Problem set # 5

Answer Key

1. **(Romer 5 ed. 8.6 - Excess Smoothness)** Suppose that $C_t = \frac{r}{1+r}\left\{A_t + \sum_{s=0}^{\infty} \frac{E_t[Y_{t+s}]}{(1+r)^s}\right\}$ and that $A_{t+1} = (1+r)(A_t + Y_t - C_t)$, where A_t denotes the wealth at time t, C_t the consumption, Y_t the income (also at time t for both), and r the interest rate which we assume constant over time. Notice that this is the same formula we discussed in the lectures when we mentioned the consumption is given by a fixed-average- fraction of the expected lifetime resources.

   (a) Show that these assumptions imply that $E_t[C_{t+1}] = C_t$ (and thus consumption follows a random walk) and that $\sum_{s=0}^{\infty} \frac{E_t[C_{t+s}]}{(1+r)^s} = A_t + \sum_{s=0}^{\infty} \frac{E_t[Y_{t+s}]}{(1+r)^s}$.

   This is important because it helps you understand why the random walk property is crucial if we want to think of consumption as a fixed and predetermined fraction of the future expected resources.

   [Hint: to obtain the formula for C_t we actually used the random walk assumption, then here you just have to verify the formula is consistent with the assumption made. A good way to go about it is to find an expression for $E_t[C_{t+1}]$ and compare it to that of C_t, when doing so don't forget to use the formula for A_{t+1} when setting C_{t+1}; for the second part, realize that $\frac{r}{1+r} = [\sum_{s=0}^{\infty} \frac{1}{(1+r)^s}]^{-1}$ and that the random walk result allows you to take in an out of the summation the consumption.]

(Ans) Get an expression for C_{t+1}

$$C_{t+1} = \frac{r}{1+r}\left\{A_{t+1} + \sum_{s=0}^{\infty} \frac{\mathbb{E}_{t+1}Y_{t+1+s}}{(1+r)^s}\right\}$$

Substitute $A_{t+1} = (1+r)(A_t + Y_t - C_t)$:

$$C_{t+1} = \frac{r}{1+r}\left\{(1+r)(A_t + Y_t - C_t) + \sum_{s=0}^{\infty} \frac{\mathbb{E}_{t+1}Y_{t+1+s}}{(1+r)^s}\right\}$$

Simplify the expression and apply the expectation operator $\mathbb{E}_t$

$$\mathbb{E}_t C_{t+1} = rA_t + rY_t - rC_t + \frac{r}{(1+r)} \sum_{s=0}^{\infty} \frac{\mathbb{E}_t Y_{t+1+s}}{(1+r)^s}$$

Where in the last term we use that $\mathbb{E}_{t+1}\mathbb{E}_t Y_{t+1+s} = \mathbb{E}_t Y_{t+1+s}$.
Subtract C_t from both sides:

$$\mathbb{E}_t C_{t+1} - C_t = rA_t + rY_t - (1+r)C_t + \frac{r}{(1+r)} \sum_{s=0}^{\infty} \frac{\mathbb{E}_t Y_{t+1+s}}{(1+r)^s}$$

Substitute C_t on the RHS, also put the $(1+r)$ of the denominator of the last term into the summation:

$$\mathbb{E}_t C_{t+1} - C_t = rA_t + rY_t - \cancel{(1+r)}\frac{r}{\cancel{1+r}}\left\{A_t + \sum_{s=0}^{\infty}\frac{E_t[Y_{t+s}]}{(1+r)^s}\right\} + r\sum_{s=0}^{\infty}\frac{\mathbb{E}_t Y_{t+1+s}}{(1+r)^{s+1}}$$

All that remains is to realize (and show) that the RHS is equal to zero, completing our proof that $\mathbb{E}_t C_{t+1} - C_t = 0$:

$$\begin{aligned}
\mathbb{E}_t C_{t+1} - C_t &= \cancel{rA_t} + rY_t - \cancel{rA_t} - r\sum_{s=0}^{\infty}\frac{E_t[Y_{t+s}]}{(1+r)^s} + r\sum_{s=0}^{\infty}\frac{\mathbb{E}_t Y_{t+1+s}}{(1+r)^{s+1}} \\
&= \cancel{rY_t} - r\cancel{\frac{\mathbb{E}_t Y_t}{(1+r)^0}} - r\sum_{s=1}^{\infty}\frac{E_t[Y_{t+s}]}{(1+r)^s} + r\sum_{s=0}^{\infty}\frac{\mathbb{E}_t Y_{t+s+1}}{(1+r)^{s+1}} \\
&= 0
\end{aligned}$$

where we also used that $r\sum_{s=1}^{\infty}\frac{E_t[Y_{t+s}]}{(1+r)^s} = r\sum_{s=0}^{\infty}\frac{\mathbb{E}_t Y_{t+s+1}}{(1+r)^{s+1}}$ (if this is not clear, plug the first terms in each summation and see how it corresponds exactly to the first terms of the other).

(b) Suppose that $\Delta Y_t = \phi\Delta Y_{t-1} + u_t$, where u is white noise. Suppose that Y_t exceeds $E_{t-1}[Y_t]$ by 1 unit (that is, suppose $u_t = 1$). By how much does consumption increase?

Here, we are just thinking in term of impulse responses in the income growth. We are particularly interested in knowing what is the response of consumption after the income turned out to be different from what expected. Why would consumption act out after the shock? you will see here that a shock today, will change the stream of expected income for some periods ahead and that should be accounted when revising our consumption.

[Hint: set an expression for $E_{t-1}[C_t]$ and obtain $C_t - E_{t-1}[C_t]$, do the same with the present value of the expected lifetime income (the Y part in the expression for consumption), this part will be affected by the shock, particularly due to its persistence over time, i.e., ϕ.]

(Ans) We obtain an expression for $E_{t-1}C_t$:

$$E_{t-1}[C_t] = \frac{r}{1+r}\left[A_t + \sum_{s=0}^{\infty}\frac{E_{t-1}[Y_{t+s}]}{(1+r)^s}\right]$$

With this expression we can get the change in the consumption after the shock:

$$\begin{aligned}
C_t - E_{t-1}[C_t] &= \frac{r}{1+r}\left[A_t + \sum_{s=0}^{\infty}\frac{E_t[Y_{t+s}]}{(1+r)^s}\right] - \frac{r}{1+r}\left[A_t + \sum_{s=0}^{\infty}\frac{E_{t-1}[Y_{t+s}]}{(1+r)^s}\right] \\
&= \frac{r}{1+r}\sum_{s=0}^{\infty}\frac{E_t[Y_{t+s}] - E_{t-1}[Y_{t+s}]}{(1+r)^s} \qquad (1)
\end{aligned}$$

Now, we know that $u_t = 1$, i.e., $Y_t - E_{t-1}[Y_t] = 1$.

For period $t+1$ we also know that $\Delta Y_{t+1} = \phi \Delta Y_t + u_{t+1}$, then it follows that:

$$E_t[Y_{t+1} - Y_t] = \phi(Y_t - Y_{t-1}) + 0 \tag{2}$$

and similarly,

$$E_{t-1}[Y_{t+1} - Y_t] = \phi(E_{t-1}[Y_t] - Y_{t-1}) + 0 \tag{3}$$

then take (2) and substract (3):

$$\begin{aligned} E_t[Y_{t+1}] - E_{t-1}[Y_{t+1}] - (\cancelto{1}{E_t[Y_t] - E_{t-1}Y_t}) &= \phi(\cancelto{1}{E_t[Y_t] - E_{t-1}Y_t}) - \cancelto{0}{\phi Y_{t-1} - Y_{t-1}} \\ E_t[Y_{t+1}] - E_{t-1}[Y_{t+1}] &= 1 + \phi \end{aligned}$$

Then: $\frac{E_t[Y_{t+1}] - E_{t-1}[Y_{t+1}]}{1+r} = \frac{1+\phi}{1+r}$. Following an analogous argument (and similar algebra steps): $\frac{E_t[Y_{t+2}] - E_{t-1}[Y_{t+2}]}{(1+r)^2} = \frac{1+\phi+\phi^2}{(1+r)^2}$, $\frac{E_t[Y_{t+3}] - E_{t-1}[Y_{t+3}]}{(1+r)^3} = \frac{1+\phi+\phi^2+\phi^3}{(1+r)^3}$, and so on so forth for any $s > 2$, i.e. $\frac{E_t[Y_{t+s}] - E_{t-1}[Y_{t+s}]}{(1+r)^s} = \frac{\sum_{j=0}^{s} \phi^j}{(1+r)^s}$.

Then:

$$\sum_{s=0}^{\infty} \frac{E_t[Y_{t+s}] - E_{t-1}[Y_{t+s}]}{(1+r)^s} = 1 + \frac{1+\phi}{1+r} + \frac{1+\phi+\phi^2}{(1+r)^2} + \frac{1+\phi+\phi^2+\phi^3}{(1+r)^3} + \cdots$$

we can rewrite the RHS of this expression, by grouping the common factors in the denominators:

$$\begin{aligned} \sum_{s=0}^{\infty} \frac{E_t[Y_{t+s}] - E_{t-1}[Y_{t+s}]}{(1+r)^s} = \left[1 + \frac{1}{1+r} + \frac{1}{(1+r)^2} + \cdots\right] + \left[\frac{\phi}{1+r} + \frac{\phi}{(1+r)^2} + \frac{\phi}{(1+r)^3} + \cdots\right] + \\ \left[\frac{\phi^2}{(1+r)^2} + \frac{\phi^2}{(1+r)^3} + \cdots\right] + \cdots \end{aligned}$$

Each of these terms on the RHS is an infinite summation of terms, each lower or equal to one, hence they are a geometric series that converge to a given value:

For simplicity, let $\gamma = \frac{1}{1+r}$, then we can find what each term converges to:

$$\left[1 + \frac{1}{1+r} + \frac{1}{(1+r)^2} + \cdots\right] = \sum_{s=0}^{\infty} \gamma^s = \frac{1}{1-\gamma}$$

$$\left[\frac{\phi}{1+r} + \frac{\phi}{(1+r)^2} + \frac{\phi}{(1+r)^3} + \cdots\right] = \gamma\phi \sum_{s=0}^{\infty} \gamma^s = \frac{\gamma\phi}{1-\gamma}$$

$$\left[\frac{\phi^2}{(1+r)^2} + \frac{\phi^2}{(1+r)^3} + \cdots\right] = (\gamma\phi)^2 \sum_{s=0}^{\infty} \gamma^s = \frac{(\gamma\phi)^2}{1-\gamma}$$

And so on, so forth, then:

$$\sum_{s=0}^{\infty} \frac{E_t[Y_{t+s}] - E_{t-1}[Y_{t+s}]}{(1+r)^s} = \frac{1}{1-\gamma}\left[1 + \gamma\phi + (\gamma\phi)^2 + (\gamma\phi)^3 + \cdots\right] = \frac{1}{1-\gamma}\frac{1}{1-\gamma\phi}$$

replacing γ back:

$$\sum_{s=0}^{\infty} \frac{E_t[Y_{t+s}] - E_{t-1}[Y_{t+s}]}{(1+r)^s} = \frac{1}{1-\gamma}\frac{1}{1-\gamma\phi} = \frac{1+r}{r}\frac{1+r}{1+r-\phi}$$

Then, substitute in (1) to obtain the change in consumption after a $u_t = 1$ shock:

$$C_t - E_{t-1}[C_t] = \frac{r}{1+r}\left[\frac{1+r}{r}\frac{1+r}{1+r-\phi}\right] = \frac{1+r}{1+r-\phi}$$

In general, if you consider a shock of a different size, you would find that the size of the consumption change is:

$$C_t - E_{t-1}[C_t] = \frac{r}{1+r}\left[\frac{1+r}{r}\frac{1+r}{1+r-\phi}u_t\right] = \frac{1+r}{1+r-\phi}u_t$$

(c) For the case of $\phi > 0$, which has a larger variance, the innovation in income, u_t, or the innovation in consumption, $C_t - E_{t-1}[C_t]$? Do consumers use saving and borrowing to smooth the path of consumption relative to income in this model? Explain.

We are interested in knowing how the volatility of the shock translates into the volatility of the consumption adjustment. During lecture and in the tech sessions, we discussed how the endogenous variables end up absorbing the time series (mean, volatility, etc.) properties of the shocks. This is no exception. Of particular interest is to know whether the consumption will over-react or under-react to the shock which is something we can know by determining whether it is more volatile or not.

[Hint: Given it is a white noise, the variance of u_t is just the parameter σ^2. Compare that to the variance of the result you got in (b)]

(Ans) Here we compare $Var(C_t - E_{t-1}[C_t])$ with $Var(u_t) = \sigma^2$.

With the result in (b) we can find the variance of the innovation in consumption (revision in consumption after the shock):

$$Var(C_t - E_{t-1}[C_t]) = Var\left(\frac{1+r}{1+r-\phi}u_t\right) = \left(\frac{1+r}{1+r-\phi}\right)^2 Var(u_t) > Var(u_t)$$

The latter shows that the consumption innovation is more volatile than that of the innovation of income (i.e. the variance of the shock) since $\left(\frac{1+r}{1+r-\phi}\right)^2 > 1$

Why consumption is overreacting if it's supposed to be smoother?

Here the increase in income (relative to what was expected) is larger over time (income is not stationary, only its change is), as income is expected to increase by even more in the future, the agent will incorporate that into its new consumption decision which implies an instantaneous overreaction to the shock.

We can give an example to make it more intuitive: if income is going up by \$10 in period 1, then by \$11 in period 2, and by \$12 in period 3 (and assume for simplicity there are no more periods considered). The consumption of period 1 would be smoother if it is increased by \$11 instantaneously (and for each period), than if we increase it by \$10, then by \$11 and finally by \$12. As a result, the initial increase of the consumption (\$11) will be larger than that of the income (\$10).

2. **(Comparison of models solutions)** Here you are going to run the Dynare files associated to two known models in the literature and compare their results.

   (a) Consider the file "Gali_2008_chapter_2.mod", it contains the code for setting up and solving the RBC model of the Chapter 2 in Jordi Gali's 2008 book[1]. This corresponds to one of the types of models we mentioned when discussing the approaches to model the aggregate supply (the one without frictions and instantaneous market clearing). Run the file and report its output and impulse responses.

(Ans) Just need to run the file (Gali_2008_chapte_2.mod).

   (b) Now, consider the file "Gali_2008_chapter_3.mod". It contains the code for setting up and solving the New-Keynesian model of the Chapter 3 in the same book. This is also one of the types of models we covered, where we add nominal rigidities to the frictionless RBC framework. Run the file and report its output and impulse responses.

(Ans) Just need to run the file (Gali_2008_chapte_3.mod).

   (c) Now, let's study the implications on money neutrality. Before checking the IRF's, what are you expecting?. Should money be neutral in either of the models? why or why not?

(Ans) Ex-ante you would expect money to be neutral in the model without frictions (price stickyness and competitive firms), and to be non-neutral in the second model with frictions (price stickyness and monopolistic competition).

   (d) Now take a look at the IRF's of each model. If the response of a variable to a certain shock is zero at all times, Dynare does not report an IRF (after all, there's no "impulse" to show in that case). That is why for either shock you may have a different number of graphs showing up. How can you relate that (the impulses showing up) to the money neutrality? [hint: focus in the case of a monetary shock, i.e., the shocks to "eps_m" for the first model and "eps_nu" for the second model. If it helps, also compare the responses in those cases with that of "real" technological shocks "eps_A" or "eps_a"]

[1] Galí, Jordi, (2008)."Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework and Its Applications" Economics Books, Princeton University Press, first edition.

(Ans) In the first case (ch.2, frictionless), after the monetary shock only the nominal variables, inflation and money growth respond. The response in the other variables is null, reflecting money is neutral (hence their IRF is not displayed). Conversely, in the case with price rigidities (ch. 3, New Keynesian model) the real variables do respond and their IRF is displayed.