Intermediate Macroeconomics

The Solow Growth Model

ECON 3311 – Spring 2025
UT Dallas

GDP $\longrightarrow$ Productive capacity, Income, Expenditure, Prices too

Growth (data)

Production function (Model)

Now: growth model

before: Output per worker → Machines per worker

But techology is key too.

Overview

In this chapter, we learn

- how **capital accumulates** over time, helping us understand **economic growth**.
- the role of the **diminishing marginal product of capital** in explaining differences in growth rates across countries.
- the principle of **transition dynamics**: the farther below its steady state a country is, the faster the country will grow.
- the limitations of capital accumulation, and how it leaves a significant part of economic growth unexplained.

↳ related to technology

Introduction

1960 $GDP_{KR} \approx GDP_{PH} \longrightarrow GDP_{KR} >> GDP_{PH}$ why?

- In 1960 South Korea and the Philippines had similar economic conditions.
- Per capita GDP was about $1,500, around 10% of the U.S. level.
- Populations of about 25 million each, with similar fractions in industry and agriculture.
- Higher education: 5% of young Koreans vs. 13% of young Filipinos attended college.

Why did South Korea's growth outpace the Philippines despite similar starting points?

- **Solow Growth Model:**
- Developed by Robert Solow in the mid-1950s.
- Basis for his Nobel Prize in 1987.
- Widely used in macroeconomics for understanding economic growth.

Setting Up the Model

Production: Same "Cobb-Douglas" setup → Output is a function of "Inputs" → e.g. Labor, K

As in the production model, the farm produces a final output good Y, using the capital stock K and labor L:

$$Y_t = F(K_t, L_t) = \bar{A}K_t^{1/3}L_t^{2/3}. \quad \Big\} \text{ CRS}$$

This case: Cobb-Douglas function exhibits constant returns to scale in K and L.

Output → Consumption, Investment, ~~Gov, NX~~

Resource constraint: $C_t + I_t = Y_t$

Explains how the farm allocates output for consumption and investment.

Capital Accumulation: $K_{t+1} = K_t + I_t - \bar{d}K_t$ → Law of motion of K

Investment I_t and depreciation $\bar{d}K_t$ add to the existing capital stock **to yield the new capital**

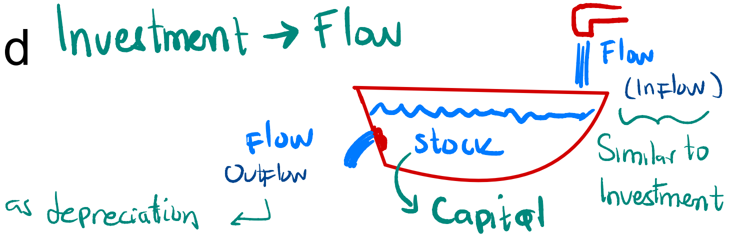
Capital → Stock

Investment → Flow

Labor L is assumed exogenous and constant.

Investment $I_t = \bar{s}Y_t$, where $\bar{s}$ is the fraction of production invested

→ s: Savings rate of economy $0 < s < 1$



Setting Up the Model

Allocation of Consumption and Investment: $I_t = \bar{s}Y_t$

Notice this implies that: $C_t = (1 - \bar{s})Y_t$

$Y_t = C_t + I_t$

$Y_t = C_t + \bar{s}Y_t \Rightarrow C_t = (1-\bar{s})Y_t$

How to adjust the consumption and investment proportions based on the value of $\bar{s}$?

The Solow Model: 5 Equations and 5 Unknowns

Unknowns/endogenous variables:	Y_t, K_t, L_t, C_t, I_t
Production function	$Y_t = \bar{A}K_t^{1/3}L_t^{2/3}$
Capital accumulation	$\Delta K_{t+1} = I_t - \bar{d}K_t$
Labor force	$L_t = \bar{L}$
Resource constraint	$C_t + I_t = Y_t$
Allocation of resources	$I_t = \bar{s}Y_t$

Parameters: $\bar{A}, \bar{s}, \bar{d}, \bar{L}, \bar{K}_0$

We produce w/ today's K (K_t) & Invest to build K for tomorrow's production (K_{t+1})

Same as

$K_{t+1} = K_t + I_t - \bar{d}K_t$

$\Delta K_{t+1} = K_{t+1} - K_t$

Prices and the Real Interest Rate

*Reminder: If MPK > Cost of Increasing K

⇒ Firms add K until MPK = Cost of K

Real Interest Rate: The amount a person can earn by saving one unit of output for a year, or the cost to borrow one unit of output for a year.

Units: Measured in units of output (or constant dollars) rather than nominal dollars.

National Income Identity: $Y_t = C_t + I_t$

Savings-Investment Relationship: $Y_t - C_t = I_t$ (saving = investment)

Real Interest Rate and Capital Market

- The real interest rate equals the rental price of capital that clears the capital market:
  - A unit of savings is used as a unit of investment which becomes a unit of capital
  - Thus, the return on savings is basically the price at which the capital can be rented
- Marginal Product: The real interest rate reflects the marginal product of capital.

1 Returns on Savings = 2 Rental Price of Capital = 3 MPK

Solving the Solow Model

Objective: To write the endogenous variables (Yt, Kt, Lt, Ct, It) as functions of the parameters of the model (or of exogenous quantities).

Complication: This needs to be done for every time period.

Not possible using algebra, but we can get close and describe the solution.

Use approximate Solution

Start by combining the investment equation with the capital accumulation:

(Plug $I_t = \bar{s} Y_t$)

$$\underbrace{\Delta K_{t+1}}_{\text{change in capital}} = \underbrace{\bar{s} Y_t - \bar{d} K_t.}_{\text{net investment}}$$

Plug $L_t = \bar{L}$

Consider the Production Function: $Y_t = \bar{A} K_t^{1/3} \bar{L}^{2/3}.$

$$\Delta K_{t+1} = \bar{s} \bar{A} K_t^{1/3} \bar{L}^{2/3} - \bar{d} K_t$$

We reduced our system (5 eqs, 5 unknowns) to one with 2 equations, 2 unknowns (Yt, Kt)

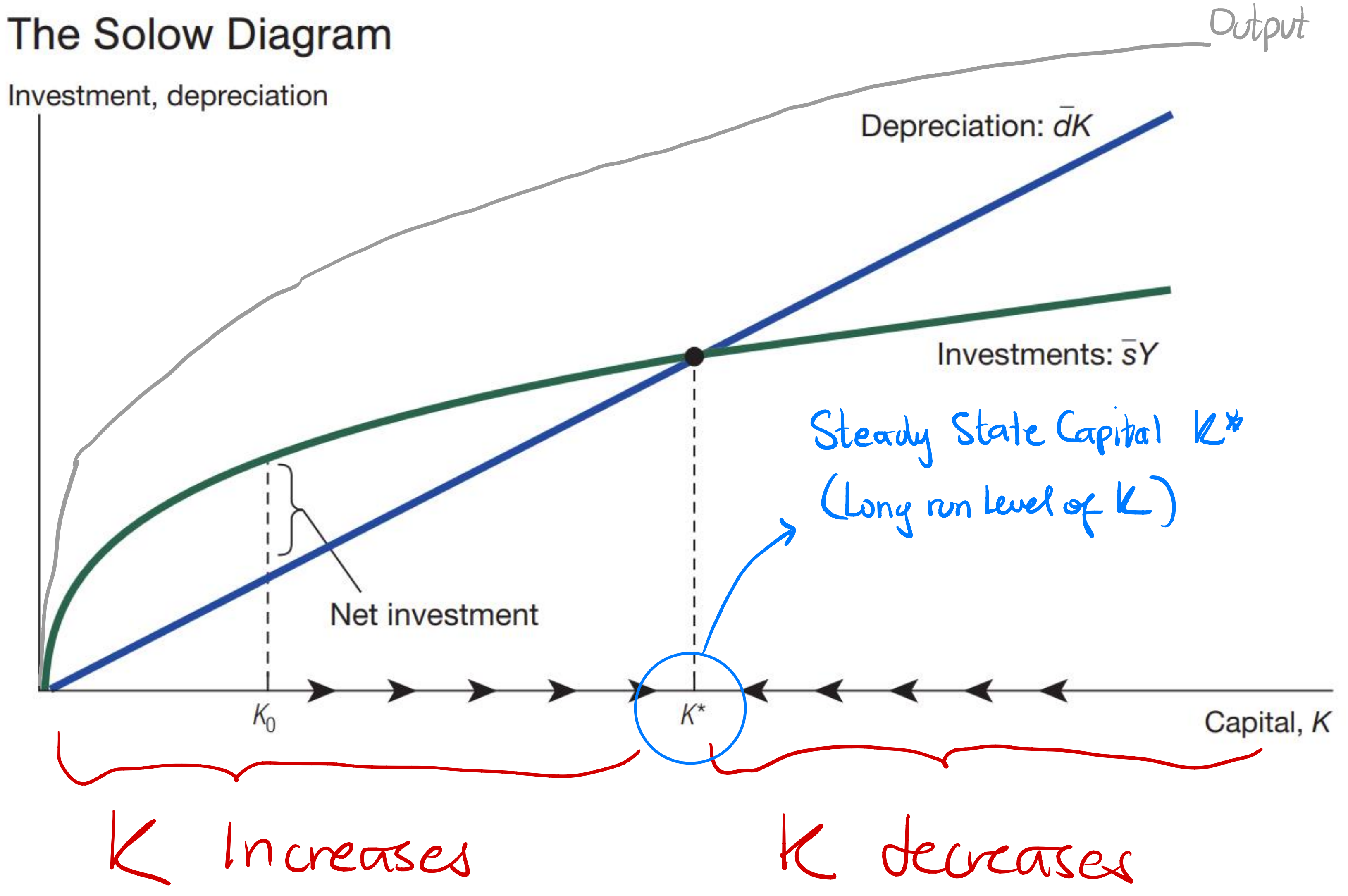
We could replace the production in the first equation and get a **dynamic** expression for capital.

The Solow Diagram

If depreciation > Investment ⇒ K stock ↓
(K destroyed) (K added)

We cannot solve the dynamic equation algebraically but we can analize it graphycally

- Curves: Investment ($\bar{s}Y$) and depreciation ($\bar{d}K$)
- These functions explain how capital stock changes over time
- **When $\bar{s}Y$ exceeds $\bar{d}K$, capital stock increases.**
- When curves intersect, i.e., $\bar{s}Y = \bar{d}K$, **steady state** is reached, and capital stock no longer changes.



Dynamics of Solow Diagram

As $\boxed{\Delta \mathrm{K}_{\mathrm{t}+1} = \bar{s}Y - \bar{d}K > 0}$ (investment larger than depreciation) we get an increase in capital stock in t+1 (next period).

If the reverse happens, next period's capital decreases.

Thus, the only way capital stays the same is if new investment exactly offsets depreciation (capital wear and tear)

Achieving Steady State:

When investment equals depreciation (sY=dK), the capital stock no longer changes, and the economy reaches a steady state (K*).

Steady state: Long-run value of the variables when they remain steady (absent other developments the economy converges here if capital accumulation is the only growth engine)

The Solow Diagram with Output

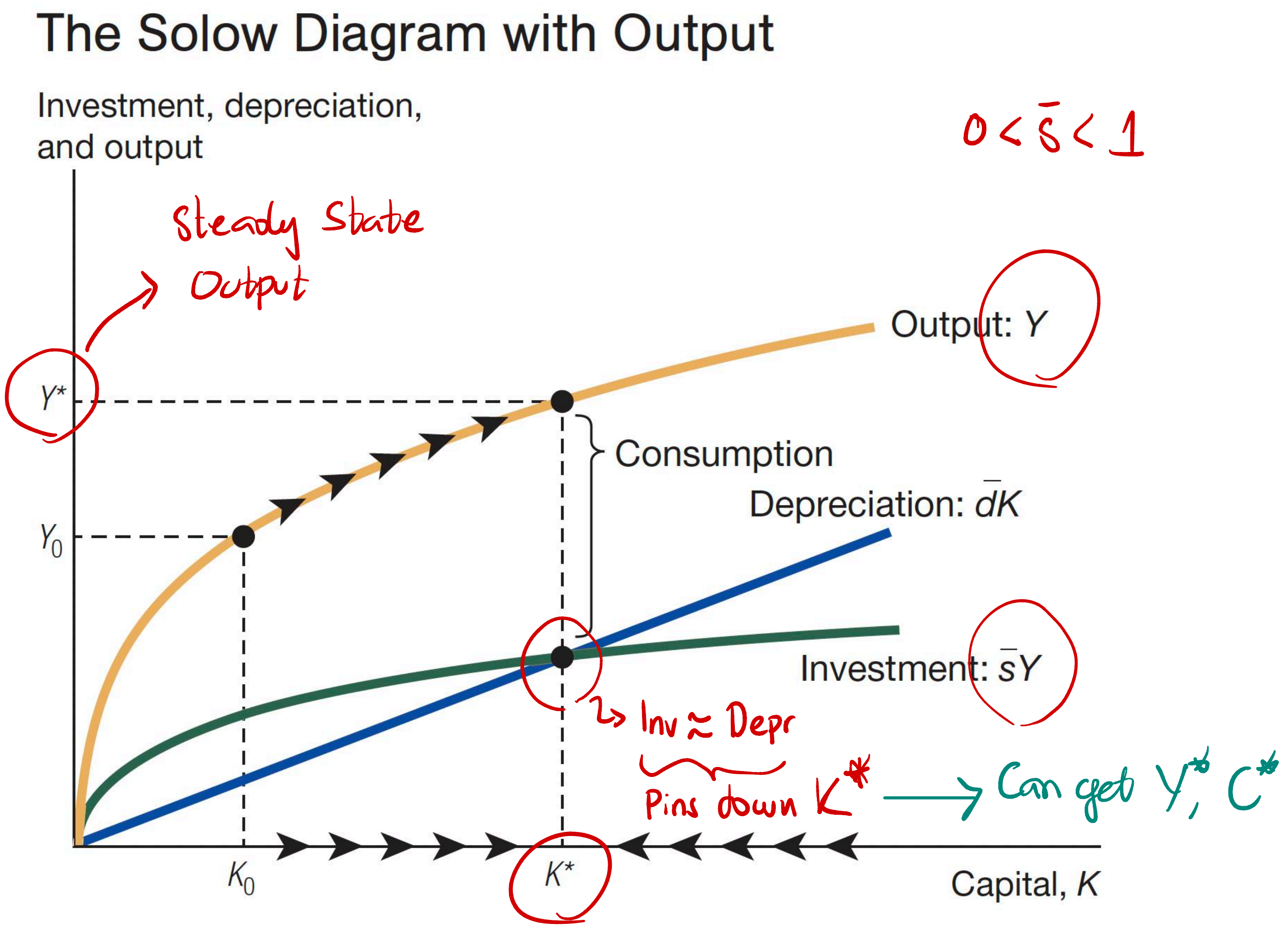
The production function is added to the Solow diagram:

$$Y_t = \bar{A}K_t^{1/3}\bar{L}^{2/3}.$$

Plotting Y as a function of K.

Output rises as the economy transits from K_0 to the steady state K^*.

Consumption is the difference between output and investment.



Solving Mathematically for the Steady State

Unlike the dynamic model, we can solve mathematically for the steady state.

(at the Steady state in t)

Why? Because the model is simplified in that case as $K_t = K_{t+1}$

$K_t = K_{t+1} = K_{t+2} = K^*$

$$\underbrace{Y_t}_{Y^*} = \bar{A} K_t^{1/3} \bar{L}^{2/3} . \quad K_t \to K^*$$

$\bar{s}\bar{A}\bar{L}^{2/3} = \bar{d}\frac{K^*}{K^{*1/3}} \longrightarrow \frac{\bar{s}\bar{A}\bar{L}^{2/3}}{\bar{d}} = K^* K^{*-1/3} = K^{*2/3}$

$\bar{s}Y^* = \bar{d}K^*$ steady-state level of capital

$\bar{s}\bar{A}K^{*1/3}\bar{L}^{2/3} = \bar{d}K^*.$ Substituting Y and K from the production function

$$\boxed{K^* = \left(\frac{\bar{s}\bar{A}}{\bar{d}}\right)^{3/2} \bar{L}.}$$ Divide by sA and L

$Y^* = \left(\frac{\bar{s}}{\bar{d}}\right)^{1/2} \bar{A}^{3/2} \bar{L}.$ Substitute this solution for K * into the equation for Y (in steady state)

$\left(\frac{1}{L}\right) \to$ $y^* \equiv \left(\frac{Y^*}{L^*}\right) = \bar{A}^{3/2}\left(\frac{\bar{s}}{\bar{d}}\right)^{1/2}$ Now we have the output per person (y) in the steady state

$$\underbrace{\Delta K_{t+1}}_{0} = \bar{s}\bar{A}K^{*1/3}\bar{L}^{2/3} - \bar{d}K^*$$

Understanding the Steady State in the Solow Model

The steady state is when capital stock K^* and production level Y^* reach a constant, unchanging level.

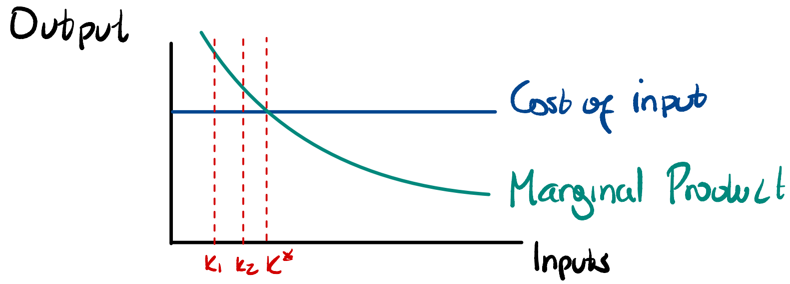
Useful to understand how the economy evolves from any starting point to a balanced long-term position.

Capital and Output: At the steady state, **investment sY^* equals depreciation dK^***.

Intuition: New capital equals "destroyed" capital. Thus, the stock won't change, and nothing else in the model changes.

Mechanism

Diminishing Returns: Additional output decreases as capital stock increases due to diminishing returns on capital.



Economic Growth in the Solow Model

Steady State: The economy eventually reaches a constant level of capital K* and output Y*

Long-Term Growth: The Solow model predicts no long-term growth, as the economy stabilizes over time.

+ **Per Capita Output and Consumption**: At the steady state, per capita output $y^* = Y^*/\bar{L}$ and per capita consumption $c^* = (1 - \bar{s})\cancel{Y}^*$ y^* are also constant.

Investment Equals Depreciation: At the steady state, annual investment $\bar{s}Y^*$ matches annual depreciation $\bar{d}K^*$ (In per capita terms: $\bar{s}y^* = \bar{d}k^*$)

Diminishing Returns: Diminishing returns on capital cause the returns on investments to decrease, ultimately halting economic growth.

Rate of growth slows down over time

⇒ Prediction: Richer countries grow at a lower rate than poorer

↳ Poorer countries are catching up

Case Study: The Family Farm

Case Description: A farm plants corn starting with a small stock of seed corn, with harvests growing annually until balancing out.

Labor Limitations and Diminishing Returns: The amount of corn a farm can harvest is limited by available labor; beyond a certain scale, diminishing returns halt growth.

How to grow beyond the current steady state?

Ans/ by increasing somehow the steady state K

Hint: Look at anything affecting the condition: $\bar{s}\bar{A}K^{1/3}\bar{L}^{2/3} = \bar{d}K$

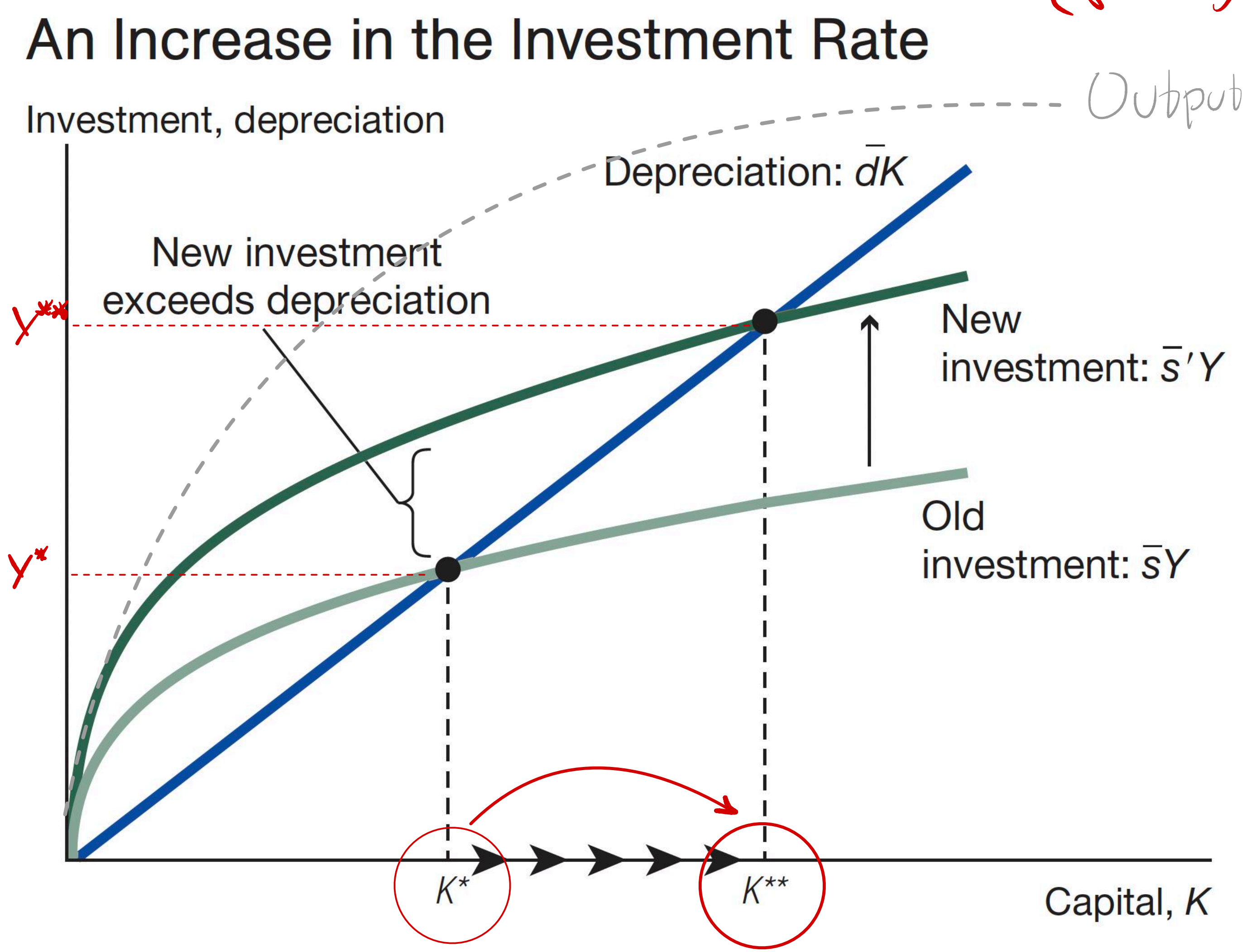
Candidates: $\begin{cases} \text{higher savings rate } \uparrow\bar{s} \\ \text{Technology improvements } (\uparrow A, \downarrow d) \end{cases}$

Changes in the investment rate s

S goes from $\bar{s}$ to $\bar{s}'$
$(\bar{s}' > \bar{s})$

Starting from K*, new investment exceeds depreciation when the investment rate rises to $\bar{s}'$.

This causes the capital stock to increase, until the economy reaches the new steady state at K**.



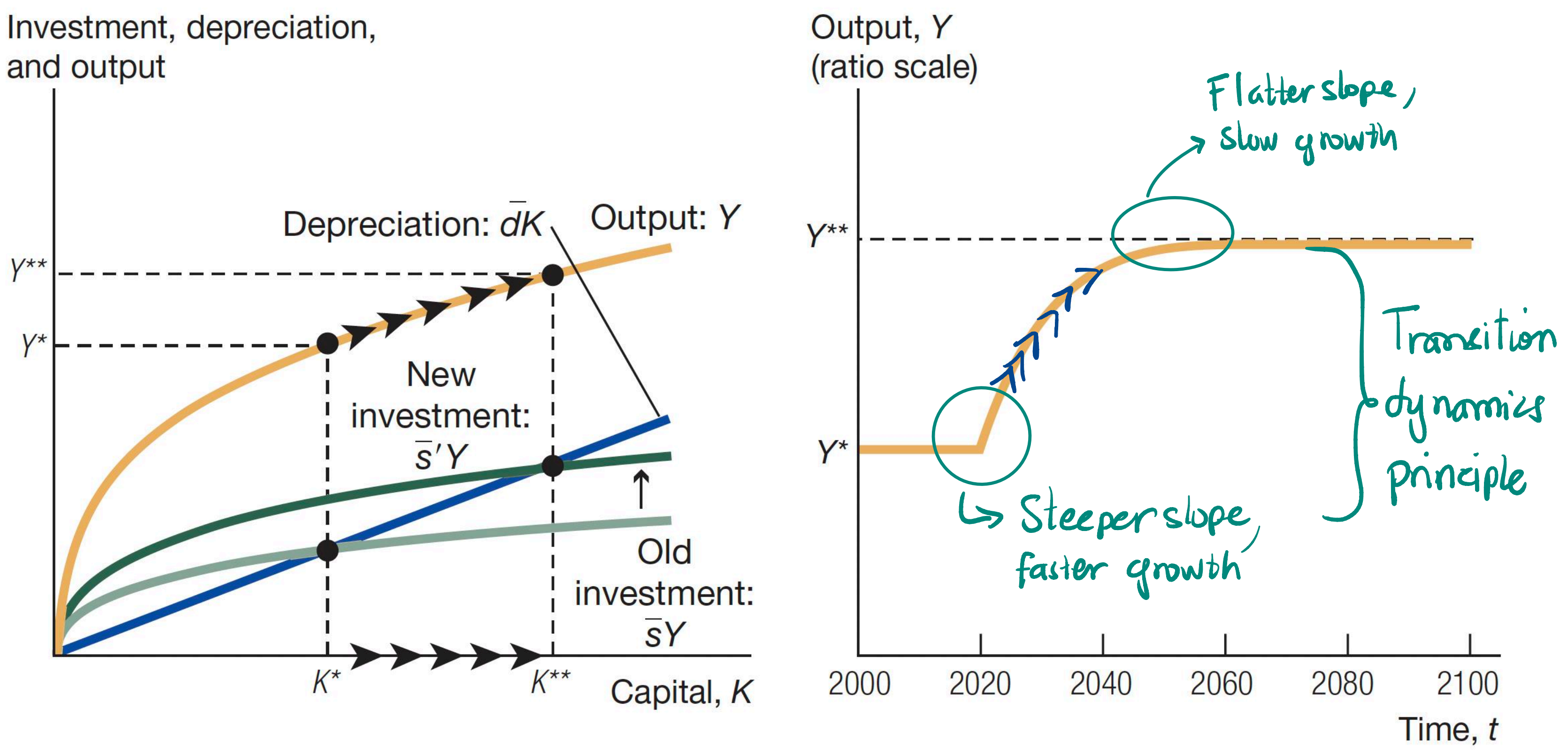
“Dynamic” behavior of Output after an Increase in $\bar{s}$

A permanent increase in the investment rate causes output to grow over time until it reaches its new steady state level Y*.

$$K^* \longrightarrow K^{**}$$

$$Y^* = F(K^*) \longrightarrow Y^{**} = F(K^{**})$$

The Behavior of Output after an Increase in $\bar{s}$



(a) The Solow diagram with output.

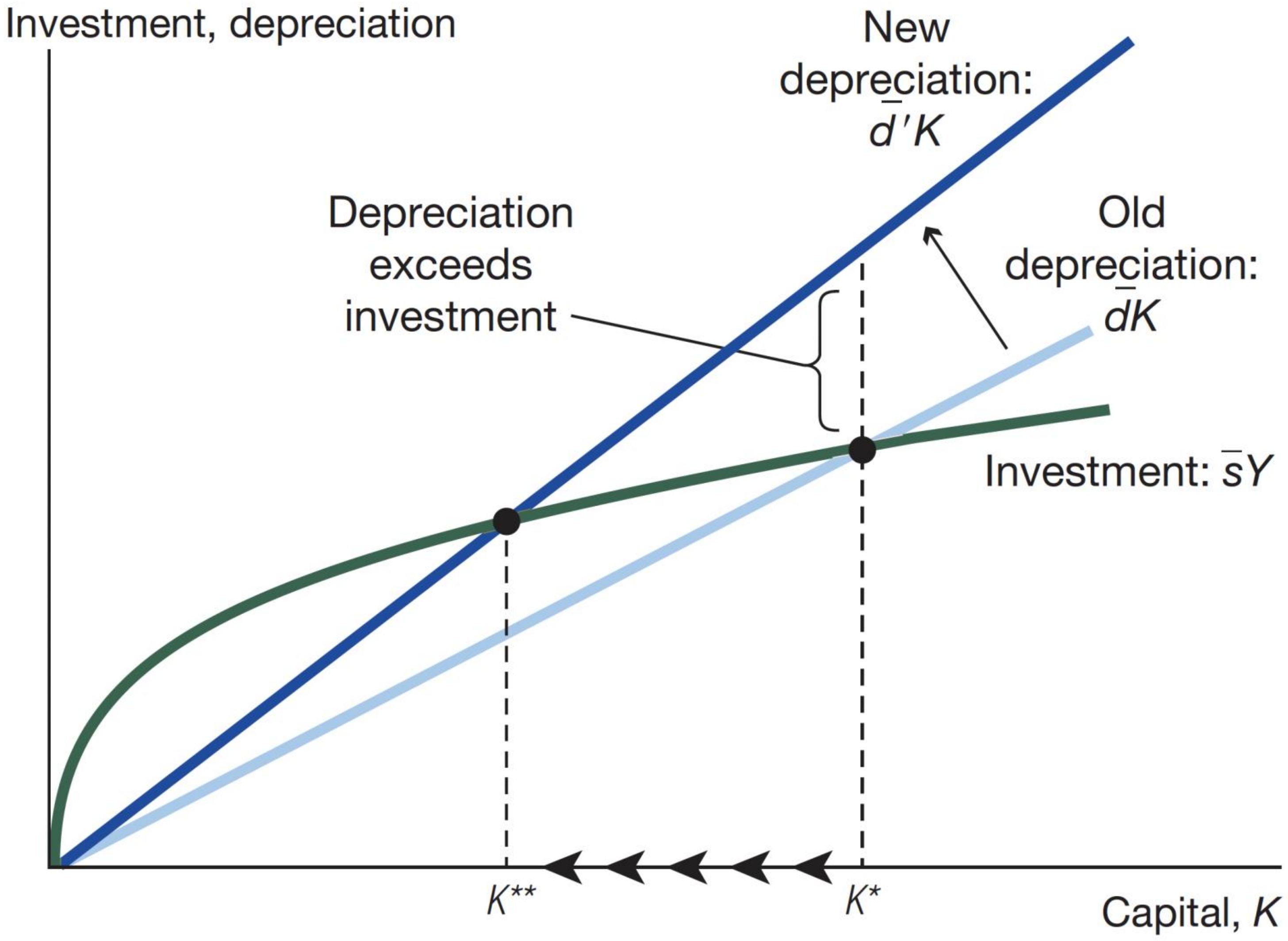
(b) Output over time.

A Rise in the Depreciation Rate

$\uparrow \bar{d}: \bar{d} \to \bar{d}' \quad (\bar{d}' > \bar{d})$

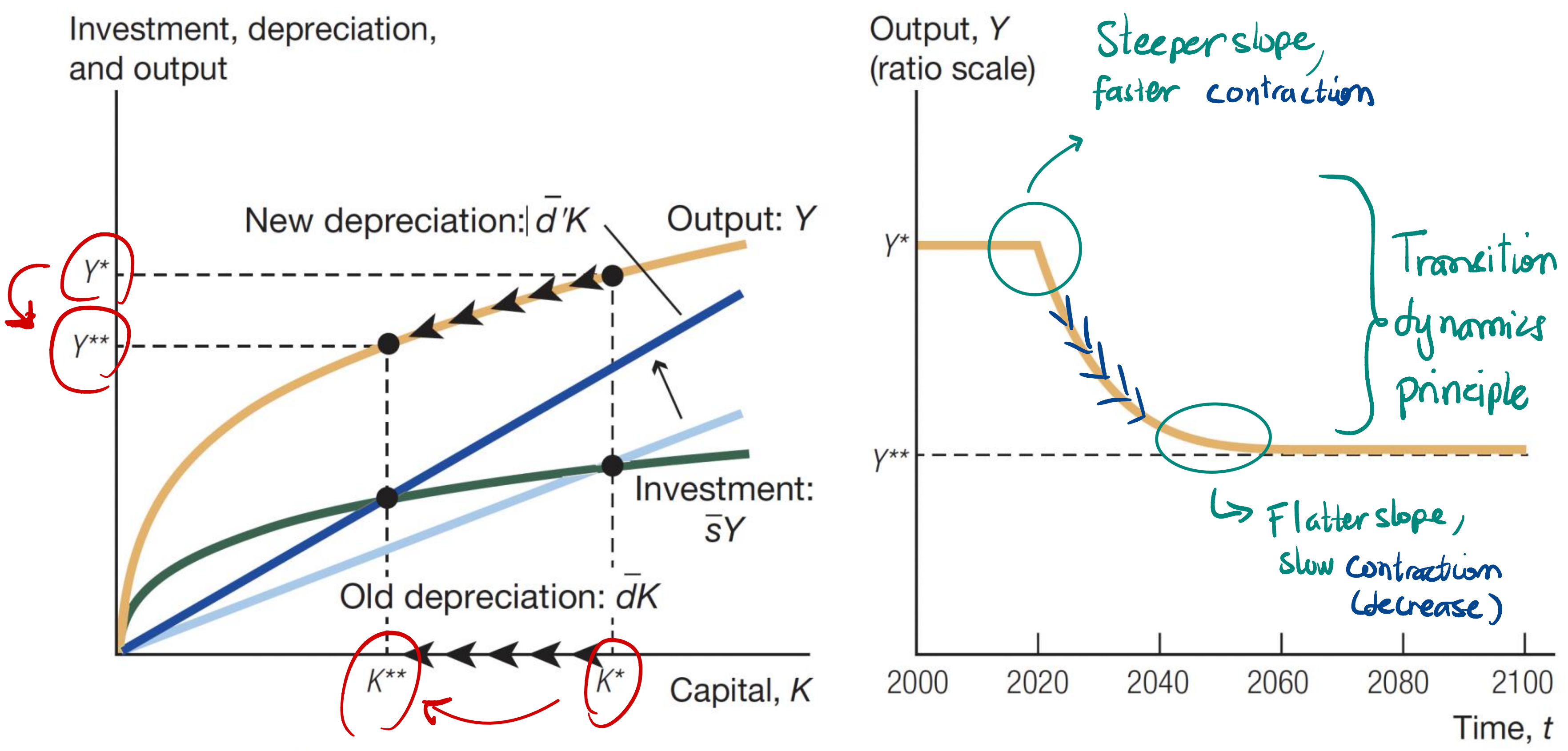
Starting from K*, depreciation exceeds new investment when the depreciation rate rises to $\bar{d}'$.

This causes the capital stock to decline until the economy reaches the new steady state at K**



Behavior of Output after an Increase in $\bar{d}$.

A permanent increase in the depreciation rate causes output to decline over time until it reaches its new steady state level Y**



(a) The Solow diagram with output.

(b) Output over time.

Transition Dynamics and Differences in Global Growth Rates

Transition Dynamics: When an economy is far from its steady state, whether above or below, **its growth rate accelerates**.

Intuition: It explains why economies starting from different initial conditions converge towards their respective steady states.

✓ **Investment Rate Change**: An increase in the investment rate leads to new investments exceeding depreciation, pushing capital stock up until it reaches a new steady state K**.

✓ **Depreciation Rate Increase**: An increase in the depreciation rate initially causes economic contraction, followed by a gradual "recovery" to a new steady state.

Shrinkage

* Technology can also affect growth (and the steady state)

Advantages of the Solow Model

Provides a **theory explaining a country's wealth in the long run** at steady state.

Relies on high investment rates, high Total Factor Productivity (TFP) levels, and low depreciation rates.

Extended models can include **other ingredients**:

Human capital investments, such as education and on-the-job training.

This is an active research area in economics!

Limitations of the Solow Model

Limitations of Physical Capital Investment

- Physical capital investment explains only a small part of income differences.
- **Differences in Total Factor Productivity (TFP) are more important** in explaining income disparities.

Savings rate (s) is key, But exogenous!

Lack of Endogenized Investment Rates

- The Solow model cannot explain why different countries have varying productivity levels and investment rates.
- **Extended models endogenize investment rates**, considering factors like patience levels and government policies (taxes and subsidies).

Lack of Long-Term Growth Theory

- Diminishing returns to capital accumulation mean **that capital accumulation alone cannot sustain long-term growth**.
- Growth eventually stops as the marginal product of capital declines.

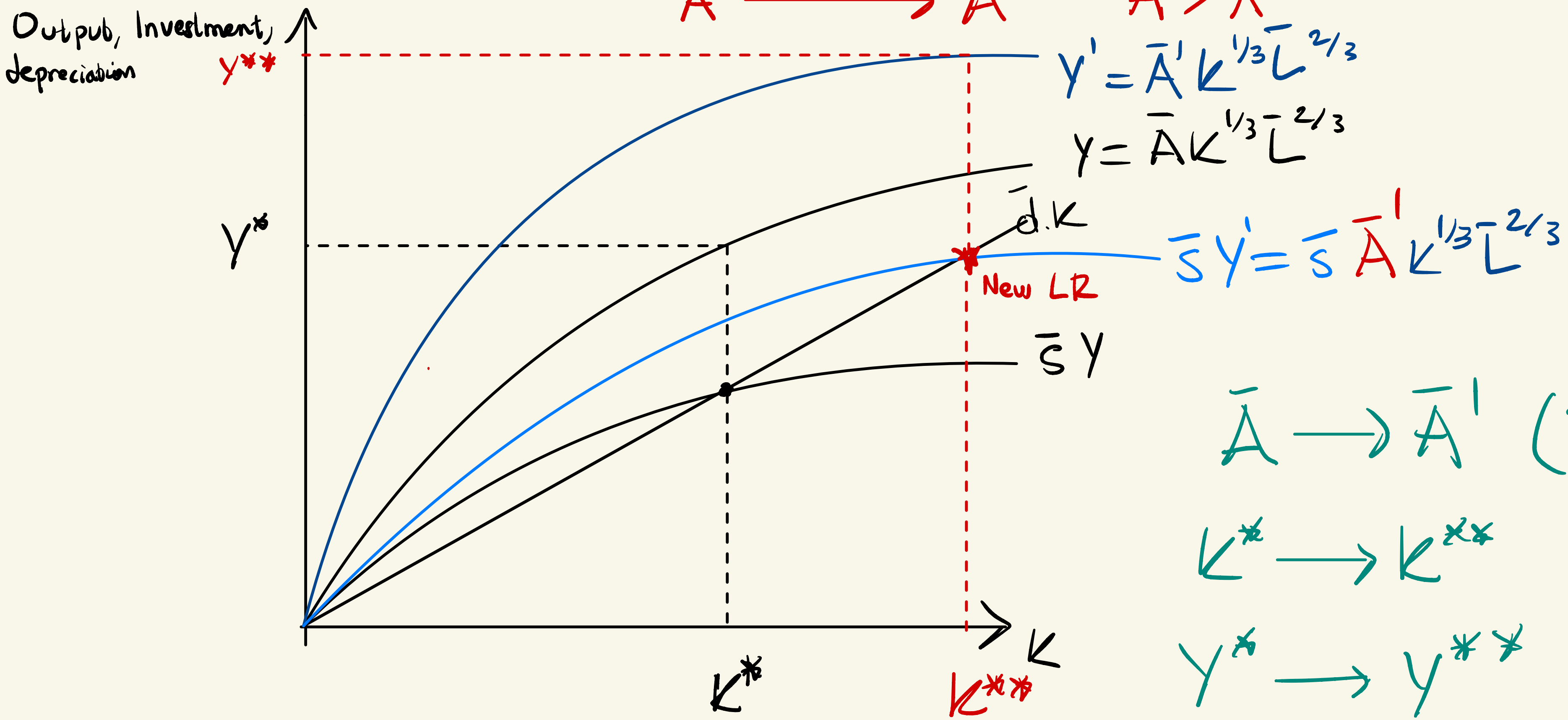
Another case: Improvement in technology

The key condition to determine K* is Investment = Depreciation

Naturally, given the assumption I = sY, other features affecting Y can also push K* towards K**

For example, technology:

$\bar{A} \longrightarrow \bar{A}'$ $\quad \bar{A}' > \bar{A}$



$\bar{A} \longrightarrow \bar{A}'$ (TFP↑)

$K^* \longrightarrow K^{**}$

$Y^* \longrightarrow Y^{**}$