Topic 4: Consumption and Aggregate Demand

We are gathering the building blocks of a standard NK DSGE model

We analyzed the supply side of the model (AS): NKPC

$$\pi_t = \beta E_t \pi_{t+1} + \lambda y_t + \varepsilon_{\pi,t}$$

OR $$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + \varepsilon_{\pi,t} \qquad \text{(NKPC)}$$

Where x_t is used to more explicitly denote **output *gap*** (output deviation from trend; what we have been calling y_t), and $\varepsilon_{\pi,t}$ is a shock to inflation dynamics (white noise, AR process)

Now we will obtain the analog to an IS (AD) equation for our model:

Focus on aggregate demand of the economy. At its simplest: **Consumption**

What's new? Emphasis on how consumption is an intertemporal decision. We trade-off current with future consumption (savings)

Once again, it's all about **dynamics and expectations** about future variables

- Due to this, the approach is the same as with the AS: incorporate 1 rational expectations and 2 micro-foundations (agent's optimal decision-making process)

  & 3 Connection to data/empirics

- From old AD: $y_t + p_t = m_t + v_t$, to what's called "**the Dynamic IS**" equation (we will see where this comes from next)

$$x_t = E_t x_{t+1} - \phi[i_t - E_t \pi_{t+1}] + \varepsilon_{x,t} \qquad \text{(Dynamic IS/ "new" AD)}$$

Where i_t : nominal interest rate (so $i_t - E_t \pi_{t+1}$: real interest rate), and $\varepsilon_{x,t}$ is a (demand) shock to output each period

Can you see where we are going?

- Remember back to Lucas model, we have AS, AD, and monetary policy
- We now have the modern version:

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + \varepsilon_{\pi,t} \qquad \text{(NKPC - AS)}$$

$$x_t = E_t x_{t+1} - \phi[i_t - E_t \pi_{t+1}] + \varepsilon_{x,t} \qquad \text{(Dyn. IS - AD)}$$

Just need to add monetary policy, e.g. the Fed sets interest rates in response to economic conditions:

$$i_t = d_\pi \pi_t + d_x x_t + \varepsilon_{x,t} \qquad \text{(MP)}$$

- The above is a system of 3 "dynamic stochastic general equilibrium" (DSGE) equations! (to solve/analyze them, we will use Dynare in Matlab)

↳ HW4, HW5
Tech Session Slides #3

Consumption:

Motivation/Big picture: Why should we care?

- C is the largest component of GDP
- Explain C = explain S => implications for K accumulation => GDP growth
- Risk: link between consumption and asset pricing

agents smooth consumption

What are we trying to explain?

- Facts: even though C is highly correlated with Y, C is much less volatile
  (e.g post-war US data: Y bounced around, C rather smooth, I volatile)
- Is C-growth predictable?: $C_t = E_t(C_{t+1})$
  Hall's random walk result: excess sensitivity; excess smoothness
- What's the link between C and asset returns?
  Is the high return observed in the equity markets consistent with optimized C-smoothing behavior? (Equity premium puzzle)

Theories and Empirical Tests: consider three broad eras

1) Keynes, Modigliani, Fisher, Friedman (pre-Rational Expectation)

- Focus on the SR and LR relationship b/w C_t and Y_t, life-time resources (e.g., Life-cycle hypothesis, Permanent Income Hypothesis)

2) Uncertainty and Rational Expectations come in:

- Stochastic models & implications (Hall's Random Walk)
- Dynamic Programming techniques come in (to deal with Expectations)
- Linearization and Certainty Equivalence

3) Beyond "Certainty Equivalence" (Post-Hall)

- 2^{nd} order effects: Var & Cov; Asset pricing; Equity Premium Puzzle

I. Keynes:

Postulate a linear relationship between aggregate C & Y:

$$C_t(Y_t) = a + bY_t$$

(ad hoc rule, no micro-foundation)

Three observations:

i) Marginal Propensity to Consume (MPC) is less than 1, i.e. $0 < b < 1$

ii) Average Propensity to Consume declines as income rises

$$APC = \frac{C}{Y} = \frac{a}{Y} + b \quad ; a \neq 0$$

iii) Interest rates don't matter much

Data:

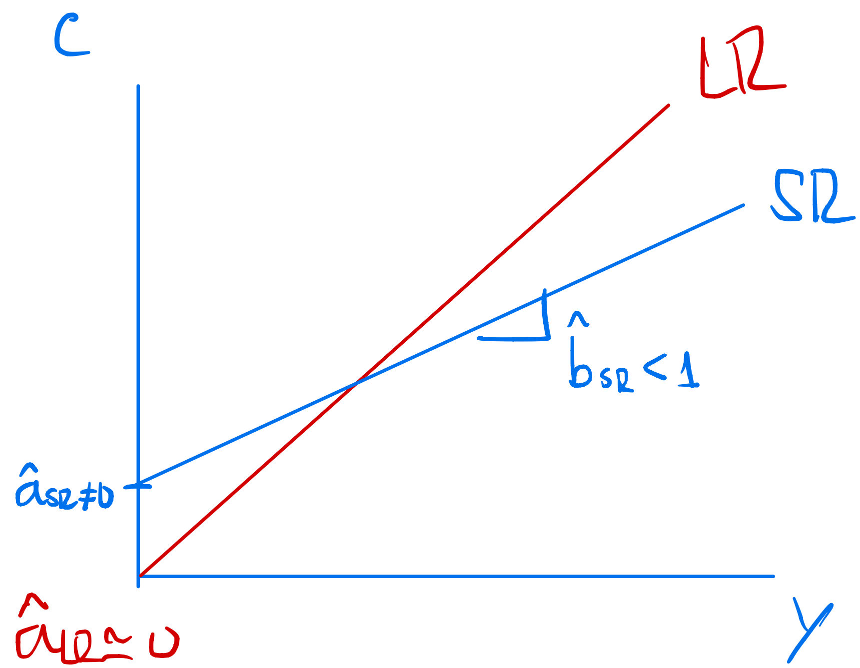
Look at a cross-section of households and their consumption pattern:

- See different pattern for SR vs. LR

SR (time series data in the 1930's): all three claims seem to hold

LR (1880 – 1970s):

(ii) no longer true



$\hat{a}_{LR} \simeq 0$

$\hat{b}_{LR} \simeq 1$

Friedman/Modigliani/Fisher's Explanation:

- C shifts over time! (or Y-C relation does)
- People make consumption decision over life-time income, so C vs. S is not a period-by-period decision.

$\left(\beta = \frac{1}{1+\rho}\right)$

ρ: Subjective discount rate

r: Interest rate

e.g. **Fisher**:

at t=0:

$$\max_{\{C_t\}_{t=0}^{T}} \sum_{t=0}^{T} \overbrace{(1+\rho)^{-t}}^{\beta^t} u(C_t) \quad (1)$$

s.t. $$\underbrace{\sum_{s=0}^{T}(1+r)^{-s}Y_s}_{\text{NPV of income}} + W_0 = \underbrace{\sum_{s=0}^{T}(1+r)^{-s}C_s}_{\text{PV of consumption spending}} \quad (2)$$

Lifetime budget Constraint

Where ρ is the subjective discount rate, r the real interest rate, and W_0 is the wealth at t=0

Set up $\underbrace{\text{lagrangian}}$, and assume $\underline{\rho = r}$ as a simplification:

$\mathcal{L} = ① - \lambda \cdot ②$ FOC $\frac{\partial \mathcal{L}}{\partial C_{t+j}} \Rightarrow U'(C_{t+j}) = \lambda \left(\frac{1+\rho}{1+r}\right)^{j}$ $\forall j$, for all $t+j$

$\rho = r \Rightarrow U'(C_{t+j}) = \lambda$ (Constant) $\Rightarrow$ Consumption (C_t) is constant ; $C_t = \bar{C}$ $\forall t$

$\Rightarrow$ replace $C_s = \bar{C}$ in budget constraint & we solve for $\bar{C}$:

$$\bar{C} = \frac{1}{\sum_{s=0}^{T}(1+r)^{-s}}\left[\sum_{s=0}^{T}(1+r)^{-s}\underbrace{Y_s}_{} + W_0\right]$$

Thus:

- C-decision is dynamic and depends on income profile over time.
  So cannot just draw inference between any contemporaneous C_t and Y_t

- Consumption is dictated by the expected income stream, not by its volatility

Replace by $E[Y_s]$ $\Rightarrow$ for Certainty Equivalence result (of R. Hall)

Modigliani: Emphasize on the life-cycle pattern of income path

Friedman (Permanent Income Hypothesis, "PIH"):

Emphasize Permanent vs. Transitory aspects of the income process

$$C = C^P + C^T$$

$$Y = \underbrace{Y^P}_{\text{Stable}} + \underbrace{Y^T}_{\text{Noisy (Shocks)}}$$

Fisher's relation is true only for the permanent component:

$$C^P = Y^P, \qquad \text{with } a = 0,\ b = 1$$

$C_S = \bar{C}$ (If permanent income doesn't change)

But empirically, we only observe C and Y we observe the sum

So when we run a regression of: $C = \hat{a} + \hat{b}Y + \epsilon$

$$b = \frac{Cov(C, Y)}{Var(Y)} \approx \frac{Cov(Y^P, C^P)}{Var(Y^P) + Var(Y^T)}$$

$C^T + C^P$ $Y^T + Y^P$ $Y^T + Y^P$

Assume C^T and Y^T are white noise: (affects covariances) $(Cov(Y^T, Y^P) = 0,\ Cov(C^T, C^P) = 0)$

Fisher

Given the structural model with $\boxed{C^P = Y^P}$, a = 0, and b = 1, we still will see the Keynesian prediction of a > 0 and b < 1 in the SR due to the presence of NOISE (temporary shocks) => “Measurement error bias”

$$\Rightarrow \hat{b} = \frac{Cov(Y^P, C^P)}{Var(Y^P) + Var(Y^T)} = \frac{Var(Y^P)}{Var(Y^P) + Var(Y^T)} < 1$$

Reconciling the SR vs. LR data patterns:

$$\hat{a} = \bar{C} - \hat{b}\bar{Y} = (1 - \hat{b})\bar{Y} \neq 0$$

- SR: Var(Y^T) more prominent (little change in Y^P)
- LR: Var(Y^P) >> Var(Y^T), so see b^{LR} close to 1

Common thread established: Consumption is an intertemporal decision

(thus C decision is "dynamic" and not just a matter of looking at current income)

With that in mind, go back to deriving the new (dynamic) AD:

Life-time Utility Maximization under Rational Expectation:

- At time t, consider a **representative agent** who sets the sequence of consumption over time (lifetime) in order to maximize overall life-time utility:

at $t=0$

$$\max_{\{c_s\}_{s=0}^{\infty}} E_t \sum_{s=t}^{\infty} \beta^{s-t} u(c_s) \quad = E_0\left[u(c_0)+\beta u(c_1)+\beta^2 u(c_2)+\dots\right]$$

Subject to:

$$\begin{cases} w_{t+1} = \tilde{y}_{t+1} + \tilde{R}_{t,t+1}\underbrace{(w_t - c_t)}_{\text{Saving}}, \ \forall t \\ c_t \in [0, \ w_t] \\ w_0 \ given \end{cases}$$

where β is the discount factor; w_t is wealth at time t; $\tilde{y}_{t+1}$ is (stochastic) income at t+1; and $\tilde{R}_{t,t+1}$ is the (stochastic) return in t+1 from savings at t ($= 1 + \tilde{r}_{t,t+1}$)

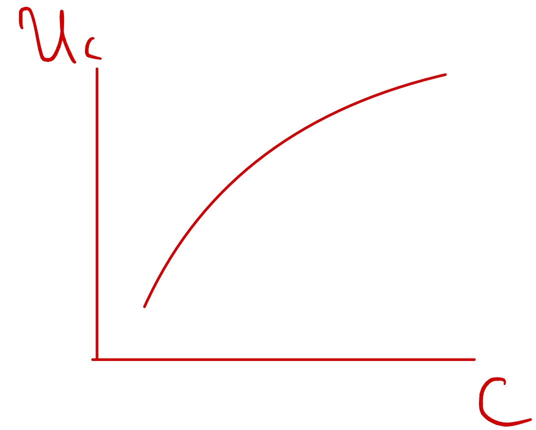
R_t, R_{t+1}, $R_{t,t+1}$ | $R_{t,t+1} = 1+r_{t,t+1}$

Set $\mathcal{L} = E_t \sum \beta^t \{u(c_t) + \lambda_t(y_{t+1} + R_{t,t+1}(w_t - c_t) - w_{t+1})\}$

get FOC wrt $[c_t]$, $[w_{t+1}]$, $[\lambda_t]$

$\rightarrow$ at the end: $u'(c_t) = \beta E_t[\underbrace{\tilde{R}_t}_{R_{t,t+1}} u'(c_{t+1})]$

Additional Assumptions:



- Concave utility function: $u' > 0; u'' < 0$

- Inada conditions: $u'(0) = \infty, u'(\infty) = 0$

- "No Ponzi game" condition: (can't die in debt/run permanent debt)

$$\lim_{T\to\infty} \left(\frac{1}{1+r}\right)^T w_T \geq 0$$

- Notation: strictly speaking $R_{t,t+1} = 1 + r_{t,t+1}$ is the gross rate set at t that pays in t+1; before we'simplified this notation and used $R_{t+1} = 1 + r_{t+1}$ (e.g., in Topic 2).
    - Moreover, we can also (and will do in the following slides) denote this rate as $R_t = 1 + r_t$; just be clear that it is the rate paid in t+1

- Note: discount factor $\beta = \frac{1}{1+\rho}$ where ρ is the discount rate

Perturbation Argument and the Euler Equation:

- Various methods to "solve" this dynamic optimization problem
- we will focus on the perturbation argument for its intuitive insight
  - In Topic 2 we used a similar intuitive approach (and result was the same)
  - We can also do the optimization problem math to get the result

The optimality condition for the optimal consumption sequence: $\{c_t^*\}_{t=0}^{\infty}$ is:

$$u'(c_t^*) = \beta E_t[u'(c_{t+1}^*)R_{t+1}] \qquad \textbf{(Euler Equation)}$$

- In words:

If consumption is allocated optimally over time, the its marginal utility at t must equal the discounted expected marginal utility at t+1 times the gross interest rate

Present Ut. of Consumption (marginal) = Present mrgnal utility of saving = Present value of Future marginal utility of consumption (times the gross interest rate)

Euler Equation: (for notation simplicity: drop the * on c's)

$$u'(c_t) = \beta E_t[u'(c_{t+1})R_t]$$

or, $$u'(c_t) = \frac{1}{1+\rho} E_t[u'(c_{t+1})(1 + r_t)]$$

Intuition:

EEq must hold in equilibrium, to see why, assume: $u'(c_t) > \beta E_t[u'(c_{t+1})R_t]$

$\Rightarrow$ Should bring resources from future into present

- One could relocate resources to increase overall utility
- Improvement: shift consumption from t+1 to t,
- If the LHS > RHS you could continue this reallocation
- This process lowers MU at t and raises MU at t+1, until LHS = RHS

$\uparrow C_t \Rightarrow u'(C_t)\downarrow \rightarrow$ LHS$\downarrow$

$\hookrightarrow \downarrow C_{t+1} \Rightarrow u'(C_{t+1})\uparrow \rightarrow$ RHS$\uparrow$

... Continue adjusting until

LHS = RHS (Euler Equation)

The Euler equation can be derived using a “**perturbation argument**”:

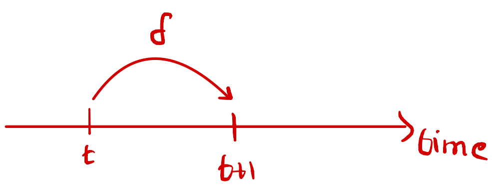
- start with the optimal sequence $\{c_t^*\}_{t=0}^{\infty}$

- imagine a small perturbation at time t to c_t^*, say by amount $-\delta$

$$\Delta c_t^* = -\delta$$

- save it and consume it next period, t+1,

$$\Delta c_{t+1}^* = \delta \mathbb{E}_t \tilde{R}_{t,t+1}$$



- keep all other c_{t+j}^* the same

- What’s the effect on the overall life-time utility?

Change at t   Change at t+1

$$\Delta \text{Utility} = -\delta u'(c_t) + \beta \delta E_t(\tilde{R}_{t,t+1} \cdot u'(c_{t+1}))$$

Putting these elements together:

Change at t Change at $t+1$

$$\Delta \text{Utility} = -\delta u'(c_t) + \beta \delta E_t(\tilde{R}_{t,t+1} \cdot u'(c_{t+1}))$$

assume δ is very small $\Rightarrow \Delta \text{Utility} \simeq 0$ $\because \{c_t^*\}$ is the optimal sequence

$$\Rightarrow -\cancel{\delta} u'(c_t) + \beta \cancel{\delta} E_t(\tilde{R}_{t,t+1} \cdot u'(c_{t+1})) = 0$$

rearrange: $u'(c_t) = \beta \mathbb{E}_t(\tilde{R}_{t,t+1} u'(c_{t+1}))$

Or, you can set the optimization and derive the Euler Equation (as in RBC topic)

Steps: Set up Lagrangian; take FOC w.r.t. [c_t], [w_{t+1}];
Realize FOC w.r.t. c_t applies for any "t" (t+1, t+2, etc.), then set it for t+1 and take the ratio of FOCs w.r.t. c_t, c_{t+1}
From FOC [w_{t+1}] solve for ratio of λ_t, λ_{t+1} and replace in expression for ratio of FOCs of consumptions.
Rearrange and obtain Euler Equation

How do we test the Euler Equation?

- Need to make some additional assumptions

$$u'(c_t) = \frac{1}{1+\rho} E_t[u'(c_{t+1})(1+r_t)]$$

1) Relying on <u>linearizing</u> the EE (by R. Hall)

→ C_t is a random walk

Key simplifying assumptions:

1) Interest rate equals to subjective discount rate (contant): $r_t = r = \rho$

Then: $$u'(c_t) = E_t[u'(c_{t+1})]$$

=> Marginal utility follows a random walk

Remember: A stochastic time series is called a **random walk process** when

$$x_{t+1} = x_t + \varepsilon_{t+1} \quad \text{or}$$

$$\Delta x_{t+1} = \varepsilon_{t+1}$$ where ε_{t+1} is i.i.d. white noise.

Equivalently under Rational Expectations: $E_t x_{t+1} = x_t$

2) **Utility function is quadratic**: $\underline{u(c_t) = \alpha + \beta c_t - \gamma c_t^2}$

$$u'(c_t) = \beta - 2\gamma c_t$$

Plug into 1) above:

$$\beta - 2\gamma c_t = \beta - 2\gamma E_t c_{t+1}$$

$$\boxed{c_t = E_t[c_{t+1}]}$$

=> Consumption follows a random walk

- Consumption follows a random walk under assumptions 1) and 2) above:

$$c_t = E_t[c_{t+1}]$$

$$= E_t[E_{t+1}[c_{t+2}]] = E_t[c_{t+2}] \quad \text{(Law of iterated expectations)}$$

$$= E_t[c_{t+j}]$$

In words…the best prediction of any future consumption is today's consumption

Consumption smoothing: optimal comsumption path is expected to be stable

Subs $W_{t+1} = \frac{W_{t+2} + y_{t+2}}{1+r} + C_{t+1}$, then subs W_{t+2},

then Subs W_{t+3},

So on, so forth

Using "repeated substitution" of the period-by-period budget constraint:

[Period by period]

$$w_{t+1} = \tilde{y}_{t+1} + (1+r)(w_t - c_t)$$

And the No Ponzi Game condition:

$$\lim_{T\to\infty} \left(\frac{1}{1+r}\right)^T w_T \geq 0$$

We get the "life-time" budget constraint:

[Lifetime]

$$\sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t c_t = w_0 + \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t \tilde{y}_t$$

Life-time budget constraint implications for current consumption:

As usual, replace future values for their expected counterparts (here for c and y)

Then, use random walk result and substitute $\underbrace{c_t = E_t c_{t+j}}_{\text{Random walk}}$:

At any time t, it is optimal (life-time expected utility maximizing) to consume:

$$c_t = \frac{1}{\sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^s}\left[w_0 + \sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^s E_t\tilde{y}_{t+s}\right]$$

- At each t: optimal to consume a fixed fraction of the expected lifetime income
- Consumption is a function of expected life stream of income
- This implies a **consumption smoothing behavior**. **Certainty equivalency**: What is relevant is the expected income, not its volatility

Food for thought:

Compare this result with the one under Fisher:

Looks identical except we now have $E_t\tilde{y}_{t+s}$ instead of $\tilde{y}_{t+s}$

(Quadratic objective function + linear constraint gives **“certainty equivalent”** result)

Other resources if interested:

- The Random Walk Hypothesis (Romer, Ch. 8.2)
- Consumption equation is the same as Equation (8.17) in Romer, except the textbook simplifies the expression by assuming r = 0, and that one lives till period T.

Extra: Algebra for the Lifetime BC

$$W_{t+1} = y_{t+1} + (1+r)(W_t - C_t) \implies W_t = \frac{W_{t+1} - y_{t+1}}{1+r} + C_t$$

this equation holds for every period and we'll use it to substitute forward the terms $W_{t+1}, W_{t+2}, W_{t+3}, W_{t+4}, \dots$

Subs $W_{t+1} = \frac{W_{t+2} - y_{t+2}}{1+r} + C_{t+1}$ in the budget constraint:

$$\frac{W_{t+2} - y_{t+2}}{1+r} + C_{t+1} = y_{t+1} + (1+r)(W_t - C_t)$$

Subs W_{t+2}:

$$\frac{W_{t+3} - y_{t+3}}{(1+r)^2} + \frac{C_{t+2}}{1+r} - \frac{y_{t+2}}{1+r} + C_{t+1} = y_{t+1} + (1+r)(W_t - C_t)$$

$$\implies \frac{W_{t+3}}{(1+r)^2} + (1+r)C_t + C_{t+1} + \frac{C_{t+2}}{1+r} = y_{t+1} + \frac{y_{t+2}}{1+r} + \frac{y_{t+3}}{(1+r)^2} + (1+r)W_t$$

Subs W_{t+3}:

$$\frac{W_{t+4}}{(1+r)^3} - \frac{y_{t+4}}{(1+r)^3} + \frac{C_{t+3}}{(1+r)^2} + (1+r)C_t + C_{t+1} + \frac{C_{t+2}}{1+r} = y_{t+1} + \frac{y_{t+2}}{1+r} + \frac{y_{t+3}}{(1+r)^2} + (1+r)W_t$$

or $$\frac{W_{t+4}}{(1+r)^3} + (1+r)C_t + C_{t+1} + \frac{C_{t+2}}{1+r} + \frac{C_{t+3}}{(1+r)^2} = y_{t+1} + \frac{y_{t+2}}{1+r} + \frac{y_{t+3}}{(1+r)^2} + \frac{y_{t+4}}{(1+r)^3} + (1+r)W_t$$

Subs W_{t+4}:

$$(1+r)C_t + C_{t+1} + \frac{C_{t+2}}{1+r} + \frac{C_{t+3}}{(1+r)^2} + \frac{C_{t+4}}{(1+r)^3} = (1+r)W_t + y_{t+1} + \frac{y_{t+2}}{1+r} + \frac{y_{t+3}}{(1+r)^2} + \frac{y_{t+4}}{(1+r)^3} + \frac{y_{t+5}}{(1+r)^4}$$

$$\implies C_t + \frac{C_{t+1}}{1+r} + \frac{C_{t+2}}{(1+r)^2} + \frac{C_{t+3}}{(1+r)^3} + \frac{C_{t+4}}{(1+r)^4} = W_t + \frac{y_{t+1}}{1+r} + \frac{y_{t+2}}{(1+r)^2} + \frac{y_{t+3}}{(1+r)^3} + \frac{y_{t+4}}{(1+r)^4} + \frac{y_{t+5}}{(1+r)^5}$$

⋮ So on,

$$\sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^s C_{t+s} = W_t + \sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^s y_{t+s}$$

Take $t=0$ (Could do this for any $t = 1, 2, \dots$)

$$\implies \sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^s C_s = W_0 + \sum_{s=0}^{\infty}\left(\frac{1}{1+r}\right)^s y_s$$ (as in the Lecture)

Empirical Testing & "Excess Sensitivity":

- The Random Walk Hypothesis of consumption, $c_t = E_t c_{t+j}$, implies that one should consumption-smooth in expectations

For example,

Assuming interest rate r and discount rate ρ are both zero, if your income is \$2K this month and you expect your income to increase to \$3K next month, and to \$4K again in month 3.

- how should you allocate your consumption across the three months to maximize your overall utility?

- Consumption-smoothing implies setting expected consumption to be 3K across all three periods

- This means: borrow \$1K today, return it in the 3rd month

Excess Sensitivity:

- What does consumption-smoothing imply about the relation between consumption growth and expected income growth between the periods?

t	1	2	3
y	2k	3k	4k
c	3k	3k	3k

$\longmapsto$ $\Delta C_{t+1} = 0$, $\Delta Y_{t+1} = 1$ } We should see in the data that $\mathrm{Corr}(\Delta \mathrm{Consump}, \Delta \mathrm{Income}) = 0$

Given consumption is smooth and income is not (varies) …

We would expect consumption and income growth to be uncorrelated

- In the data, however, we consistently see consumption growth to be positively correlated with income growth. We call this the **Excess Sensitivity** of consumption growth (to income growth)

[Summary] Euler Equation & Earlier Tests:

1) Hall's "Excess Sensitivity"

$$u'(c_t) = \frac{1}{1+\rho} E_t[u'(c_{t+1})(1+r_t)]$$

Relying on <u>linearizing</u> the EE

Key simplifying assumptions of Hall:

1) Non-stochastic r_t, & $r_t = r = \rho$

2) $u(c_t)$ is quadratic

⇨ C follows a random walk

⇨ C-smoothing over life-time, C = fraction of expected lifetime income

⇨ Empirically, see excess sensitivity

[Summary – cont.]

Big Picture Goal: understanding "C", aggregate consumption patterns

Approach:

- formalize a framework with broad, general assumptions

  (RA, RE, life-time utility optimization, time-separable utility; no externalities, no frictions, no information problem…etc. to start with)

- Under such general settings, get **Euler Equation** (note: perturbation argument)
- Test it empirically to see if it's a good model
- Add additional assumptions to derive testable implications
- Are they supported in data? If not, systematically examine where the problems may be and propose alternatives (debugging process)
- Repeat until satisfactory outcome (most research agendas on-going…)

Note $E_t[e^{\ln[\cdot]}]$

We use: $\ln(1+x) \approx x$ for x small

$\ln\left(\frac{C_{t+1}}{C_t}\right) = \ln C_{t+1} - \ln C_t = \Delta \ln C_{t+1}$

$\ln(x^\alpha) = \alpha \ln x$

More generally to Linearization of the Euler Equation

(assume a more general u() function and allow interest rate to change)

γ: Coef of Risk Aversion

Assume:

$u(c) = \frac{c^{1-\gamma} - 1}{1-\gamma}$ (if $\gamma = 1 \rightarrow u(c) = \ln c$)

i) CRRA utility

$u'(c) = c^{-\gamma} \xrightarrow{\text{Plug in Euler Eq}} E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\left(\frac{1+r_t}{1+\rho}\right)\right] = 1 \rightarrow 1 = E_t\{e^{r_t - \rho - \gamma \Delta \ln C_{t+1}}\}$

ii) r_t is known at time t (deterministic & time-varying)

iii) $\Delta ln(C_t)$ is Normal:

(Fact: $X \sim N(\mu, \sigma^2) \rightarrow Ee^x = e^{EX + \frac{1}{2}\sigma^2}$)

$$1 = e^{[E_t r_t - \rho - \gamma E_t \Delta \ln C_{t+1} + \frac{1}{2}\gamma^2 Var_t(\Delta \ln C_{t+1})]}$$

apply logs: $0 = E_t r_t - \rho - \gamma E_t \Delta \ln C_{t+1} + \frac{1}{2}\gamma^2 Var_t(\Delta \ln C_{t+1})$

rearrange:

$$E_t \Delta ln C_{t+1} = \frac{1}{\gamma}\underbrace{(E_t r_t - \rho)}_{} + \underbrace{\frac{1}{2}\gamma Var_t(\Delta ln C_{t+1})}_{}$$

expected growth rate of C

↑ Expected Returns
↑ Savings
↑ Consumption growth

Precautionary Savings
the more uncertain consumption is the more you save today (boosts future cons. growth)

iv) Rational expectation

$$\Delta lnC_{t+1} = \frac{1}{\gamma}(r_t - \rho) + \frac{1}{2}\gamma Var_t(\Delta lnC_{t+1}) + \varepsilon_{t+1}$$

Where ε_{t+1} is orthogonal to any information known at date t

v) **Precautionary savings term:** let $Var_t(\Delta lnC_{t+1})$ is constant over time,

$$\Delta lnC_{t+1} = \alpha + \frac{1}{\gamma}r_t + \varepsilon_{t+1} \qquad (*)$$

Note:

- α absorbs both precautionary savings and ρ/γ
- C in general is NOT a RW.
  - If assuming $r = \rho$, Logarithm of C is a RW.
- C depends on r_t

Intuition: when r_t is high (i.e. price of C_t relative to C_{t+1} is high; giving up C_t trades off more C_{t+1}) => people save more => higher growth rate of C

Again, remember what the assumptions are, beyond Euler:

1) CRRA utility $U(\cdot)$

2) r_t is known at time t (deterministic and time-varying)

3) ΔlnC_t is Normally distributed.

$$E_t \Delta lnC_{t+1} = \frac{1}{\gamma}(E_t r_t - \rho) + \underbrace{\frac{1}{2}\gamma Var_t(\Delta lnC_{t+1})}_{\text{Prec. Savings}}$$

4) Rational expectation:

$$\Delta lnC_{t+1} = \underbrace{\frac{1}{\gamma}}_{\to \text{EIS}}(r_t - \rho) + \frac{1}{2}\gamma Var_t(\Delta lnC_{t+1}) + \varepsilon_{t+1}$$

Where ε_{t+1} is orthogonal to any information known at date t

5) Precautionary savings term: $Var_t(\Delta lnC_{t+1})$ is constant over time:

$$\Delta lnC_{t+1} = \alpha + \frac{1}{\gamma} r_t + \varepsilon_{t+1} \quad (*)$$

Empirical Test of this linearized Euler: $\Delta \ln C_{t+1} = \alpha + \frac{1}{\gamma} r_t + \varepsilon_{t+1}$

2 folds:

1) Estimate Elast. of Intertemporal Substitution: EIS

$$EIS = \frac{\partial \Delta \ln C_t}{\partial r_t} \xrightarrow{\text{Under CRRA}} EIS = \frac{1}{\gamma}$$

2) Test orthogonality restriction: $I_t \perp \varepsilon_{t+1}$ (where I_t: Info set available at t)

test X_t = Income growth (expected, current, lagged)

↳ See whether there is still "Excess Sensitivity"

$E_t \Delta \ln Y_{t+1} \longrightarrow \Delta \ln C_{t+1}$?

Add $\boxed{\beta \Delta lnY_{t+1}}$ into the above equation (RHS), and regression results show:

$\frac{1}{\gamma} \approx 0$ e.g. [0,0.2] in Hall (1988)

That is, not much evidence that that interest rates are relevant for consumption decision (similar to Keynes' postulate)

If $\frac{1}{\gamma}$ small $\Rightarrow \gamma$ huge

But do such low estimates make sense? (CRRA parameter would be huge!)

$$\Delta \ln C_{t+1} = \alpha + \beta \Delta \ln y_{t+1} + \frac{1}{\gamma} r_t + \varepsilon_{t+1}$$

- β in [0.1,0.8]>0 ln

  e.g. Cambell-Mankiw (1989); Shea (1995): Excess sensitivity again

- Could it be due to the CRRA utility functional form? (where CRRA = 1/EIS)

  From Topic 2 (RBC): it's troublesome that risk aversion and IES are tied to the same parameter in the CRRA utility

What we have seen so far:

Different sets of assumptions imply slight variations, but so far, all models predict "consumption smoothing" … but, consumption is not as smooth as theory predicts

theory

↳ empirically: Excess Sensitivity

Big Question:

Why is consumption growth correlated with expected income growth?

(or why does $E_t \Delta ln Y_{t+1}$ predict $\Delta ln C_{t+1}$? – **Excess Sensitivity**)

Possible reasons (each with vast literature):

$\uparrow$ higher consumption $\rightarrow$ lower leisure $\rightarrow$ $\uparrow$ Labor Supply $\rightarrow$ $\uparrow$ Income

1) *Leisure and consumption are substitutes* (Heckman (1974), Aguiar and Hurst (2005,2007))

2) *Life-cycle story*: households support more dependents in mid-life when income is highest (Browning (1992), Attanasio (1995))

3) *Precautionary savings term is not constant over time* as we assumed, so regression is biased (omitted variable bias): Dynan (1993), Carroll (1994)

$\frac{1}{2} r \, Var_t(\Delta \ln C_{t+1})$ Not constant

4) *Alternative preference specifications*:

$u(C_t, C_{t-1})$ $u(C_t, \bar{C})$

- Non-additively separable utility (Habit formation), or "keeping up with the Joneses"
- Need more general $u()$ that does not impose CRRA = 1/EIS

5) Liquidity constraints and Impatience => Buffer stock models

- Carroll (1992), Deaton (1991), Laibson (1997), Zeldes (1989)
- Consumers face borrowing constraints so can't smooth C as they wish

  (capital market is imperfect)
- Consumers are impatient ($\rho > r$) and want immediate gratification
- Result: a "buffer stock" type of consumption pattern: accumulate small stock of assets to buffer transitory income shocks, then after that, consume Y_t

6) Non or sub-rationality

- People don't really optimize but follow rules of thumb when making consumption decision (Thaler, Cambell-Mankiw)

7) r_t is stochastic

Stochastic Rate of Return & Consumption CAPM:

Stochastic

Consider an economy with multiple assets, each offer stochastic return, $\tilde{r}^i_t$
(note: r_t is the return between t and t+1)
Note the Euler condition has to hold for each asset i

$$u'(c_t) = \beta E_t[R^i_t u'(c_{t+1})]$$
$$u'(c_t) = \beta E_t[R^j_t u'(c_{t+1})]$$

$$E_t\left\{\boxed{\frac{u'(c_{t+1})}{u'(c_t)}\frac{(1+\tilde{r}^i_t)}{1+\rho}}\right\} = 1$$

$\hookrightarrow \tilde{M}_t \equiv$ Stochastic Discount Factor

$$E_t\{\tilde{M}_t(1+\tilde{r}^i_t)\} = 1 \quad \forall i$$ (Now we use $E(xy) = E(x)E(y) + Cov(x,y)$)

$$E_t(\tilde{M}_t)E_t(1+r^i_t) + Cov(1+\tilde{r}^i_t, \tilde{M}_t) = 1$$

$$\tilde{M}_t = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}$$

$$E_t(1+r^i_t) = \frac{1}{E_t[\tilde{M}_t]}\left[1 - \beta Cov\left(1+\tilde{r}^i_t, \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right)\right]$$

$(1+\Delta\ln C_{t+1})^{-\gamma} \approx 1 - \gamma\Delta\ln C_{t+1}$

$$E(1+\tilde{r}^i_t) = \frac{1}{(1+\rho)EM_t}[1+\rho+\gamma Cov(\tilde{r}^i_t, \Delta\ln C_{t+1})]$$

$\hookrightarrow \beta$

β^{-1}

More detailed algebra steps

$\left(\beta = \frac{1}{1+\rho} \equiv \right.$ Standard discount Factor $\left. \right)$

$$E_t \left\{ \frac{u'(C_{t+1})}{u'(C_t)} \frac{(1+\tilde{r}_t^i)}{1+\rho} \right\} = 1 \quad \forall\, i, \qquad M_t = \frac{u'(C_{t+1})}{u'(C_t)} \frac{1}{1+\rho} \equiv \text{Stochastic discount Factor}$$

$$\Rightarrow E_t\{\tilde{M}_t(1+\tilde{r}_t^i)\} = 1 \Rightarrow E_t(\tilde{M}_t)E_t(1+\tilde{r}_t^i) + Cov(1+\tilde{r}_t^i, \tilde{M}_t) = 1$$

(uses that: $Cov(x,y) = E(xy) - E(x)E(y)$)

With CRRA: $\tilde{M}_t = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma}$

$$\Rightarrow E_t(1+\tilde{r}_t^i) = \frac{1}{E_t(\tilde{M}_t)} \left[1 - \beta\, Cov\left(1+\tilde{r}_t^i, \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \right) \right]$$

use that $\frac{C_{t+1}}{C_t} \approx 1 + \Delta \ln C_{t+1}$ (i) $\frac{X_{t+1}}{X_t} \approx 1 + \%\text{growth in } X \approx (1+g_x)$. (ii) $\ln(1+g_x) \approx g_x$. (iii) Percentage growth $\equiv g_x \approx \ln X_{t+1} - \ln X_t = \Delta \ln X_{t+1}$

$$E_t(1+\tilde{r}_t^i) = \frac{1}{E_t(\tilde{M}_t)} \left[1 - \beta\, Cov\left(1+\tilde{r}_t^i, (1+\Delta \ln C_{t+1})^{-\gamma} \right) \right]$$

$$= \frac{1}{E_t(\tilde{M}_t)} \left[1 - \beta\, Cov\left(1+\tilde{r}_t^i, 1-\gamma \Delta \ln C_{t+1} \right) \right]$$

Uses binomial approximation: $(1+x)^{\alpha} \approx 1 + \alpha x$

$$= \frac{1}{1+\rho} \frac{1}{E_t(\tilde{M}_t)} \left[1+\rho - Cov\left(\tilde{r}_t^i, -\gamma \Delta \ln C_{t+1} \right) \right] = \frac{1}{(1+\rho)E_t(\tilde{M}_t)} \left\{ 1+\rho + \gamma\, Cov\left(\tilde{r}_t^i, \Delta \ln C_{t+1} \right) \right\}$$

Consumption Capital Asset Pricing Model (C-CAPM):

$$E(1+\tilde{r}_t^i) = \frac{1}{(1+\rho)EM_t}[1+\rho+\underbrace{\gamma}_{②}\overbrace{Cov(\tilde{r}_t^i, \Delta \ln C_{t+1})}^{①}]$$

1) Expected rate of return depends on the covariance of the return and the growth rate of consumption

Intuition: consider two assets: *i* and *j*, each with stochastic payoffs

Cov > 0

For *i*, the cov term is positive and large, i.e., *i* pays off well in booms or states of nature when C is high (MU_C low)

Cov < 0

Asset *j* pays off well in recessions, when C is low (c-growth small),when MU_C is high, i.e. cov < 0 or low.

Asset *j* would be a preferred asset as it functions as a hedge for consumption risk (pays well when one most needs it.)

i.e. people would be willing to hold asset *j* even if its expected payoff is not as high (i.e. asset *j* is more “expensive”)

=> $\boldsymbol{E_t(1+\tilde{r}_t^i) > E_t(1+\tilde{r}_t^j)}$ must hold …

… in order to compensate for the undesirable payoff timing.

2) γ: the more risk averse people are, the higher the expected return has to be.

Example: 2 states w/ 50-50 Probability of each in t+1

1/2 employed (C_{t+1} high)

1/2 unemployed (C_{t+1} low)

3 assets:

Asset 1: Pays \$100 Regardless $Cov(r_t, \Delta \ln C_{t+1}) = 0$

$\boxed{\text{Asset 2: Pays } \begin{cases} \$200 \text{ if employed} \\ \$0 \text{ if Unemployed} \end{cases} \quad Cov(r_t, \Delta \ln C_{t+1}) > 0}$ → Least desirable

Asset 3: Pay $\begin{cases} \$0 \text{ if employed} \\ \$200 \text{ if Unemployed} \end{cases}$ $Cov < 0$

Consumption Capital Asset Pricing Model (C-CAPM) - summary:

Consumer has access to n risky assets, each with return $\boldsymbol{r_t^i}$:

- Optimal portfolio choice: (expected return consistent with Euler eq.)

$$1 = E_t[(1+\tilde{\mathrm{r}}_{\mathrm{t}}^{\mathrm{i}})M_{t+1})] \quad \text{where } Mt \equiv \frac{1}{1+\rho}\frac{U'(C_{t+1})}{U'(C_t)}$$

- Euler Equation holds for all assets

- Assuming CRRA utility with coefficient $\boldsymbol{\gamma}$:

$$1 = \frac{1}{1+\rho}E_t[(1+\tilde{\mathrm{r}}_{\mathrm{t}}^{\mathrm{i}})(\frac{C_{t+1}}{C_t})^{-\gamma}]$$

- Assume joint conditional log-normality of asset return and consumption growth, we can approximate the above with:

$$\boxed{E(1+\tilde{r}_t^{i}) \approx 1+\rho+\gamma Cov(\tilde{r}_t^{i},\Delta \ln C_{t+1})-\frac{\gamma(1+\gamma)}{2}Var(\Delta \ln C_{t+1})+\gamma E(\Delta \ln C_{t+1})}$$

- Concept of risk: Covariance

If a stock pays off well only when the economy is doing well (C high, MU low), it is a "risky" stock with <u>little hedging value</u> …

…so it needs to offer a higher return as a compensation.

$i = f, e$ $\quad$ f: Risk Free (bonds)
e: Equity (Stocks)

Equity Premium Puzzle:

Consider 2 assets:

1) a risk free asset offering return: r^f (gov. bond)

2) a risky asset (equity) with r^e. (S&P 500)

Use the above C-CAPM equation for both and subtract:

$$E(1+r_t^e) - E(1+r_t^f) = \cancel{1+\rho} + \gamma Cov(\tilde{r}_t^e, \Delta \ln C_{t+1}) - \cancel{\frac{\gamma(1-\gamma)}{2} Var(\Delta \ln C_{t+1})} + \cancel{\gamma E_t(\Delta \ln C_{t+1})}$$
$$- \left(\cancel{1+\rho} + \gamma \underbrace{Cov(\tilde{r}_t^f, \Delta \ln C_{t+1})}_{\approx 0} - \cancel{\frac{\gamma(1-\gamma)}{2} Var(\Delta \ln C_{t+1})} + \cancel{\gamma E_t(\Delta \ln C_{t+1})} \right)$$

$$\underbrace{E_t[r_t^e] - r^f}_{\text{Equity premium}} \approx \gamma Cov_t(r_t^e, \Delta \ln C_{t+1})$$

Same idea: the more r co-moves with consumption, the less good a hedge it is

=> less desirable/cheaper and must offer a higher expected return

Mehra and Prescott (1985):

- Data from 1880-1979
- Risky return is measured by the average return on the US stock market,
- "Safe" return = the return to short-term government bills/bonds

Find that average Equity Premium $\approx 6\%$; $Cov_t(\tilde{r}_t^e, \Delta \ln C_{t+1}) \approx 0.0003 - 0.0024$

$\Rightarrow \quad \hat{\gamma} = \dfrac{0.06}{Cov_t(r_t^e, \Delta \ln C_{t+1})} \approx [25, 200]$ vs. $\underbrace{\gamma \simeq 1-5}$ Expected from theory

Terminal Value of $1 Invested in Stocks and Bonds

	Stocks		T-bills	
Investment period	Real	Nominal	Real	Nominal
1802–2004	$655,348.00	$10,350,077.00	$293.00	$4,614.00
1926–2004	$238.30	$2,533.43	$1.54	$17.87

Puzzle: With observed equity premium, the estimated risk aversion $\gamma \in [25,200]$ whereas the one we expect to see (from theory and micro data estimates) is $\gamma \in [1,5]$

Hard to know why? … flawed theories, bad assumptions, consumption, or stock returns are poorly measured … these and many other potential explanations are still under study

The puzzle in general is robust across sample series and across countries:

U.S. Equity Premium Using Different Data Sets

Data set	Real return on a market index (%)	Real return on a relatively riskless security (%)	Equity premium (%)
	Mean	Mean	Mean
1802–2004 (Siegel)	8.38	3.02	5.36
1871–2005 (Shiller)	8.32	2.68	5.64
1889–2005 (Mehra–Prescott)	7.67	1.31	6.36
1926–2004 (Ibbotson)	9.27	0.64	8.63

Equity Premium for Selected Countries

Equity Premium for Selected Countries

Country	Period	Mean real return: Market index (%)	Mean real return: Relatively riskless security (%)	Equity premium (%)
United Kingdom	1900–2005	7.4	1.3	6.1
Japan	1900–2005	9.3	−0.5	9.8
Germany	1900–2005	8.2	−0.9	9.1
France	1900–2005	6.1	−3.2	9.3
Sweden	1900–2005	10.1	2.1	8.0
Australia	1900–2005	9.2	0.7	8.5
India	1991–2004	12.6	1.3	11.3

Source: Dimson et al. (2002) and Mehra (2007) for India.

$\gamma \to (1, 5)$

Why is the high $\boldsymbol{\gamma}$ a problem? (unreasonably risk-averse agents!)

Mankiw-Zeldes (1991): $\gamma \to (28, 200)$

What value of X would make you indifferent between the following two gambles?

Gamble 1:	\$50K	with probability 0.5
	\$100K	with probability 0.5
Gamble 2:	\$X	with probability 1.0

Solve for X s.t.

$u(X) = \frac{1}{2}u(50K) + \frac{1}{2}U(100K)$

Map X to γ

X	γ	
70k	1.1	Consistent w/ theory
65k	3	
58k	5	
51.8k	20	Consistent w/ data
51.5k	25	

What's driving this high γ?

- Empirically: C simply doesn't move around enough, so the covariance term is small, implying a high-risk aversion to match with the equity premium.

FEDERAL RESERVE BANK of ST. LOUIS
CENTRAL TO AMERICA'S ECONOMY®

REGIONAL ECONOMIST I THIRD QUARTER 2017

https://www.stlouisfed.org/publications/regional-economist/third-quarter-2017/household-participation-in-stock-market-varies-widely-by-state

Household Participation in Stock Market Varies Widely by State

YiLi Chien , Paul Morris

From 1928 to 2016, the average annual stock return was about 8 percentage points higher than the return on three-month Treasury bills. This leads to sizable return gaps over time: $100 investments in stocks and in Treasury bills in 1928 would have yielded nearly $329,000 and $2,000, respectively, 88 years later.[1]

Given the high return of stocks, it is puzzling that many households do not participate in the stock market and, hence, forgo the high return. In addition, the nonparticipation behavior is at odds with modern portfolio theory. The theory implies that all households should invest at least a fraction of their wealth in stocks in order to take advantage of the equity premium. However, the data show that many households do not participate in financial markets.

The inability of modern portfolio theory to explain what is observed in the data leads to a "participation puzzle." A common explanation of this puzzle is the individual participation cost, which includes both monetary and nonmonetary costs. The monetary costs are relatively straightforward, including transaction or brokerage fees. The nonmonetary costs are broadly defined to be the cognitive and time costs of understanding the investment object or processing previous experiences with stock markets. The participation cost, especially the nonmonetary costs, could vary widely across the population.

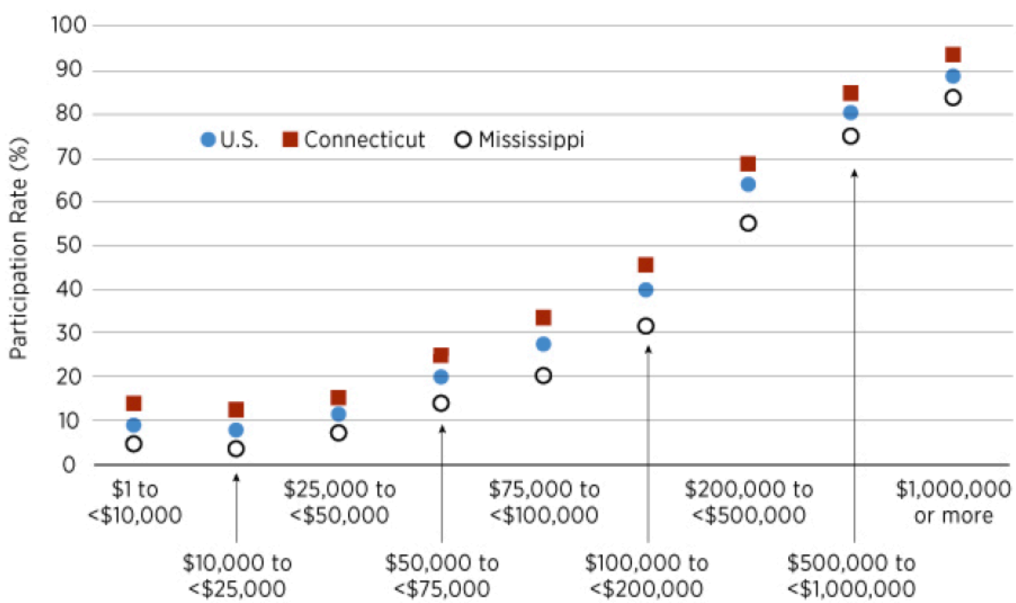
Wide Disparities among States

The data show a large variation in stock market participation rates across the United States. The disparities are sizable, with rates ranging from 10.5 percent in Mississippi to 26.6 percent in Connecticut. This seems reasonable, as the average household income is higher in Connecticut than in Mississippi and the existing literature shows that the participation rate increases with income.[2]

However, this is not the whole story. Even when controlling for household income level, the large variation in participation across states prevails. Figure 1 plots the participation rates for Connecticut, Mississippi and the United States across different income groups. The participation gap remains large for each group, indicating that household income level does not entirely lead to differences in participation rates.

Figure 1

Stock Market Participation across Income Groups



SOURCES: IRS' 2014 individual income and tax data and authors' calculations.

NOTE: Connecticut and Mississippi had the highest and lowest rates, respectively, for overall stock market participation among the states.

■ FEDERAL RESERVE BANK OF ST. LOUIS

SUMMARY: EPP & Consumption Capital Asset Pricing Model (C-CAPM):

$$\mathrm{E}(1+\tilde{\mathrm{r}}_{\mathrm{t}}^{\mathrm{i}}) \approx 1+\rho+\gamma Cov(\tilde{\mathrm{r}}_{\mathrm{t}}^{\mathrm{i}},\Delta\ln C_{t+1})-\frac{\gamma(1+\gamma)}{2}Var(\Delta\ln C_{t+1})+\gamma E(\Delta\ln C_{t+1})$$

Consumer has access to n risky assets, each with return r_t^i:

Optimal portfolio choice:

$$1=E_t[(1+\tilde{\mathrm{r}}_{\mathrm{t}}^{\mathrm{i}})M_t)] \quad \text{where } Mt \equiv \frac{1}{1+\rho}\frac{U'(C_{t+1})}{U'(C_t)}$$

Euler equation holds for all assets

Assuming CRRA Utility with risk aversion coefficient γ

Assume joint conditional log-normality of asset return and consumption growth

Equity Premium Puzzle:

Consider 2 assets: 1) a risk free asset offering return: r^f … and 2) a risky asset (equity) with r^e

$$\mathrm{E}(1+\tilde{\mathrm{r}}_{\mathrm{t}}^{e}) - \mathrm{r}_t^f = \gamma Cov(\tilde{\mathrm{r}}_{\mathrm{t}}^{e},\Delta\ln C_{t+1})$$

Mehra and Prescott (1985) and follow-up work shows that:

Observed equity premium (around 6%) implies too risk-averse agents in this model

(then, model is flawed and cannot explain the observed equity returns) → Extensions, fixes: in group presentations

Having studied consumption we proceed with how to link it with aggregate demand in our simplified framework

(we know AD is not only given by consumption but work with this simplification for now)

Towards our simple Dynamic Stochastic General Equilibrium macro model:

- **AS: New Keynesian Phillips curve (NKPC)**

((Calvo handout or Rotemberg Slides)

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + \varepsilon_{\pi,t} \qquad \text{(NKPC)}$$

- **AD: Dynamic IS** equation:

(From Euler Eq)

$$x_t = E_t x_{t+1} - \phi[i_t - E_t \pi_{t+1}] + \varepsilon_{x,t} \qquad \text{(AD)}$$

- **Monetary policy:** Taylor interest rates rule:

$$i_t = d_\pi \pi_t + d_x x_t + \varepsilon_{x,t} \qquad \text{(MP)}$$

How to get the AD: Dynamic IS equation?

$$x_t = E_t x_{t+1} - \phi[i_t - E_t \pi_{t+1}] + \varepsilon_{x,t} \qquad \text{(AD)}$$

- **Depart from Euler Equation:** $C_t^{-\gamma} = \beta E_t [C_{t+1}^{-\gamma} R_{t+1}]$

- **Log-linearize** (and assume joint log normality or RHS variables and homoskedasticity)

  re-set model in terms of $c_t = \frac{C_t - \bar{C}_t}{\bar{C}_t}$

  $c_t = E_t c_{t+1} - \frac{1}{\gamma} E_t r_{t+1}$

- **Closed Economy Assumption** (with only consumption)

  We assume no gov. spending, no capital, no net exports => AD = C
  (i.e. C + I + G + NX = C)
  Put that together with equilibrium condition supply = Y, demand = AD

  $y_t = c_t$ or $x_t = c_t \Rightarrow x_t = E_t x_{t+1} - \frac{1}{\gamma} E_t (r_{t+1})$

- **Replace Real Int. Rate by Nominal – Inflation** (and add demand shock)

  $r_{t+1} = i_t - E_t \pi_{t+1} \Rightarrow \underline{x_t = E_t x_{t+1} - \frac{1}{\gamma} E_t (i_t - E_t \pi_{t+1}) + \varepsilon_{x,t}}$ NK-IS

[Extra] Don't forget key assumptions behind these results (monopolistic competition, price stickiness)

Sticky Price Model (Gali (2008) Chapter 3)

Representative household chooses consumption, labor, and bond-holding to maximize infinite life time utility, subject to period-wise budget constraint.

This is the same model but with "elastic labor" or "variable labor supply"

$$\max_{\{C_t, N_t\}} E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\phi}}{1+\phi} \right]$$

$$\text{s.t. } P_t C_t + \frac{B_t}{1+i_t} = B_{t-1} + w_t N_t$$

B_t : Units of bond purchased at a price $Q_t = \frac{1}{1+i_t}$ that pays \$1 at $t+1$

Maximization gives the F.O.C.'s that are rearranged give the Euler Eq:

[Ct], [Bt]: yields Euler Equation (after rearranging these two FOCs)

$$E_t\left\{\beta(1+i_t)\left[\frac{C_{t+1}}{C_t}\right]^{-\sigma}\left[\frac{P_t}{P_{t+1}}\right]\right\} = 1$$

[Nt]: (New condition, shows up as households have one more choice variable)

$$-\frac{u_{N_t}}{u_{C_t}} = \frac{w_t}{P_t}$$

- The first condition is the **Inter-temporal choice** (consumption vs. savings)
- The second the **Intra-temporal one** (consumption vs. leisure)

Log-linearize the first equation (Euler Equation) around Steady State:

$\log C_t - \log \bar{C}$

$$c_t = E_t c_{t+1} - \frac{1}{\sigma}[i_t - E_t \pi_{t+1} - \rho]$$

(Key) Market Clearing Condition:

$$c_t = y_t$$

Can Define Output Gap and Set the Equation in Terms of it:

$x_t = y_t - \bar{y}_t$ or assume y_t is already a gap measure as it's already in log-linear terms

New forward looking IS curve, or AD equation:

$$x_t = E_t x_{t+1} - \phi[i_t - E_t \pi_{t+1}] + \varepsilon_{x,t} \qquad \text{(AD)}$$

This is the same equation as in Gali Ch. 3 but allowing for a AD shock (see the chapter for a full derivation of that model)

Notes and interpretations:

We derived the New IS (AD) and NK Phillips curve (AS):

$$x_t = E_t x_{t+1} - \frac{1}{\sigma}[i_t - E_t \pi_{t+1}] + g_t \qquad \text{(NIS)}$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + \underbrace{u_t}_{\text{Cost Push shock}} \qquad \text{(NKPC)}$$

1) Output gap: $x_t = y_t - y_t^N$ where y_t^N is output level under flexible price and can be derived from production side:

From Euler and letting $y_t = c_t$

$$y_t - y_t^N = E_t[y_{t+1} - y_{t+1}^N] - \frac{1}{\sigma}[i_t - \rho - E_t \pi_{t+1}] + E_t \Delta y_{t+1}^N$$

$$x_t = E_t[x_{t+1}] - \frac{1}{\sigma}[i_t - E_t \pi_{t+1} - r_t^N]$$

Where $r_t^N = \rho + \sigma E_t \Delta y_{t+1}^N$ is the natural rate of interest

Then, the NIS can be alternatively written as: $x_t = E_t[x_{t+1}] - \frac{1}{\sigma}[i_t - E_t \pi_{t+1} - r_t^N] + \xi_t$

2) Note that inflation is forward-looking (as from Calvo handout):

Inflation today depends on today's expected discounted future marginal cost and mark-ups

3) Higher nominal ridigity implies inflation is less sensitive to x_t

(low κ; or low λ in Calvo handout)

$$\theta \uparrow, \kappa \downarrow \Rightarrow \pi \text{ less sensitive to } x_t$$

Know that **cost-push (supply) shock $\boldsymbol{u_t}$** can be motivated by time-varying mark-up, imperfection in the labor market, labor income tax, etc.

Key Questions (EC)

- What are the main differences (or different emphases) between Neoclassical Synthesis and the modern New Neoclassical Synthesis we've been building in this course?

- What is the relationship between the AD equation we built and Y = C + I + G + NX?

Algebra for log-linearizing Euler Equation.

Depart from Euler Equation: $C_t^{-\gamma} = \beta E_t[C_{t+1}^{-\gamma} R_{t+1}]$

assume RHS variables are jointly log-normal & homoskedastic

Take logs: $-\gamma \log C_t = \log\beta + \log E_t[C_{t+1}^{-\gamma} R_{t+1}]$

W/ log normality: $\log(E_t(x_{t+1})) = E_t[\log(x_{t+1})] + \frac{1}{2}Var_t[\log(x_{t+1})]$

$$\Rightarrow \log E_t[C_{t+1}^{-\gamma} R_{t+1}] = E_t[\log(C_{t+1}^{-\gamma} R_{t+1})] + \frac{1}{2}Var_t[\log(C_{t+1}^{-\gamma} R_{t+1})]$$

$$= -\gamma E_t \log C_{t+1} + E_t \log R_{t+1} + \frac{\gamma^2}{2}\sigma^2_{t,\log C_{t+1}} + \frac{1}{2}\sigma^2_{t,\log R_{t+1}} - \gamma\,\sigma_{t,\log C_{t+1},\log R_{t+1}}$$

$$\Rightarrow -\gamma \log C_t = \log\beta + E_t(-\gamma\log C_{t+1} + \log R_{t+1}) + \overbrace{\frac{\gamma^2}{2}\sigma^2_{t,\log C_{t+1}} + \frac{1}{2}\sigma^2_{t,\log R_{t+1}} - \gamma\,\sigma_{t,\log C_{t+1},\log R_{t+1}}}^{\text{Constant}}$$

Substract Equation For Steady State

$$-\gamma\, d\log C_t = -\gamma E_t\, d\log C_{t+1} + E_t\, d\log R_{t+1}$$

$$d\log z_t = \hat{z}_t = \frac{z_t - z^{ss}}{z^{ss}} \qquad \left(r_{t+1} = \frac{R_{t+1} - R^{ss}}{R^{ss}} \right)$$

$$\Rightarrow -\gamma \hat{C}_t = -\gamma E_t \hat{C}_{t+1} + E_t r_{t+1}$$

$$\hat{C}_t = E_t \hat{C}_{t+1} - \frac{1}{\gamma} E_t r_{t+1}$$