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Mathematical and Statistical Foundations for Data Science(CMPINF 2105)
"Pen and Paper" Homework 2: Linear Regression & Random Samples (Modules 3 & 4)

1. Given the overdetermined system of linear equations:

$$2x_1 + 3x_2 = 5$$
$$4x_1 + 5x_2 = 11$$
$$6x_1 + 7x_2 = 17$$

   Use the Gram matrix approach to find the least squares solution to this system.

2. Consider the following data points for a simple linear regression problem:

x	y
1	2
2	3
3	5
4	7
5	8

   Find the best-fit line $y = \beta_0 + \beta_1 x$ by finding β_0 and β_1.

3. Suppose you have tabular data given by:

x_i	y_i
0	2.0
0.1	2.12
0.2	2.28
0.3	2.48
0.4	2.72
0.5	3.0
0.6	3.32
0.7	3.68
0.8	4.08
0.9	4.52

   Assume the data follows the model

$$y \approx \beta_0 + \beta_1 x + \beta_2 x^2$$

   Find the least squares estimates for β_0, β_1, and β_2.

4. Consider the following discrete random variable X with the probability distribution given below:

x	$P(X = x)$
1	0.2
2	0.5
3	0.3

Calculate the expected value $E[X]$ of the random variable X.

5. Suppose you have a population with an unknown distribution that has an expected value of $E[D] = 50$ and a standard deviation of $\sigma_D = 10$. You draw a sample s of size $n = 100$ from this population. According to the Central Limit Theorem, what is the expected distribution of the sample mean $E[s]$?

6. A factory produces light bulbs that have lifetimes following a distribution with $E[D] = \sigma_D = 800$ hours. If you take a sample s of 100 light bulbs, what is the probability that their average lifetime $E[s]$ is between 720 and 880 hours?

① $2x_1 + 3x_2 = 5$

$4x_1 + 5x_2 = 11$

$6x_1 + 7x_2 = 17$

$M = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix}, \quad y = \begin{bmatrix} 5 \\ 11 \\ 17 \end{bmatrix}$

$M^T = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix}$

$M \cdot x = y$

$M^T \cdot M \cdot x = M^T \cdot y$

$(M^TM)^{-1} \cdot M^TM \cdot x = (M^TM)^{-1} \cdot M^Ty$

$x = (M^TM)^{-1} \cdot M^Ty$

$= \begin{bmatrix} 83/24 & -17/6 \\ -17/6 & 7/3 \end{bmatrix} \cdot \begin{bmatrix} 156 \\ 189 \end{bmatrix}$

$= \dfrac{12948}{24} - \dfrac{3213}{6}(4)$

$\dfrac{-2652}{6} + \dfrac{1323}{3}(2)$

$= \dfrac{12948 - 3213(4)}{24}$

$\dfrac{-2652 + 1323(2)}{6}$

$= \begin{matrix} \dfrac{96}{24} \\ \\ \dfrac{-6}{6} \end{matrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ ✓

X1 $= 4$, X2 $= -1$

$M^T \cdot M = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix}$

$= \begin{matrix} 2\cdot2 + 4\cdot4 + 6\cdot6 & 2\cdot3 + 4\cdot5 + 6\cdot7 \\ 3\cdot2 + 5\cdot4 + 7\cdot6 & 3\cdot3 + 5\cdot5 + 7\cdot7 \end{matrix}$

$= \begin{matrix} 4 + 16 + 36 & 6 + 20 + 42 \\ 6 + 20 + 42 & 9 + 25 + 49 \end{matrix}$

$= \begin{bmatrix} 56 & 68 \\ 68 & 83 \end{bmatrix}$

$(M^TM)^{-1} = \left[\begin{array}{cc|cc} 56 & 68 & 1 & 0 \\ 68 & 83 & 0 & 1 \end{array}\right]$

$R_1 = R_1/56$

$R_2 = R_2/83$

$M^T \cdot y = \begin{bmatrix} 2\cdot5 + 4\cdot11 + 6\cdot17 \\ 3\cdot5 + 5\cdot11 + 7\cdot17 \end{bmatrix} = \begin{bmatrix} 156 \\ 189 \end{bmatrix}$

$= \begin{bmatrix} 1 & \overset{=17/14}{68/56} & 1/56 & 0 \\ \frac{68}{83} & 1 & 0 & 1/83 \end{bmatrix}$

$R_2 = R_2 - \frac{68}{83} R_1$

$\begin{bmatrix} 1 & 17/14 & 1/56 & 0 \\ 0 & 3/581 & \frac{-17}{1162} & 1/83 \end{bmatrix}$

$R_2 = R_2 \cdot \frac{581}{3}$

$\begin{bmatrix} 1 & 17/14 & 1/56 & 0 \\ 0 & 1 & \frac{-17}{6} & \frac{7}{3} \end{bmatrix}$

$\frac{-17}{1162_2} \cdot \frac{581}{3} = \frac{-17}{6} \quad 1/83 \cdot \frac{581^{7}}{3}$

$R_1 = R_1 - 17/14 R_2$

$1 - \frac{68}{83}\left(\frac{17}{14}\right)$

$1 - \frac{1156}{1162}$

$\frac{6/2}{1162/2} = \frac{3}{581}$

$0 - \frac{68}{83}\left(\frac{1}{56}\right)$

$\frac{-68}{4648} \div 4 = \frac{-17}{1162}$

$(M^TM)^{-1} = \begin{bmatrix} 1 & 0 & 83/24 & -17/6 \\ 0 & 1 & \frac{-17}{6} & \frac{7}{3} \end{bmatrix}$

$= 1/56 - \frac{17}{14}\left(-\frac{17}{6}\right)$

$= \frac{84}{84} \frac{1}{56} + \frac{289}{84} \cdot \frac{56}{56} = \frac{84 + 289 \cdot 56}{84 \cdot 56}$

$= \frac{16268}{4704} \div \frac{196}{196} = \frac{83}{24}$

$0 - \frac{17}{14_2}\left(\frac{7}{3}\right) = -\frac{17}{6}$

(2) Data points

x	y
1	2
2	3
3	5
4	7
5	8

Find best fit line

$y = B_0 \cdot 1 + B_1 \cdot x$

$$M = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix}, \quad M^T = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$M^T \cdot M = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix}$$

$$= \begin{matrix} 1 + \overset{4}{2 \cdot 2} + \overset{9}{3 \cdot 3} + \overset{16}{4 \cdot 4} \cdot \overset{25}{5 \cdot 5} & 1+2+3+4+5 \\ 1+2+3+4+5 & 1+1+1+1+1 \end{matrix}$$

$$= \begin{bmatrix} 55 & 15 \\ 15 & \cancel{4}5 \end{bmatrix}$$

$$(M^T \cdot M)^{-1} = \left[\begin{array}{cc|cc} 55 & 15 & 1 & 0 \\ 15 & 5 & 0 & 1 \end{array}\right]$$

$M^T \cdot y =$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \\ 8 \end{bmatrix} = \begin{matrix} 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 5 + 4 \cdot 7 + 5 \cdot 8 \\ 2+3+5+7+8 \end{matrix} = \begin{bmatrix} 91 \\ 25 \end{bmatrix}$$

$$x = (M^T \cdot M)^{-1} \cdot M^T y$$

$R_1 = R_1 / 55$

$$= \begin{bmatrix} 1 & \frac{15}{55} & 1/55 & 0 \\ 15 & 5 & 0 & 1 \end{bmatrix}$$

$R_2 = R_2 - 15 R_1$

$$= \begin{bmatrix} 1 & 3/11 & 1/55 & 0 \\ 0 & 10/11 & -\frac{3}{11} & 1 \end{bmatrix}$$

$5 - 15(15/55)$

$$5 - \frac{225}{55} = 5 - \frac{45}{11} = \frac{55-45}{11} = \frac{10}{11}$$

$0 - 15/55$

$R_2 = R_2 \cdot 11/10$

$$= \begin{bmatrix} 1 & 3/11 & 1/55 & 0 \\ 0 & 1 & -3/10 & 11/10 \end{bmatrix}$$

$R_1 = R_1 - 3/11 R_2$

$$(M^T \cdot M)^{-1} = \begin{bmatrix} 1 & 0 & 1/10 & -3/10 \\ 0 & 1 & -3/10 & 11/10 \end{bmatrix}$$

$$(2)\ 1/55 - \frac{3}{11}\left(\frac{-3}{10}\right) = \frac{2+9}{110} = \frac{\cancel{11}}{\cancel{110}} \frac{1}{10}$$

$$0 - \frac{3}{\cancel{11}} \cdot \frac{\cancel{11}}{10} = \frac{-3}{10}$$

$$x = \begin{bmatrix} 1/10 & \frac{-3}{10} \\ \frac{-3}{10} & 11/10 \end{bmatrix} \cdot \begin{bmatrix} 91 \\ 25 \end{bmatrix}$$

$$= \begin{matrix} 91/10 - \frac{\overset{25 \cdot 3}{75}}{10} \\ \frac{\overset{91 \cdot 3}{-273}}{10} + \frac{\overset{11 \cdot 25}{275}}{10} \end{matrix} = \begin{matrix} \frac{16}{10} \\ \frac{2}{10} \end{matrix}$$

$$= \begin{bmatrix} 8/5 \\ 1/5 \end{bmatrix}$$

$B_1 = 8/5$ or 1.6

$B_0 = .2$ or $1/5$

$y = 1/5 + 8/5 x$ OR

$= 1.6x + 0.2$

③ $M = \begin{bmatrix} 0 & 0 & 1 \\ 0.01 & 0.1 & 1 \\ 0.04 & 0.2 & 1 \\ 0.09 & 0.3 & 1 \\ 0.16 & 0.4 & 1 \\ 0.25 & 0.5 & 1 \\ 0.36 & 0.6 & 1 \\ 0.49 & 0.7 & 1 \\ 0.64 & 0.8 & 1 \\ 0.81 & 0.9 & 1 \end{bmatrix}$ 10×3

$M^T = \begin{bmatrix} 0 & 0.01 & 0.04 & 0.09 & 0.16 & .25 & .36 & .49 & .64 & .81 \\ 0 & 0.1 & 0.2 & 0.3 & .4 & .5 & .6 & .7 & .8 & .9 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ 3×10

$y = \begin{bmatrix} 2 \\ 2.12 \\ 2.28 \\ 2.48 \\ 2.72 \\ 3.0 \\ 3.32 \\ 3.68 \\ 4.08 \\ 4.52 \end{bmatrix}$

$Mx = y$

$M^T \quad\quad M^T$

$\cancel{(M^TM)^{-1}} \cdot \cancel{M^TM}x = (M^TM^{-1})M^Ty$

$x = (M^TM)^{-1}M^Ty$

$M^Ty = 0\cdot 2 + 0.01(2.12) + 0.04(2.28) + .09(2.48) + .16(2.72 + .25(3) + .36(3.32) + .49(3.68) + .64(4.08) + .81(4.52)$

$0(2) + 0.1(2.12) + .2(2.28) + .3(2.48) + .4(2.72) + .5(3) + .6(3.32) + .7(3.68) + .8(4.08) + .9(4.52)$

$2 + 2.12 + 2.28 + 2.48 + 2.72 + 3 + 3.32 + 3.68 + 4.08 + 4.52 = 30.2$

$.0212 + .0912 + .2232 + .4352 + .75 + 1.1952 + 1.8032 + 2.6112 + 3.6612 = \boxed{10.7916}$

$.212 + .456 + .744 + 1.088 + 1.5 + 1.992 + 2.576 + 3.264 + 4.068 = \boxed{15.9}$

$M^Ty = \begin{bmatrix} 10.7916 \\ 15.9 \\ 30.2 \end{bmatrix}$

~~.022~~

$M^TM = 0 + 0.01^2 + 0.04^2 + .09^2 + .16^2 + .25^2 + .36^2 + .49^2 + .64^2 + .81^2 = 1.5333$

$0 + 0.01(.1) + .04(.2) + .09(.3) + .16(.4) + .25(.5) + .36(.6) + .49(.7) + .64(.8) + .81(.9) = 2.025$

$0 + 0.1^2 + .2^2 + .3^2 + .4^2 + .5^2 + .6^2 + .7^2 + .8^2 + .9^2 = 2.85$

$0 + 0.01 + .04 + .09 + .16 + .25 + .36 + .49 + .64 + .81 = 2.85$

$.1 + .2 + .3 + .4 + .5 + .6 + .7 + .8 + .9 = 4.5$

$1+1+1+1+1+1+1+1+1+1 = 10$

$= \begin{bmatrix} 1.533 & 2.025 & 2.85 \\ 2.025 & 2.85 & 4.5 \\ 2.85 & 4.5 & 10 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1.53 & 2.025 & 2.85 & 1 & 0 & 0 \\ 2.025 & 2.85 & 4.5 & 0 & 1 & 0 \\ 2.85 & 4.5 & 10 & 0 & 0 & 1 \end{array}\right]$$

$R_1 = R_1/1.533$

$$\left[\begin{array}{cccccc} 1 & 1.3209 & 1.859 & .6523157 & 0 & 0 \\ \underset{0}{\cancel{2.025}} & \underset{.1751775}{\cancel{2.85}} & \underset{.735525}{\cancel{4.5}} & \underset{-1.3209}{\cancel{0}} & 1 & 0 \\ \underset{0}{\cancel{2.85}} & \underset{.735435}{\cancel{4.5}} & \underset{4.70185}{\cancel{10}} & \underset{-1.859}{\cancel{0}} & 0 & 1 \end{array}\right]$$

$R_2 = R_2 - 2.025 R_1$

$R_3 = R_3 - 2.85 R_1$

$R_2 = R2/.1751775$

$$\left[\begin{array}{cccccc} 1 & \overset{0}{\cancel{1.3209}} & \overset{-3.61153}{\cancel{1.859}} & \overset{10.61236}{\cancel{.6523157}} & \overset{-7.540}{\cancel{0}} & 0 \\ 0 & 1 & 4.19874 & -7.54035 & 5.7084 & 0 \\ 0 & \underset{0}{\cancel{.735435}} & \underset{1.6139}{\cancel{4.70185}} & \underset{3.68643}{\cancel{-1.859}} & \underset{-4.1981}{\cancel{0}} & 1 \end{array}\right]$$

$R_1 = R_1 - 1.3209 R_2$

$R_3 = R_3 - .735435 R_2$

$R_3 = R_3/1.6139$

$$\left[\begin{array}{cccccc} 1 & 0 & \overset{0}{\cancel{-3.61153}} & \overset{18.94}{\cancel{10.61236}} & \overset{-17.05}{\cancel{-7.540}} & \overset{2.28}{\cancel{0}} \\ 0 & 1 & \underset{0}{\cancel{4.19874}} & \overset{-17.05}{\cancel{-7.54035}} & \overset{16.56}{\cancel{5.7084}} & \overset{-2.60}{\cancel{0}} \\ 0 & 0 & 1 & 2.28 & -2.60 & 0.6196 \end{array}\right]$$

$R_1 = R_1 + 3.61153 R_3$

$R_2 = R_2 - 4.19874 R_3$

$$(MTM)^{-1} \quad \begin{bmatrix} 18.94 & -17.05 & 2.28 \\ -17.05 & 16.56 & -2.60 \\ 2.28 & -2.60 & 0.62 \end{bmatrix}$$

$$x = (M^T M)^{-1} \cdot M^T y$$

$$= \begin{bmatrix} 18.94 & -17.05 & 2.28 \\ -17.05 & 16.56 & -2.60 \\ 2.28 & -2.60 & 0.62 \end{bmatrix} \begin{bmatrix} 10.7916 \\ 15.9 \\ 30.2 \end{bmatrix}$$

$$= \begin{matrix} 18.94(10.7916) - 17.05(15.9) + 2.28(30.2) \\ -17.05(10.7916) + 16.56(15.9) - 2.60(30.2) \\ 2.28(10.7916) + \overset{-2.60}{\cancel{16.56}}(15.9) + 0.62(30.2) \end{matrix}$$

$$= \cancel{2.2153} \begin{bmatrix} 2 \\ 0.80 \sim 1 \\ 2 \end{bmatrix}$$

I rounded up a lot of my decimals so I rounded up my results!

Sorry ♥

$$\boxed{\begin{matrix} B_0 = 2, \; B_1 = 1, \; B_2 = 2 \\ y = 2.0 + 1.0x + 2.0x^2 \end{matrix}}$$

(4)

x	$P(X=x)$
1	0.2
2	0.5
3	0.3

calculate

$E[X]$ of random variable X

$$E[x] = \sum_{x \in X} x \cdot p(x)$$

$$= 1 \cdot 0.2 + 2 \cdot 0.5 + 3 \cdot 0.3$$

$$= 0.2 + 2\left(\tfrac{1}{2}\right) + 3\left(\tfrac{3}{10}\right)$$

$$= 0.2 + 1 + 0.9$$

$\boxed{E[x] = 2.1}$ is the expected value of the random variable X

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(5) $E[D] = 50$, $n = 100$

$\sigma_D = 10$

The expected distribution of the sample mean $E[S] = E[D]$, (Because of a large sample size of 100)

so $E[s] = E[D] = 50$

$$\sigma_s = \frac{\sigma_D}{\sqrt{n}} = \frac{10}{\sqrt{100}} = \frac{10}{10} = 1$$

The expected distribution of sample mean, S is:

$$S \sim \mathcal{N}(50, 1)$$

(6) Factory produces lightbulbs that have lifetimes following a distribution ~~with~~ $E[D] = \sigma_D = 800$ hrs

Sample of 100 lightbulbs, what is probability that average lifetime $E[S] > 720$ and $E[S] < 880$ hrs?

$$E[S] = E[D] = 800$$

$$\sigma_S = \frac{\sigma_D}{\sqrt{n}} = \frac{800}{\sqrt{100}} = \frac{80\cancel{0}}{1\cancel{0}} = 80$$

$$S \sim N(800, 80)$$

convert 720 and 880 to z-scores to get probabilities

$$S = 720 \longrightarrow z = \frac{720-800}{80} = \frac{80}{80} = 1$$

$$S = 880 \longrightarrow z = \frac{880-800}{80} = \frac{80}{80} = 1$$

$$P(z \le 1) = .84134$$

$$P(z \le -1) = .15866$$

$$P(720 \le S \le 880) = .84134 - .15866$$

$$= .68268 \approx 68.27\%$$

There is 68.27% probability that the avg. lifetime of a randomly chosen sample of 100 bulbs is between 720 & 880 hours.