ECON 6356
International Finance and Macroeconomics

Lecture 2 (part 2): A Small Open Economy Model with Capital

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slides

chapter 3

an open economy with capital

These slides are an adjusted version of the materials for Chapter 3 of the OEM book provided by the authors

Motivation

here: Introduction of capital ⇒ we introduce investment

In this chaper we introduce production and physical capital accumulation.

Doing so will allow us to address two important issues. One is that for the most commonly used stationary specifications of the shock process—namely, AR(1) specifications—the endowment economy model presented in Chapter 2 fails to predict the observed countercyclicality of the trade balance and the current account (documented in Chapter 1).

Second, the assumption that output is an exogenously given stochastic process—maintained until now—is unsatisfactory if the goal is to understand observed business cycles. to explain.

$\delta = 0 \Rightarrow K_{t+1} = (1-\cancel{\delta})K_t + I_t$

To allow for a full characterization of the equilibrium dynamics using pen and paper we abstract from depreciation and uncertainty, and assume, as in Chapter 2, that $\beta(1+r) = 1$. In later chapters we will relax these assumptions.

W/ K the tb_t can become countercyclical (w/ AR(1) income)
↳ given productivity shocks the MPK is expected to be high later
⇒ agents will increase Investment ⇒ ↑ Demand = (C+I) ↑

Intuition

Why allowing for production and capital accumulation might induce a countercyclical trade balance in the model -even for AR(1) shock processes?

⇒ ↓ Savings ⇒ ↓ ca, ↓ tb

Suppose persistent AR(1) productivity shocks are the main source of uncertainty. Then the marginal product of capital is expected to be high not just in the period of the shock but also in the next couple of periods. Thus the economy has an incentive to invest more to take advantage of the higher productivity of capital.

This increase in domestic demand might be so large that total domestic demand, consumption plus investment, rises by more than output, resulting in a countercyclical impact response of the trade balance.

We will derive the following 2 principles:

Principle I: The more persistent productivity shocks are, the more likely an initial deterioration of the trade balance will be.

Principle II: The more pronounced are capital adjustment costs, the smaller will be the initial trade balance deterioration in response to a productivity shock.

3.1 Model

Small open economy, no uncertainty, no depreciation.

Preferences:

Max $\{c_t, d_t, i_t, k_{t+1}\}$

$$\sum_{t=0}^{\infty} \beta^t U(c_t) \tag{3.1}$$

Sequential budget constraint of the household:

$$c_t + i_t + (1+r)d_{t-1} = y_t + d_t \tag{3.2}$$

new part

Interpretation: LHS displays the uses of wealth: purchases of consumption goods (c_t); purchases of investment goods (i_t); payment of principal and interest on outstanding debt ($(1+r)d_{t-1}$). RHS displays the sources of wealth: output (y_t) and new debt issuance (d_t).

Production function:

$$y_t = A_t F(k_t) \tag{3.3}$$

$A_t =$ exogenous and deterministic productivity factor
$F(\cdot) =$ increasing and concave production function
$k_t > 0$ physical capital, determined in $t-1$

↳ time of usage (Not when it is determined)

Law of motion of capital:

$$k_{t+1} = k_t + i_t \tag{3.4}$$

No-Ponzi game constraint:

$$\lim_{j\to\infty} \frac{d_{t+j}}{(1+r)^j} \leq 0 \tag{3.5}$$

Lagrangian of household's problem:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ U(c_t) + \lambda_t \left[\overbrace{\underbrace{A_t F(k_t)}_{y_t} + d_t - c_t - (k_{t+1} - k_t) - (1+r)d_{t-1}}^{\text{replacing } i_t = k_{t+1} - k_t \text{ in BC}} \right] \right\}.$$

The first-order conditions corresponding to c_t, d_t, k_{t+1}, and λ_t, respectively, are

$[c_t]$:
$$U'(c_t) = \lambda_t, \tag{3.6}$$

$\frac{\partial \mathcal{L}}{\partial d_t} = 0$ $[d_t]$:
$$\lambda_t = \beta(1+r)\lambda_{t+1}, \tag{3.7}$$

$[k_{t+1}]$:
$$\lambda_t = \beta \lambda_{t+1}[A_{t+1}F'(k_{t+1}) + 1], \tag{3.8}$$

and

$[\lambda_t]$:
$$A_t F(k_t) + d_t = c_t + k_{t+1} - k_t + (1+r)d_{t-1}. \tag{3.9}$$

Household optimization implies that the borrowing constraint holds with equality (transversality condition):

TVC binds
$$\lim_{t\to\infty} \frac{d_t}{(1+r)^t} = 0. \tag{3.10}$$

Assume that

$$\beta(1+r) = 1 \qquad (*)$$

This assumption together with (3.6) and (3.7) implies that consumption is constant

w/ $\beta(1+r)=1$ and no uncertainty $\Rightarrow$ $$c_{t+1} = c_t \qquad (3.11)$$

As we will see shortly, consumption is again determined by non-financial permanent income net of interest on initial debt outstanding.

$\beta(1+r)=1$

Assumption (*) and equilibrium condition (3.8) imply that

$$r = A_{t+1}F'(k_{t+1}) \tag{3.12}$$

In eqm it's the same to save through debt (r) or via investment $(A_{t+1}F'(k_{t+1}))$

Households invest in physical capital in period t until the expected marginal product of capital in period $t+1$ equals the rate of return on foreign debt.

It follows from this equilibrium condition that next period's level of physical capital, k_{t+1}, is an increasing function of the future expected level of productivity, A_{t+1}, and a decreasing function of the interest rate r.

from (3.12): k_{t+1} is a decreasing f/n of $\frac{r}{A_{t+1}}$ (or increasing in A_{t+1}/r):

$$k_{t+1} = \kappa\left(\frac{A_{t+1}}{r}\right), \tag{3.14}$$

with $\kappa' > 0$.

Future K is a f/n of future productivity and cost of debt

⇒ Investment Grows w/ Productivity

(adds downward pressure to TB, CA allowing them to become countercyclical)

Set Intert. BC to charact. eq: Set Lifetime BC as before & use constant Consumption result to solve for C_t

To characterize the equilibrium, work (again) with the intertemporal budget constraint. Write the sequential budget constraint for period $t+j$:

BC_{t+j}
$$A_{t+j}F(k_{t+j}) + d_{t+j} = c_{t+j} + k_{t+j+1} - k_{t+j} + (1+r)d_{t+j-1}$$

Divide by $(1+r)^j$ and sum for $j=0$ to $j=J$.

$\frac{1}{1+r}$ & Sum over j:
$$\sum_{j=0}^{J} \frac{A_{t+j}F(k_{t+j})}{(1+r)^j} + \frac{d_{t+J}}{(1+r)^J} = \sum_{j=0}^{J} \frac{c_{t+j} + k_{t+j+1} - k_{t+j}}{(1+r)^j} + (1+r)d_{t-1}$$

Now use the fact that in eqm consumption is constant over time, (3.11), and rearrange terms

Subst $C_{t+j} = C_t$
$$c_t \sum_{j=0}^{J} \frac{1}{(1+r)^j} + (1+r)d_{t-1} = \sum_{j=0}^{J} \frac{A_{t+j}F(k_{t+j}) - (k_{t+j+1} - k_{t+j})}{(1+r)^j} + \underbrace{\cancel{\frac{d_{t+J}}{(1+r)^J}}}_{=0 \text{ as } J\to\infty}$$

Take limit for $J \to \infty$ and use the transversality condition (3.10) to obtain (3.13) again $c_t = y^p_t - rd_{t-1}$

$$c_t + rd_{t-1} = y_t^p \equiv \frac{r}{1+r} \sum_{j=0}^{\infty} \frac{A_{t+j}F(k_{t+j}) - \overbrace{(k_{t+j+1} - k_{t+j})}^{\text{New: } i_t}}{(1+r)^j} \qquad (3.13)$$

Interpretation: The right-hand side of (3.13) is the household's nonfinancial permanent income, y_t^p. (It is a natural generalization of a similar expression obtained in the endowment economy, see equation 2.10).

Here, nonfinancial permanent income is given by a weighted average of present and future expected output net of investment expenditure. Thus, equilibrium condition (3.13) states that each period households allocate their nonfinancial permanent income to consumption and to servicing their debt.

(3.13) $c_t = y_t^p - rd_{t-1}$

(3.12) $r = A_{t+1}F'(k_{t+1})$

A **perfect-foresight equilibrium** is a value c_0 and a sequence $\{k_{t+1}\}_{t=0}^{\infty}$ satisfying (3.13) evaluated at $t = 0$, and (3.12) for all $t \geq 0$, given the initial stock of physical capital, k_0, the initial net external debt position, d_{-1}, and the sequence of productivity $\{A_t\}_{t=0}^{\infty}$.

This is a system we can fully characterize with pen and paper.

[Obtain eq values for c_t from (3.11), i_t from (3.4), y_t from (3.3), and d_t from (3.2)]

(3.11) $c_{t+1} = c_t$ (3.4) $k_{t+1} = k_t + i_t$ (3.3) $y_t = A_t F(k_t)$

(3.2) $c_t + i_t + (1+r)d_{t-1} = y_t + d_t$

Note that k_t for $t > 0$ is a function of the exogenous variable A_t only. Thus permanent income, y_t^p, is a function of productivity only and is increasing in present and future values of productivity.

Now higher i_t s.t.
$\Delta(c_t + i_t) > \Delta y_t$
does the job of generating a countercyclical tb, ca.

$\uparrow \; -(\uparrow) \; -(\uparrow)$

Trade balance: $tb_t = y_t \overbrace{- c_t - i_t}^{\text{total Demand (or absorption)}}$

New to this definition in the model w/ capital

To obtain the prediction of a countercyclical trade balance response it is no longer required that consumption increases by more than one-for-one with output.

As long as domestic absorption, $c_t + i_t$, increases by more than output, the model will predict a countercyclical trade balance response.

Next we study adjustment to permanent and temporary productivity shocks and ask whether the model predicts a countercyclical trade balance response.

3.2. Steady State Equilibrium

Suppose $A_t = \bar{A}$ for all $t \geq 0$, and $k_0 = \bar{k} \equiv \kappa\left(\frac{\bar{A}}{r}\right)$.

By (3.14), $k_t = \bar{k}$ for all $t > 0$

By (3.11) and (3.13), $c_t = \bar{c} \equiv -rd_{-1} + \bar{A}F(\bar{k})$

and $d_t = d_{-1}$ for all $t \geq 0$

Output: $y_t = \bar{y} \equiv \bar{A}F(\bar{k})$

Trade balance: $tb_t = \bar{tb} \equiv rd_{-1}$

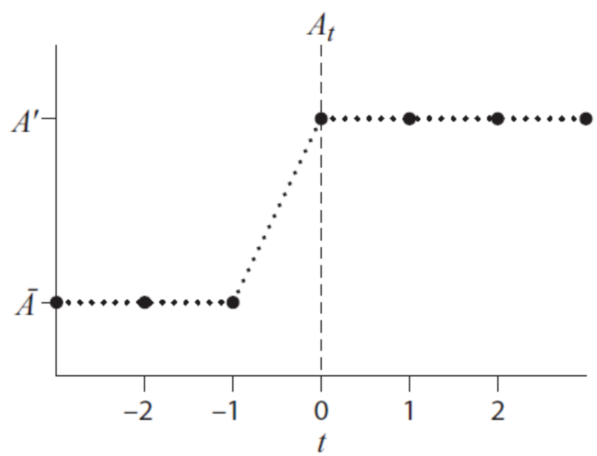
$\bar{tb} = \underbrace{\bar{y}}_{\bar{A}F(\bar{k})} - \underbrace{\bar{c}}_{\bar{A}F(\bar{k}) - rd_{-1}} - \underbrace{\bar{i}}_{=0}$

Current account: $ca_t = d_{t-1} - d_t = 0$

3.3 Adjustment to a Permanent Unanticipated Increase in Productivity

Experiment: In period 0 it is learned that A_t increases from $\bar{A}$ to $A' > \bar{A}$ for all $t \geq 0$. Prior to period 0, A_t was expected to be $\bar{A}$ forever.

$$A_t = \begin{cases} \bar{A} & \text{for } t \leq -1 \\ A' > \bar{A} & \text{for } t \geq 0 \end{cases} .$$



Adjustment of Capital and Investment

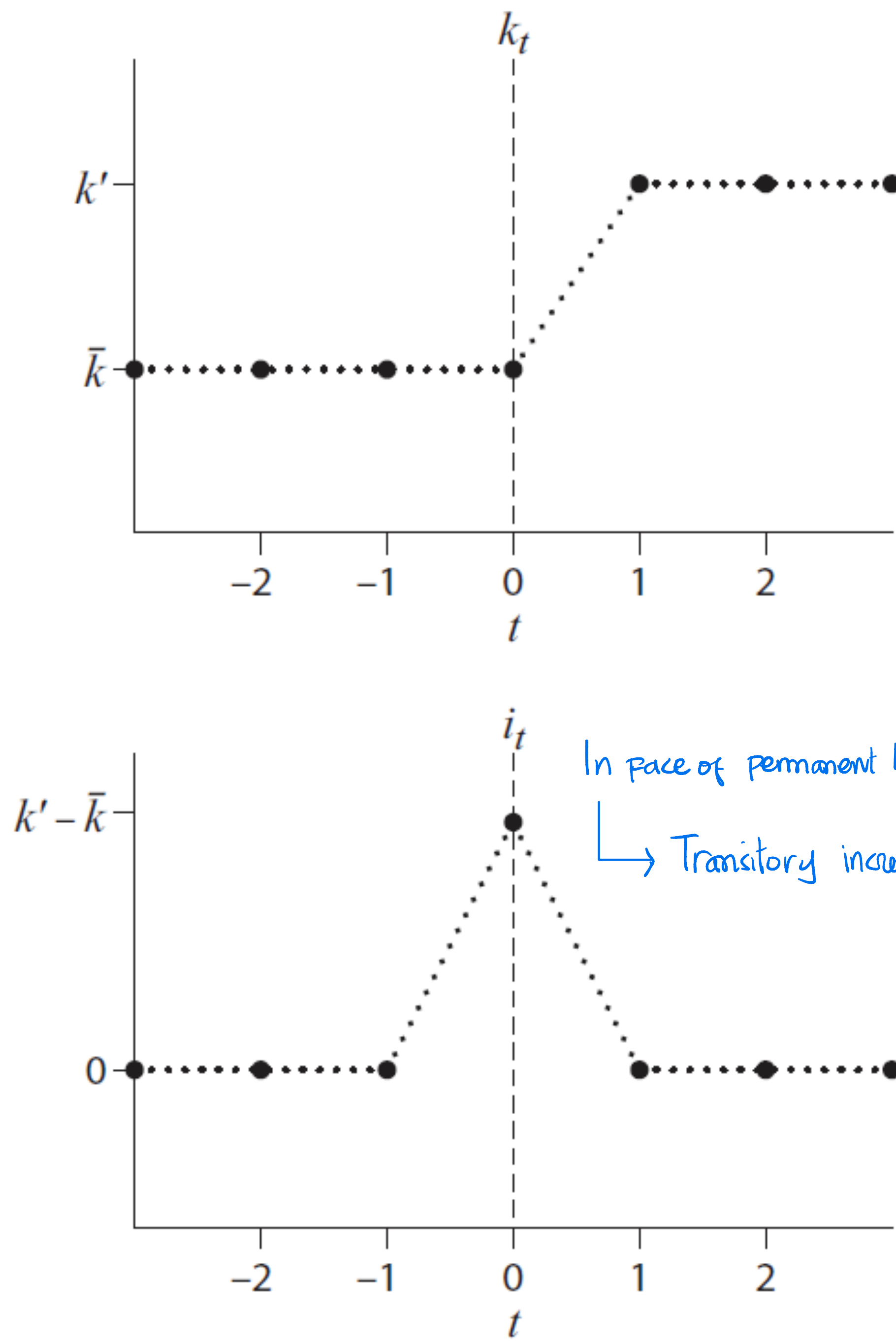
For $t > 0$, by (3.14)

$$k_t = k' \equiv \kappa\left(\frac{A'}{r}\right) > \bar{k}$$

Thus positive investment in period 0 and zero investment thereafter.

$$t = 0 : i_0 = k' - \bar{k} > 0$$

$$t > 0 : i_t = 0$$



In face of permanent Increase in Productivity
↳ Transitory increase in Investment

$AF(K) \xrightarrow{\text{at shock period}} A \to A' \Rightarrow A'F(K)$

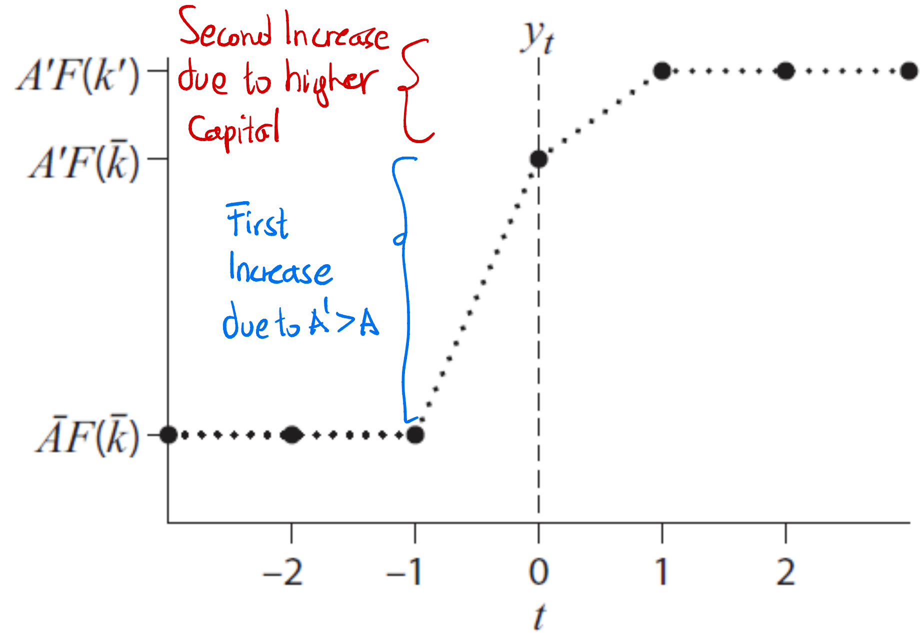
$\xrightarrow{\text{The period after}} K \to K' \Rightarrow A'F(K')$

Adjustment of Output

Output increases in period 0 because A_0 rises
and then again in period 1 because k_1 is larger:

$$t = 0 : y_0 = A'F(\bar{k}) > \bar{A}F(\bar{k}) = \bar{y}$$

$$t > 0 : y_t = A'F(k') > A'F(\bar{k}) = y_0$$



New: Second Increase in output

Due to capital accumulation which is traced back to Investment

What about consumption? Intuitively, it should increase. How to show that it does? By (3.11)

$$c_t = c_0$$

for all $t \geq 0$. Thus, we only need to find c_0.

By (3.13), $c_0 = y_0^p - rd_{-1}$. If permanent income in period 0 rises, so does consumption. Thus, let's find first the adjustment in y_0^p.

New Cons

Consumption increases: $c' = -r\bar{d} + A'F(\bar{k}) + \frac{1}{1+r}\left[A'[F(k') - F(\bar{k})] - A'F'(k')[k' - \bar{k}] \right]$

$> -r\bar{d} + A'F(\bar{k}) > -r\bar{d} + \bar{A}F(\bar{k}) = \bar{c}$

↳ Previous Consumption

From period 1 on, $i_t = 0$, thus using the definition of y_0^p we have:

$$\begin{aligned}
y_0^p &= \frac{r}{1+r}\left[(A'F(\bar{k}) - k' + \bar{k}\right] + \frac{1}{1+r}A'F(k') \\
&= A'F(\bar{k}) + \frac{1}{1+r}\left[A'F(k') - A'F(\bar{k}) - r(k' - \bar{k})\right] \\
&= A'F(\bar{k}) + \frac{1}{1+r}\left[A'F(k') - A'F(\bar{k}) - A'F'(k')(k' - \bar{k})\right] \\
&> A'F(\bar{k})(= y_0) \\
&> \bar{A}F(\bar{k})(= y_{-1}^p).
\end{aligned}$$

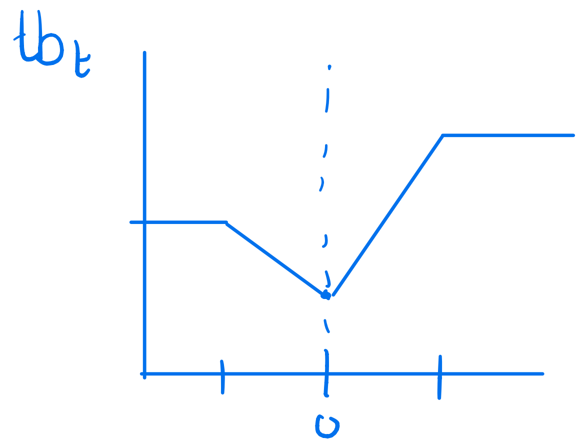
(The 1st inequality follows from $F(\cdot)$ being increasing and concave and $k' > \bar{k}$)

Because in period 0 permanent income exceeds current income, we have that c_0 increases by more than y_0. This by itself—that is, ignoring the increase in i_0—leads to a negative trade balance response in period 0.

By contrast, in the endowment economy of Chapter 2 a once-and-for-all increase in the endowment leaves the trade balance unchanged. The intuition for this result is that the path of output is upward sloping in the economy with capital in response to the permanent shock.

Adjustment of the Trade Balance

We just established that the trade balance deteriorates in period 0. But what about period 1. In period 1, output is higher than in period 0, consumption is the same, and investment is lower. Thus clearly $tb_1 > tb_0$.



For $t > 0$, $tb_t = tb' > tb_0$.

Is tb' greater or less than tb_{-1}? By (3.2) for $t > 0$

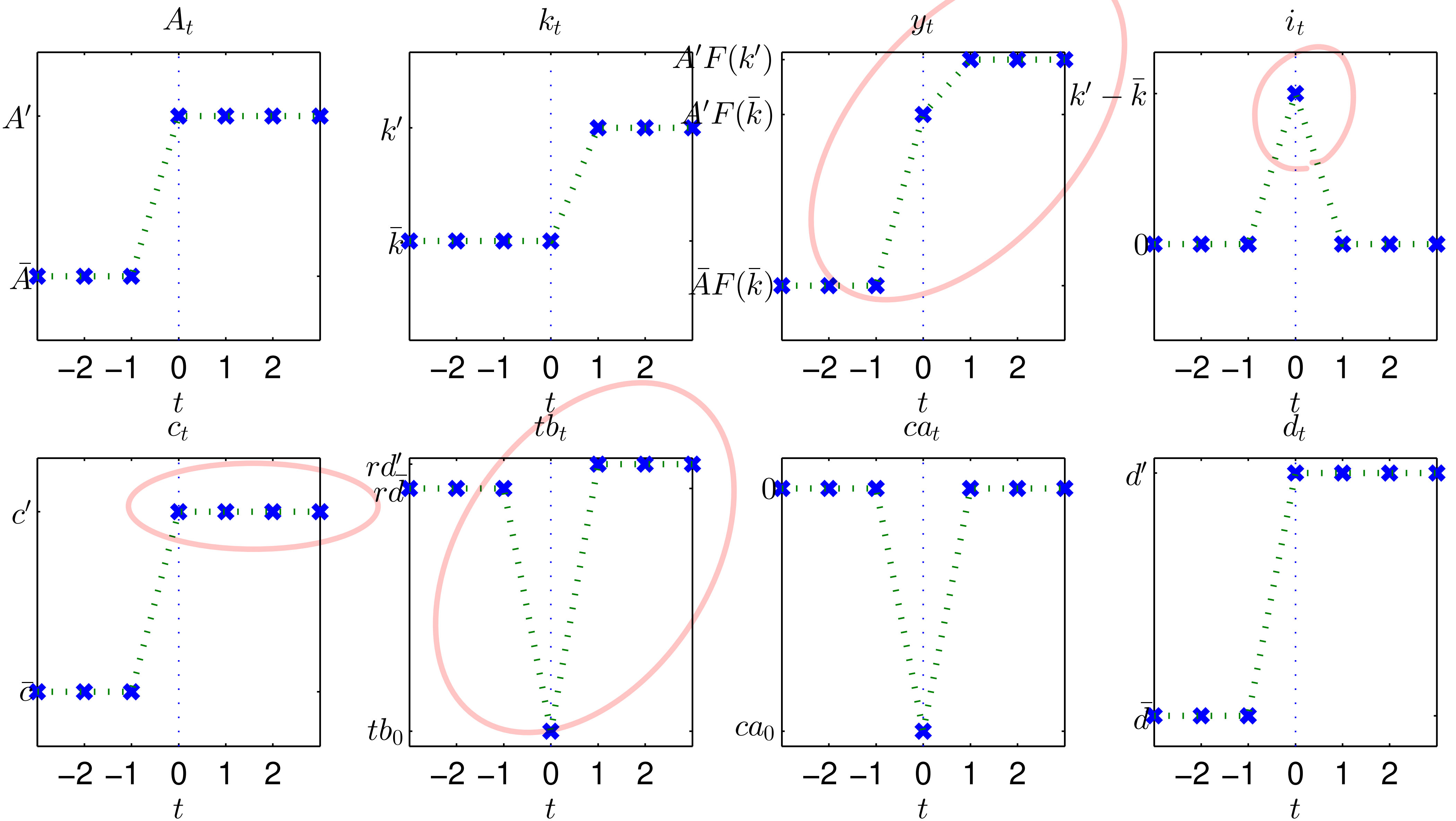
$$d_t = (1 + r)d_{t-1} - tb'$$

This will satisfy (3.10) only if

$$tb' = rd_0$$

where $d_0 = d_{-1} + y_0^p - y_0 > d_{-1}$. The new level of debt is permanently higher than it was prior to the productivity shock and therefore the trade balance, which is used to service the interest on the debt, must also be permanently higher.

Summary of Adjustment to Permanent Productivity Shock



3.4 Adjustment to Temporary Productivity Shocks

Experiment: In period 0 it is learned that $A_0 = A' > A_{-1} = \bar{A}$ and that $A_t = \bar{A}$ for all $t > 0$.

$$A_t = \begin{cases} \bar{A} & \text{for } t \leq -1 \\ A' > \bar{A} & \text{for } t = 0 \\ \bar{A} & \text{for } t > 0 \end{cases}$$

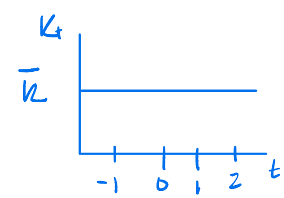
Key: here K_0 is predetermined based on productivity before change $A_{-1}=\bar{A}$

$\Rightarrow$ K does not change

Since increase in A is temporary agents expect A to revert; then when deciding K_1 they still set it as $\bar{K}$

By (3.12)

$$k_t = \bar{k}; \text{ for all } t > 0$$



given constant K the dynamics of i_t are unaffected.

By (3.4)

$$i_t = 0; \text{ for all } t \geq 0$$

By (3.3)

$$y_0 = A'F(\bar{k}) > \bar{y}; \quad \text{and } y_t = \bar{y} = \bar{A}F(\bar{k}); \text{ for all } t > 0$$

Note that the adjustment to a purely temporary shock in the economy with capital is thus the same as the adjustment to a purely temporary endowment shock in the economy without capital studied in Chapter 2.

By (3.11)

$$c_t = c_0 \quad \text{for all } t \geq 0$$

By (3.13)

$$c_0 = -rd_{-1} + \bar{A}F(\bar{k}) + \frac{r}{1+r}\left(A'F(\bar{k}) - \bar{A}F(\bar{k})\right)$$

Recalling that $c_{-1} = -r\bar{d} + \bar{A}F(\bar{k})$ and that $d_{-1} = \bar{d}$ yields

$$c_0 - c_1 = \frac{r}{1+r}\left(A'F(\bar{k}) - \bar{A}F(\bar{k})\right) > 0$$

Thus consumption increases by only a small fraction of the increase in income.

From the definition of the trade balance we have

$$tb_0 - tb_{-1} = (y_0 - y_{-1}) - (c_0 - c_{-1}) - (i_0 - i_{-1}) = \frac{1}{1+r}(y_0 - y_{-1}) > 0$$

$\Rightarrow$ procyclical trade balance adjustment in period 0.

For $t > 0$: c_t, y_t, i_t are all constant. Hence tb_t is also constant. At what level?

By same argument as above,

$$tb_t = tb' = rd_0; \text{ and } d_t = d_0; \quad \forall t > 0$$

Because c_0 increases by less than y_0 and i_0 is unchanged (at zero), it must be that $d_0 < d_{-1} = \bar{d}$. It follows that

$$tb' < tb_{-1} < tb_0$$

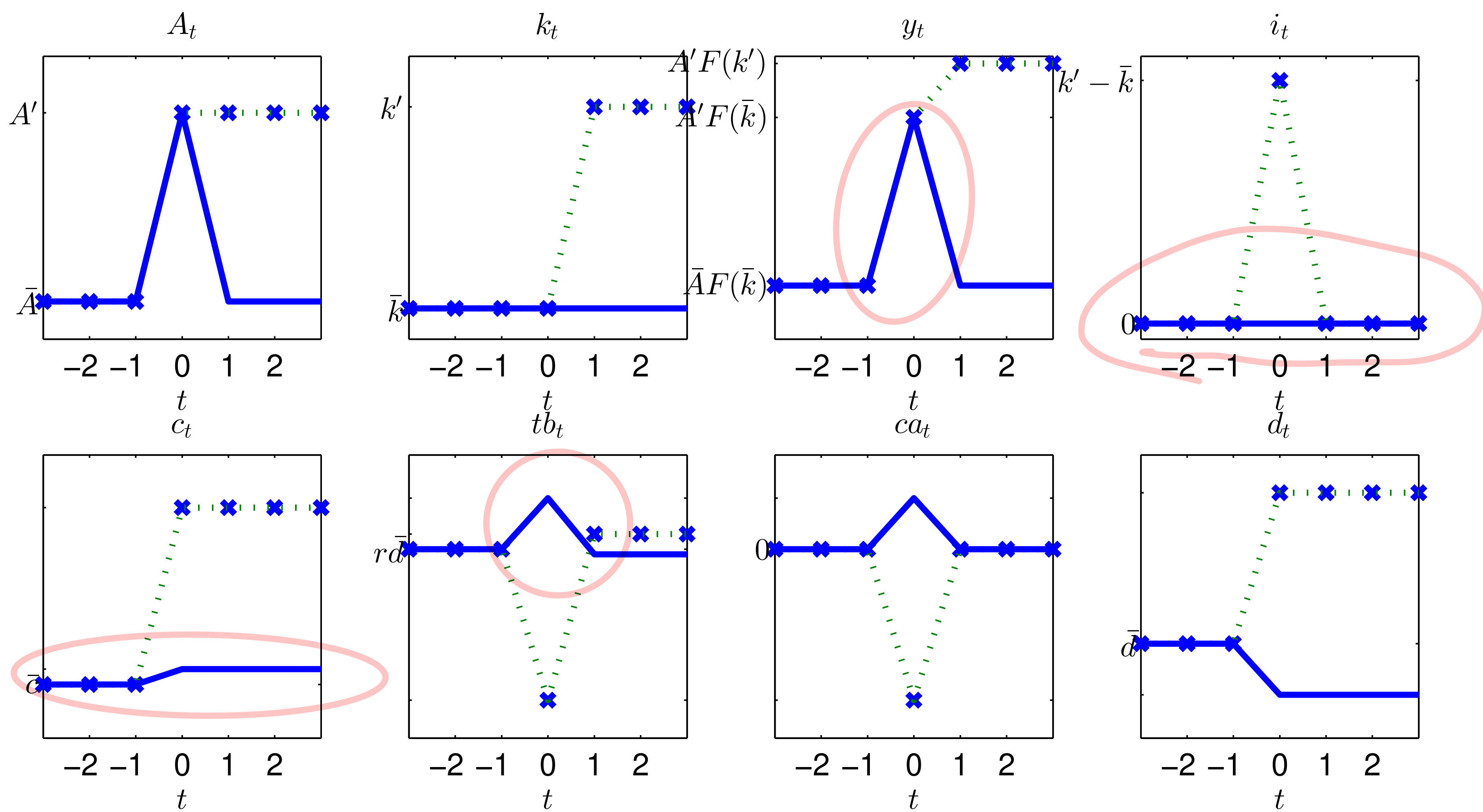
Finally, the adjustment of the current acount is

$$ca_0 - ca_{-1} = tb_0 - tb_{-1} > 0$$

and

$$ca_t = 0; \quad \forall t > 0$$

Adjustment to Temporary and Permanent Productivity Increases



: Temporary Productivity Increase x. . . x. . . x. . . : Permanent Productivity Increase

Principle I: The more persistent productivity shocks are, the more likely an initial deterioration of the trade balance will be.

3.5 Capital Adjustment Costs

Motivation: Capital adjustment costs are a standard feature of open economy business cycle models. They are used to ensure that the predicted volatility of investment relative to the volatility of output does not exceed the observed one.

With adjustment costs, investment will be spread out over a number of periods.

Consequences adjustment of trade balance in $t = 0$: Increase in investment in period 0 will be lower, and, increase in permanent income will be lower (output increases more slowly to new permanently high level) and so will be the consumption response $\Rightarrow$ more muted trade balance response

We will show that:

Principle II: The more pronounced are capital adjustment costs, the smaller will be the initial trade balance deterioration in response to a positive and persistent productivity shock.

$$\text{Capital adjustment costs } = \frac{1}{2}\,\frac{i_t^2}{k_t}$$

• If $i_t = 0$, then adj costs are nil.

• adj costs are convex in i_t

• these are actual resources lost!

• Slope of adjustment costs: $\frac{\partial \frac{i_t^2}{2k_t}}{\partial i_t} = \frac{i_t}{k_t}$

• in our model in steady state $i_t = 0$, so adjustment costs and marginal adjustment costs are nil in steady state.

With adjustment costs the sequential budget constraint becomes:

$$c_t + i_t + \frac{1}{2}\frac{i_t^2}{k_t} + (1+r)d_{t-1} = A_t F(k_t) + d_t \tag{3.16}$$

Lagr. Mult of BC: λ_t

Tobin's q

Lagr. Mult of K dynamics: $\lambda_t \cdot q_t$

Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ U(c_t) + \lambda_t \left[A_t F(k_t) + d_t - (1+r)d_{t-1} - c_t - i_t - \frac{1}{2}\frac{i_t^2}{k_t} + q_t(k_t + i_t - k_{t+1}) \right] \right\}$$

BC

Optimality conditions: (3.4), (3.5) holding with equality, (3.6), (3.7), (3.16), and,

$k_{t+1} = k_t + i_t$ ↳ TVC ↳ $u'(c_t) = \lambda_t$ ↳ $\lambda_t = \beta(1+r)\lambda_{t+1}$

$[i_t]$:
$$1 + \frac{i_t}{k_t} = q_t \tag{3.17}$$

$[k_{t+1}]$:
$$\lambda_t q_t = \beta \lambda_{t+1} \left[q_{t+1} + A_{t+1} F'(k_{t+1}) + \frac{1}{2}\left(\frac{i_{t+1}}{k_{t+1}}\right)^2 \right] \tag{3.18}$$

q_t = Tobin's q, shadow price of capital in terms of consumption goods

Again assume that $\beta(1+r) = 1$, then (3.18) can be written as

$$\overbrace{(1+r)q_t}^{\text{return on Debt}} = \overbrace{\underbrace{A_{t+1}F'(k_{t+1})}_{\text{Extra Output}} + \underbrace{q_{t+1}}_{\text{New price of Capital}} + \underbrace{\frac{1}{2}\left(\frac{i_{t+1}}{k_{t+1}}\right)^2}_{\text{Reduction in Inv. costs}}}^{\text{Return on Physical Asset}} \tag{3.19}$$

Interpretation: Suppose you have q_t units of consumption goods. LHS is the return if those are invested in bonds. RHS is the return if those are invested in capital, which is the marginal product of capital, the undepreciated capital, and the reduction in investment adjustment costs.

As in the case without adjustment costs, we can separate c_t dynamics from k_t or i_t dynamics.

Solving the sequential budget constraint (3.16) forward and using the no-Ponzi-game constraint (3.5) holding with equality yields

New

$$c_t = -rd_{t-1} + \frac{r}{1+r}\sum_{j=0}^{\infty}\frac{A_{t+j}F(k_{t+j}) - i_{t+j} - \frac{1}{2}(i_{t+j}^2/k_{t+j})}{(1+r)^j}.$$

Households split their nonfinancial permanent income, given by the second term on the right-hand side, to service their outstanding debt and to consume.

The definition of nonfinancial permanent income is adapted to include adjustment costs as one additional component of domestic absorption subtracted from the flow of output. The right-hand side of the above expression is known as *permanent income* and is given by the sum of net investment income ($-rd_{t-1}$) and nonfinancial permanent income.

Simplify FOCs & get a 2x2 first order equations in k_t, q_t

Dynamics of the Capital Stock:

$k_{t+1} = k_t + i_t$ $\quad 1 + \frac{i_t}{k_t} = q_t$ $\quad (1+r) q_t = A_{t+1} F'(k_{t+1}) + q_{t+1} + \frac{1}{2}\left(\frac{i_{t+1}}{k_{t+1}}\right)^2$

Combine (3.4), (3.17), and (3.19), to obtain two first-order, nonlinear difference equations in k_t and q_t:

Combine eqs to drop i_t

$$k_{t+1} = q_t k_t \tag{3.20}$$

$$q_t = \frac{A_{t+1} F'(q_t k_t) + (q_{t+1} - 1)^2/2 + q_{t+1}}{1+r} \tag{3.21}$$

SS: $q^{ss} = 1$

$\cancel{1} + r = \underbrace{\bar{A} F'(k^{ss})}_{} + \cancel{1}$

$r = \bar{A} F'(k^{ss})$

Steady state solution: (q, k)

Suppose $A_t = \bar{A}$ for all t

By (3.20),

$$q = 1$$

And using this result in (3.21)

$$r = \bar{A}F'(k)$$

$\rightarrow$ investment adjustment costs play no role for long run values of k and q

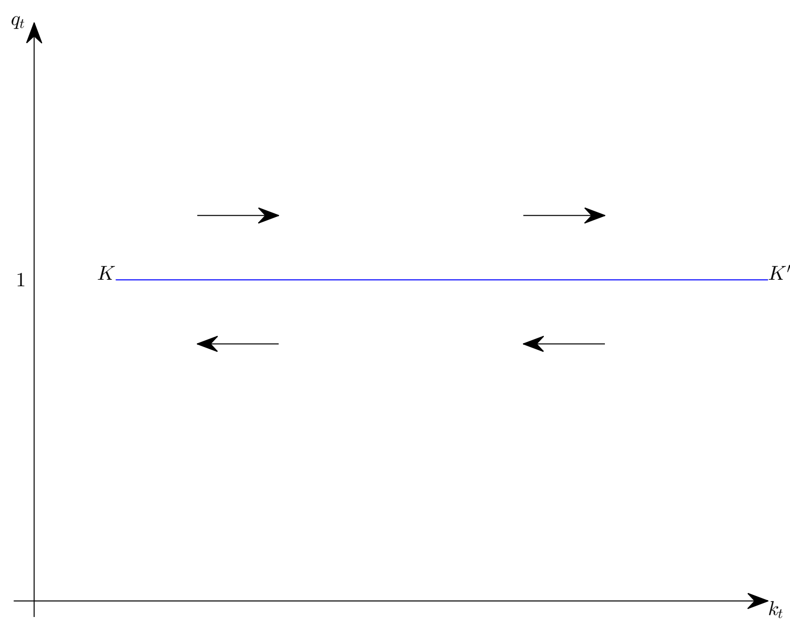
but they do play a role for the short-run dynamics, which we will analyze next using a phase diagram

Let's plot the locus of pairs (k_t, q_t) such that $k_{t+1} = k_t$. Call it the $\overline{KK'}$ locus. By (3.20) if

$$\begin{array}{ll} q_t > 1, & k_{t+1} > k_t \\ q_t = 1, & k_{t+1} = k_t \\ q_t < 1, & k_{t+1} < k_t \end{array}$$

$k_{t+1} = q_t \cdot k_t$

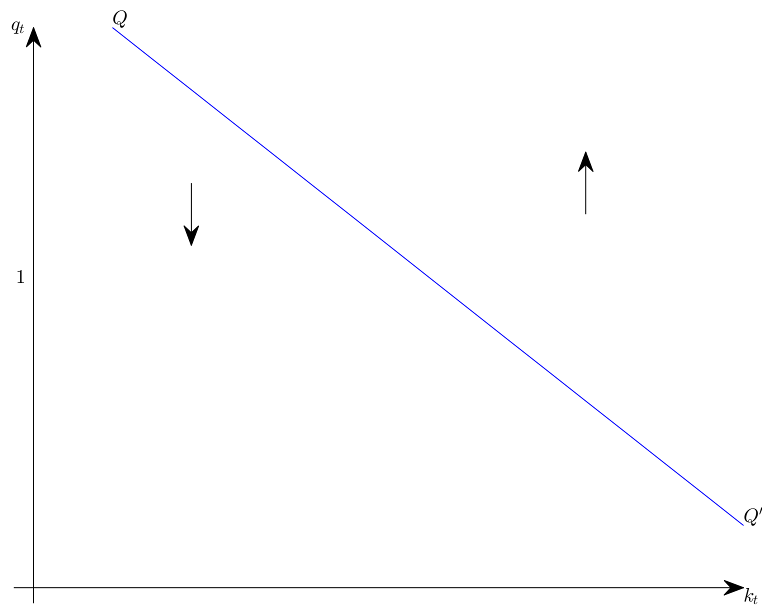
$\hookrightarrow q_t = \frac{k_{t+1}}{k_t}$



Assume that $A_t = \bar{A}$ for all t. Plot the locus of pairs (k_t, q_t) such that $q_{t+1} = q_t$ in a neighborhood around $q_t = 1$. (This is a local analysis.) Call this the $\overline{QQ'}$ locus. By (3.21), the $\overline{QQ'}$ locus is given by

$$rq_t = \bar{A}F'(q_t k_t) + (q_t - 1)^2/2 \qquad (+)$$

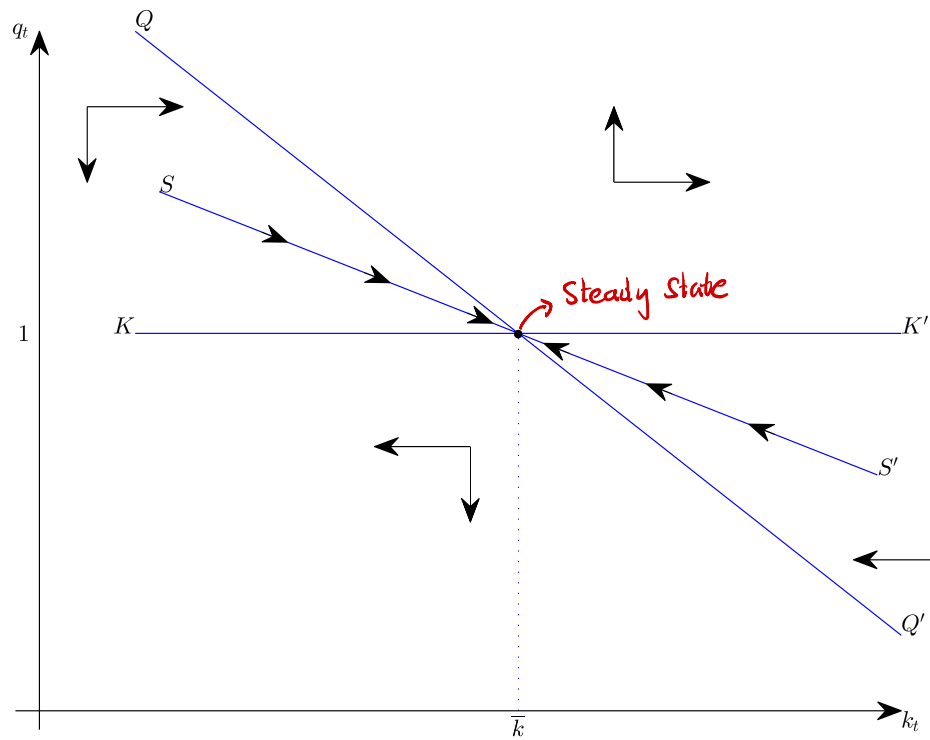
$$\text{If} \quad \begin{cases} (k_t, q_t) \text{ above } \overline{QQ'}, & q_{t+1} > q_t \\ (k_t, q_t) \text{ on } \overline{QQ'}, & q_{t+1} = q_t \\ (k_t, q_t) \text{ below } \overline{QQ'}, & q_{t+1} < q_t \end{cases}$$



In (3.21) make $q_{t+1} = q_t$
(we have a negative slope as $F'(\cdot)$ decreases w/ k_t)

Now, if $q_{t+1} > q_t \Rightarrow$ the eq. above (+) should include an extra positive term (thus we'd be above the blue line in the plot)

This yields the phase diagram:



- The intersection of $\overline{KK'}$ and $\overline{QQ'}$ is the steady state pair $(k, q) = (\bar{k}, 1)$
- The locus $\overline{SS'}$ is the saddle path.
- Given the initial capital stock, k_0, Tobin's q, q_0, jumps to the saddle path, and (k_t, q_t) converge monotonically to $(\bar{k}, 1)$.

Experiment 1: Adjustment to a temporary productivity shock. $\rightarrow$ identical to the economy without capital adjustment costs, as there is no reason to adjust the capital stock. (results as in Section 3.4).

Experiment 2: Adjustment to a permanent productivity shock.
In period 0 it is learned that A_t increases from $\bar{A}$ to $A' > \bar{A}$ for all $t \geq 0$. Prior to period 0, A_t was expected to be $\bar{A}$ forever.

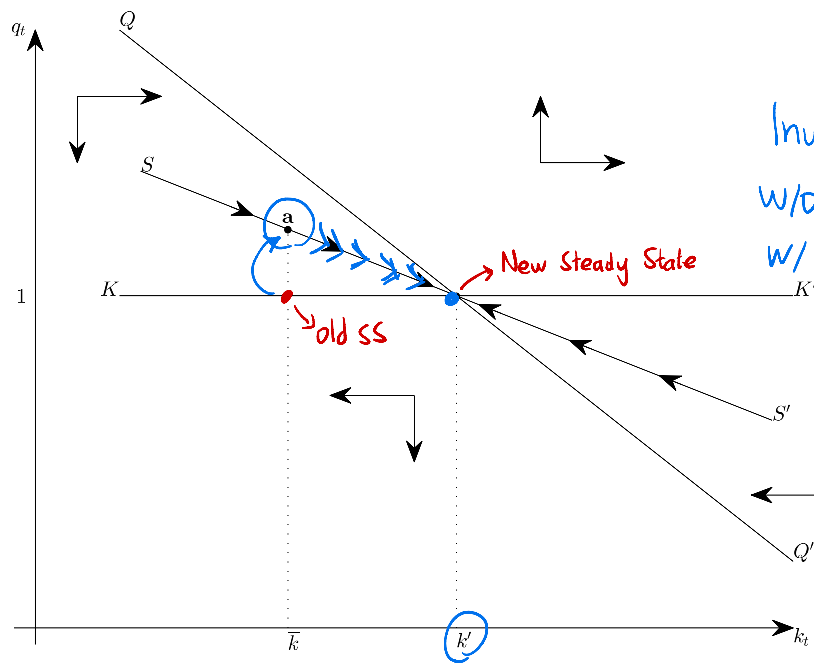
$$A_t = \begin{cases} \bar{A} & \text{for } t \leq -1 \\ A' > \bar{A} & \text{for } t \geq 0 \end{cases} .$$

How can we capture this in the phase diagram?

The $\overline{KK'}$ locus does not change. But the $\overline{QQ'}$ locus changes.

The new locus is implicitly given by $rq_t = A'F'(q_t k_t) + (q_t - 1)^2/2$. This means that the $\overline{QQ'}$ locus shifts up and to the right. The new steady state is $(k_t, q_t) = (k', 1)$, where k' solves $r = A'F'(k')$. The initial capital stock is $k_0 = \bar{k}$, hence $k_0 < k'$.

The dynamics of the capital stock can be read of the graph below.



Investment on impact:

w/o adj costs: $i_0 = k' - \bar{k}$

w/ adj costs: $i_0 < k' - \bar{k}$ $\Rightarrow$ tb_0 lowers by less

In period 0 the economy jumps to point a, where $q_0 > 1$ and $k_0 = \bar{k}$. That is, capital converges monotonically to k' from below and Tobin's q converges monotonically to 1 from above. Investment is positive during the entire transition, but, importantly, $i_0 < k' - \bar{k}$. It follows that domestic absorption increases by less on impact in the presence of capital adjustment costs. And thus, the deterioration of the trade balance in response to a positive permanent productivity shock is smaller on impact. We summarize these results as follows:

Principle II: The more pronounced are capital adjustment costs, the smaller will be the initial trade balance deterioration in response to a positive and persistent productivity shock.

Alternative method: Log-linearization

Thus far, to determine the dynamics in the model with capital adjustment costs we used a phase diagram. The phase diagram is a convenient graphical tool to analyze dynamics qualitatively. Specifically, we used the phase diagram to establish that if k_0 is below steady state, then:

- the model is saddle path stable
- the price of capital converges to its steady state value from above
- capital converges to its steady state value from below.
- investement is positive along the entire transition.
- capital adjustment costs dampen the trade balance deterioration in response to a permanent productivity increase.

We now consider an alternative method to determine whether the model is saddle path stable and to characterize the adjustment of the economy when k_0 is below its steady state value.

usual variable: x_t

log linear variables: $\ln x_t - \ln x^{ss} = \hat{x}_t$

Characterization of Adjustment using a Log-linear approximation

We wish to characterize the dynamics of q_t and k_t described by

$$k_{t+1} = q_t k_t \tag{3.20R}$$

$$q_t = \frac{A_{t+1} F'(q_t k_t) + (q_{t+1} - 1)^2/2 + q_{t+1}}{1 + r} \tag{3.21R}$$

$k_t =$ endogenous predetermined variable
$q_t =$ endogenous nonpredetermined variable
$A_t =$ exogenous variable

Consider the dynamics around the steady state associated with $A_t = A' > \bar{A}$ for all $t \geq 0$. The steady state solution to (3.20) and (3.21) is

$$q^{ss} = 1$$

$$k^{ss} = k'$$

where k' is the solution to $r = A'F'(k')$.

Let

$$\boxed{\begin{aligned} \widehat{q}_t &\equiv \ln \frac{q_t}{q^{ss}} \\ \widehat{k}_t &\equiv \ln \frac{k_t}{k^{ss}} \end{aligned}}$$

Log-linearize (3.20) and (3.21) around the point $(q_t, k_t) = (1, k')$

Since this is the first time we use this technique, we will explain each step. Take logs of (3.20), then take the total differential:

Subtr same equation but in Steady State

$$
\begin{aligned}
\ln k_{t+1} &= \ln k_t + \ln q_t \\
(\ln k_{t+1} - \ln k^{ss}) &= (\ln k_t - \ln k^{ss}) + (\ln q_t - \ln q^{ss}) \\
\widehat{k}_{t+1} &= \widehat{q}_t + \widehat{k}_t
\end{aligned}
\tag{3.20'}
$$

Applying the same steps to (3.21) is a little more complicated. To make the presentation clearer, let $x_{t+1} = A'F'(k_{t+1}) + (q_{t+1} - 1)^2/2 + q_{t+1}$. (Note that $x^{ss} = 1 + r$). With this notation in hand, after takings logs of both sides, (3.21) becomes

$$\ln(1 + r) + \ln q_t = \ln x_{t+1}$$

Take total differential with respect to $\ln q_t$ and $\ln x_{t+1}$

$$
\begin{aligned}
\ln q_t - \ln q^{ss} &= \ln x_{t+1} - \ln x^{ss} \\
\widehat{q}_t &= \widehat{x}_{t+1}
\end{aligned}
$$

Since we want to do approximation (on RHS) wrt $\ln(k_{t+1})$ and not k_{t+1} we should take derivatives wrt $\ln(k_{t+1})$, then here is key to replace k_{t+1} for $\exp(\ln(k_{t+1}))$ when taking the derivatives:

$$\frac{\partial \ln x_{t+1}}{\partial \ln k_{t+1}} = \frac{1}{x_{t+1}} \cdot \frac{\partial (A'F'(\exp(\ln(k_{t+1}))) + \overbrace{\ldots}^{\text{other terms w/o } k_{t+1}})}{\partial \ln k_{t+1}} = \frac{1}{x_{t+1}} \cdot A'F''(\exp(\ln(k_{t+1}))) . \exp(\ln(k_{t+1}))$$

$$= \frac{1}{x_{t+1}} A'F''(k_{t+1}).k_{t+1} \Big|_{k^{ss}} = \frac{1}{x^{ss}} A'F''(k^{ss}).k^{ss}$$

To find $\widehat{x}_{t+1}$ proceed as follows

$$x_{t+1} = A'F'(k_{t+1}) + (q_{t+1} - 1)^2/2 + q_{t+1}$$

apply logs:

$$\ln x_{t+1} = \ln[A'F'(k_{t+1}) + (q_{t+1} - 1)^2/2 + q_{t+1}]$$

Totally differentiate

i.e. do a First Order Taylor approximation of $f(\cdot) = \ln(x_{t+1})$ around x^{ss}; but with $\ln(k_{t+1})$, $\ln(q_{t+1})$ terms on the RHS (see note on annotated slides (above))

$$\ln x_{t+1} - \ln x^{ss} = \frac{1}{[A'F'(k^{ss}) + (q^{ss} - 1)^2/2 + q^{ss}]}$$

$$\times \left(A'F''(k^{ss})k^{ss}(\ln k_{t+1} - \ln k^{ss}) + (q^{ss} - 1)q^{ss}(\ln q_{t+1} - \ln q^{ss}) + q^{ss}(\ln q_{t+1} - \ln q^{ss})\right)$$

$$\widehat{x}_{t+1} = \frac{1}{1+r}\left(A'F''(k^{ss})k^{ss}\widehat{k}_{t+1} + \widehat{q}_{t+1}\right)$$

Let

$$\epsilon_{F'} \equiv -\frac{F''(k^{ss})k^{ss}}{F'(k^{ss})} > 0$$

$$(1+r)\widehat{x}_{t+1} = -r\epsilon_{F'}\widehat{k}_{t+1} + \widehat{q}_{t+1}$$

The log-linearized version of (3.21) then is

$$(1+r)\widehat{q}_t = -r\epsilon_{F'}\widehat{k}_{t+1} + \widehat{q}_{t+1}$$

log linearized!

After some rearranging and substituting we have that the log-linearization of (3.20) and (3.21) around the steady state $(q^{ss}, k^{ss}) = (1, k')$ is

$$\begin{bmatrix} \widehat{k}_{t+1} \\ \widehat{q}_{t+1} \end{bmatrix} = M \begin{bmatrix} \widehat{k}_t \\ \widehat{q}_t \end{bmatrix}; \quad \boxed{M = \begin{bmatrix} 1 & 1 \\ r\epsilon_{F'} & 1 + r + r\epsilon_{F'} \end{bmatrix}} \qquad (***)$$

If we knew the initial values k_0 and q_0 we could trace out the dynamics. We do know k_0 as it is an initial condition. But we do not know the initial value of Tobin's q, q_0. To obtain it, we impose a terminal condition, we require that the economy converges back to the steady state. Thus our question becomes, does there exist such a solution and if so, is it unique. We are interested in solutions such that

$$\lim_{t\to\infty} \begin{bmatrix} \widehat{k}_t \\ \widehat{q}_t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which says that $k_t \to k^{ss} = k'$ and $q_t \to q^{ss} = 1$.

By (***)

$$\lim_{t\to\infty} \begin{bmatrix} \widehat{k}_t \\ \widehat{q}_t \end{bmatrix} = \lim_{t\to\infty} M^t \begin{bmatrix} \widehat{k}_0 \\ \widehat{q}_0 \end{bmatrix}$$

• If both eigenvalues of M lie outside the unit circle, then no equilibrium converging to the steady state exists.

• If both eigenvalue of M lie inside the unit circle, then for any initial value of q_0, an equilibrium converging to the steady state exists, that is, the equilibrium is locally indeterminate.

Blanchard - Kahn Conditions

• If one eigenvalue of M lies inside the unit circle and one outside, then a unique value for q_0 exists, such that the equilibrium converges to the steady state given some k_0 in the neighborhood of the steady state.

Let λ_1 and λ_2 be the eigenvalues of M. Then the equilibrium is locally unique iff

$$|\lambda_1| > 1 \quad \text{and} \quad |\lambda_2| < 1$$

Is this eigenvalue condition satisfied in our economy? Yes, it is. To see this note, use that in general for any matrix

$$det(M) = \lambda_1 \lambda_2; \quad \text{and} \quad trace(M) = \lambda_1 + \lambda_2$$

In our case,

$$det(M) = 1 + r > 1; \quad \text{and} \quad trace(M) = 1 + 1 + r + r\epsilon_{F'} > 2 + r$$

(from here it follows that both eigenvalues are positive (or have positive real parts) and that at least one is greater than one in modulus. In turn this implies that if an equilibrium of the type we are looking for exists, then it would be unique.)

To find whether it exists, let's first consider the case that the eigenvalues are real. Make a graph with λ_1 on the x-axis and λ_2 on the y-axis and plot: 1.) $\lambda_2 = \frac{1+r}{\lambda_1}$ and 2.) $\lambda_2 = 2 + r + r\epsilon_{F'} - \lambda_1$. These lines must intersect twice in the positive quadrant because 1.) is positive, decreasing, and becomes arbitrarily large as $\lambda_1 \to 0$ from above and converges to zero as $\lambda_1 \to \infty$ and at the same time 2.) is positive and finite for $\lambda_1 = 0$, decreasing, and converges to $-\infty$ as $\lambda_1 \to \infty$. The question is: is one intersection at $\lambda_1 < 1$ and the second at $\lambda_1 > 1$? At $\lambda_1 = 1$ 2.) is: $1 + r + r\epsilon_{F'}$, which is greater than 1. It then follows that the conditions for uniqueness are satisfied. That is, we have shown that $\lambda_1 > 1$ and $0 < \lambda_2 < 1$.

How to find q_0? Let

$$y_t = \begin{bmatrix} \widehat{k}_t \\ \widehat{q}_t \end{bmatrix}$$

Premultiply (***) by the left eigenvector of M associated with the unstable eigenvalue, λ_1, denoted v_1

$$v_1 y_{t+1} = v_1 M y_t = \lambda_1 v_1 y_t$$

Let $\tilde{y}_t = v_1 y_t$. Then $\tilde{y}_t = \lambda_1^t \tilde{y}_0$. Because $|\lambda_1| > 1$, $\lim_{t\to\infty} \tilde{y}_t = 0$ only if $\tilde{y}_0 = 0$, that is, if

$$0 = v_1 \begin{bmatrix} \widehat{k}_0 \\ \widehat{q}_0 \end{bmatrix} = v_1^1 \widehat{k}_0 + v_1^2 \widehat{q}_0$$

$$\widehat{q}_0 = -\frac{v_1^1}{v_1^2}\widehat{k}_0 = -(1-\lambda_2)\widehat{k}_0$$

The last equality follows from

$$\begin{aligned}
\begin{bmatrix} v_1^1 & v_1^2 \end{bmatrix} M =& \lambda_1 \begin{bmatrix} v_1^1 & v_1^2 \end{bmatrix} \\
v_1^1 + v_1^2(1 + r + r\epsilon_{F'}) =& \lambda_1 v_1^2 \\
v_1^1 + v_1^2(trace(M) - 1) =& \lambda_1 v_1^2 \\
v_1^1 + v_1^2(\lambda_1 + \lambda_2 - 1) =& \lambda_1 v_1^2 \\
\frac{v_1^1}{v_1^2} =& (1 - \lambda_2)
\end{aligned}$$

Now use $\widehat{q}_t = -(1-\lambda_2)\widehat{k}_t$ in (3.20') to obtain:

$$\begin{aligned}\widehat{k}_{t+1} =& \widehat{k}_t + \widehat{q}_t \\ =& (1 - 1 + \lambda_2)\widehat{k}_t \\ =& \lambda_2 \widehat{k}_t.\end{aligned}$$

Summary of dynamics:

$$\begin{aligned}\widehat{k}_t =& \lambda_2^t \widehat{k}_0 \\ \widehat{q}_t =& -(1-\lambda_2)\lambda_2^t \widehat{k}_0\end{aligned}$$

• unique saddle path stable eqm exists locally in the neighborhood around (q^{ss}, k^{ss}).

• The adjustment to a permanent increase in productivity induces capital to converge monotonically from below, Tobin's q to converge monotonically from above.

• Because the increase in capital is spread out of many periods, investment is also positive for many periods and because the total increase in capital is independent of the size of the adjustment cost, it follows that adjustment costs dampen the initial increase in investment, and hence the initial deterioration in the trade balance and the current account. (Principle II)