Econ 5322 - MS Macroeconomics Spring 2025
Granados

Midterm Exam

Answer Key

Answer the following questions. Show and explain your work instead of only writing the final answers. Read the entire exam before starting to answer it..

1. (20 points) Suppose the economy's representative household maximizes the expected intertermporal utility function,

$$\mathbb{E}_t \sum_{s=t}^{\infty} \beta^{s-t} u(C_s)$$

subject to the constraints given by the law motion of capital,

$$K_{t+1} = (1-\delta)K_t + I_t$$

and the budget constraint:

$$C_t + I_t = \tilde{r}_t K_t + w_t$$

where $\mathbb{E}_t$ denotes the expectation operator conditional on information available at time t, $\beta \in (0,1)$ is the discount factor, C_t is consumption in period t, K_{t+1} is capital at the beginning of period $t+1$, $\delta \in (0,1)$ is the rate of capital depreciation, I_t is investment in period t, $\tilde{r}_t$ is the rental rate on capital, w_t is the real wage, and labor supplly has been normalized to 1. Households rent capital to firms in a perfectly competitive market. Also, the investment is given by $I_t = K_{t+1} - (1-\delta)K_t$.

The production function is $Y_t = A_t^{\alpha} K_t^{1-\alpha}$, where $\alpha \in (0,1)$ and A_t is an exogenous technology shock

(a) Denote with R_{t+1} the return to capital accumulation at $t+1$. What is the expression for R_{t+1}? Explain.

(Ans) We can obtain the return on capital from the budget costraint, and being careful to replace the investment as it is also related to the capital stock in two periods:

$$C_t + I_t = \tilde{r}_t K_t + w_t$$
$$C_t + K_{t+1} - (1-\delta)K_t = \tilde{r}_t K_t + w_t$$
$$C_t + K_{t+1} = (1 + \tilde{r}_t - \delta)K_t + w_t$$

We can see in the budget constraint how the return on capital accumulated by t is the term that factors out K_t, now we are asked about such return for K_{t+1}, thus we forward one period:

$$R_{t+1} = (1 + \tilde{r}_{t+1} - \delta)$$

Moreover, we also have that $\tilde{r}_t$ is the marginal product on capital such that: $\tilde{r}_t = \frac{\partial Y_t}{\partial K_t} = (1-\alpha)A_t^{\alpha}K_t^{-\alpha} = (1-\alpha)\left(\frac{A_t}{K_t}\right)^{\alpha}$.

Then: $R_{t+1} = (1-\alpha)\left(\frac{A_t}{K_t}\right)^{\alpha} + 1 - \delta$

Intuition: δ is the rate of capital depreciation, $\tilde{r}_t$ is the rental rate the households receive from firms that rent its capital (and is also the marginal product of capital in this case). Then, the gross return on capital R_{t+1} is equal to one plus marginal return gained by renting the capital, minus the depreciation of such stock (also remember that a gross return is equal to one plus the marginal return).

(b) Explain how to obtain the Euler equation for capital accumulation intuitively.

(Ans) Suppose a household can use one dollar to buy consumption in t or invest it in an asset that will generate the uncertain gross return R_{t+1} at t+1.

If the household uses the dollar to buy consumption, the benefit given by the increment in utility from consumption today is $u'(C_t)$.

If the household invests the dollar with the return R_{t+1} at $t+1$, the benefit given by the increment generated by the extra consumption at $t+1$ is $u'(C_{t+1})$

The household is uncertain about C_{t+1} and R_{t+1}, so they form an expectation $\mathbb{E}_t\left[u'(C_{t+1})R_{t+1}\right]$ based on the information at t

In comparing the benefit of today's consumption, the household will discount the future benefit with a discount factor β.

At an optimum, the cost of investing one unit of consumption today in capital accumulation must be equal to the expected discounted marginal utility value of the gross return from investing one unit of consumption good in capital accumulation. That is:

$$u'(C_t) = \beta\mathbb{E}_t\left[u'(C_{t+1})R_{t+1}\right]$$

2. (20 points) In the same setup as in Question 1, assume that the period utility function takes the form

$$u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}, \quad \text{with } \gamma > 0$$

Assume log-normality and homoskedasticity. Log-normality implies that, given a variable X_{t+1},

$$\log[\mathbb{E}_t(X_{t+1})] = \mathbb{E}_t[\log X_{t+1}] + \frac{1}{2}Var_t[\log X_{t+1}]$$

(a) Show that log-normality and homoskedasticity imply that the log-linear version of the Euler equation you obtained in Question 1 has the form:

$$\mathbb{E}_t(c_{t+1} - c_t) = \sigma\mathbb{E}_t r_{t+1}$$

where $c_t \equiv d\log C_t$ (difference of log with respect to steady state value), $r_t = d\log R_t$, and $\sigma = 1/\gamma$

(ans) We have that,

$$u'(C_t) = \beta \mathbb{E}_t\left[u'(C_{t+1})R_{t+1}\right]$$
$$C_t^{-\gamma} = \beta \mathbb{E}_t\left[C_{t+1}^{-\gamma}R_{t+1}\right]$$

Taking logs,

$$-\gamma \log C_t = \log\beta + \log\left[\mathbb{E}_t\left(C_{t+1}^{-\gamma}R_{t+1}\right)\right]$$

by log-normality of $C_{t+1}^{-\gamma}R_{t+1}$

$$\log\left[\mathbb{E}_t\left(C_{t+1}^{-\gamma}R_{t+1}\right)\right] =$$
$$= \mathbb{E}_t\left[\log\left(C_{t+1}^{-\gamma}R_{t+1}\right)\right] + \frac{1}{2}Var_t\left[\log\left(C_{t+1}^{-\gamma}R_{t+1}\right)\right]$$
$$= -\gamma\mathbb{E}_t\left[\log C_{t+1}\right] + \mathbb{E}_t\left[\log R_{t+1}\right] + \frac{\gamma^2}{2}\sigma^2_{t,\log C_{t+1}} + \frac{1}{2}\sigma^2_{t,\log R_{t+1}} - \gamma\sigma_{t,\log C_{t+1},\log R_{t+1}}$$

where $\sigma^2_{t,X}$ refers to the variance of X at t, and $\sigma_{t,X,Y}$ to the covariance between X and Y in period t. We differentiate this last expression with respect to its steady state ("barred") counterpart:

$$-\gamma d\log C_t \approx -\gamma\mathbb{E}_t(d\log C_{t+1}) + \mathbb{E}_t(d\log R_{t+1})$$

In this expression we were able to cancel out the last three terms involving variances and covariances because we assumed homoskedasticity, i.e., they are constant over time, and then are the same in the expression above as in the steady-state expression we are subtracting. We rearrange:

$$\mathbb{E}_t(d\log C_{t+1} - d\log C_t) \approx \frac{1}{\gamma}\mathbb{E}_t(d\log R_{t+1})$$

Finally, we use that $r_{t+1} = d\log R_{t+1}$, $x_t \approx d\log X_t$, and $\sigma = \frac{1}{\gamma}$:

$$\mathbb{E}_t(c_{t+1} - c_t) = \sigma\mathbb{E}_t r_{t+1}$$

(b) What is σ? What does it measure?

(Ans) σ is the intertemporal elasticity of substitution. It measures the responsiveness of consumption to a change in the return to asset accumulation.

(optional) At the same time with a CRRA utility function (as in this problem) it will be given as the inverse of the coefficient of (relative) risk aversion $1/\gamma$.

3. (20 points) Let the solution of the real business cycle model be described by the relations:

$$
\begin{aligned}
k_{t+1} &= \eta_{kk}k_t + \eta_{ka}a_t \\
c_t &= \eta_{ck}k_t + \eta_{ca}a_t \\
y_t &= \eta_{yk}k_t + \eta_{ya}a_t
\end{aligned}
$$

with

$$a_t = \phi a_{t-1} + \varepsilon$$

where lower-case letters denote percentage deviations of their corresponding variables from their steady-state levels, the η's are elasticities obtained with the method of undetermined coefficients, and $\phi \in (0, 1)$

(a) What is the intuition for this solution? Put differently, why do we guess that the solution for endogenous variables depends on k_t and a_t?

(Ans) Optimal behavior in this model maps the state of the economy (k_t and a_t) into the variables that are endogenous (c_t, y_t). The solution also maps the state at time t into the choice of assets entering at $t + 1$ (k_{t+1}).

(b) Why can we guess that this is the unique solution of the log-linearized model?

(Ans) After we verified the determinacy of the system, we can tell whether it's solution is unique; k_t is a predetermined value and c_t is not predetermined. If one eigenvalue of a matrix constructing from the log-linearized model lies outside the unit circle and one inside the unit circle, we can tell there is a unique solution.

(c) Suppose there is a technology innovation $\varepsilon_0 = 1$ at time $t = 0$, followed by no other innovation in subsequent periods. Use the relations above to compute the responses of capital, consumption, and output to the innovation in periods 0, 1, 2.

(Ans) We can use the fact that $k_0 = 0$ since it was determined at $t = -1$, i.e., before the innovation to technology took place.

Time (t)	ε_t	a_t	k_t	c_t	y_t
-1	0	0	0	0	0
0	1	1	0	η_{ca}	η_{ya}
1	0	ϕ	η_{ka}	$\eta_{ca}\phi + \eta_{ck}\eta_{ka}$	$\eta_{ya}\phi + \eta_{yk}\eta_{ka}$
2	0	ϕ^2	$\phi\eta_{ka} + \eta_{kk}\eta_{ka}$	$\eta_{ca}\phi^2 + \eta_{ck}(\phi\eta_{ka} + \eta_{kk}\eta_{ka})$	$\eta_{ya}\phi^2 + \eta_{yk}(\phi\eta_{ka} + \eta_{kk}\eta_{ka})$

(d) What is the response of technology to the innovation if $\phi = 1$? Do capital, consumption, and output return to the initial steady state in this case? Why?

(Ans) if $\phi = 1$, technology innovations have permanent effects, and the economy does not return to the original steady-state.

Output will then converve to a new, permanently higher (or lower) steady-state path after a one-time positive (or negative) technology shock with $\phi = 1$.

To compare this to the case depicted in (c), rather than seeing a decreasing pattern in a_t (1, ϕ, ϕ^2, ...) that converges to zero, we would see a constant pattern that never decreases $(1, 1, 1, \ldots)$ and the rest of the system will also reflect such dynamics.

4. (25 points) The first-order condition for optimal labor supply in the real business cycle model with variable labor implies:

$$U_{1-N_t}(C_t, 1-N_t) = w_t U_{C_t}(C_t, 1-N_t)$$

where C_t is consumption in period t, $1 - N_t$ is the leisure, and w_t is the real wage.

(a) Explain this condition intuitively

(Ans) The marginal utility of leisure must equal how much marginal utility of consumption the real wage earned by supplying an extra unit of labor is generated.

This relationship must hold in equilibrium, i.e., at the optimum, otherwise it would make sense to supply less or more labor (i.e., to get more or less leisure).

Intuitively, in equilibrium the marginal utility generated by increasing leisure must equal that generated by increasing labor time (labor supplied).

(b) Suppose the production function is $Y_t = N_t$ and the labor market is perfectly competitive. What is the value on the real wage? Why?

(Ans) In such case the profit made by the representative firm is given by: $P_t Y_t - W_t N_t = (P_t - W_t) N_t$; where P_t is the price of the good produced by the firm and W_t is the nominal wage.

Thus the profit maximization problem of the firm is:

$$\max_{N_t} P_t Y_t - W_t N_t$$

The first-order condition with respect to the labor demanded by the firm is:

$$P_t = W_t$$

That is real wage equals one: $w_t = \frac{W_t}{P_t} = 1$ Or, given the assumption of perfect competition, the real wage equals the marginal product of labor (MPL): $w_t = \frac{\partial Y_t}{\partial N_t} = 1$

Now suppose the representative household is subject to an additional constraint for its consumption:

a *cash-in-advance* constraint (CIA). The constraint in terms of the model is set as "nominal consumption cannot exceed the amount of money in the household" and can be seen as a friction that, in principle, may prevent agents from making the same consumption decisions as in the frictionless case (where they could expend beyond their cash, e.g., by borrowing money).

The first-order condition for optimal labor supply becomes:

$$U_{1-N_t}(C_t, 1-N_t) = w_t\left(1 + \frac{i_t}{1+i_t}\right)U_{C_t}(C_t, 1-N_t)$$

where i_t is the nominal interest rate.

(c) Suppose production and the labor market assumptions are as above, and that a benevolent central banker (social planner) is choosing monetary policy to maximize welfare. What is the optimal interest rate that the central bank would choose? What is the intuition for the result?

[Hint: in thinking about this, you don't really need to know about the CIA model itself, but to remember the main lesson about efficiency in the RBC model in an environment without frictions and what this implies for policymaking.

For the same token, do not try to set a CIA model, the only result you need from it to answer is the first-order condition above]

(Ans) We know that the baseline RBC model outlines a frictionless economy, i.e., one without distortions. In that case the private agents achieve an equilibrium that is efficient, that is, that a benevolent social planner (e.g., a Central Banker in our setup) cannot improve on with any policymaking.

Given this, we know that, at best, we would like to achieve the equilibrium allocation of the frictionless economy (i.e., one where $U_{1-N_t}(C_t, 1-N_t) = w_t U_{C_t}(C_t, 1-N_t)$).

In the other case, there is a different allocation in the private equilibrium due to the CIA constraint. In such allocation we have,

$$U_{1-N_t}(C_t, 1-N_t) = w_t\left(1 + \frac{i_t}{1+i_t}\right)U_{C_t}(C_t, 1-N_t)$$

Thus, the distortion preventing this economy from achieving the best possible allocation is reflected in the factor $\left(1 + \frac{i_t}{1+i_t}\right)$. If it would be equal to 1 the private equilibrium achieved in both cases would be the same.

Thus, the social planner acknowledges this and wants to eliminate the distortion by setting $i_t = 0$. That is the optimal policy.

Intuition: the central banker wants to lower the opportunity cost of holding money, which in practice will work as a "tax" on the consumption-leisure decisions of private agents, prompting them to supply more labor.

5. (15 points) The Stochastic Growth Model or Real Business Cycle model in its basic version is a good

benchmark but can be seen as too simplistic to explain more realistic features. The approach to deal with this has consisted in enriching the model with more nuanced features, such as new variables, mechanisms and even frictions.

(a) Briefly explain the issues that are observed in the basic version of the model.

(Ans) First, this version of the model is not able to generate persistent economic effects from transitory shocks; in fact, the capital accumulation dynamics will have a big impact on the economy only if the shock itself is very persistent.

Second, this model is unable to deliver strong changes in the expected returns of capital after a technological shock. This happens because most of the expected return on capital consists of the undepreciated share of capital is not affected by the shock (it is always $1 - \delta$) (optional as it is not mentioned in the slides: To fix this, more nuanced adjustment mechanisms of capital could be introduced).

A third one is mentioned in (b).

To put it simply, this model is too smooth and too frictionless, at least compared to reality; and although we are not building a model to replicate reality (or it becomes too complicated), we still would like to say something insightful about some economic phenomena we observed in our daily lives and consider relevant, e.g., the role that monetary, and other policies can have in the well-being of the population.

In light of these type of issues, including the fact that policy does not play any role when in reality it does, more nuanced versions of the model are in order. However, the basic RBC is an important benchmark that is useful to have as a departure point.

(b) What is the motivation behind including variable labor in the model? What is achieved in this version of the model relative to the more basic RBC model?

(Ans) The RBC that relies only on capital accumulation cannot generate output responses to a shock that are of a larger magnitude than the shock itself. Thus, slower than normal growth is not able to deliver a recession in the model, and instead, the only way to have an economic contraction would be through an actual technological destruction which is unrealistic.

By including variable labor, i.e., allowing the agents to optimally determine how much to work and how much to use for leisure, this issue is addressed.

(c) Does monetary policy have any meaningful role or effect in the basic version of the model? Refer to how the private equilibrium (the one achieved by decentralized agents) compares to the one that can be achieved by a benevolent social planner.

(Ans) The basic RBC model has no role for any policymaking. That means that there is no policy that can improve on the private equilibrium reached by the agents of the economy in a decentralized fashion.

In terms of allocations, this implies that the optimal private equilibrium allocation is identical to the one achieved by optimal policymaking, that in this case is characterized by a social planner that does not make use of its available tools (e.g., not setting any taxes or changing interest rates).