

## Problem set # 1

### Answer Key

1. **(Simple model, rational expectations)** Consider an economy that is described by the following conditions:

$w_t = \mathbb{E}_{t-1} p_t$	(1) [one period wage stickyness]
$y_t = -(w_t - p_t)$	(2) [Aggregate Supply, AS]
$y_t + p_t = m_t + \tilde{v}_t$	(3) [Aggregate Demand, AD]
$m_t = b(L)v_{t-1}$	(4) [Deterministic Monetary Policy Rule]

With  $\tilde{v}_t = D(L)v_{t-1} + \eta_t$ ,  $\eta_t \sim N(0, \sigma^2)$ .

Where  $y_t$ : Output in period  $t$ ,  $w_t$ : wages,  $p_t$ : price level,  $\mathbb{E}_{t-1}[\cdot]$ : expectation operator with information up to  $t - 1$ ,  $\tilde{v}_t$ : observed aggregate demand shock,  $\eta$ : stochastic or surprise component of the AD shock,  $v_t$ : deterministic component of the AD shock.

(Here  $D(L)$ ,  $b(L)$  correspond to polynomials of coefficients with the  $L$  denoting a "lag" operator such that  $L^k x_t = x_{t-k}$ . For the solution of this problem you can abstract from these details and assume that  $D(L)$ ,  $b(L)$  are coefficients).

- (a) Determine which variables are endogenous and which are exogenous.

(Ans) Exogenous variables: the shocks  $\tilde{v}_t$  and  $\eta_t$  are exogenous, meaning they are taken as given and do not change depending on the economic decisions of the agents (or policy makers). This implies that  $v_{t-1}$  is also exogenous<sup>1</sup>

Everything else will be endogenous. Then:

Endogenous variables:  $y_t, p_t, w_t$ , and hence  $\mathbb{E}_{t-1} p_t$ . Also,  $m_t$  is endogenous (as in decided by the agents that formulate policy). However, notice that  $m_t$  is already "solved" in the sense that is already given in terms of exogenous.

- (b) Solve for the endogenous that is an expectation ( $\mathbb{E}_{t-1} p_t$ ) as a function of the exogenous variables.

(Ans) Solution for  $\mathbb{E}_{t-1} p_t$ :

In (3) solve for  $p_t$ :

$$p_t = -y_t + m_t + \tilde{v}_t$$

<sup>1</sup>Given it's predetermined by  $t$ , i.e. defined in  $t - 1$ , or already defined by  $t$ . I am trying to list here all of the possible ways to refer to these type of variables.

Then replace  $y_t$  from (2) and  $m_t$  from (4):

$$p_t = w_t - p_t + b(L)v_{t-1} + \tilde{v}_t$$

Replace  $w_t$  from (1) and  $\tilde{v}_t$ :

$$p_t = \mathbb{E}_{t-1}p_t - p_t + b(L)v_{t-1} + D(L)v_{t-1} + \eta_t$$

Apply the expectation operator conditional to  $t - 1$ ,  $\mathbb{E}_{t-1}[\cdot]$  to both sides:

$$\mathbb{E}_{t-1}p_t = \mathbb{E}_{t-1}[\mathbb{E}_{t-1}p_t - p_t] + \mathbb{E}_{t-1}[b(L)v_{t-1} + D(L)v_{t-1}] + \mathbb{E}_{t-1}[\eta_t]$$

Notice that:  $\mathbb{E}_{t-1}[\mathbb{E}_{t-1}p_t - p_t] = \mathbb{E}_{t-1}[\mathbb{E}_{t-1}p_t] - \mathbb{E}_{t-1}[p_t] = \mathbb{E}_{t-1}[p_t] - \mathbb{E}_{t-1}[p_t] = 0$ . Also:  $\mathbb{E}_{t-1}[b(L)v_{t-1} + D(L)v_{t-1}] = b(L)v_{t-1} + D(L)v_{t-1}$  since by  $t - 1$  those variables are already known, and  $\mathbb{E}_{t-1}[\eta_t] = 0$  as specified above in the distribution of  $\eta_t$ . Therefore:

$$\mathbb{E}_{t-1}p_t = b(L)v_{t-1} + D(L)v_{t-1}$$

When you check this solution, make sure to notice how applying the expectation operator to get rid of terms allowed to simplify the equation and pinpoint the solution.

(c) Solve for the remaining endogenous variables

(Ans) The remaining endogenous variables are  $y_t, p_t, w_t, m_t$ . However,  $w_t = \mathbb{E}_{t-1}p_t$  which we just solved for and  $m_t$  is already a function of the exogenous terms. Then, we only have to solve for  $y_t$  and  $p_t$  at this point.

Notice that now, (2), (3) represent a 2 by 2 system of equations, i.e., 2 equations, with 2 unknowns:  $p_t, y_t$ .

Then we can replace all of the other terms in (2), (3) and solve for  $p_t$  and  $y_t$ .

Replace  $y_t$  from (2) in (3) and also (4) and  $\tilde{v}_t$

$$-w_t + y_t + p_t = b(L)v_{t-1} + D(L)v_{t-1} + \eta_t$$

Replace  $w_t = \mathbb{E}_{t-1}p_t = b(L)v_{t-1} + D(L)v_{t-1}$ :

$$-(b(L)v_{t-1} + D(L)v_{t-1}) + 2p_t = b(L)v_{t-1} + D(L)v_{t-1} + \eta_t$$

Solving for  $p_t$ :

$$p_t = b(L)v_{t-1} + D(L)v_{t-1} + \frac{\eta_t}{2}$$

Finally, in (2) substitute  $p_t, w_t$ :

$$y_t = -(w_t - p_t)$$

$$\begin{aligned}
&= -(\mathbb{E}_{t-1}p_t - p_t) \\
&= -(b(L)v_{t-1} + D(L)v_{t-1} - b(L)v_{t-1} - D(L)v_{t-1} - \frac{\eta_t}{2}) \\
&= \frac{\eta_t}{2}
\end{aligned}$$

2. **(Empirical exercise)** In this exercise you will update the macroeconomic time series plots from lecture 1 for the US economy<sup>2</sup>. Download the quarterly GDP data and another one of the following variables: total consumption, durable goods consumption, non-durable goods consumption, investment, employment.

For this exercise, you needed to download the data from the Fred, and run the code provided. Here I will show you how your data should look like for Matlab to read it, and also the lines you should have changed for it to work.

**Important:** If you want to understand what a code does, run it line by line paying attention to the result after each instruction. People reporting issues usually run the whole file which made difficult for them to identify where the error (if any) is.

- (a) Using the code "ps1q2\_code" (change the corresponding lines so that it runs in your pc and be sure to include "hpfilter.m" in your working folder) plot the log of the level and the trend in one graph and the cycle in another graph. Do this for each of the two variables.

(Ans) The data is read with this command:

```
data = xlsread('filename.xlsx',1) ;
gdp = data(:,2); % 2 refers to the column of the variable
con = data(:,3); % consumption (or other variable), 3 refers to column
```

That means that the data had to be provided in an excel file (you needed to change the name). Furthermore, the GDP should be in the second column and the other variable in the third column. This is an example (I use the Australian data for my example here, not exactly the same one as in the homework - US):

<sup>2</sup>the series can be found at the Federal Reserve Economic Data (<https://fred.stlouisfed.org/>). Keep in mind that the variables you are interested in are the real ones (not nominal).

	A	B	C
1	Date	RGDP_AUS	RCONS_AUS
2	1960-01-01	14.9593485022868	14.36899804
3	1960-04-01	15.3725184914262	14.55122526
4	1960-07-01	15.4071916632273	14.73787741
5	1960-10-01	15.3767687511954	14.62202435
6	1961-01-01	15.4230742128910	14.43215405
7	1961-04-01	15.2539586136547	14.47479442
8	1961-07-01	15.1447940469519	14.56771823
9	1961-10-01	15.3170414165445	14.66506698
10	1962-01-01	15.7469887468780	15.15060395
11	1962-04-01	16.0487571904888	15.34530146
12	1962-07-01	16.1785019623897	15.46557947
13	1962-10-01	16.4688618139885	15.60476405
14	1963-01-01	16.8493719122702	15.89479898
15	1963-04-01	16.6299242894516	16.02070874
16	1963-07-01	17.3124265292267	16.49820036
17	1963-10-01	17.6833176185571	16.8240371
18	1964-01-01	17.6549079874684	16.91615637
19	1964-04-01	18.0906714627493	17.18245751
20	1964-07-01	18.2027441019257	17.49863983

First things first, let us take a look at the data. If you work with a data set, **Always take a look at it and make sure what you see makes sense.**

In this particular case, we are plotting the real GDP and a real macro variable in levels (in my case consumption). Hence we should see variables that take only on positive values and don't seem to go back to their means. If you have issues involving complex numbers it is because you are not using the data in levels but in growth rates. Then when taking the logs of each variable in lines 13 and 14 the command won't work (you cannot take logarithms of negative numbers, some growth rates are negative):

```
gdp_hp = hpfilter(log(gdp),1600); % Trend
con_hp = hpfilter(log(con),1600);

gdp_cyc = log(gdp) - gdp_hp; % Cyclical component (detrending)
con_cyc = log(con) - con_hp;
```

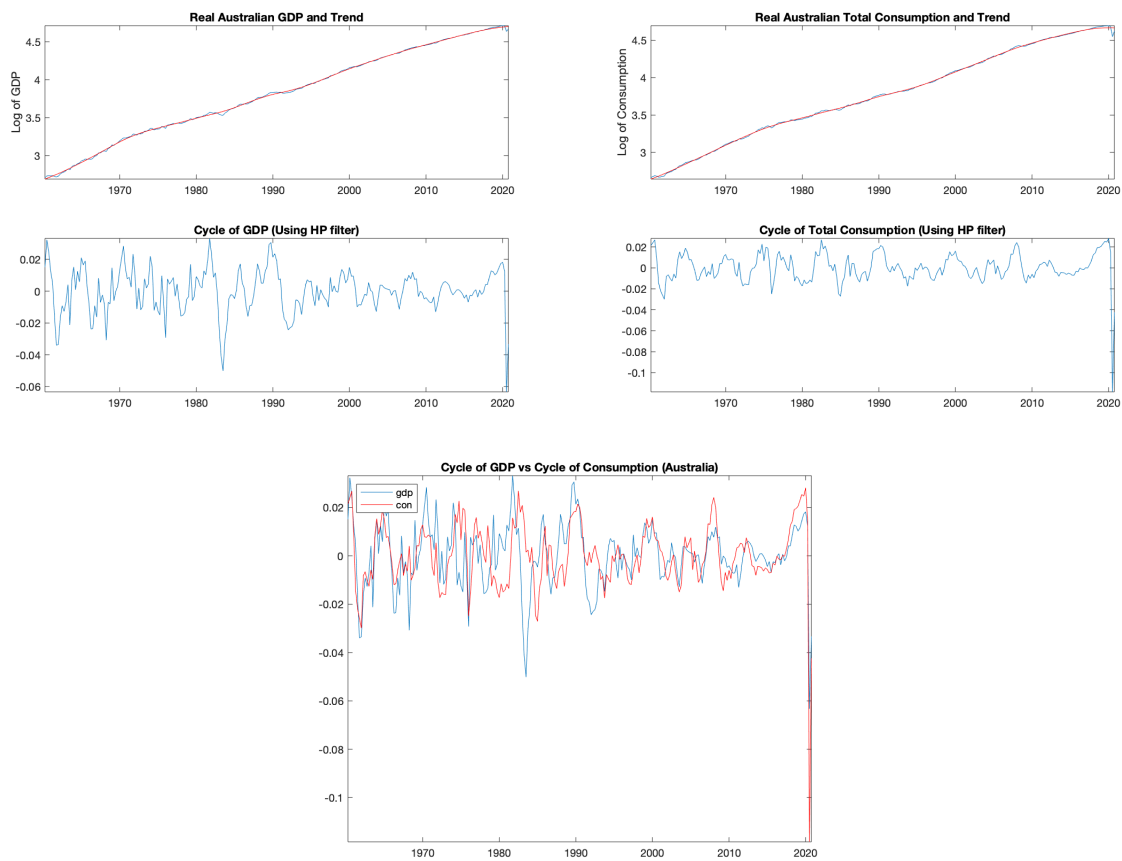
Finally, the next instruction consisted in creating the date variable for the plots (x-axis). This is done with:

```
x = 1960.25:0.25:2020.75; %date variable for plots
```

Here it is critical for  $x$  to have the same size as the data you are plotting. To check the sizes (or length of each variable) you can type this in the Command window:

```
>> length(gdp_cyc)
ans =
243
>> length(x)
ans =
243
```

The output should have been plots similar to these (this is the Australian example, in your case you did the same but the US data):



- (b) For the cycle of each variable compute the standard deviation, the correlation with the output, and the autocorrelation.

(Ans) The standard deviation of the cycle, and the correlation with the output were obtained with the following commands:

```
>> std(gdp_cyc)
std(con_cyc)
ans =
0.0135
ans =
0.0140
>> corr1=corr(gdp_cyc,gdp_cyc)
corr2=corr(gdp_cyc,con_cyc)
corr1 =
1.0000
corr2 =
0.5432
```

The standard deviations imply that for my data (Australia) the volatility of the consumption is similar to that of the Output (similar findings are obtain for US data, although it may

depend on whether consumption of durables goods is included or not). The second part implies that there is a positive correlation between the GDP and consumption (not a perfect correlation, where the value would be 1, but still positive and large). The latter implies that the consumption is a procyclical variable.

Finally the autocorrelation is obtained with:

```
>> cor_gdp = autocorr(gdp_cyc) %first order corr will be the second value
cor_con = autocorr(con_cyc)
cor_gdp =
1.0000
0.6481
.
.
.
```

You were asked for the first order autocorrelation, i.e., the correlation between  $y_t$  and its lag one period before  $y_{t-1}$ . That will be given by the second value (in my case 0.6481).

- (c) What can you interpret about these variables after updating? Do you think we are still going through a Great Moderation of the business cycles as in the early 2000's?

(Ans) Lastly, you were asked to interpret and guess whether we're still going through a Great Moderation Phase. It does not seem to be the case: Not only did we experience the Great Financial Crisis of 2008, but also we are currently went through a deep recession, also denominated the Great Lockdown.

3. **(Autorregressive processes)** The following process, denoted AR(1), describes the behavior of  $y_t$ :

$$y_t = \alpha y_{t-1} + \epsilon_t \quad (1)$$

$$\epsilon_t \stackrel{i.i.d.}{\sim} N(0, \sigma^2) \quad (2)$$

With  $|\alpha| < 1$ . Notice the error term ( $\epsilon_t$ ) is a white noise process, exogenous and uncorrelated with current and past realizations of  $y$ . Also, this AR(1) process is characterized by having well defined moments (expectation, variance, etc.) that do not depend on  $t$  (i.e.  $E[y_t] = E[y_{t-1}]$  and the same for higher order moments).

- (a) Using the expectation operator ( $E[\cdot]$ ) obtain the expected value of  $y_t$

(Ans) Expected value of  $y_t$ :

Apply the expectation operator to (1):

$$E[y_t] = E[\alpha y_{t-1}] + E[\epsilon_t]$$

Now, remember the expectation does not depend on  $t$ , i.e., is constant, let's call that expectation  $\mu_y$ . Then  $E[y_t] = E[y_{t-1}] = \mu_y$ . Also, remember that we can factor the constant  $\alpha$  from the expectation and that we can replace  $E[\epsilon_t] = 0$ :

$$\mu_y = \alpha\mu_y + 0$$

solve for  $E[y_t] = \mu_t$ :

$$\mu_y = \frac{1}{1 - \alpha} 0 = 0$$

- (b) Obtain the variance of  $y_t$  (Hint: you can find the variance of a random variable  $X$  as  $V(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$ ).

(Ans) Variance of  $y_t$ :

Approach 1: apply variance operator

$$\begin{aligned} \text{Var}[y_t] &= \text{Var}[\alpha y_{t-1} + \epsilon_t] \\ &= \text{Var}[\alpha y_{t-1}] + \text{Var}[\epsilon_t] + 2\text{Cov}(\alpha y_{t-1}, \epsilon_t) \\ &= \alpha^2 \text{Var}[y_{t-1}] + \sigma_\epsilon^2 + 0 \end{aligned}$$

Where in the second line we use that  $\text{Var}(A + B) = \text{Var}(A) + \text{Var}(B) + 2\text{Cov}(A, B)$ , also the fact that a constant is factored out squared from a variance ( $\text{Var}(\alpha X) = \alpha^2 \text{Var}(X)$  where  $X$  is a random variable), and the fact that the error  $\epsilon_t$  is **independent**, and identically distributed (i.i.d.), meaning that the covariance with  $y_{t-1}$  is zero.

Finally, the variance of  $y_t$  will also not depend on  $t$ , i.e.,  $\text{Var}(y_t) = \text{Var}(y_{t-1}) = \sigma_y^2$ .

Replacing and solving for  $\sigma_y^2$ :

$$\sigma_y^2 = \frac{\sigma_\epsilon^2}{1 - \alpha^2}$$

Approach 2: Find  $E[y_t^2]$  and replace in variance formula:  $\text{Var}(y_t) = E[y_t^2] - (E[y_t])^2$

Before we do this, notice that since the expectation of the error is zero, it follows that  $\text{var}(\epsilon_t) = E[\epsilon_t^2] = \sigma_\epsilon^2$ .

Now let's find  $E[y_t^2]$ :

$$\begin{aligned} E[y_t^2] &= E[(\alpha y_{t-1} + \epsilon_t)(\alpha y_{t-1} + \epsilon_t)] \\ &= E[\alpha^2 y_{t-1}^2 + 2\alpha y_{t-1} \epsilon_t + \epsilon_t^2] \\ &= \alpha^2 E[y_{t-1}^2] + \cancel{2\alpha E[y_{t-1} \epsilon_t]} + \sigma_\epsilon^2 \end{aligned}$$

Where in the last line, we used for the second to last term that the error ( $\epsilon_t$ ) is independent of  $y_{t-1}$  and hence their joint expectation is zero, and for the last term that the expected

value of the error squared is the variance.

We have that the expected value of the square of  $y$  will also not depend on time:  $E[y_t^2] = E[y_{t-1}^2]$ . Replacing this and solving for  $E[y_t^2]$  we have:

$$E[y_t^2] = \frac{1}{1 - \alpha^2} \sigma_\epsilon^2$$

Finally, notice that this is already the variance of  $y_t$  given that the expected value of  $y_t$  is zero:

$$Var(y_t) = E[y_t^2] - (E[y_t])^2 = E[y_t^2] - 0 = \frac{1}{1 - \alpha^2} \sigma_\epsilon^2$$

(c) find the covariance of  $y_t$  with  $y_{t-1}$ . (Hint: in general  $Cov(AB) = \mathbb{E}[AB] - \mathbb{E}[A]\mathbb{E}[B]$ )

(Ans) Covariance of  $y_t$  and  $y_{t-1}$ :

With the results we have so far we can set the formula of the covariance and easily obtain its value:

$$\begin{aligned} Cov(y_t, y_{t-1}) &= E[y_t y_{t-1}] - \cancel{E[y_t]}^0 \cancel{E[y_{t-1}]}^0 \\ &= E[(\alpha y_{t-1} + \epsilon_t) y_{t-1}] \\ &= \alpha E[y_{t-1}^2] + \cancel{E[y_{t-1} \epsilon_t]}^0 \\ &= \alpha \frac{1}{1 - \alpha^2} \sigma_\epsilon^2 = \frac{\alpha}{1 - \alpha^2} \sigma_\epsilon^2 \end{aligned}$$

In the first line, we replaced the expected value of  $y_t$  and  $y_{t-1}$  (both zero). Then in the second to last line we drop the expected product of the error in  $t$  with  $y_{t-1}$  (zero due to independence of the error with previous observations of  $y$ ).