Extra: Algebra for the Lifetime BC

$$W_{t+1} = y_{t+1} + (1+r)(W_t - C_t) \implies W_t = \frac{W_{t+1} - y_{t+1}}{1+r} + C_t$$

this equation holds for every period and we'll use it to substitute forward the terms $W_{t+1}, W_{t+2}, W_{t+3}, W_{t+4}, \ldots$

Subs $W_{t+1} = \frac{W_{t+2} - y_{t+2}}{1+r} + C_{t+1}$ in the budget constraint:

$$\frac{W_{t+2} - y_{t+2}}{1+r} + C_{t+1} = y_{t+1} + (1+r)(W_t - C_t)$$

Subs W_{t+2}:

$$\frac{W_{t+3} - y_{t+3}}{(1+r)^2} + \frac{C_{t+2}}{1+r} - \frac{y_{t+2}}{1+r} + C_{t+1} = y_{t+1} + (1+r)(W_t - C_t)$$

$$\implies \frac{W_{t+3}}{(1+r)^2} + (1+r)C_t + C_{t+1} + \frac{C_{t+2}}{1+r} = y_{t+1} + \frac{y_{t+2}}{1+r} + \frac{y_{t+3}}{(1+r)^2} + (1+r)W_t$$

Subs W_{t+3}:

$$\frac{W_{t+4}}{(1+r)^3} - \frac{y_{t+4}}{(1+r)^3} + \frac{C_{t+3}}{(1+r)^2} + (1+r)C_t + C_{t+1} + \frac{C_{t+2}}{1+r} = y_{t+1} + \frac{y_{t+2}}{1+r} + \frac{y_{t+3}}{(1+r)^2} + (1+r)W_t$$

or $$\frac{W_{t+4}}{(1+r)^3} + (1+r)C_t + C_{t+1} + \frac{C_{t+2}}{1+r} + \frac{C_{t+3}}{(1+r)^2} = y_{t+1} + \frac{y_{t+2}}{1+r} + \frac{y_{t+3}}{(1+r)^2} + \frac{y_{t+4}}{(1+r)^3} + (1+r)W_t$$

Subs W_{t+4}:

$$(1+r)C_t + C_{t+1} + \frac{C_{t+2}}{1+r} + \frac{C_{t+3}}{(1+r)^2} + \frac{C_{t+4}}{(1+r)^3} = (1+r)W_t + y_{t+1} + \frac{y_{t+2}}{1+r} + \frac{y_{t+3}}{(1+r)^2} + \frac{y_{t+4}}{(1+r)^3} + \frac{y_{t+5}}{(1+r)^4}$$

$$\implies C_t + \frac{C_{t+1}}{1+r} + \frac{C_{t+2}}{(1+r)^2} + \frac{C_{t+3}}{(1+r)^3} + \frac{C_{t+4}}{(1+r)^4} = W_t + \frac{y_{t+1}}{1+r} + \frac{y_{t+2}}{(1+r)^2} + \frac{y_{t+3}}{(1+r)^3} + \frac{y_{t+4}}{(1+r)^4} + \frac{y_{t+5}}{(1+r)^5}$$

⋮

So on,

$$\sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s C_{t+s} = W_t + \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s y_{t+s}$$

Take $t=0$ (Could do this for any $t = 1, 2, \ldots$)

$$\implies \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s C_s = W_0 + \sum_{s=0}^{\infty} \left(\frac{1}{1+r}\right)^s y_s$$ (as in the Lecture)

More detailed algebra steps (CAPM from Euler Eq)

$\left(\beta = \frac{1}{1+\rho} \equiv \text{Standard discount Factor} \right)$

$$E_t \left\{ \frac{u'(C_{t+1})}{u'(C_t)} \frac{(1+\tilde{r}_t^i)}{1+\rho} \right\} = 1 \quad \forall\, i, \qquad M_t = \frac{u'(C_{t+1})}{u'(C_t)} \frac{1}{1+\rho} \equiv \text{Stochastic discount Factor}$$

$$\Rightarrow E_t \left\{ \tilde{M}_t (1+\tilde{r}_t^i) \right\} = 1 \Rightarrow E_t(\tilde{M}_t) E_t(1+\tilde{r}_t^i) + Cov(1+\tilde{r}_t^i, \tilde{M}_t) = 1 \left(\begin{array}{l} \text{uses that:} \\ Cov(x,y) = E(xy) - E(x)E(y) \end{array} \right)$$

With CRRA: $\tilde{M}_t = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma}$

$$\Rightarrow E_t(1+\tilde{r}_t^i) = \frac{1}{E_t(\tilde{M}_t)} \left[1 - \beta\, Cov\left(1+\tilde{r}_t^i, \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right) \right]$$

use that $\frac{C_{t+1}}{C_t} \approx 1 + \Delta \ln C_{t+1}$ (i) $\frac{X_{t+1}}{X_t} \approx 1 + \%\text{ growth in } X \approx (1+g_x)$. (ii) $\ln(1+g_x) \approx g_x$. (iii) Percentage growth $\equiv g_x \approx \ln X_{t+1} - \ln X_t = \Delta \ln X_{t+1}$

$$E_t(1+\tilde{r}_t^i) = \frac{1}{E_t(\tilde{M}_t)} \left[1 - \beta\, Cov\left(1+\tilde{r}_t^i, (1+\Delta \ln C_{t+1})^{-\gamma}\right) \right]$$

$$= \frac{1}{E_t(\tilde{M}_t)} \left[1 - \beta\, Cov\left(1+\tilde{r}_t^i, 1 - \gamma \Delta \ln C_{t+1}\right) \right]$$

Uses binomial approximation: $(1+x)^{\alpha} \approx 1 + \alpha x$

$$= \frac{1}{1+\rho} \frac{1}{E_t(\tilde{M}_t)} \left[1 + \rho - Cov\left(\tilde{r}_t^i, -\gamma \Delta \ln C_{t+1}\right) \right] = \frac{1}{(1+\rho) E_t(\tilde{M}_t)} \left\{ 1 + \rho + \gamma\, Cov\left(\tilde{r}_t^i, \Delta \ln C_{t+1}\right) \right\}$$

Algebra for log-linearizing Euler Equation.

Depart from Euler Equation: $C_t^{-\gamma} = \beta E_t[C_{t+1}^{-\gamma} R_{t+1}]$

assume RHS variables are jointly log-normal & homoskedastic

Take logs: $-\gamma \log C_t = \log\beta + \log E_t[C_{t+1}^{-\gamma} R_{t+1}]$

W/ log normality: $\log(E_t(X_{t+1})) = E_t[\log(X_{t+1})] + \frac{1}{2}Var_t[\log(X_{t+1})]$

$$\Rightarrow \log E_t[C_{t+1}^{-\gamma} R_{t+1}] = E_t[\log(C_{t+1}^{-\gamma} R_{t+1})] + \frac{1}{2}Var_t[\log(C_{t+1}^{-\gamma} R_{t+1})]$$

$$= -\gamma E_t \log C_{t+1} + E_t \log R_{t+1} + \frac{\gamma^2}{2}\sigma^2_{t,\log C_{t+1}} + \frac{1}{2}\sigma^2_{t,\log R_{t+1}} - \gamma\sigma_{t,\log C_{t+1},\log R_{t+1}}$$

$$\Rightarrow -\gamma\log C_t = \log\beta + E_t(-\gamma\log C_{t+1} + \log R_{t+1}) + \underbrace{\frac{\gamma^2}{2}\sigma^2_{t,\log C_{t+1}} + \frac{1}{2}\sigma^2_{t,\log R_{t+1}} - \gamma\sigma_{t,\log C_{t+1},\log R_{t+1}}}_{\text{Constant}}$$

Substract Equation for Steady State

$$-\gamma\, d\log C_t = -\gamma E_t\, d\log C_{t+1} + E_t\, d\log R_{t+1}$$

$$d\log Z_t = \hat{z}_t = \frac{z_t - z^{ss}}{z^{ss}} \qquad \left(r_{t+1} = \frac{R_{t+1} - R^{ss}}{R^{ss}}\right)$$

$$\Rightarrow -\gamma\hat{C}_t = -\gamma E_t\hat{C}_{t+1} + E_t r_{t+1}$$

$$\hat{C}_t = E_t\hat{C}_{t+1} - \frac{1}{\gamma}E_t r_{t+1}$$