alpha} + 1 - \delta$) and by time variation in the consumption-output ratio (or the savings rate).
- *Exact* analytical solution of the model is possible only in the unrealistic special case in which capital depreciates fully in one period ($\delta = 1$) and agents have log utility ($\gamma = 1$), so that the consumption-output ratio (and therefore the savings rate) is always constant.
- You find the details on how this special case works in Appendix B.
- $\delta = 1$ and $\gamma = 1$ are extremely restrictive hypotheses. In all other cases, the model features a mixture of multiplicative and additive terms that make an exact solution impossible.
- How do we proceed? ... we must approximate the model linearly somehow

↳ We are going to recast the model so that the resulting system of variables is actually linear

original variables: X_t recasted variables: $\log X_t - \log \bar{X}_t \approx \frac{X_t - \bar{X}_t}{\bar{X}_t}$

Log-Linearization

- Solution approach: seek an *approximate* analytical solution by transforming the model into a system of *approximate* log-linear difference equations.

- In doing so, we are going to rely on the following result: For small deviations of the variable X_t from its steady state:

$$\frac{dX_t}{\overline{X}_t} = \frac{X_t - \overline{X}_t}{\overline{X}_t} \approx d\log X_t = \log X_t - \log \overline{X}_t,$$

and we are going to define:

$C_t - \overline{C}_t = dC_t$

$\log C_t - \log \overline{C}_t = c_t$

$$x_t \equiv \frac{dX_t}{\overline{X}_t}.$$

- Now, interpret all lower-case variables below as (zero-mean) percentage deviations from the steady state of the model that we obtained above.

In a nutshell: we will recast the model so it's set in terms of lower case (log-deviation) variables. Doing this will yield a linear model we can actually solve.

Log-Linearization, Continued

- From the production function,

$$y_t = \alpha a_t + (1-\alpha)k_t. \tag{18}$$

This one is easy: Just take logs of the production function and remember that $N_t = 1$; (18) holds exactly, it is not an approximation.

$$Y_t = A_t^{\alpha} K_t^{1-\alpha}$$

$$\log Y_t = \alpha \log A_t + (1-\alpha)\log K_t$$

$$\log \bar{Y}_t = \alpha \log \bar{A}_t + (1-\alpha)\log \bar{K}_t$$

$$\Rightarrow \log Y_t - \log \bar{Y}_t = \alpha \log A_t + (1-\alpha)\log K_t - \left(\alpha \log \bar{A}_t + (1-\alpha)\log \bar{K}_t\right)$$

$$= \alpha(\log A_t - \log \bar{A}_t) + (1-\alpha)(\log K_t - \log \bar{K}_t)$$

$$y_t = \alpha . a_t + (1-\alpha)k_t$$

Log-Linearization, Continued

- Things are harder for equations that are not log-linear.

- For example:

$$C_t + K_{t+1} = (1-\delta)K_t + Y_t. \tag{19}$$

- John Campbell (1994, Journal of Monetary Economics) used Taylor expansions to approximate the model. That article is a standard reference on how to find the approximation.

- I find it more transparent and efficient to proceed as follows.

$$\bar{C}_t + \bar{K}_{t+1} = (1-\delta)\bar{K}_t + \bar{Y}_t$$

$$C_t - \bar{C}_t + K_{t+1} - \bar{K}_{t+1} = (1-\delta)(K_t - \bar{K}_t) + Y_t - \bar{Y}_t$$

$$dC_t + dK_{t+1} = (1-\delta)\, dK_t + dY_t$$

$$\bar{C}_t \frac{dC_t}{\bar{C}_t} + \bar{K}_{t+1} \frac{dK_{t+1}}{\bar{K}_{t+1}} = (1-\delta)\bar{K}_t \frac{dK_t}{\bar{K}_t} + \bar{Y}_t \frac{dY_t}{\bar{Y}_t}$$

$$\bar{C}_t . c_t + \bar{K}_{t+1} . k_{t+1} = (1-\delta)\bar{K}_t . k_t + \bar{Y}_t . y_t$$

divide by $\bar{K}_t$:

$$\frac{\bar{C}_t}{\bar{K}_t} . c_t + \underbrace{\frac{\bar{K}_{t+1}}{\bar{K}_t}}_{G} . k_{t+1} = (1-\delta)\cancel{\frac{\bar{K}_t}{\bar{K}_t}}^{\,1} . k_t + \frac{\bar{Y}_t}{\bar{K}_t} . y_t$$

Log-Linearization, Continued

- The differential of (19) is:

$$dC_t + dK_{t+1} = (1-\delta)dK_t + dY_t.$$

- Thus,

$$\overline{C}_t \frac{dC_t}{\overline{C}_t} + \overline{K}_{t+1} \frac{dK_{t+1}}{\overline{K}_{t+1}} = (1-\delta)\overline{K}_t \frac{dK_t}{\overline{K}_t} + \overline{Y}_t \frac{dY_t}{\overline{Y}_t},$$

or

$$\frac{\overline{C}_t}{\overline{K}_t} c_t + \frac{\overline{K}_{t+1}}{\overline{K}_t} k_{t+1} = (1-\delta) k_t + \frac{\overline{Y}_t}{\overline{K}_t} y_t. \tag{20}$$

Log-Linearization, Continued

- Now, we know that

$$\frac{\overline{K}_{t+1}}{\overline{K}_t} = G \approx 1 + g.$$

Also,

$$\frac{\overline{Y}_t}{\overline{K}_t} \approx \frac{r+\delta}{1-\alpha}$$

- Then, a steady-state version of (19) implies:

$$\underbrace{\frac{\overline{K}_{t+1}}{\overline{K}_t}}_{G} = (1-\delta) + \frac{\overline{Y}_t}{\overline{K}_t} - \frac{\overline{C}_t}{\overline{K}_t}.$$

- Using $\frac{\overline{K}_{t+1}}{\overline{K}_t} \approx 1 + g$ and $\frac{\overline{Y}_t}{\overline{K}_t} \approx \frac{r+\delta}{1-\alpha}$ and solving for $\frac{\overline{C}_t}{\overline{K}_t}$ yields:

$$\frac{\overline{C}_t}{\overline{K}_t} \approx \frac{r+\delta}{1-\alpha} - (g+\delta)$$

Note: this application is another reason why we took the work of finding the steady-state ratios before. We know those already as long as we have the parameters!

Log-Linearization, Continued

$y_t = \alpha a_t + (1-\alpha) k_t$

- Therefore, substituting these results into (20), we can rewrite it as:

$$\left(\frac{r+\delta}{1-\alpha} - (g+\delta)\right) c_t + (1+g)k_{t+1} = (1-\delta)k_t + \frac{r+\delta}{1-\alpha} y_t,$$

or:

$$(1+g)k_{t+1} = (1-\delta)k_t + \frac{r+\delta}{1-\alpha} y_t + \left(g+\delta - \frac{r+\delta}{1-\alpha}\right) c_t.$$

This is a linear equation in the variables k_{t+1}, k_t, y_t, and c_t—the percentage deviations of the variables K_{t+1}, K_t, Y_t, and C_t from their steady-state levels!

- Moreover, substituting $y_t = \alpha a_t + (1-\alpha)k_t$, we have:

$$k_{t+1} = \frac{1+r}{1+g} k_t + \frac{\alpha(r+\delta)}{(1-\alpha)(1+g)} a_t + \left[\frac{g+\delta}{1+g} - \frac{r+\delta}{(1+g)(1-\alpha)}\right] c_t.$$

$k_{t+1} = \varphi_1 \cdot k_t + \varphi_2 a_t + \varphi_3 c_t$ (linear expression)

Log-Linearization, Continued

- Let

$$\lambda_1 \equiv \frac{1+r}{1+g} \quad \text{and} \quad \lambda_2 \equiv \frac{\alpha\,(r+\delta)}{(1-\alpha)\,(1+g)}.$$

Then:

$$k_{t+1} = \lambda_1 k_t + \lambda_2 a_t + (1-\lambda_1-\lambda_2)\,c_t. \tag{21}$$

At the benchmark parameter values,

$$\lambda_1 = 1.01, \quad \lambda_2 = .08, \quad \text{and} \quad 1-\lambda_1-\lambda_2 = -.09.$$

- To understand these coefficients, note that

$$1-\lambda_1-\lambda_2 = -\frac{\overline{C}_t}{\overline{Y}_t}\frac{\overline{Y}_t}{\overline{K}_t}(1+g)^{-1} = -(.118)(.745)(1.005)^{-1}.$$

  - This is the negative of the steady-state ratio of this period's consumption to next period's capital stock
    - A $\$1$ increase in consumption today lowers tomorrow's capital stock by $\$1$, but a 1 percent increase in consumption this period lowers next period's capital stock by only $.09$ percent because in steady state one period's consumption is only $.09$ times as big as the next period's capital stock.

Log-Linearization, Continued

- Now focus on the Euler equation:

$$C_t^{-\gamma} = \beta E_t \left(C_{t+1}^{-\gamma} R_{t+1} \right).$$

- Assume that the variables on the right-hand side are *jointly log-normal and homoskedastic.*
  - The first assumption means that the log-variables are normally distributed and the second means that they have constant second moments (variances and covariances).
  - The assumptions are consistent with a log-normal productivity shock being the source of fluctuations and with the approximations we use.
  - Making assumptions like these here is necessary because now we are dealing with an expected value $E_t()$ that implies thinking about uncertainty

$$-\gamma \log C_t = \log \beta + \log E_t \left(C_{t+1}^{-\gamma} R_{t+1} \right)$$

Log-Linearization, Continued

- Taking logs of both sides of the Euler equation:

$$-\gamma \log C_t = \log \beta + \log \left[E_t \left(C_{t+1}^{-\gamma} R_{t+1} \right) \right]. \tag{22}$$

- Now let's deal with the expected value. Log-normality implies the following property:

$$\log \left[E_t \left(X_{t+1} \right) \right] = E_t \left[\log \left(X_{t+1} \right) \right] + \frac{1}{2} var_t \left[\log \left(X_{t+1} \right) \right].$$

- Therefore:

$E_t \left[-\gamma \log C_{t+1} + \log R_{t+1} \right]$

$$\begin{aligned} \log \left[E_t \left(C_{t+1}^{-\gamma} R_{t+1} \right) \right] &= E_t \left[\log \left(C_{t+1}^{-\gamma} R_{t+1} \right) \right] + \frac{1}{2} var_t \left[\log \left(C_{t+1}^{-\gamma} R_{t+1} \right) \right] \\ &= -\gamma E_t \left(\log C_{t+1} \right) + E_t \left(\log R_{t+1} \right) \\ &\quad + \frac{\gamma^2}{2} \sigma^2_{t, \log C_{t+1}} + \frac{1}{2} \sigma^2_{t, \log R_{t+1}} - \gamma \sigma_{t, \log C_{t+1}, \log R_{t+1}}, \end{aligned}$$

where for any variable X_{t+1}, $\sigma^2_{t, \log X_{t+1}}$ denotes the conditional variance at time t of $\log X_{t+1}$, and $\sigma_{t, \log C_{t+1}, \log R_{t+1}}$ denotes the conditional covariance at time t of $\log C_{t+1}$ and $\log R_{t+1}$.

$Var(\alpha X + \beta Y) = \alpha^2 Var(X) + \beta^2 Var(Y) + 2\alpha\beta Cov(X, Y)$

Log-Linearization, Continued

- Hence, (22) becomes:

$$-\gamma \log C_t = \log \beta + \overbrace{E_t \left(-\gamma \log C_{t+1} + \log R_{t+1}\right) + \frac{\gamma^2}{2}\sigma^2_{t,\log C_{t+1}} + \frac{1}{2}\sigma^2_{t,\log R_{t+1}} - \gamma \sigma_{t,\log C_{t+1},\log R_{t+1}}}^{\log\left[E_t\left(C_{t+1}^{-\gamma} R_{t+1}\right)\right]}.$$

- Now differentiate this equation (w.r.t. to its steady state analog) the to obtain:

$$-\gamma \log C_t - (-\gamma \log \bar{C}_t) = -\gamma d \log C_t \approx -\gamma E_t \left(d \log C_{t+1}\right) + E_t \left(d \log R_{t+1}\right).$$

- Why did second moments disappear?

- Remember: We are assuming that variables are homoskedastic. Hence, conditional second moments are constant, and they drop out when we differentiate!

$$-\gamma \underbrace{d \log C_t}_{c_t} \approx -\gamma E_t(\underbrace{d \log C_{t+1}}_{c_{t+1}}) + E_t(\underbrace{d \log R_{t+1}}_{r_{t+1}})$$

$$-\gamma . c_t = -\gamma E_t c_{t+1} + r_{t+1}$$

$$\gamma E_t(c_{t+1} - c_t) = r_{t+1}$$

Log-Linearization, Continued

- Given

$$x_t \equiv \frac{dX_t}{\overline{X}_t} = \frac{X_t - \overline{X}_t}{\overline{X}_t} \approx d \log X_t = \log X_t - \log \overline{X}_t,$$

  it finally follows that we can write the Euler equation in log-linear form as:

$$\gamma E_t \left(c_{t+1} - c_t \right) \approx E_t r_{t+1}$$

  where $r_{t+1} = d \log R_{t+1}$.

- Or, recalling $\sigma = \frac{1}{\gamma}$,

$$E_t \left(c_{t+1} - c_t \right) \approx \frac{1}{\gamma} E_t r_{t+1} = \sigma E_t r_{t+1}. \tag{23}$$

  – The intertemporal elasticity of substitution σ measures the responsiveness of consumption to a change in the return to asset accumulation.

Log-Linearization, Continued

- Now,

$$R_{t+1} = (1-\alpha)\left(\frac{A_{t+1}}{K_{t+1}}\right)^{\alpha} + 1 - \delta.$$

- Recall $d\log R_{t+1} \approx \frac{dR_{t+1}}{\overline{R}_{t+1}}$. Then:

$$dR_{t+1} = (1-\alpha)\,\alpha\frac{(dA_{t+1})\,\overline{K}_{t+1} - (dK_{t+1})\,\overline{A}_{t+1}}{\overline{K}_{t+1}^{2}}\left(\frac{\overline{A}_{t+1}}{\overline{K}_{t+1}}\right)^{\alpha-1},$$

or:

$$\overline{R}_{t+1}r_{t+1} \approx \alpha\,(1-\alpha)\,\frac{\overline{A}_{t+1}}{\overline{K}_{t+1}}\,(a_{t+1}-k_{t+1})\left(\frac{\overline{A}_{t+1}}{\overline{K}_{t+1}}\right)^{\alpha-1} = \alpha\,(1-\alpha)\,(a_{t+1}-k_{t+1})\left(\frac{\overline{A}_{t+1}}{\overline{K}_{t+1}}\right)^{\alpha}.$$

Log-Linearization, Continued

- Recall also:

$$\frac{\overline{A}_{t+1}}{\overline{K}_{t+1}} \approx \left(\frac{r+\delta}{1-\alpha}\right)^{1/\alpha}.$$

- Then:

$$\overline{R}_{t+1} \approx (1-\alpha)\frac{r+\delta}{1-\alpha} + 1 - \delta = 1 + r.$$

Log-Linearization, Continued

- So,

$$(1+r)\, r_{t+1} \approx \alpha\,(1-\alpha)\,(a_{t+1}-k_{t+1})\,\frac{r+\delta}{1-\alpha},$$

and

$$\underline{r_{t+1} \approx \lambda_3\,(a_{t+1}-k_{t+1})\,,} \tag{24}$$

with:

$$\lambda_3 \equiv \frac{\alpha\,(r+\delta)}{1+r}.$$

  – The same result can be obtained by taking the differential of

$$\log R_{t+1} = \log\left[1-\delta+(1-\alpha)\left(\frac{A_{t+1}}{K_{t+1}}\right)^{\alpha}\right].$$

  You should try doing it as an exercise.

Log-Linearization, Continued

- At the benchmark parameter values, $\lambda_3 = .03$.
  - This coefficient is very small. One way to understand this is to note that changes in technology have only small proportional effects on the one-period return on capital because capital depreciates only slowly, so most of the return R is undepreciated capital rather than marginal output from the Cobb-Douglas production function.
  - Alternatively, we can note that $r_{t+1} \approx R_{t+1} - 1 \approx (1-\alpha)\left(\frac{A_{t+1}}{K_{t+1}}\right)^{\alpha}$ when δ is negligible. In this case, a 1 percent increase in the technology-capital ratio raises r_{t+1} by about α percent. But α percent of r_{t+1} is only αr_{t+1} percentage points.
- Equations (23) and (24) together imply:

$$E_t\left(c_{t+1} - c_t\right) \approx \sigma\lambda_3 E_t\left(a_{t+1} - k_{t+1}\right). \tag{25}$$

Log-Linearization, Continued

- To close the model, we only need to specify a process for the technology shock a_t, the percentage deviation of A_t from its steady-state level $\overline{A}_t$: $a_t = \frac{A_t - \overline{A}_t}{\overline{A}_t}$.

- We assume an $AR(1)$ process:

$$a_t = \phi a_{t-1} + \varepsilon_t, \quad -1 \leq \phi \leq 1. \tag{26}$$

- We assume that the innovations to technology, ε_t, are normally distributed and such that $E_{t-1}(\varepsilon_t) = 0$.

- The $AR(1)$ coefficient ϕ measures the *persistence of technology shocks*, with the extreme case $\phi = 1$ being a random walk for technology.

Log-Linearization, Continued

- We did a ton of math and it looked awfully complicated, but look what we have now: Equations (21), (25), and(26) form a system of linear expectational difference equations in (the percentage deviations from the steady state of) technology, capital, and consumption!

- In other words, we boiled down the non-linear model we started from to the linear system:

$$\boxed{\begin{aligned} k_{t+1} &\approx \lambda_1 k_t + \lambda_2 a_t + (1-\lambda_1-\lambda_2)\, c_t, \\ E_t\left(c_{t+1}-c_t\right) &\approx \sigma\lambda_3 E_t\left(a_{t+1}-k_{t+1}\right), \\ a_t &= \phi a_{t-1} + \varepsilon_t. \end{aligned} \qquad (27)}$$

- The parameter of these equations include λ_1, λ_2 and λ_3 (where $\lambda_1 \equiv \frac{1+r}{1+g}$, $\lambda_2 \equiv \frac{\alpha(r+\delta)}{(1-\alpha)(1+g)}$, and $\lambda_3 \equiv \frac{\alpha(r+\delta)}{1+r}$), $\sigma = \frac{1}{\gamma}$, the $AR(1)$ coefficient ϕ that measures the persistence of technology shocks, and the variance of the technology innovations ε_t.

Rather than a very complicated system of non-linear equations we ended up with system of 3 linear equations!

What is left is to reorganize this simpler system of equations in a way that we end up with a solved system where we have a decision variable in terms of the states of the economy

We do that by assuming that form exists (controls as linear function of states) and then using the system above (27) to characterize the coefficients of such system

The Calibration Approach to RBC Analysis

Persistence of a_t

- How to get the values of the parameters (and thus coefficients) of the model?

EEq parameter

σ_ε

- In Campbell's interpretation, the "calibration" approach to real business cycle analysis takes λ_1, λ_2, and λ_3 as known, and searches for values of σ and ϕ (and a variance for the technology innovation, ε) to match the moments of observed macroeconomic the series.
  - If λ_1, λ_2, and λ_3 are not taken as given, one can search for values of all the structural parameters of the model–including those of which λ_1, λ_2, and λ_3 are functions–to match moments of observed data.
- One can then verify if the values of parameters such that the model matches moments of the data are reasonable.
- An alternative interpretation of the calibration approach runs in the opposite direction and asks the following questions:
  - Given reasonable values of the structural parameters of the model and a process for a_t that is roughly consistent with the data, how far are the moments of endogenous variables implied by the model from the moments of actual data?
  - How far are the *impulse responses* (the responses of endogenous variables to exogenous shocks) from those implied by the data?

Determinacy of the Solution

- We are very interested in the *impulse responses*: how does the system of variables react when there is an exogenous shock (e.g., technological)?
- To compute impulse responses (*i.e.*, the responses of consumption and capital to technology innovations) or the second-moment properties (*i.e.*, the variances or covariances of consumption and capital implied by assumptions on the variance of technology innovations) to compare them to properties of the data, we must solve the system (27).
- But whenever we solve a system of linear, expectational, difference equations such as (27), in principle we need to check that there is a *unique* solution, *i.e.*, that the solution (if it exists) is *determinate*.
  - If the solution is indeterminate, the economy is subject to fluctuations that are not caused by changes in the fundamentals–*sunspot fluctuations*.
  - We will not study how to prove that the system (27) has a unique solution. You find the information in Appendix C. For our purposes, trust that the system does have a unique solution.

Determinacy of the Solution, Continued

- Important: A very interesting branch of macroeconomics that time limitations do not allow us to study focuses precisely on what happens in model-economies that do not have a unique solution (for dynamics around the steady state or even for the steady state itself).
- These models are best suited to capture John Maynard Keynes' idea of *animal spirits*, fluctuations in sentiment that trigger economic fluctuations and that are not captured by so-called fundamental-driven fluctuations we are focusing on. (Fundamental in the sense that technology is among the "fundamentals" exogenous drivers of the model.)
- This is a fascinating, very important branch of macroeconomics. With some of its major contributors being Roger Farmer and Karl Shell.

The Method of Undetermined Coefficients

- Once we trust that the system (27) has a unique solution, it can be solved with a method known as the method of undetermined coefficients.
- Let η_{zx} denote the partial elasticity of variable z with respect to variable x.
- Guess that the solution for consumption takes the form:

$$c_t = \eta_{ck} k_t + \eta_{ca} a_t, \tag{28}$$

where η_{ck} and η_{ca} are unknown but assumed constant.

- We verify the guess by finding values of η_{ck} and η_{ca} that satisfy the restrictions of the model (27)

The Method of Undetermined Coefficients, Continued

- **Note 1:** The guess (28) is consistent with the logic of a technique for solving dynamic models known as dynamic programming: Optimal behavior maps the *state of the economy at time* t (described in our model by capital—k_t—and technology—a_t) into the variables that are endogenous during that period (in this case, consumption during that period—c_t).
  - Think about it: From the perspective of households and firms in the economy, what can summarize the state (initial conditions) of the economy at the time they make a decision?
  - Well, that's the capital stock they entered the period with and the current realization of the technology.
  - So, given a set of linear equations that we are trying to solve, we are going to guess that the solution is a linear function of the endogenous state (capital) and the exogenous one (technology).

The Method of Undetermined Coefficients, Continued

- **Note 2:** Having verified determinacy allows us to be confident that the solution we are guessing (referred to also as the *minimum state vector solution*, since it depends on the smallest number of state variables) is the unique solution of the system (27).
  - If the solution were indeterminate, (28) would be only one of the possible solutions of (27)
  - Alternative solutions would exist—among them, solutions that map so-called non-fundamental states (such as the color of sunspots...) into the endogenous variables.

The Method of Undetermined Coefficients, Continued

- Given our guessed solution for consumption and equation (21), we can write the guessed solution for capital as:

$$k_{t+1} = \eta_{kk} k_t + \eta_{ka} a_t, \tag{29}$$

with:

$$\eta_{kk} \equiv \lambda_1 + (1 - \lambda_1 - \lambda_2)\,\eta_{ck}, \quad \eta_{ka} \equiv \lambda_2 + (1 - \lambda_1 - \lambda_2)\,\eta_{ca}.$$

- **Note 3:** As expected (again, consistent with the logic of dynamic programming), the solution maps the state at time t into the choice of assets entering $t+1$ (k_{t+1}).

Eq (21): $k_{t+1} = \lambda_1 k_t + \lambda_2 a_t + (1 - \lambda_1 - \lambda_2) c_t \Longrightarrow k_{t+1} = \eta_{kk} \cdot k_t + \eta_{ka} a_t$

Subs $c_t = \eta_{ck} k_t + \eta_{ca} a_t$

The Method of Undetermined Coefficients, Continued

- **Note 4:** As noted above, the solution in (28) and (29) is also called the *minimum state variable solution*, in the sense that it is the solution that expresses the endogenous variables as functions of the minimum state vector—the vector consisting of the endogenous, predetermined state variable k_t and of the exogenous state variable a_t.
  - The concept of minimum state variable (MSV) solution has been proposed by Bennett McCallum as the only solution that is relevant in practice even in situations in which there is indeterminacy.
  - Many scholars see determinacy of the equilibrium (which ensures that the MSV solution is the unique equilibrium) as an important, desirable property of macroeconomic models.
  - Others, like Roger Farmer and Karl Shell, view allowing for multiple possible solutions (and therefore multiple possible equilibria) as central to understanding fluctuations.

$c_{t+1} = \eta_{ck} k_{t+1} + \eta_{ca} a_{t+1}$

$c_t = \eta_{ck} k_t + \eta_{ca} a_t$

The Method of Undetermined Coefficients, Continued

- Substituting the conjectured solution into $E_t\left(c_{t+1} - c_t\right) = \sigma\lambda_3 E_t\left(a_{t+1} - k_{t+1}\right)$ yields:

$$\eta_{ck}\left(k_{t+1} - k_t\right) + \eta_{ca} E_t\left(a_{t+1} - a_t\right) = \sigma\lambda_3 E_t a_{t+1} - \sigma\lambda_3 k_{t+1} \tag{30}$$

  (k_{t+1} is known at time t, when it is determined).

- Substitute (29) into (30) and use $E_t a_{t+1} = \phi a_t$. The result (taking the definitions of η_{kk} and η_{ka} into account) is an equation in only the two state variables, k_t and a_t:

$$\begin{aligned} & \eta_{ck}\left[\lambda_1 - 1 + \left(1 - \lambda_1 - \lambda_2\right)\eta_{ck}\right] k_t + \eta_{ck}\left[\lambda_2 + \left(1 - \lambda_1 - \lambda_2\right)\eta_{ca}\right] a_t + \eta_{ca}\left(\phi - 1\right) a_t \\ = \ & \sigma\lambda_3\phi a_t - \sigma\lambda_3\left[\lambda_1 + \left(1 - \lambda_1 - \lambda_2\right)\eta_{ck}\right] k_t - \sigma\lambda_3\left[\lambda_2 + \left(1 - \lambda_1 - \lambda_2\right)\eta_{ca}\right] a_t. \end{aligned} \tag{31}$$

Now we can equal the coefficients on the RHS of the equation (31) for a variable (e.g. of a_t) to the coefficients on the LHS. They must be equal! and thus lead us to something we can use to solve for the elasticities.

The Method of Undetermined Coefficients, Continued

- To solve this equation, we first equate coefficients on k_t to find η_{ck} and then equate coefficients on a_t to find η_{ca}, given η_{ck}.

- Equating coefficients on k_t gives the quadratic equation:

$$Q_2\eta_{ck}^2 + Q_1\eta_{ck} + Q_0 = 0, \tag{32}$$

with:

$$\begin{aligned}
Q_2 &\equiv 1 - \lambda_1 - \lambda_2, \\
Q_1 &\equiv \lambda_1 - 1 + \sigma\lambda_3 \left(1 - \lambda_1 - \lambda_2\right), \\
Q_0 &\equiv \sigma\lambda_3\lambda_1.
\end{aligned}$$

The Method of Undetermined Coefficients, Continued

- The quadratic formula gives two solutions to (32).
- With the benchmark set of parameter values, one of these is positive.
  - Equation (21), with $\lambda_1 > 1$, shows that η_{ck} must be positive for the steady state to be locally stable.
    - If $\eta_{ck} < 0$, then $\lambda_1 + (1 - \lambda_1 - \lambda_2)\,\eta_{ck} > 1$, which implies $\eta_{kk} > 1$ in (29), or an unstable steady state to which the economy never returns after shocks.
  - Hence, the positive solution is the appropriate one:

$$\eta_{ck} = \frac{1}{2Q_2}\left(-Q_1 - \sqrt{Q_1^2 - 4Q_0Q_2}\right).$$

  - Intuitively: It makes economic sense that, if the economy invested more during period $t-1$, and therefore it enters period t with more capital, consumption during period t is higher.
- Note that η_{ck} depends only on σ and the λ parameters and is invariant to the persistence of the technology shock, ϕ.

The Method of Undetermined Coefficients, Continued

- The solution of the model is then completed by finding η_{ca} as:

$$\eta_{ca} = \frac{-\eta_{ck}\lambda_2 + \sigma\lambda_3\left(\phi - \lambda_2\right)}{\phi - 1 + \left(1 - \lambda_1 - \lambda_2\right)\left(\eta_{ck} + \sigma\lambda_3\right)}.$$

- To obtain this, equate coefficients on a_t at the left and right side of equation (31), substitute the solution for η_{ck}, and solve the resulting equation for η_{ca}.

The Method of Undetermined Coefficients, Continued

- To summarize, we have:

$$\begin{aligned} a_t &= \phi a_{t-1} + \varepsilon_t, \\ k_{t+1} &= \eta_{kk} k_t + \eta_{ka} a_t, \\ c_t &= \eta_{ck} k_t + \eta_{ca} a_t, \end{aligned}$$

  as **solution of the model**, with the parameters obtained above.

- These equations make it possible to study the dynamics of consumption, capital, and technology following an innovation to the latter.

- In other words, the equations can be used to analyze the response of the economy to technology shocks, *i.e.*, to perform *impulse response analysis*.

The Method of Undetermined Coefficients, Continued

- Given numerical values for parameters, we can use the equations to compute the paths of technology, capital, and consumption over time in response to an initial innovation to technology $\varepsilon_0 = 1$ at time $t = 0$ (assuming that the economy was in steady state until—and including—period $t = -1$).
  - We will want to use the fact that $k_0 = 0$ (since it was determined at $t = -1$, before the innovation to technology).

- We will explore how to do this using a simple Excel spreadsheet that will illustrate how the consumption elasticities, η_{ck} and η_{ca}, and the capital elasticities, η_{kk} and η_{ka}, derived from them determine the dynamic behavior of our model economy.

- For those of you who have taken time series econometrics, Appendix D goes over some time series implications of the model.

A Summary of the Dynamic Properties of the Model

- Three characteristics of the fixed-labor model deserve note.
- **First**, analysis of impulse responses shows that *capital accumulation has an important effect on the dynamics of the economy only when the underlying technology shock is persistent*, lasting long enough for significant changes in capital to occur.
- *The stochastic growth model–or at least this version–is unable to generate persistent effects from transitory shocks*.

A Summary of the Dynamic Properties of the Model, Continued

- To understand this point, recall the solution equations:

$$
\begin{aligned}
c_t &= \eta_{ck} k_t + \eta_{ca} a_t, \\
y_t &= \eta_{yk} k_t + \eta_{ya} a_t, \\
k_{t+1} &= \eta_{kk} k_t + \eta_{ka} a_t.
\end{aligned}
$$

- Persistence in the dynamics of consumption and other endogenous variables follows from their dependence on the exogenous state a_t and on the endogenous, predetermined state k_t.

- The persistence of technology (ϕ) is an exogenous parameter.

- Therefore, the persistence of dynamics that comes from dependence on a_t is exogenous.

- Instead, we refer to the persistence that arises as a consequence of dependence on the endogenous state k_t as *endogenous persistence*.

- However, if ϕ is small, the technology shock does not last long enough to generate significant changes in capital, and the effect of capital dynamics on the economy is consequently small, so that the deviation of consumption and output from the steady state becomes very small once a_t has returned to the steady state.

A Summary of the Dynamic Properties of the Model, Continued

- **Second**, *technology shocks do not have strong effects on realized or expected returns on capital*.

- The reason is that the gross rate of return on capital largely consist of undepreciated capital rather than the net output that is affected by technology shocks.

- The realized return on capital equals λ_3, and $\lambda_3 = .03$ at benchmark parameter values.

- Thus, a 1 percent technology shock changes the realized return on capital on impact by only 3 basis points (12 at annual rate).

- The expected return on capital is even more stable (constant if the representative agent is risk neutral) because capital accumulation lowers the marginal product of capital one period after a positive technology shock occurs, partially offsetting any persistent effects of the shock.

A Summary of the Dynamic Properties of the Model, Continued

- **Third**, *capital accumulation does not generate a short or long-run "multiplier"* in the sense of an output response to a technology shock that is larger (in percentage terms) than the underlying shock itself.

- This means that slower-than-normal technology growth can generate only slower-than-normal output growth and not actual declines in output.

- The model with fixed labor supply can explain output declines only by appealing to implausible declines in the *level* of technology.

  - To fix this we can set the model with elastic supply, i.e., $N_t < 1$, implying that agents can choose a portion of time for working and another for leisure
    - This is done in the next 10-15 slides that are left for your own review if interested

Model with Variable Labor Supply

- We now move to a model in which labor supply is allowed to vary over time and is determined endogenously.

- The production function is unchanged:

$$Y_t = (A_t N_t)^{\alpha} K_t^{1-\alpha}. \tag{33}$$

- Also the law of motion for capital remains:

$$K_{t+1} = (1-\delta) K_t + Y_t - C_t. \tag{34}$$

- However, we now assume that the period utility function is:

$$u(C_t, 1-N_t) = \log C_t + \theta \frac{(1-N_t)^{1-\gamma_n}}{1-\gamma_n}. \tag{35}$$

Variable Labor Supply, Continued

$$u\left(C_t, 1-N_t\right) = \log C_t + \theta \frac{\left(1-N_t\right)^{1-\gamma_n}}{1-\gamma_n}.$$

- The total amount of time available to agents in each period is normalized to 1. Thus, $1-N_t$ is leisure in period t.

- Utility is additively separable in consumption and leisure. Robert King, Charles Plosser, and Sergio Rebelo showed in a 1988 *Journal of Monetary Economics* article that log utility from consumption is required to obtain constant steady-state labor supply (i.e., balanced growth) when utility is additively separable over consumption and leisure.

  – The balanced growth requirement does not restrict the form of the utility function for leisure.

- Power utility nests several cases in the literature (for example, log when $\gamma_n = 1$, linear when $\gamma_n = 0$). Let $\sigma_n \equiv \frac{1}{\gamma_n}$ denote the elasticity of intertemporal substitution for leisure.

Variable Labor Supply, Continued

- The Euler equation for consumption is still:

$$C_t^{-1} = \beta E_t \left(C_{t+1}^{-1} R_{t+1} \right) . \tag{36}$$

- But now:

$$R_{t+1} = (1-\alpha)\left(\frac{A_{t+1}N_{t+1}}{K_{t+1}}\right)^{\alpha} + 1 - \delta. \tag{37}$$

- And the key new feature of the model is that there is now a *static* first-order condition for the optimal choice of leisure relative to consumption at each point in time.

Variable Labor Supply, Continued

- We refer to this first-order condition as the *labor-leisure tradeoff*: the agent needs wage income to consume, but labor has a utility cost.
- Intuitively, it must be that, for the household to be optimizing, the marginal utility of leisure equals the real wage evaluated in terms of the marginal utility of consumption.
  - *i.e.*, the marginal utility of leisure must equal how much marginal utility of consumption the real wage earned by supplying an extra unit of labor generates:

$$\theta \left(1 - N_t\right)^{-\gamma_n} = C_t^{-1} w_t. \tag{38}$$

  - *Ceteris paribus*, if consumption increases, its marginal utility (and thus the marginal utility of wage income to buy consumption) decreases, and so does labor supply.
- Condition (38) also states that the marginal rate of substitution between leisure and consumption has to be equal to the real wage.
- You can obtain this condition by solving a modified version of the household's maximization problem with period utility (35) and $N_t \neq 1$. Maximizing with respect to N_t gives (38).

Variable Labor Supply, Continued

- With competitive markets, the real wage equals the marginal product of labor:

$$w_t = \alpha A_t^\alpha \left(\frac{K_t}{N_t}\right)^{1-\alpha}. \tag{39}$$

- Thus, in an efficient economy, the marginal rate of substitution between leisure (or labor) and consumption in household utility has to be equal to the marginal rate at which labor is transformed into output in firm production.

- Combining (38) and (39) yields the labor market clearing condition that determines equilibrium employment:

$$\theta \left(1 - N_t\right)^{-\gamma_n} = \alpha \frac{A_t^\alpha}{C_t} \left(\frac{K_t}{N_t}\right)^{1-\alpha}. \tag{40}$$

The Steady State with Variable Labor Supply

- It turns out that the analysis of the steady state from the model with fixed labor carries over directly to the variable-labor model.
- It is still the case that, in a steady state with $\frac{\overline{A}_{t+1}}{\overline{A}_t} = G$, $G^{\gamma} = \beta R$, or $G = \beta R$, as $\gamma = 1$ in this model. Thus, $g = \log \beta + r$.
- The steady-state values of the ratios $\frac{A_t}{K_t}$, $\frac{Y_t}{K_t}$, and $\frac{C_t}{Y_t}$ can be obtained following similar steps to those above.
  - See Appendix E slides for some details.

The Log-Linear Model with Variable Labor Supply

- We can linearize the model's equations around the steady sate as we did for the fixed-labor model, using $d \log X_t \approx \frac{dX_t}{\overline{X}_t} = x_t$.

- The log-linear version of the capital accumulation equation, using

$$y_t = \alpha \left(a_t + n_t\right) + \left(1 - \alpha\right) k_t,$$

  is:

$$k_{t+1} \approx \lambda_1 k_t + \lambda_2 \left(a_t + n_t\right) + \left(1 - \lambda_1 - \lambda_2\right) c_t, \tag{41}$$

  with λ_1 and λ_2 the same as in the fixed-labor model. (See the Appendix F slides.)

- The interest rate is:

$$r_{t+1} \approx \lambda_3 \left(a_{t+1} + n_{t+1} - k_{t+1}\right), \tag{42}$$

  with λ_3 the same as before.

- Linearizing the Euler equation, using the log-normality and homoskedasticity assumptions, and using (42) yields:

$$E_t \left(c_{t+1} - c_t\right) \approx \lambda_3 E_t \left(a_{t+1} + n_{t+1} - k_{t+1}\right). \tag{43}$$

The Log-Linear Model with Variable Labor Supply, Continued

- Now focus on (40). Taking logs:

$$\log\theta - \gamma_n \log(1 - N_t) = \log\alpha + \alpha\log A_t - \log C_t + (1-\alpha)\log K_t - (1-\alpha)\log N_t.$$

- Then:

$$-\gamma_n d\log(1 - N_t) = \alpha d\log A_t - d\log C_t + (1-\alpha) d\log K_t - (1-\alpha) d\log N_t.$$

- Observe that:

$$d\log(1 - N_t) = -\frac{dN_t}{1-\overline{N}} = -\frac{\overline{N}}{1-\overline{N}}\frac{dN_t}{\overline{N}} = -\frac{\overline{N}}{1-\overline{N}}n_t.$$

- Thus,

$$\gamma_n \frac{\overline{N}}{1-\overline{N}} n_t \approx \alpha a_t + (1-\alpha)(k_t - n_t) - c_t,$$

or:

$$n_t \approx \frac{1-\overline{N}}{\overline{N}}\sigma_n\left[\alpha a_t + (1-\alpha)(k_t - n_t) - c_t\right]. \qquad (44)$$

The Log-Linear Model with Variable Labor Supply, Continued

- $\overline{N}$ solves a non-linear equation shown in Appendix E.
- If we assume that households allocate on average one-third of their time to market activities, then $\overline{N} = \frac{1}{3}$ and $\frac{1-\overline{N}}{\overline{N}} = 2$.
- We take this as benchmark (and the equation in Appendix E can be used to solve for the combinations (γ_n, θ) such that $\overline{N} = \frac{1}{3}$ with the other parameters at their benchmark values).
- Equation (44) can be rewritten as:

$$n_t \left[1 + \frac{(1-\alpha)\left(1-\overline{N}\right)}{\overline{N}}\sigma_n\right] \approx \frac{1-\overline{N}}{\overline{N}}\sigma_n \left[\alpha a_t + (1-\alpha) k_t - c_t\right],$$

or:

$$n_t \left[\frac{\overline{N} + (1-\alpha)\left(1-\overline{N}\right)\sigma_n}{\overline{N}}\right] \approx \frac{1-\overline{N}}{\overline{N}}\sigma_n \left[\alpha a_t + (1-\alpha) k_t - c_t\right],$$

which implies:

$$n_t \approx \mu \left[(1-\alpha) k_t + \alpha a_t - c_t\right], \qquad (45)$$

with

$$\mu = \mu(\sigma_n) \equiv \frac{\left(1-\overline{N}\right)\sigma_n}{\overline{N} + (1-\alpha)\left(1-\overline{N}\right)\sigma_n}.$$

The Log-Linear Model with Variable Labor Supply, Continued

- μ measures the responsiveness of labor supply to shocks that change the real wage or consumption, taking into account the fact that, if labor supply increases, the real wage is driven down.

- To see this, observe that

$$\begin{aligned} w_t &= \alpha A_t^{\alpha} \left(\frac{K_t}{N_t} \right)^{1-\alpha} \Rightarrow \\ \omega_t &= \alpha a_t + (1-\alpha)(k_t - n_t) = (1-\alpha) k_t + \alpha a_t - (1-\alpha) n_t, \end{aligned}$$

  where $\omega_t \equiv \frac{dw_t}{\overline{w}}$.

- Because of this effect, even when utility from leisure is linear ($\sigma_n \to \infty$), μ is finite and equal to $\frac{1}{1-\alpha}$ (as implied by $\omega_t = (1-\alpha) k_t + \alpha a_t - (1-\alpha) n_t$).

- As the curvature of the utility function for leisure increases, μ falls and becomes zero when $\gamma_n \to \infty$ ($\sigma_n \to 0$). This corresponds to the fixed-labor-supply model we studied before.

- Note that the value of $\overline{N}$ affects only the relation between σ_n and μ and not any other aspect of the model.

The Log-Linear Model with Variable Labor Supply, Continued

- Substituting equation (45) into equations (41) and (43) and maintaining the assumption $a_t = \phi a_{t-1} + \varepsilon_t$, $E_{t-1}(\varepsilon_t) = 0$, returns a system of equations in capital, consumption, and technology similar to the one we studied when labor was fixed at 1.

- The system can be solved using the same method of undetermined coefficients.

- Assuming that there is a unique solution, we can conjecture the MSV solution:

$$c_t = \eta_{ck} k_t + \eta_{ca} a_t. \tag{46}$$

  - A good exercise would be to verify determinacy of the equilibrium as done in the case of fixed labor supply.

- η_{ck} solves a quadratic equation of the type:

$$Q_2 \eta_{ck}^2 + Q_1 \eta_{ck} + Q_0 = 0,$$

  where the coefficients Q_2, Q_1, and Q_0 are now more complicated than before. (See Appendix G.)

- As before, we pick the positive solution. The solution for η_{ca} can be obtained straightforwardly from η_{ck} and the other parameters. These solutions are the same as in the fixed-labor-supply model when labor supply is completely inelastic, so that $\mu = 0$.

Dynamics with Variable Labor Supply

- The dynamics of the economy take the same form as in the fixed-labor model. Once again:

$$k_{t+1} = \eta_{kk} k_t + \eta_{ka} a_t. \tag{47}$$

(The definitions/expressions for η_{kk} and η_{ka} are in Appendix G.)

- Substituting (46) into (45) yields:

$$n_t = \mu \left[(1-\alpha) k_t + \alpha a_t - \eta_{ck} k_t - \eta_{ca} a_t \right] = \eta_{nk} k_t + \eta_{na} a_t, \tag{48}$$

with $\eta_{nk} \equiv \mu (1 - \alpha - \eta_{ck})$ and $\eta_{na} \equiv \mu (\alpha - \eta_{ca})$.

- Increases in capital raise the real wage by a factor $1-\alpha$ (recall $\omega_t = (1-\alpha) k_t + \alpha a_t - (1-\alpha) n_t$).

- This stimulates labor supply.

- But capital also increases consumption by a factor η_{ck}, and this can have an offsetting effect.

- Similarly, increases in technology raise the real wage by a factor α, but the stimulating effect on labor supply is offset by the effect η_{ca} of technology on consumption.

Dynamics with Variable Labor Supply, Continued

- Finally, using $y_t = \alpha\left(a_t + n_t\right) + \left(1-\alpha\right)k_t$ and substituting for n_t from (48) gives:

$$y_t = \eta_{yk}k_t + \eta_{ya}a_t, \tag{49}$$

  with $\eta_{yk} \equiv 1-\alpha+\alpha\mu\left(1-\alpha-\eta_{ck}\right)$, $\eta_{ya} \equiv \alpha+\alpha\mu\left(\alpha-\eta_{ca}\right)$.

- As before, if we use the lag operator notation, it turns out that output is an $ARMA(2,1)$ process. (See Appendix D for time series implications of the model)

- However, capital and technology now affect output both directly (with coefficients α and $1-\alpha$, respectively) and indirectly through labor supply.

- The initial response to a technology shock is now $\alpha+\alpha\mu\left(\alpha-\eta_{ca}\right)$ rather than α.

- Thus, the variable-labor model can produce an amplified output response to technology shocks, even in the very short run.

Dynamics with Variable Labor Supply, Continued

- *At the benchmark parameter values*, if $\sigma_n = 0$, the model reduces to the fixed-labor case: $\eta_{nk} = \eta_{na} = 0$.
- As σ_n increases, η_{nk} becomes increasingly negative, while η_{na} becomes increasingly positive.
- Thus, an increase in capital lowers the work effort because it increases consumption more than the real wage.
- A positive technology shocks increases the work effort.
- η_{nk} is independent of the persistence parameter ϕ, but η_{na} declines with ϕ.
- The reason is that, if an innovation has persistent effects on technology, it increases consumption more than a transitory shock (η_{ca} increases with ϕ),
- The increase in consumption lowers the marginal utility of income and reduces the work effort.
- Put another way, *transitory shocks produce a stronger intertemporal substitution effect in labor supply* (if ϕ is small, η_{na} is large).

Dynamics with Variable Labor Supply, Continued

- Campbell analyzes a number of special cases and properties of the model, along with the responses of the return on capital and the real wage to shocks.

- It turns out that the responses of the return on capital to capital and technology are $\lambda_3\left(\eta_{nk}-1\right)$ and $\lambda_3\left(1+\eta_{na}\right)$, respectively.

- These responses remain very small also in the variable-labor model.

- It is possible to check that:

$$\omega_t = y_t - n_t, \tag{50}$$

  so that $\eta_{\omega k} = \eta_{yk} - \eta_{nk}$ and $\eta_{\omega a} = \eta_{ya} - \eta_{na}$.

- $\eta_{\omega a}$ is smallest when utility is linear in leisure ($\gamma_n = 0$). In this case, $\eta_{\omega a} = \eta_{ca}$, because linear utility from leisure generates a constant wage-consumption ratio.

- $\eta_{\omega a}$ rises as labor supply becomes less elastic (*i.e.*, as γ_n increases, or σ_n decreases).

Dynamics with Variable Labor Supply, Continued

- Variable labor supply has important implications for the short-run elasticity of output with respect to technology, η_{ya}.
- When labor supply is fixed, $\eta_{ya} = \alpha = .667$.
- With variable labor supply, $\eta_{ya} = \alpha + \alpha\mu\,(\alpha - \eta_{ca})$, which can exceed 1 (it falls with ϕ, however, and it cannot exceed $.99$ when $\phi = 1$).
- This is important, because an elasticity greater than 1 allows absolute declines in output to be generated by positive but slower-than-normal growth in technology, which is surely more plausible than the notion of absolute declines in technology.

RBC Wrapping Up

- This concludes our analysis of the real business cycle model.
- As we noted, key assumptions and results of the framework are not supported by evidence, but the model gives us a methodological starting point and conceptual foundation.
  - Chapter 14 of Sanjay Chugh's textbook (Modern Macroeconomics) gives you a diagrammatic analysis of the RBC model and its properties.
- We will begin departing from this basic framework by introducing the consequences of monopoly power next (that implies nominal rigidities).

Appendix A: Risk Aversion and the Equity Premium Puzzle

- Consider the utility function $u = \frac{c^{1-\gamma}}{1-\gamma}$, which we are using in the RBC model and we will use again in many other models. $\gamma = -\frac{cu''(c)}{u'(c)}$ is called *coefficient of relative risk aversion*.

- **Interpretation** (Pratt, 1964, *Econometrica*)**:** Suppose we offer two alternatives to a consumer who starts off with risk-free consumption level c: (s)he can receive $c - \pi$ with certainty or a lottery paying $c - y$ with probability $.5$ and $c + y$ with probability $.5$.

- For given values of c and y, we want to find the value of $\pi = \pi(y, c)$ that leaves the consumer indifferent between the two choices (the maximum amount the consumer is willing to pay in order to avoid the bet).

- That is, we want to find $\pi(y, c)$ such that:

$$u\left[c - \pi\left(y, c\right)\right] = .5u\left(c + y\right) + .5u\left(c - y\right)$$

- For given c and y, this non-linear equation can be solved for π.

Appendix A: Risk Aversion and the Equity Premium Puzzle, Continued

- Alternatively, for small y, use Taylor expansions and a local argument:
  - Expansion of $u(c-\pi)$:

$$u(c-\pi) = u(c) - \pi u'(c) + O\left(\pi^2\right), \tag{51}$$

    where we let $c-\pi$ be the variable x in our expansion of $f(x) = u(c-\pi)$ around $x_0 = c$ and $O(\cdot)$ means terms of order at most $(\cdot)$.

  - Expansion of $u(c+\widetilde{y})$:

$$u(c+\widetilde{y}) = u(c) + \widetilde{y}u'(c) + \frac{1}{2}\widetilde{y}^2 u''(c) + O\left(\widetilde{y}^3\right),$$

    where $\widetilde{y}$ is the random variable that takes value y with probability $.5$ and $-y$ with probability $.5$, and we let $c+\widetilde{y}$ be the variable x in our expansion of $f(x) = u(c+\widetilde{y})$ around $x_0 = c$.

  - We consider a second-order expansion here due to the randomness of $\widetilde{y}$, which requires us to include second moments in the expansion.

Appendix A: Risk Aversion and the Equity Premium Puzzle, Continued

- Taking expectations of both sides of this equation yields:

$$Eu\left(c+\widetilde{y}\right)=u\left(c\right)+\frac{1}{2}y^{2}u''\left(c\right)+o\left(y^{2}\right), \tag{52}$$

where $o\left(\cdot\right)$ means terms of smaller order than $\left(\cdot\right)$.

- Equating (51) and (52) and ignoring higher-order terms gives:

$$\pi\left(y,c\right)\approx\frac{1}{2}y^{2}\left[\frac{-U''\left(c\right)}{U'\left(c\right)}\right],$$

or, if $u\left(c\right)=\frac{c^{1-\gamma}}{1-\gamma}$,

$$\pi\left(y,c\right)\approx\frac{1}{2}y^{2}\frac{\gamma}{c},$$

which can be rearranged to

$$\frac{\pi\left(y,c\right)}{y}\approx\frac{1}{2}\gamma\frac{y}{c}.$$

- This tells us that the *premium* that the consumer is willing to pay to avoid a fair bet of size y is (approximately) equal to $\frac{1}{2}\gamma$ times the ratio between the size of the bet and the consumer's initial level of consumption. γ characterizes the consumer's attitude toward uncertainty and is key to determine the premium (s)he is willing to pay to avoid it.

Appendix A: Risk Aversion and the Equity Premium Puzzle, Continued

- Now, think of confronting someone with initial consumption of $\$50,000$ per year with a 50-50 chance of winning or losing y dollars.
- Consider $y = 10, 100, 1000, 5000.$ How much would the person be willing to pay to avoid that risk?
- Based on $\pi = \frac{1}{2}\gamma\frac{y^2}{c}$:

	y	**10**	**100**	**1000**	**5000**
	2	.002	.2	20	500
γ	**5**	.005	.5	50	1250
	10	.01	1	100	2500

- A common reaction to these premia is that for γ as high as 5, they are too big. This motivates most macroeconomists' view that γ should not be much higher than 2 or 3.

Appendix A: Risk Aversion and the Equity Premium Puzzle, Continued

- Mehra and Prescott (1985, *Journal of Monetary Economics*) consider data on average yields on relatively riskless bonds and risky equity in the U.S. for the period 1889 -1978.

- The average real yield on the S&P 500 index was 7 percent. The average yield on short-term debt was only 1 percent, *i.e.*, there was an equity premium of 6 percent.

- Let $1 + r^i_{t+1}$ denote the real rate of return on asset i between t and $t+1$, $i = b$ for bonds, $i = s$ for stocks and look at the summary statistics below:

	Mean	**Var-Cov**		
		$1 + r^s_{t+1}$	$1 + r^b_{t+1}$	$\frac{c_{t+1}}{c_t}$
$1 + r^s_{t+1}$	1.070	.0274	.00104	.00219
$1 + r^b_{t+1}$	1.010		.00308	-.000193
$\frac{c_{t+1}}{c_t}$	1.018			.00127

- The presence of an equity premium is consistent with the theory: Stocks are riskier than bonds and therefore agents require a premium in order to hold them.

- But is a 6 percent spread justifiable within basic models given actual riskiness of stocks and bonds?

- No—and addressing this puzzle resulted in its own literature in macro-finance.

Appendix B: A Log-Linear Model of Fluctuations

- To see what happens when $\gamma = 1$ and $\delta = 1$, observe that, with $\gamma = 1$, the Euler equation becomes:

$$C_t^{-1} = \beta E_t \left(C_{t+1}^{-1} R_{t+1} \right) \tag{53}$$

- Recall that the economy we are modeling is such that:

$$Y_t = C_t + I_t, \quad \text{or} \quad 1 = \frac{C_t}{Y_t} + \frac{I_t}{Y_t}.$$

- Let

$$\frac{I_t}{Y_t} \equiv \tilde{s}_t = \text{saving rate.}$$

- Then,

$$\frac{C_t}{Y_t} = 1 - \tilde{s}_t, \quad \text{or} \quad C_t = (1 - \tilde{s}_t) Y_t.$$

Appendix B: A Log-Linear Model of Fluctuations, Continued

- Thus, (53) implies:

$$-\log(1-\tilde{s}_t) - \log Y_t = \log\beta + \log\left\{E_t\left[\frac{R_{t+1}}{(1-\tilde{s}_{t+1})Y_{t+1}}\right]\right\} \tag{54}$$

- $Y_t = A_t{}^{\alpha}K_t^{1-\alpha}$ and $\delta = 1$ imply:

$$R_{t+1} = (1-\alpha)\left(\frac{A_{t+1}}{K_{t+1}}\right)^{\alpha} = (1-\alpha)\frac{Y_{t+1}}{K_{t+1}}.$$

- Also, $\delta = 1$ implies $K_{t+1} = Y_t - C_t = \tilde{s}_t Y_t$. Hence,

$$R_{t+1} = \frac{1-\alpha}{s_t}\left(\frac{Y_{t+1}}{Y_t}\right).$$

- Thus, (54) reduces to:

$$\begin{aligned} -\log(1-\tilde{s}_t) - \log Y_t &= \log\beta + \log\left\{ E_t \left[\frac{(1-\alpha)}{\tilde{s}_t(1-\tilde{s}_{t+1})Y_t} \right] \right\} \\ &= \log\beta + \log(1-\alpha) - \log\tilde{s}_t - \log Y_t + \log\left[E_t\left(\frac{1}{1-\tilde{s}_{t+1}} \right) \right], \end{aligned}$$

which implies:

$$\log\tilde{s}_t - \log(1-\tilde{s}_t) = \log\beta + \log(1-\alpha) + \log\left[E_t\left(\frac{1}{1-\tilde{s}_{t+1}} \right) \right]. \tag{55}$$

- Because technology and capital do not enter (55), there is a constant value of $\tilde{s}_t$, $\widehat{s}$, that satisfies (55).

Appendix B: A Log-Linear Model of Fluctuations, Continued

- To verify this, note that, if $\tilde{s}_t = \widehat{s}\ \forall t$, then

$$E_t\left(\frac{1}{1-\tilde{s}_{t+1}}\right) = \frac{1}{1-\widehat{s}},$$

  and (55) becomes:

$$\log \widehat{s} = \log \beta + \log(1-\alpha),$$

  or

$$\widehat{s} = \beta(1-\alpha).$$

- Now, if $s_t = \widehat{s} = \beta(1-\alpha)$, it follows that:

$$\frac{C_t}{Y_t} = 1 - \beta(1-\alpha),$$

  and:

$$K_{t+1} = \beta(1-\alpha)Y_t.$$

Appendix B: A Log-Linear Model of Fluctuations, Continued

- Given the production function $Y_t = A_t{}^{\alpha} K_t^{1-\alpha}$, $K_t = \beta(1-\alpha)Y_{t-1}$ yields:

$$Y_t = A_t{}^{\alpha} \left[\beta(1-\alpha)\right]^{1-\alpha} Y_{t-1}^{1-\alpha},$$

  or

$$\log Y_t = (1-\alpha)\log \widehat{s} + (1-\alpha)\log Y_{t-1} + \alpha \log A_t. \tag{56}$$

- Equation (56) implies that, *given assumptions on the process for* $\log A_t$*, it is the possible to obtain exact solutions for the paths of all endogenous variables*:
  - Given assumptions on $\log A_t$, we can use (56) to reconstruct the exact path of $\log Y_t$.
  - We can then use $\log K_{t+1} = \log\left[\beta(1-\alpha)\right] + \log Y_t$ and $\log C_t = \log\left[1-\beta(1-\alpha)\right] + \log Y_t$ to reconstruct the exact paths of capital and consumption.
    - Since capital is predetermined (capital at $t+1$ is chosen at t), K_{t+1} is a function of Y_t.)
    - Recall that the assumptions on utility and constraints that ensure a unique solution for the competitive equilibrium/planner's problem are satisfied here.
    - Hence, $\widehat{s}$ is the unique optimal saving rate when $\delta = 1$ and $\gamma = 1$, so that we are assured that the model can be solved exactly under these assumptions.

Appendix C: Determinacy of the Solution

- To verify determinacy, we proceed as follows.
- Focus on the endogenous variables (consumption and capital) and on a perfect foresight version of (27).
- We do not need to worry about the exogenous shock variable a_t and about the expectation operator when verifying determinacy. Use the symbol $=$ instead of $\approx$ to simplify notation.

Appendix C: Determinacy of the Solution, Continued

- We can write:

$$
\begin{aligned}
k_{t+1} &= \lambda_1 k_t + (1 - \lambda_1 - \lambda_2)\, c_t, \\
c_{t+1} &= c_t - \sigma\lambda_3 k_{t+1} = c_t - \sigma\lambda_3 \left[\lambda_1 k_t + (1 - \lambda_1 - \lambda_2)\, c_t\right] \\
&= -\sigma\lambda_1\lambda_3 k_t + \left[1 - (1 - \lambda_1 - \lambda_2)\,\sigma\lambda_3\right] c_t.
\end{aligned}
$$

- Or, in matrix form:

$$
\begin{bmatrix} k_{t+1} \\ c_{t+1} \end{bmatrix} = M \begin{bmatrix} k_t \\ c_t \end{bmatrix}, \quad M \equiv \begin{bmatrix} \lambda_1 & 1 - \lambda_1 - \lambda_2 \\ -\sigma\lambda_1\lambda_3 & 1 - (1 - \lambda_1 - \lambda_2)\,\sigma\lambda_3 \end{bmatrix}. \tag{57}
$$

Appendix C: Determinacy of the Solution, Continued

- Blanchard and Kahn (1980, *Econometrica*) showed that, for a system of linear, expectational difference equations such as (57) to have a unique solution, the number of eigenvalues of the matrix M that lie (strictly) outside the unit circle must be equal to the number of non-predetermined variables in the vector $\left[\begin{array}{cc} k_t & c_t \end{array}\right]'$.

- Capital at time t was chosen at time $t-1$. Hence, k_t is a predetermined variable. Consumption–c_t–is not predetermined. Therefore, we need an eigenvalue of M outside the unit circle and one inside for the system (57) (and (27)) to have a determinate solution.

- To calculate the eigenvalues of M, we must solve:

$$\det\left[\begin{array}{cc} \lambda_1 - q & 1-\lambda_1-\lambda_2 \\ -\sigma\lambda_3\lambda_1 & 1-(1-\lambda_1-\lambda_2)\,\sigma\lambda_3 - q \end{array}\right] = q^2 - \left[1+\lambda_1 - (1-\lambda_1-\lambda_2)\,\sigma\lambda_3\right] q + \lambda_1 = 0. \quad (58)$$

Appendix C: Determinacy of the Solution, Continued

- We can try to solve equation (58) by brute force or we can be smart. The latter method is best. :-)
- Consider:

$$J\left(q\right)=q^{2}-\left[1+\lambda_{1}-\left(1-\lambda_{1}-\lambda_{2}\right)\sigma\lambda_{3}\right]q+\lambda_{1}.$$

- $J\left(q\right)$ is a parabola. It is strictly convex, since $J''\left(q\right)=2>0$.
- Graph $J\left(q\right)$. If the parabola intersects the q-axis once inside the unit circle and once outside, we are done: The matrix M has an eigenvalue outside and an eigenvalue inside the unit circle, and our system of expectational difference equations has a unique solution.
  - The solution is stable (the model displays the desired property that variables return to the steady state after temporary shocks) if the eigenvalue inside the unit circle is strictly inside.

Appendix C: Determinacy of the Solution, Continued

- To graph $J(q)$, compute:

$$\begin{aligned} J(0) &= \lambda_1 > 0, \\ J(1) &= (1-\lambda_1-\lambda_2)\,\sigma\lambda_3, \\ J(-1) &= 2(1+\lambda_1) - (1-\lambda_1-\lambda_2)\,\sigma\lambda_3, \\ \lim_{q\to-\infty} J(q) &= \lim_{q\to+\infty} J(q) = +\infty. \end{aligned}$$

- Using the expressions for λ_1, λ_2, and λ_3, you can verify that:

$$J(1) = -\frac{\sigma\alpha(r+\delta)\left[r+\alpha\delta-g(1-\alpha)\right]}{(1+r)(1-\alpha)(1+g)} < 0 \Leftrightarrow r+\alpha\delta > g(1-\alpha).$$

- To simplify our analysis, assume that structural parameter values are such that $r+\alpha\delta > g(1-\alpha)$. Note that:

$$J(1) < 0 \Rightarrow J(-1) > 0.$$

Appendix C: Determinacy of the Solution, Continued

- Compute also:

$$J'(q) = 2q - [1 + \lambda_1 - (1 - \lambda_1 - \lambda_2)\,\sigma\lambda_3]\,.$$

- Hence:

$$\begin{aligned} J'(0) &= -[1 + \lambda_1 - (1 - \lambda_1 - \lambda_2)\,\sigma\lambda_3]\,, \\ J'(1) &= 1 - \lambda_1 + (1 - \lambda_1 - \lambda_2)\,\sigma\lambda_3, \\ J'(-1) &= -3 - \lambda_1 + (1 - \lambda_1 - \lambda_2)\,\sigma\lambda_3. \end{aligned}$$

- Note that $J(1) = (1 - \lambda_1 - \lambda_2)\,\sigma\lambda_3 < 0$ implies

$$J'(0) < 0 \quad \text{and} \quad J'(-1) < 0.$$

Appendix C: Determinacy of the Solution, Continued

- Therefore, the graph of $J(q)$ is strictly positive and decreasing at $q = -1$ and $q = 0$.
- It crosses the q-axis *once* between 0 and 1 (this is a consequence of $J(0) > 0$, $J'(0) < 0$, and $J(1) < 0$).
- At $q = 1$, $J(q)$ may be increasing or decreasing, but, regardless of the sign of $J'(1)$, the second intersection of $J(q)$ with the q-axis must happen to the right of 1.
  - The fact that $J(q)$ is a parabola, *i.e.*, it switches from decreasing to increasing only once, and $J(1) < 0$ rule out a second intersection to the left of 1.
- It follows that the roots of $J(q) = q^2 - [1 + \lambda_1 - (1 - \lambda_1 - \lambda_2)\sigma\lambda_3]q + \lambda_1 = 0$ lie one inside and one outside the unit circle.
- Therefore, the eigenvalues of M are one inside and one outside the unit circle, and the system of expectational difference equations (27) has a determinate solution.
  - If $J(q)$ never intersects the q-axis, the eigenvalues of M are complex. In this case, verifying determinacy would involve checking the norm of the eigenvalues. We focus on the real case for simplicity.

Appendix D: Some Time Series Implications of the Basic Model

- Now define the lag operator L by:

$$Lx_t = x_{t-1}.$$

- Using the lag operator, the log-linearized solution for capital, $k_{t+1} = \eta_{kk} k_t + \eta_{ka} a_t$, can be written as:

$$k_{t+1} = \frac{\eta_{ka}}{1 - \eta_{kk} L} a_t. \tag{59}$$

- Using the same notation, the $AR(1)$ technology process that we have assumed can be written as:

$$a_t = \frac{1}{1 - \phi L} \varepsilon_t. \tag{60}$$

Appendix D: Some Time Series Implications of the Basic Model, Continued

- Equations (59) and (60) imply that capital follows an $AR(2)$ process:

$$k_{t+1} = \frac{\eta_{ka}}{(1-\eta_{kk}L)(1-\phi L)}\varepsilon_t, \tag{61}$$

or:

$$\begin{aligned}
(1-\eta_{kk}L)(1-\phi L)\,k_{t+1} &= \eta_{ka}\varepsilon_t \Rightarrow \\
\left[1-(\phi+\eta_{kk})\,L+\phi\eta_{kk}L^2\right]k_{t+1} &= \eta_{ka}\varepsilon_t \Rightarrow \\
k_{t+1}-(\phi+\eta_{kk})\,k_t+\phi\eta_{kk}k_{t-1} &= \eta_{ka}\varepsilon_t \Rightarrow
\end{aligned}$$

$$k_{t+1} = (\phi+\eta_{kk})\,k_t - \phi\eta_{kk}k_{t-1} + \eta_{ka}\varepsilon_t.$$

Appendix D: Some Time Series Implications of the Basic Model, Continued

$$k_{t+1} = (\phi + \eta_{kk})\, k_t - \phi\eta_{kk}k_{t-1} + \eta_{ka}\varepsilon_t.$$

- Two points on this:
  - (*a*) The roots of the capital stock process are η_{kk} and ϕ, which are both *real* numbers.
    - Thus, the model does not produce oscillating responses to shocks (which would happen with complex roots).
  - (*b*) The shock to capital at $t+1$ is the technology innovation realized at time t.
    - The capital stock is known one period in advance as it is an endogenous *state* variable, determined by lagged investment and a non-stochastic depreciation rate.

Appendix D: Some Time Series Implications of the Basic Model, Continued

- Recall $y_t = (1-\alpha)k_t + \alpha a_t$.
- With fixed labor supply, $\eta_{yk} = 1-\alpha$ and $\eta_{ya} = \alpha$. Substitute (60) and (61) into

$$y_t = (1-\alpha)Lk_{t+1} + \alpha a_t.$$

- It follows that:

$$\begin{aligned} y_t &= \frac{(1-\alpha)\eta_{ka}L}{(1-\eta_{kk}L)(1-\phi L)}\varepsilon_t + \frac{\alpha}{1-\phi L}\varepsilon_t \\ &= \frac{\alpha + [(1-\alpha)\eta_{ka} - \alpha\eta_{kk}]L}{(1-\eta_{kk}L)(1-\phi L)}\varepsilon_t. \end{aligned} \tag{62}$$

- Technology innovations affect output both directly ($\frac{\alpha}{1-\phi L}\varepsilon_t$) and indirectly, through their impact on capital accumulation ($\frac{(1-\alpha)\eta_{ka}L}{(1-\eta_{kk}L)(1-\phi L)}\varepsilon_t$).

Appendix D: Some Time Series Implications of the Basic Model, Continued

- The sum of the two effects is what we call an $ARMA(2,1)$ process for output:

$$(1-\eta_{kk}L)\,(1-\phi L)\,y_t = \alpha\varepsilon_t + \left[(1-\alpha)\eta_{ka} - \alpha\eta_{kk}\right] L\varepsilon_t,$$

  or

$$y_t = (\phi+\eta_{kk})\,y_{t-1} - \phi\eta_{kk}y_{t-2} + \alpha\varepsilon_t + \left[(1-\alpha)\eta_{ka} - \alpha\eta_{kk}\right]\varepsilon_{t-1},$$

  where $(\phi+\eta_{kk})\,y_{t-1} - \phi\eta_{kk}y_{t-2}$ is the $AR(2)$ component of the process and $\alpha\varepsilon_t + \left[(1-\alpha)\eta_{ka} - \alpha\eta_{kk}\right]\varepsilon_{t-1}$ is the $MA(1)$ part.

- The process for consumption comes from substituting (59) and (60) into $c_t = \eta_{ck}k_t + \eta_{ca}a_t$:

$$\begin{aligned} c_t &= \frac{\eta_{ck}\eta_{ka}L}{(1-\eta_{kk}L)\,(1-\phi L)}\varepsilon_t + \frac{\eta_{ca}}{1-\phi L}\varepsilon_t \\ &= \frac{\eta_{ca} + (\eta_{ck}\eta_{ka} - \eta_{ca}\eta_{kk})\,L}{(1-\eta_{kk})\,(1-\phi L)}\varepsilon_t. \end{aligned} \tag{63}$$

- This too is an $ARMA(2,1)$ process:

$$c_t = (\phi+\eta_{kk})\,c_{t-1} - \phi\eta_{kk}c_{t-2} + \eta_{ca}\varepsilon_t + (\eta_{ck}\eta_{ka} - \eta_{ca}\eta_{kk})\,\varepsilon_{t-1}.$$

Appendix D: Some Time Series Implications of the Basic Model, Continued

- Note that capital, output, and consumption processes all have the same autoregressive roots η_{kk} and ϕ.
- Thus, we can reconstruct the entire path of the dynamic responses of k, y, and c to a technology innovation at an initial point in time (impulse responses).
- Generally, we will let the computer do this job for us and plot the responses.
  - A set of Matlab codes written by Harald Uhlig of the University of Chicago in 1999 essentially implements the method of undetermined coefficients.
- However, there are cases—like the basic RBC model—in which models are sufficiently simple that we can solve for the elasticities η with pencil and paper, and, as noted above, we can calculate impulse responses using Excel.

Appendix D: Some Time Series Implications of the Basic Model, Continued

- Of curse, the nature of the response (size of initial movement, shape, speed of return to the steady state–if this happens) depends on parameter values.
  - If $\phi = 1$, technology innovations have permanent effects, and the economy does not return to the original steady state.
  - Output converges to a new, permanently higher (or lower) steady-state path after a one-time positive (or negative) technology shock with $\phi = 1$.
- Campbell's paper analyzes the consequences of different parameter values for the elasticities η and the characteristics of impulse responses.
  - Note that Campbell allows for $\beta > 1$, which is not the usual assumption.
  - Read this part of Campbell's paper for more information.

Appendix D: Using the Time Series Process Equations to Obtain Impulse Responses

- Suppose the economy was in steady state until time 0, and suppose $\varepsilon_0 > 0$, $\varepsilon_t = 0 \ \forall t > 0$.
- We can use the equations we obtained above for the processes followed by capital, output, and consumption to compute the responses to the innovation ε_0.
- At time $t = 0$:

$$
\begin{aligned}
k_0 &= 0 \text{ (because capital is predetermined)}, \\
y_0 &= \alpha \varepsilon_0, \\
c_0 &= \eta_{ca} \varepsilon_0.
\end{aligned}
$$

- At time $t = 1$:

$$
\begin{aligned}
k_1 &= \eta_{ka} \varepsilon_0, \\
y_1 &= (\phi + \eta_{kk})\, y_0 + [(1-\alpha)\eta_{ka} - \alpha \eta_{kk}]\, \varepsilon_0 \\
&= (\phi + \eta_{kk})\, \alpha \varepsilon_0 + [(1-\alpha)\eta_{ka} - \alpha \eta_{kk}]\, \varepsilon_0 \\
&= [\alpha \phi + (1-\alpha)\eta_{ka}]\, \varepsilon_0, \\
c_1 &= (\phi + \eta_{kk})\, c_0 + (\eta_{ck}\eta_{ka} - \eta_{ca}\eta_{kk})\, \varepsilon_0 \\
&= (\phi + \eta_{kk})\, \eta_{ca} \varepsilon_0 + (\eta_{ck}\eta_{ka} - \eta_{ca}\eta_{kk})\, \varepsilon_0 \\
&= (\eta_{ck}\eta_{ka} + \eta_{ca}\phi)\, \varepsilon_0,
\end{aligned}
$$

Appendix D: Using the Time Series Process Equations to Obtain Impulse Responses, Continued

- At time $t = 2$:

$$
\begin{aligned}
k_2 &= (\phi + \eta_{kk})\, k_1 - \phi \eta_{kk} k_0 \\
&= (\phi + \eta_{kk})\, \eta_{ka} \varepsilon_0, \\
y_2 &= (\phi + \eta_{kk})\, y_1 - \phi \eta_{kk} y_0, \text{ where the solutions for } y_1 \text{ and } y_0 \text{ are above,} \\
c_2 &= (\phi + \eta_{kk})\, c_1 + \phi \eta_{kk} c_0, \text{ where the solutions for } c_1 \text{ and } c_0 \text{ are above.}
\end{aligned}
$$

- And so on...

- As I suggested above, you can calculate exactly the same impulse responses directly using the equations:

$$
\begin{aligned}
k_{t+1} &= \eta_{kk} k_t + \eta_{ka} a_t, \\
y_t &= (1 - \alpha)\, k_t + \alpha a_t, \\
c_t &= \eta_{ck} k_t + \eta_{ca} a_t, \\
a_t &= \phi a_{t-1} + \varepsilon_t.
\end{aligned}
$$

- Try it as an exercise: Set $\varepsilon_0 > 0$, $\varepsilon_t = 0 \; \forall t > 0$ in the equations above, figure out the paths of k, y, and c, and verify that they coincide with those in the Excel example that used the process equations for these variables.

Appendix E: The Steady State with Variable Labor Supply

- $$\overline{R}_{t+1} = R = (1-\alpha)\left(\frac{\overline{A}_{t+1}\overline{N}_{t+1}}{\overline{K}_{t+1}}\right)^{\alpha} + 1 - \delta = (1-\alpha)\left(\frac{\overline{A}_{t+1}\overline{N}}{\overline{K}_{t+1}}\right)^{\alpha} + 1 - \delta \Rightarrow$$

$$\begin{aligned}
\left[\frac{\frac{G}{\beta} - (1-\delta)}{1-\alpha}\right]^{\frac{1}{\alpha}} &= \frac{\overline{A}_{t+1}\overline{N}}{\overline{K}_{t+1}} \Rightarrow \\
\frac{\overline{A}_{t+1}\overline{N}}{\overline{K}_{t+1}} &\approx \left(\frac{r+\delta}{1-\alpha}\right)^{\frac{1}{\alpha}} \Rightarrow \\
\frac{\overline{A}_{t+1}}{\overline{K}_{t+1}} &\approx \frac{1}{\overline{N}}\left(\frac{r+\delta}{1-\alpha}\right)^{\frac{1}{\alpha}}.
\end{aligned}$$

- Now, (40) implies:

$$\theta\left(1-\overline{N}\right)^{-\gamma_n} = \alpha\left(\frac{\overline{A}_t}{\overline{K}_t}\right)^{\alpha}\left(\frac{\overline{K}_t}{\overline{C}_t}\right)\overline{N}^{-(1-\alpha)}. \tag{64}$$

Appendix E: The Steady State with Variable Labor Supply, Continued

- So, recalling $\frac{\overline{A}_t}{\overline{K}_t} \approx \frac{1}{\overline{N}} \left(\frac{r+\delta}{1-\alpha}\right)^{\frac{1}{\alpha}}$,

$$\theta \left(1 - \overline{N}\right)^{-\gamma_n} \approx \alpha \overline{N}^{-\alpha} \frac{r+\delta}{1-\alpha} \left(\frac{\overline{K}_t}{\overline{C}_t}\right) \overline{N}^{-(1-\alpha)}. \tag{65}$$

- Also, (34) implies:

$$\begin{aligned} \frac{\overline{K}_{t+1}}{\overline{K}_t} &= 1 - \delta + \frac{\overline{Y}_t}{\overline{K}_t} - \frac{\overline{C}_t}{\overline{K}_t}, \\ 1 + g &= 1 - \delta + \frac{\left(\overline{A}_t \overline{N}\right)^{\alpha} \overline{K}_t^{1-\alpha}}{\overline{K}_t} - \frac{\overline{C}_t}{\overline{K}_t}, \end{aligned} \tag{66}$$

or:

$$\frac{\overline{C}_t}{\overline{K}_t} = \left(\frac{\overline{A}_t}{\overline{K}_t}\right)^{\alpha} \overline{N}^{\alpha} - (g + \delta).$$

Appendix E: The Steady State with Variable Labor Supply, Continued

- But, using $\frac{\overline{A}_t}{\overline{K}_t} \approx \frac{1}{\overline{N}}\left(\frac{r+\delta}{1-\alpha}\right)^{\frac{1}{\alpha}}$,

$$\begin{aligned}\frac{\overline{C}_t}{\overline{K}_t} &\approx \left(\frac{1}{\overline{N}}\right)^{\alpha}\frac{r+\delta}{1-\alpha}\overline{N}^{\alpha} - (g+\delta) = \frac{r+\delta-(1-\alpha)(g+\delta)}{1-\alpha} \\ &= \frac{r+\delta-g-\delta+\alpha g+\alpha\delta}{1-\alpha} = \frac{r+\alpha\delta-g(1-\alpha)}{1-\alpha},\end{aligned} \tag{67}$$

  which shows that $\frac{\overline{C}_t}{\overline{K}_t}$ is the same as in the fixed-labor-supply model, and using (66) shows that $\frac{\overline{Y}_t}{\overline{K}_t}$ is also the same as before.

- Equations (65) and (67) imply:

$$\theta\left(1-\overline{N}\right)^{-\gamma_n} \approx \alpha\overline{N}^{-1}\frac{r+\delta}{r+\alpha\delta-g(1-\alpha)}. \tag{68}$$

- Thus, $\overline{N}$ solves the non-linear equation (68).

Appendix F: Log-Linearizing the Variable-Labor Supply Model

- From the production function:

$$y_t = \alpha \left(a_t + n_t\right) + \left(1-\alpha\right) k_t.$$

- The law of motion of capital implies:

$$dK_{t+1} = \left(1-\delta\right) dK_t + dY_t - dC_t,$$

or:

$$\frac{\overline{K}_{t+1}}{\overline{K}_t}\frac{dK_{t+1}}{\overline{K}_{t+1}} = \left(1-\delta\right)\frac{dK_t}{\overline{K}_t} + \frac{\overline{Y}_t}{\overline{K}_t}\frac{dY_t}{\overline{Y}_t} - \frac{\overline{C}_t}{\overline{K}_t}\frac{dC_t}{\overline{C}_t},$$

from which:

$$\left(1+g\right) k_{t+1} = \left(1-\delta\right) k_t + \frac{\overline{Y}_t}{\overline{K}_t} y_t - \frac{\overline{C}_t}{\overline{K}_t} c_t,$$

where we showed before that $\frac{\overline{Y}_t}{\overline{K}_t}$ and $\frac{\overline{C}_t}{\overline{K}_t}$ are the same as in the fixed-labor model.

Appendix F: Log-Linearizing the Variable-Labor Supply Model, Continued

- Using the log-linear production function yields:

$$k_{t+1} \approx \lambda_1 k_t + \lambda_2 (a_t + n_t) + (1 - \lambda_1 - \lambda_2) c_t,$$

with λ_1 and λ_2 the same as before.

- Finally,

$$\begin{aligned} dR_{t+1} &= (1-\alpha)\alpha \frac{\left(dA_{t+1}\overline{N} + dN_{t+1}\overline{A}_{t+1}\right)\overline{K}_{t+1} - d\overline{K}_{t+1}\left(\overline{A}_{t+1}\overline{N}\right)}{\overline{K}_{t+1}^2} \left(\frac{\overline{A}_{t+1}\overline{N}}{\overline{K}_{t+1}}\right)^{\alpha-1} \\ &= (1-\alpha)\alpha \frac{\overline{A}_{t+1}\overline{N}}{\overline{K}_{t+1}} (a_{t+1} + n_{t+1} - k_{t+1}) \left(\frac{\overline{A}_{t+1}\overline{N}}{\overline{K}_{t+1}}\right)^{\alpha-1} \\ &= (1-\alpha)\alpha \left(\frac{\overline{A}_{t+1}\overline{N}}{\overline{K}_{t+1}}\right)^{\alpha} (a_{t+1} + n_{t+1} - k_{t+1}). \end{aligned}$$

- Then, using our results above,

$$r_{t+1} \approx \frac{(1-\alpha)\alpha\frac{r+\delta}{1-\alpha}(a_{t+1} + n_{t+1} - k_{t+1})}{(1-\alpha)\frac{r+\delta}{1-\alpha} + 1 - \delta} = \lambda_3 (a_{t+1} + n_{t+1} - k_{t+1}),$$

where $\lambda_3 \equiv \frac{\alpha(r+\delta)}{1+r}$, as in the fixed-labor model.

Appendix G: Solving the Variable Labor-Supply Model

- Recall equations (41), (43), and (45):

$$\begin{aligned} k_{t+1} &\approx \lambda_1 k_t + \lambda_2 \left(a_t + n_t\right) + \left(1 - \lambda_1 - \lambda_2\right) c_t, \\ E_t \left(c_{t+1} - c_t\right) &\approx \lambda_3 E_t \left(a_{t+1} + n_{t+1} - k_{t+1}\right), \\ n_t &\approx \mu \left[\left(1-\alpha\right) k_t + \alpha a_t - c_t\right] \end{aligned}$$

- Substitute (45) into (41):

$$k_{t+1} \approx \lambda_1 k_t + \lambda_2 a_t + \lambda_2 \mu \left(1-\alpha\right) k_t + \lambda_2 \mu \alpha a_t - \lambda_2 \mu c_t + \left(1 - \lambda_1 - \lambda_2\right) c_t,$$

or:

$$k_{t+1} \approx \left[\lambda_1 + \lambda_2 \mu \left(1-\alpha\right)\right] k_t + \lambda_2 \left(1 + \mu\alpha\right) a_t + \left[1 - \lambda_1 - \lambda_2 \left(1+\mu\right)\right] c_t. \tag{69}$$

- Substitute (45) into (43):

$$\begin{aligned} E_t \left(c_{t+1} - c_t\right) &\approx \lambda_3 E_t \left[a_{t+1} + \mu \left(1-\alpha\right) k_{t+1} + \mu\alpha a_{t+1} - \mu c_{t+1} - k_{t+1}\right] \\ &= \lambda_3 E_t \left\{\left[\mu \left(1-\alpha\right) - 1\right] k_{t+1} + \left(1 + \mu\alpha\right) a_{t+1} - \mu c_{t+1}\right\}. \end{aligned} \tag{70}$$

Appendix G: Solving the Variable Labor-Supply Model, Continued

- Guess $c_t = \eta_{ck} k_t + \eta_{ca} a_t$ and substitute into (69):

$$\begin{aligned} k_{t+1} &\approx [\lambda_1 + \lambda_2 \mu (1-\alpha)] k_t + \lambda_2 (1 + \mu\alpha) a_t \\ &\quad + [1 - \lambda_1 - \lambda_2 (1+\mu)] \eta_{ck} k_t + [1 - \lambda_1 - \lambda_2 (1+\mu)] \eta_{ca} a_t \\ &= \eta_{kk} k_t + \eta_{ka} a_t, \end{aligned}$$

with

$$\begin{aligned} \eta_{kk} &\equiv \lambda_1 + \lambda_2 \mu (1-\alpha) + [1 - \lambda_1 - \lambda_2 (1+\mu)] \eta_{ck}, \\ \eta_{ka} &\equiv \lambda_2 (1 + \mu\alpha) + [1 - \lambda_1 - \lambda_2 (1+\mu)] \eta_{ca}. \end{aligned}$$

- Substitute $k_{t+1} = \eta_{kk} k_t + \eta_{ka} a_t$ and $c_t = \eta_{ck} k_t + \eta_{ca} a_t$ into (70).

- Use $a_t = \phi a_{t-1} + \varepsilon_t$, so that $E_t (a_{t+1}) = \phi a_t$, and the fact that k_{t+1} is known at time t.

Appendix G: Solving the Variable Labor-Supply Model, Continued

- Then,

$$\begin{aligned}&\eta_{ck}\left(k_{t+1}-k_t\right)+\eta_{ca}\left(\phi a_t-a_t\right)\\ \approx\ &\lambda_3\left\{\left[\mu\left(1-\alpha\right)-1\right]k_{t+1}+\left(1+\mu\alpha\right)\phi a_t-\mu\eta_{ck}k_{t+1}-\mu\eta_{ca}\phi a_t\right\},\end{aligned}$$

or:

$$\begin{aligned}&\eta_{ck}\left(\eta_{kk}-1\right)k_t+\eta_{ck}\eta_{ka}a_t+\eta_{ca}\left(\phi-1\right)a_t\\ \approx\ &\lambda_3\left\{\left[\mu\left(1-\alpha\right)-1-\mu\eta_{ck}\right]\left(\eta_{kk}k_t+\eta_{ka}a_t\right)+\left(1+\mu\alpha-\mu\eta_{ca}\right)\phi a_t\right\},\end{aligned}$$

and η_{ck} solves:

$$\eta_{ck}\left(\eta_{kk}-1\right)=\lambda_3\left[\mu\left(1-\alpha\right)-1-\mu\eta_{ck}\right]\eta_{kk}.$$

Appendix G: Solving the Variable Labor-Supply Model, Continued

- Recalling $\eta_{kk} \equiv \lambda_1 + \lambda_2\mu(1-\alpha) + [1-\lambda_1-\lambda_2(1+\mu)]\,\eta_{ck}$, this equation becomes:

$$\begin{aligned} & \eta_{ck}\left\{\lambda_1 + \lambda_2\mu(1-\alpha) + [1-\lambda_1-\lambda_2(1+\mu)]\,\eta_{ck}\right\} - \eta_{ck} \\ = \; & \lambda_3\left[\mu(1-\alpha) - 1 - \mu\eta_{ck}\right]\left\{\lambda_1 + \lambda_2\mu(1-\alpha) + [1-\lambda_1-\lambda_2(1+\mu)]\,\eta_{ck}\right\}, \end{aligned}$$

which has the form:

$$Q_2\eta_{ck}^2 + Q_1\eta_{ck} + Q_0 = 0,$$

with:

$$\begin{aligned} Q_2 &\equiv (1+\lambda_3\mu)\left[1-\lambda_1-\lambda_2(1+\mu)\right], \\ Q_1 &\equiv (1+\lambda_3\mu)\left[\lambda_1+\lambda_2(1-\alpha)\mu\right] - \lambda_3\left[\mu(1-\alpha)-1\right]\left[1-\lambda_1-\lambda_2(1+\mu)\right] - 1, \\ Q_0 &\equiv -\lambda_3\left[\mu(1-\alpha)-1\right]\left[\lambda_1+\lambda_2(1-\alpha)\mu\right]. \end{aligned}$$

Appendix G: Solving the Variable Labor-Supply Model, Continued

- Finally, η_{ca} solves:

$$\begin{aligned} & \eta_{ck}\eta_{ka} + \eta_{ca}\left(\phi - 1\right) \\ = \; & \lambda_3\left[\mu\left(1-\alpha\right) - 1 - \mu\eta_{ck}\right]\eta_{ka} + \lambda_3\left(1 + \mu\alpha - \mu\eta_{ca}\right)\phi, \end{aligned}$$

  or:

$$\begin{aligned} & \eta_{ca}\left(\phi - 1\right) + \lambda_3\mu\phi\eta_{ca} \\ = \; & \lambda_3\left[\mu\left(1-\alpha\right) - 1 - \mu\eta_{ck}\right]\eta_{ka} + \lambda_3\left(1 + \mu\alpha\right)\phi - \eta_{ck}\eta_{ka}. \end{aligned}$$

- But $\eta_{ka} \equiv \lambda_2\left(1+\mu\alpha\right) + \left[1 - \lambda_1 - \lambda_2\left(1+\mu\right)\right]\eta_{ca}$. Hence, substituting and rearranging,

$$\eta_{ca} = \frac{\left(1+\alpha\mu\right)\left\{\lambda_3\phi - \lambda_2\left[\eta_{ck}\left(1+\lambda_3\mu\right) - \lambda_3\left[\mu\left(1-\alpha\right)-1\right]\right]\right\}}{\left[1-\lambda_1-\lambda_2\left(1+\mu\right)\right]\left\{\eta_{ck}\left(1+\lambda_3\mu\right) - \lambda_3\left[\mu\left(1-\alpha\right)-1\right]\right\} - \left[1-\phi\left(1+\lambda_3\mu\right)\right]}.$$