Intermediate Macroeconomics

A Model of Production

ECON 3311 – Fall 2024

UT Dallas

Introduction

before:

Measuring Output

- What is Macroeconomics
- GDP and its components
- Growth

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Modeling → Now: How to model GDP

In this chapter, we will examine:

- How to set up and solve a macroeconomic model
- The purpose of a **production function** and its use for understanding differences in GDP per capita across countries
- The role of **capital per person and technology** in explaining differences in economic growth
- The relevance of **'returns to scale'** and **'diminishing marginal products'**
- How to examine data through the lens of a macroeconomic model

Introduction

A model is a mathematical representation of an economic phenomena we want to study

- A model consists of equations and unknowns with real-world interpretations
- The goal is to have the model be simple enough to interpret and use, but still explain an important real-world aspect of the macroeconomy
- Its nature is hypothetical and is not meant to be highly realistic but useful
  - Trade-off: “KISS” principle (parsimony is paramount)
  - “All models are wrong, but some are useful” (George E. P. Box)

Steps to building and using a model:

1) Document facts → Data, economic variable, etc.
2) Build a model to understand the facts
3) Examine the model performance and effectiveness

Model of Production

Suppose we are attempting to explain GDP (Y) and have a single good that is being produced in a closed economy

- There are a fixed number of people $\bar{L}$ and a fixed number of capital (machines) $\bar{K}$ in the economy
- The production function is going to depict how many units of output (Y) can be produced given any number of inputs of *K* and *L*:

$$Y = F(\overset{}{K}, L)$$

+ +

Increases w/ L, K
w/ L=0 → Y=0
depicts (for Y>0)
L>0, K>0

As the values of K and L change, the amount the firm can produce changes

An example of a production function is: $Y = \bar{A}K^{1/3}L^{2/3}$

Total Factor Productivity
(Measures efficiency in using inputs)

$\bar{A}$ is a **productivity** parameter: As it increases, the firm can produce more with the same amount of *K* and *L*

$\uparrow\bar{A} \Rightarrow Y\uparrow$

Returns to Scale

We can plug in different values of K and L to see how the amount of production changes:

Suppose $K = 8, L = 27$, and $\bar{A} = 1$: $\quad Y = 1 * 8^{\frac{1}{3}} 27^{\frac{2}{3}} = 18$

Increasing $\bar{A}$ will allow the firm to produce more with the same amount of inputs – this is what **technological progress** is.

This type of production function is called Cobb-Douglas and is of the form:

$$Y = \bar{A} K^{\alpha} L^{\beta}$$ (in logs: $\ln Y = \bar{a} + \alpha \ln K + \beta \ln L$)

α and β represent the degree to which Y increases when K and L are increased, respectively

α and β are also known as output elasticities

Returns to Scale

$$\alpha+\beta \begin{cases} >1 & \text{Increasing Returns to Scale} \\ =1 & \text{Constant Returns to scale} \\ <1 & \text{Decreasing returns to scale} \end{cases}$$

Cobb-Douglas production functions have several useful properties:

One of these properties is the **sum of the exponents on K and L**

If α and β sum to 1 then the production function displays **constant returns to scale**

Meaning: the factor by which inputs change is the same by which output will change

(before $K=8$, $L=27$) $\xrightarrow{\text{Double}}$ ($K=16$, $L=54$)

That is, if we double K and L (e.g., to 16 and 54), output will double to 36:

$$Y = 1 * 16^{1/3} 54^{2/3} = 36 \quad \text{(before } Y=18\text{)}$$

Y exhibits CRS: $\alpha+\beta = \frac{1}{3}+\frac{2}{3} = \frac{3}{3} = 1$

Think of a company copying its factory to another location and being able to produce twice as much

Double the inputs → Double the GDP } Constant Returns to Scale

Returns to Scale

Increasing returns to scale:

$\alpha + \beta = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} > 1$ IRS

Suppose the production function is: $Y = \bar{A}K^{\frac{2}{3}}L^{\frac{2}{3}}$

In this case, the exponents on *K* and *L* sum to more than 1 and the percentage by which output changes is greater than the percentage by which K and L change

Suppose $K = 8, L = 8,$ and $\bar{A} = 1$: $Y = 1 * 8^{\frac{2}{3}}8^{\frac{2}{3}} = 16$

If we double both inputs to 16: $\underline{Y = 1 * 16^{\frac{2}{3}}16^{\frac{2}{3}} = 40.32} > \underbrace{2.16 = 32}$

GDP more than doubled

(due to IRS)

When inputs double in this case, output more than doubles

Note that the question is not whether output will increase, as output will always increase if we increase the inputs, **but by how much**

Returns to Scale

Decreasing returns to scale:

$\alpha + \beta = \frac{1}{3} + \frac{1}{3} = \frac{2}{3} < 1$ DRS

Suppose the production function is $Y = \bar{A}K^{\frac{1}{3}}L^{\frac{1}{3}}$

In this case, the exponents sum to **less than 1** and the percentage by which output changes is less than the percentage by which K and L change

Suppose $K = 8, L = 8,$ and $\bar{A} = 1$: $Y = 1 * 8^{\frac{1}{3}}8^{\frac{1}{3}} = 4$

If we double both inputs to 16: $Y = 1 * 16^{\frac{1}{3}}16^{\frac{1}{3}} = 6.35 < 8 \Rightarrow$ Decreasing Returns to scale

When inputs double, output less than doubles

*Note that we use these definitions of returns to scale for any proportional change in the inputs, not just a doubling of them

↳ Scales: 1.1, 1.5, 2, 3, etc.

Returns to Scale

Determine whether the following production functions exhibit constant, increasing, or decreasing returns to scale:

$$Y = \bar{A}K^{1/8}L^{3/4} \rightarrow \alpha + \beta = \frac{1}{8} + \frac{3}{4} = \frac{1}{8} + \frac{6}{8} = \frac{7}{8} < 1$$

DRS

$$Y = \bar{A}K^{5/8}L^{1/2} \rightarrow \alpha + \beta = \frac{5}{8} + \frac{1}{2} = \frac{5}{8} + \frac{4}{8} = \frac{9}{8} > 1$$

IRS

Allocating Resources

$F(\cdot) \to$ technology

How does a firm decide how many units of labor to hire and how many units of capital to use?

We assume firms want to maximize profits and therefore we need to set up its **profit function:**

$$\max_{\boxed{K,L}} \underbrace{\Pi}_{\text{Profit}} = \underbrace{F(K,L)}_{\text{Revenue}} - \underbrace{wL - rK}_{\text{Cost of Production}}$$

↳ wage ↳ rental price of capital

↳ Firm wants to pick optimal K, L

We will assume that the firm operates in a perfectly competitive market, and therefore *w*, *r*, and P (the price the product can be sold for) are constant

We can take the derivative of the profit function with respect to K and with respect to L and set it equal to 0 in order to find the optimal values of K and L

$$\frac{\partial \Pi}{\partial k} = 0 \qquad \frac{\partial \Pi}{\partial L} = 0$$

} FONC "First Order Necessary Conditions"

↳ $\frac{\partial Y}{\partial k} - r = 0 \to MPK = r$; $MPL = w$

Allocating Resources

We find that the firm will increase K and L until their marginal products are equal to their respective input prices

If we assume decreasing marginal product of labor and capital, then at some point it will no longer be profitable to hire more labor or more capital

Using the previous production function: $\underline{Y = \bar{A}K^{1/3}L^{2/3}}$

$$MPK = \frac{\partial Y}{\partial K} = \bar{A}\frac{1}{3}K^{1/3-1}L^{2/3} = \frac{1}{3}\bar{A}\left(\frac{L}{K}\right)^{2/3} = \frac{1}{3}\bar{A}\,\frac{L^{2/3}}{K^{2/3}}\cdot\frac{K^{1/3}}{K^{1/3}} = \frac{1}{3}\frac{Y}{K}$$

$$MPL = \frac{\partial Y}{\partial L} = \bar{A}\frac{2}{3}K^{1/3}L^{2/3-1} = \frac{2}{3}\bar{A}\left(\frac{K}{L}\right)^{1/3} = \frac{2}{3}\bar{A}\,\frac{K^{1/3}}{L^{1/3}}\cdot\frac{L^{2/3}}{L^{2/3}} = \frac{2}{3}\frac{Y}{L}$$

This diminishing marginal product can be seen both graphically and mathematically

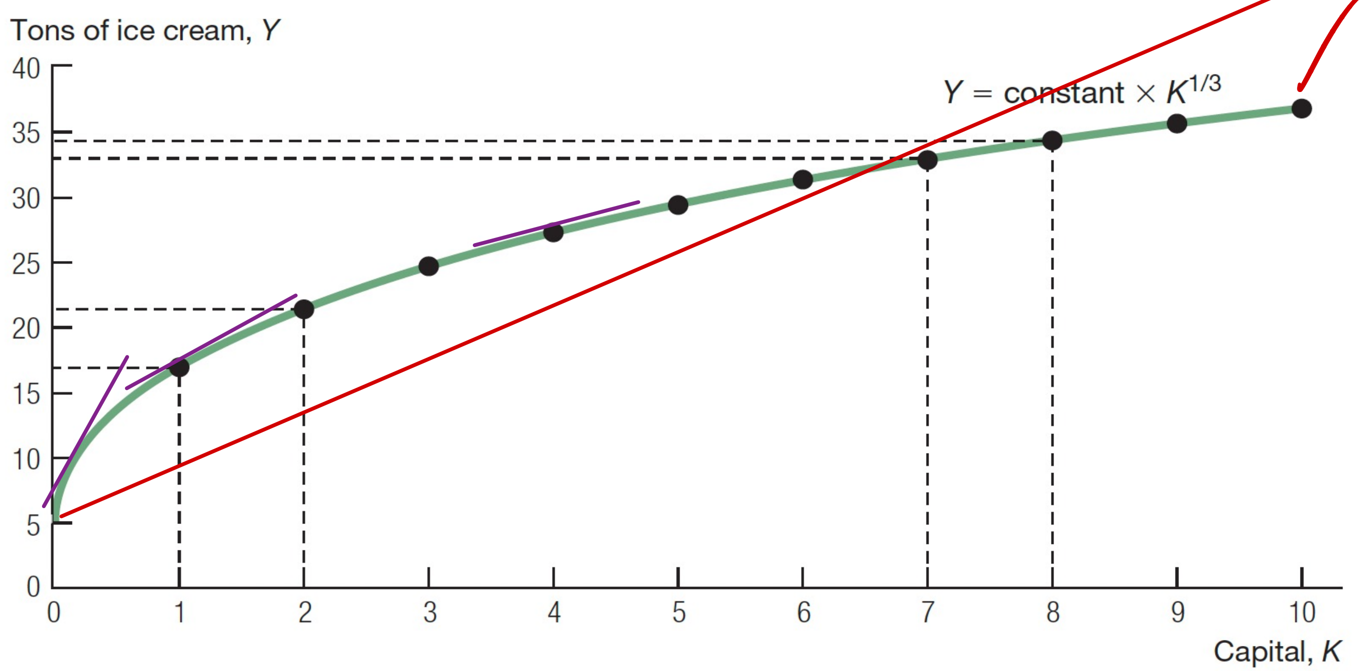
Diminishing marginal product

Diminishing marginal product can be confirmed by the second derivative being negative:

Here: this case

Graphically, production increases, but at a decreasing rate:

The Diminishing Marginal Product of Capital in Production

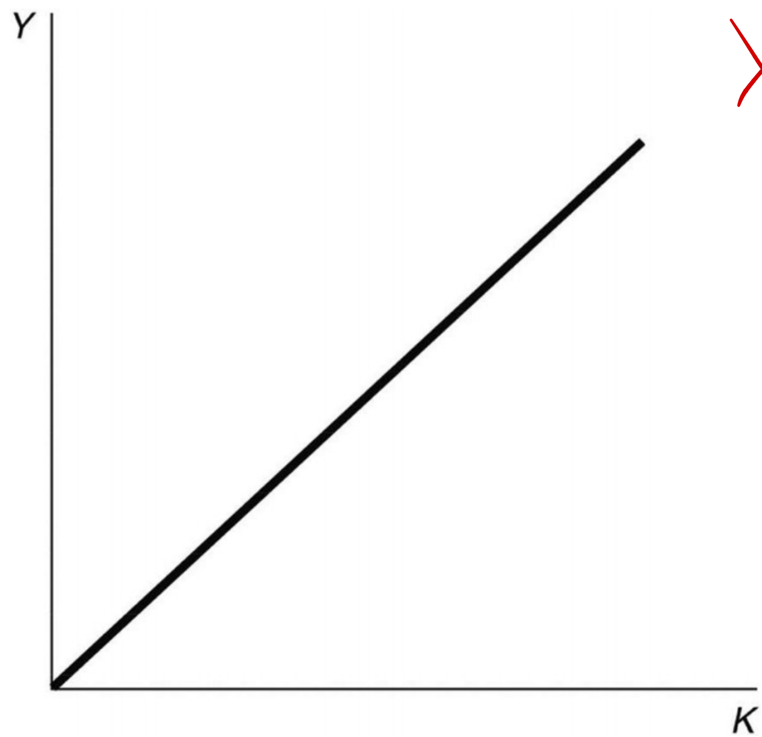


$MPK = \frac{1}{3}\frac{Y}{K}$

Diminishing marginal product

What assumption does this graph violate regarding the usual relationship in a production function that we expect between K and Y?

(Hint: How the production function shape differs from the one in the last slide?)



X this function does not display marginal decreasing Productivity of capital

Solving the model: General equilibrium

We have several variables in our model which can be solved for by using the following equations:

The production function: $Y = \bar{A}K^{\overbrace{1/3}^{\alpha}}L^{\overbrace{2/3}^{\beta}}$

The rule for hiring capital: $MPK = r$

The rule for hiring labor: $MPL = \cancel{W}$ w (lowercase)

Microfoundations (Come from Firm's Profit maximization)

Supply equals the demand for labor: $L = \bar{L}$

Supply equals the demand for capital: $K = \bar{K}$

The model parameters (exogenous variables) are: $\bar{A}, \bar{K}, \bar{L}$

The unknowns (endogenous variables) are: w, r, L, K, Y

For simplification, we assume that the number of firms in the economy is 1

Solving the model: General equilibrium

Mathematically, we will have the endogenous variables as functions of the exogenous variables and the parameters of the model:

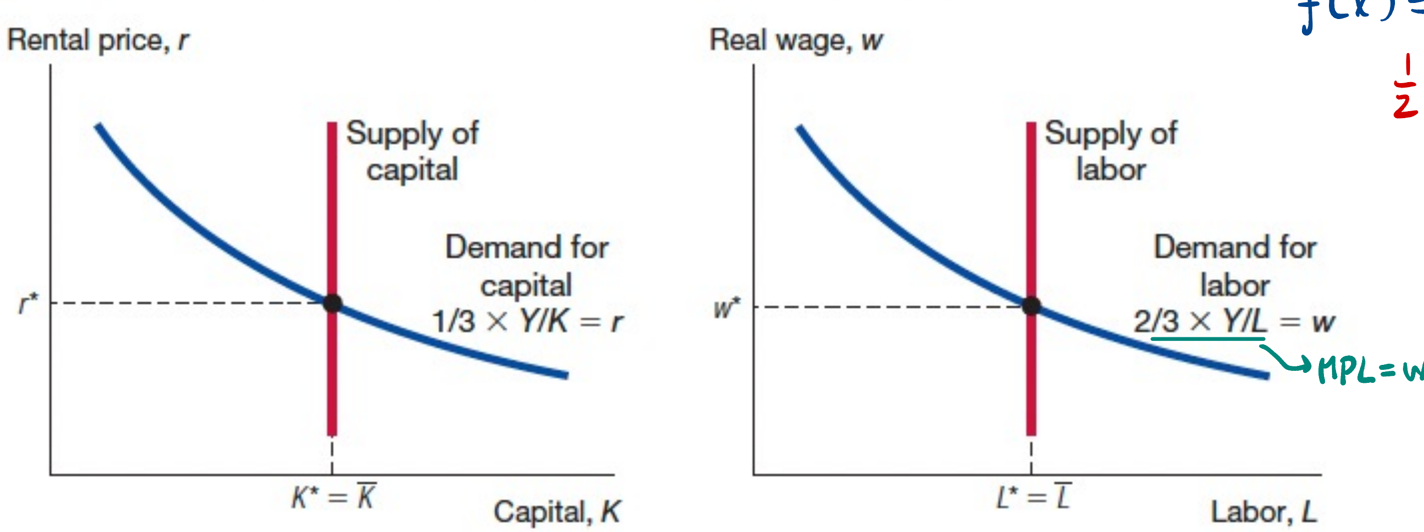
*: Solution

Capital	$K^* = \bar{K}$
Labor	$L^* = \bar{L}$
Rental rate	$r^* = \frac{1}{3} \cdot \frac{Y^*}{K^*} = \frac{1}{3} \cdot \bar{A} \cdot \left(\frac{\bar{L}}{\bar{K}}\right)^{2/3}$
Wage	$w^* = \frac{2}{3} \cdot \frac{Y^*}{L^*} = \frac{2}{3} \cdot \bar{A} \cdot \left(\frac{\bar{K}}{\bar{L}}\right)^{1/3}$
Output	$Y^* = \bar{A}\bar{K}^{1/3}\bar{L}^{2/3}$

$\frac{1}{3}\bar{A}\left(\frac{L}{K}\right)^{2/3} = \frac{1}{3}\frac{Y}{K}$

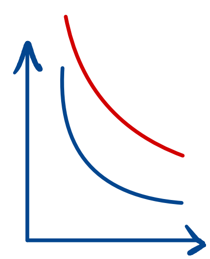
$\frac{2}{3}\bar{A}\left(\frac{K}{L}\right)^{1/3} = \frac{2}{3}\frac{Y}{L}$

Supply and Demand in the Capital and Labor Markets



(a) The capital market (b) The labor market

$f(x) = \frac{\cancel{1}^{\,2}}{x}$ $\frac{1}{2}$



Implications of the model

Cobb Douglas (Production function)

The total amount of output in the economy is determined by the total amount of capital and the total amount of labor available in the economy

Output per worker can also be calculated: Equal to the income per worker

- The equilibrium wage is proportional to the output per worker
- The equilibrium return on capital is proportional to the output per unit of capital

Moreover, we can determine the portion of production that is paid to labor and the portion of production that is paid to capital

Their sum is equal to the total production of the economy

Technology: $Y^* = \bar{A}\bar{K}^{1/3}\bar{L}^{2/3}$

Labor Income:

$w^* L^* = \frac{2}{3}\frac{Y^*}{\cancel{L^*}} \cdot \cancel{L^*} = \frac{2}{3} Y^*$

Capital Income

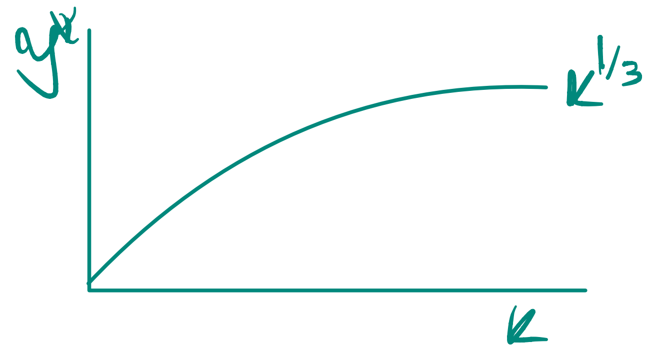
$r^* K^* = \frac{1}{3} Y^*$

Analyzing the production model

$F(K,L) = F\left(\frac{K}{L}, \overset{1}{\frac{L}{L}}\right)$

$= F\left(\frac{K}{L}\right)$

We can use the model to solve for output per person and get an indication of how the level of welfare in an economy is determined



→ Lowercase variables ≈ Per worker

y is equal to output per person $y = Y/L$

k is equal to capital per person $k = K/L$

Output per person in equilibrium is therefore:

$$\bar{A}k^{1/3} = y^* = \frac{Y^*}{L^*} = \frac{\bar{A}\bar{K}^{1/3}\bar{L}^{2/3}}{\bar{L}} = \bar{A}\left(\frac{\bar{K}}{\bar{L}}\right)^{1/3} = \bar{A}\bar{k}^{1/3} = y^*$$

We see that output per person depends on two terms: $\bar{A}$ and k

*Note that there are diminishing returns to capital

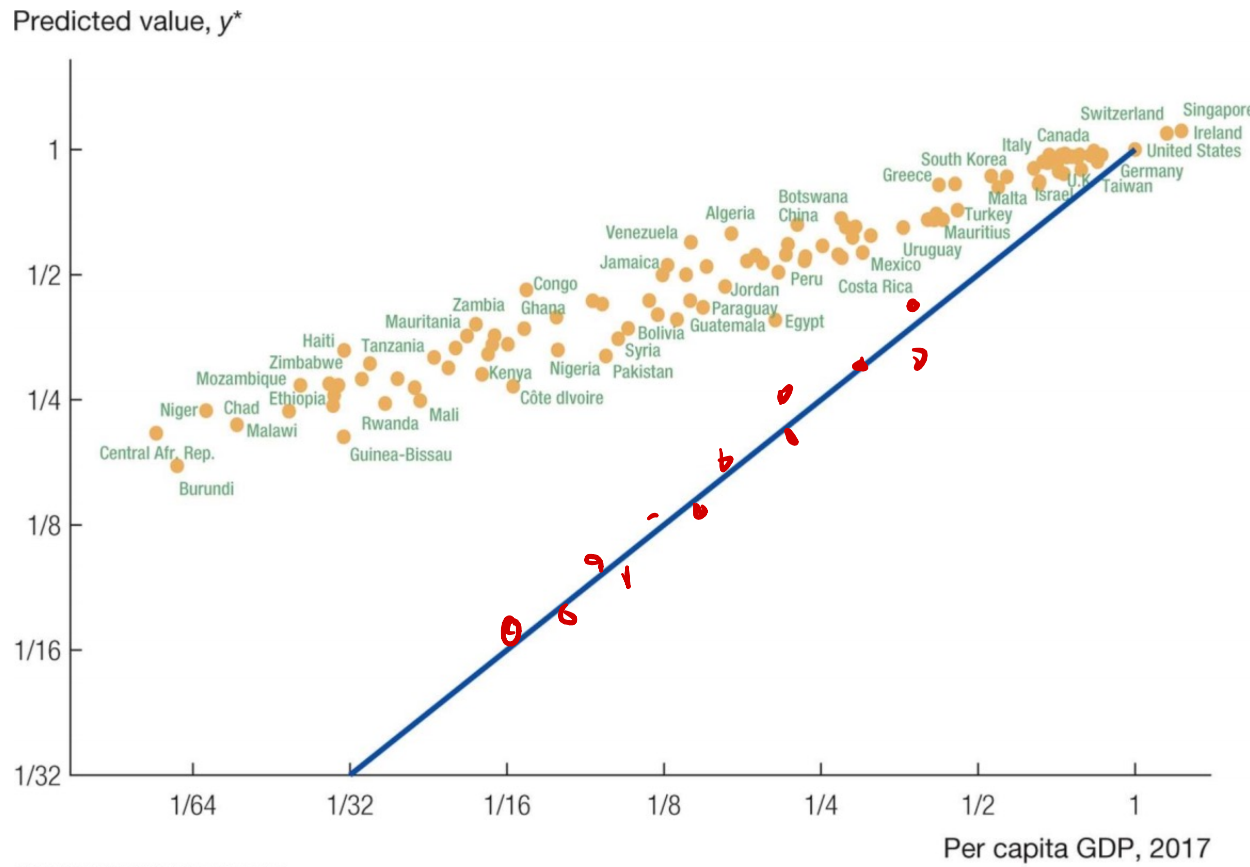
Analyzing the production model

Does the model fit with the data?

We can compare an economy's capital stock to its level of GDP per capita using the result we obtained earlier: $y^* = \bar{A}\bar{k}^{1/3}$.

If $\boxed{\bar{A} = 1}$ then $y^* = \bar{k}^{1/3}$

the graph on the right shows the predicted per capita GDP and actual per capita GDP for a large sample of countries



(if the prediction and the actual value would be the same, the dots would fall on the 45 degree line)

Analyzing the production model

$\underbrace{\bar{A}}_{\text{TFP}} \underbrace{K^{1/3} L^{2/3}}_{\text{Factors}} = Y$

The model does not fit the data very well. In particular, most countries are not as rich as the model predicts

GDP

This leaves the parameter $\bar{A}$ as a possible reason for differences in outcomes between countries $\bar{A}$ is referred to as **Total Factor Productivity (TFP)** as it represents how efficient countries are in using their K and L

Because there is no way to directly measure TFP, we will assume our model is correct and then see what level of TFP makes our equation hold

Solving for $\bar{A}$ we obtain: $\bar{A} = \frac{y^*}{\bar{k}^{1/3}}$ } Using $y = \bar{A}k^{1/3}$

For instance, if we plug in the values for Italy (and set the US A = 1) we obtain:

Actual values: $k = 0.804, y = 0.680$
Predicted: $y^* = 0.804^{1/3} = 0.930$

Implied TFP measures in different countries

$0.680 = \bar{A}^{Italy} \times 0.930 \Rightarrow \bar{A}^{Italy} = 0.73$

We can perform a similar calculation for other countries and obtain a measure of their implied TFP $\bar{A}$

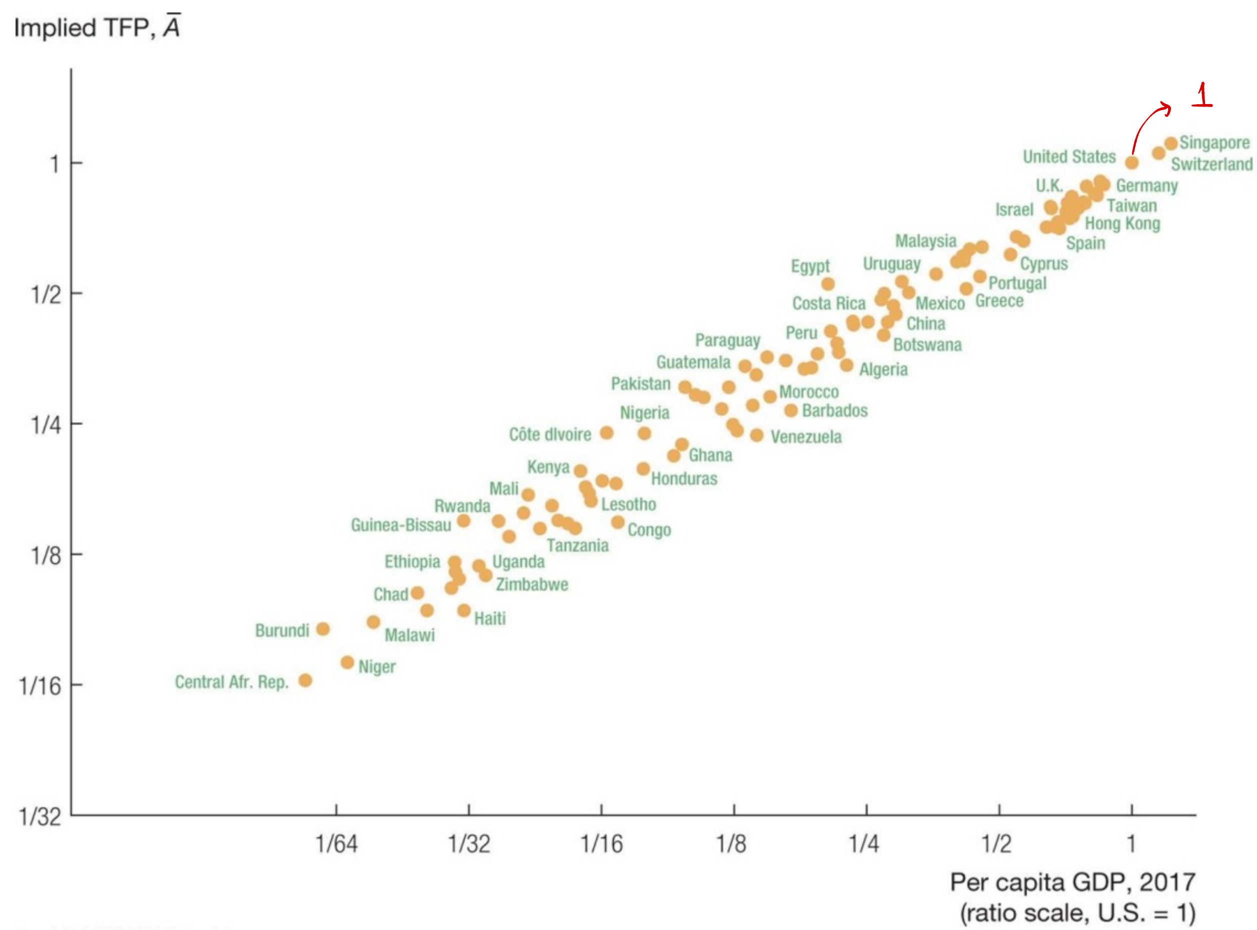
assuming our model is correct

Country	Per capita GDP	$\bar{k}^{1/3}$	implied TFP
United States	1.000	1.000	1.000
Switzerland	1.151	1.094	1.052
UK	0.714	0.885	0.807
Japan	0.734	0.976	0.752
Italy	0.680	0.930	0.731
Spain	0.640	0.901	0.710
China	0.279	0.651	0.429
Brazil	0.252	0.587	0.429
South Africa	0.214	0.559	0.383
India	0.117	0.433	0.270
Burundi	0.015	0.173	0.084

Plot of TFP in large sample of countries

There is a high correlation between per capita GDP and TFP

Rich countries have generally high levels of TFP and poorer countries tend to have lower levels of TFP



What explains differences in incomes?

It is important to determine what causes income differences between countries

Is it capital per person (k) or TFP (A)?

$$\frac{\left(\frac{1}{5}\sum_{\text{5 Richest}} y\right)}{\left(\frac{1}{5}\sum_{\text{5 Poorest}} y\right)} \approx 64 \approx \underbrace{5}_{\text{Due to Capital Per Worker differences}} \times \overbrace{13}^{\text{Due to technology differences}}$$

We can gain insight into this by examining the five richest countries and five poorest countries:

Per capita GDP of the five richest countries is 64 times that of the five poorest.

- Differences in capital per person explain a factor of 5 of this difference
- Differences in TFP explain a factor of 13 of this difference

Therefore, while rich countries possess more capital, **TFP plays an even more important role** in explaining these differences

What explains TFP differences between countries?

TFP is considered a residual that captures those factors that affect output that are ‘omitted’ from the production function

Three factors commonly deemed to explain these differences in TFP:

1. Human Capital: Stock of skills that individuals accumulate that makes them more productive

Education is a major source of human capital: college degrees, literacy, computer knowledge, etc.

One way to incorporate human capital is therefore through years of education

For instance, suppose the return to a year of schooling is 10%

A difference of five years in average education between two countries would then relate to a 50% difference if we only consider education differences

While this explains some of the TFP differences, it does not explain the 13-fold difference we saw earlier

What explains TFP differences between countries?

2. Technology: Some countries use more advanced technology in production than other countries

Think about the following examples in making higher-income countries more productive:

- Access to mobile phones to quickly communicate with others
- Access via smartphones to functions that a computer can do – bank transactions, GPS, etc.
- Robots to perform the tasks that humans do – assembly lines, bomb disposal, etc.
- Artificial intelligence
- Smartwatches to monitor biometric data
- Self-checkout stands

What explains TFP differences between countries?

3. Institutions – Institutions are the rules, norms, and beliefs that shape behavior in a country. Property rights, legal system, social norms, etc.

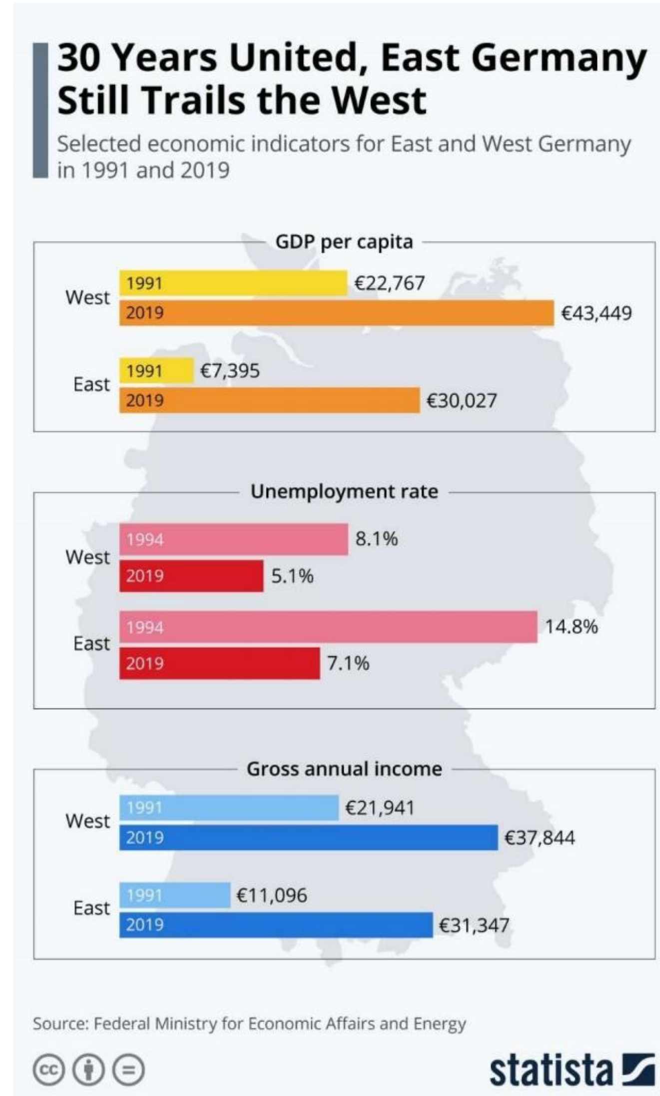
Form of government, economic system, and legal systems are important institutions

One way to study the importance of institutions is to exploit natural experiments, where two countries very similar countries differ in how they establish institutions

The splitting of Korea and the splitting of Germany in the late 1940s provides two of these examples: Previously unified countries were now split, with each adopting very different institutions

Differences in outcomes between these two countries can be attributed (largely) to differences in institutions

Differences in institutions



Conclusion

- GDP per capita varies substantially between countries
- Using the **Cobb-Douglas production function**, we can get a sense of the factors behind this difference
- Equilibrium in the production model we solved indicates that output per person depends on **Total Factor Productivity (*A*) and capital per person (*k*)**
- Output per capita differs across countries by much more than differences in capital per person predicts, indicating that TFP varies significantly between countries
- Some reasons for differences in TFP are: **human capital differences, technological differences, and institutional differences**