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CMPINF 2105

"Pen and Paper" HW 1: Linear Algebra & Linear Systems

(Modules 1 & 2)

① $\vec{a} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

$$\vec{a} + \vec{b} = \begin{bmatrix} 2-3 \\ -1+4 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \checkmark$$

$$\vec{a} - \vec{b} = \begin{bmatrix} 2+(+3) \\ -1-4 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix} \checkmark$$

$$2\vec{a} - 2\vec{b} = \begin{bmatrix} 2\cdot 2 \\ 2\cdot(-1) \end{bmatrix} - \begin{bmatrix} 2(-3) \\ 2\cdot 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4+(+6) \\ -2-8 \end{bmatrix} = \begin{bmatrix} 10 \\ -10 \end{bmatrix} \checkmark$$

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② Dot product of vectors $\vec{c}$ and $\vec{d}$

$$\vec{c} = \begin{bmatrix} 6 \\ 0 \\ -2 \end{bmatrix}, \quad \vec{d} = \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}$$

$$\vec{c}\cdot\vec{d} = \underset{1\times 3}{\begin{bmatrix} 6 & 0 & -2 \end{bmatrix}} \cdot \underset{3\times 1}{\begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}} = \overset{6\cdot(-1)+0\cdot 4-2(-1)}{(-6+0+2)} = \boxed{-4} \checkmark$$

$$= 6\cdot(-1) + 0(4) - 2(-1) = -6+0+2 = -4 \checkmark$$

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③ $M = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix}$, $N = \begin{bmatrix} 1 & 0 \\ -3 & 4 \end{bmatrix}$

$$M+N = \begin{bmatrix} 2+1 & 3+0 \\ 5-3 & -1+4 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 2 & +3 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 2 & 3 \end{bmatrix} \checkmark$$

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④ $$3\cdot P = 3\cdot\begin{bmatrix} 7 & -2 \\ -6 & 3 \end{bmatrix} = \begin{bmatrix} 7\cdot 3 & -2\cdot 3 \\ -6\cdot 3 & 3\cdot 3 \end{bmatrix} = \begin{bmatrix} 21 & -6 \\ -18 & 9 \end{bmatrix} \checkmark$$

$P = \begin{bmatrix} 7 & -2 \\ -6 & 3 \end{bmatrix}$

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⑤ $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$$A\cdot\vec{b} = \underset{2\times 2}{\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 3 \\ -1 \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} \overset{6}{2\cdot 3} + \overset{-1}{1(-1)} \\ \underset{-3}{-1\cdot 3} + \underset{-3}{(3)(-1)} \end{bmatrix}} = \begin{bmatrix} 5 \\ -6 \end{bmatrix} = \begin{bmatrix} 5 \\ -6 \end{bmatrix} \checkmark$$

(6) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ matrix dot product of A & B

$B = \begin{bmatrix} -1 & 3 \\ 4 & -2 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} -1 & 3 \\ 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1(-1)+2\cdot4 & 1\cdot3+2(-2) \\ 3(-1)+4\cdot4 & 3\cdot3+4(-2) \end{bmatrix}$$

$$= \begin{bmatrix} -1+8 & 3-4 \\ -3+16 & 9-8 \end{bmatrix} = \begin{bmatrix} 7 & -1 \\ 13 & 1 \end{bmatrix} \checkmark$$

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(7) $M = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

7a) Find eigenvalues of M

We will need to find the determinant of $(\lambda \cdot 1 - A)$ to get eigenvalues.

$$A\vec{v} = \lambda \cdot \vec{v}$$

$$A\vec{v} = \lambda \cdot 1 \cdot \vec{v}$$

$$0 = \lambda \cdot 1 \cdot \vec{v} - A\vec{v}$$

$$= \vec{v}(\lambda \cdot 1 - A)$$

$$= \vec{v}\left(\lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)$$

$$= \vec{v}\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)$$

$$= \vec{v}\left(\begin{bmatrix} \lambda-2 & 0-1 \\ 0-1 & \lambda-2 \end{bmatrix}\right)$$

Determinant of

$$|\lambda \cdot 1 - A| = (\lambda-2)(\lambda-2) - (-1)(-1)$$

$$= \lambda^2 - 2\lambda - 2\lambda + 4 - 1$$

$$= \lambda^2 - 4\lambda + 3$$

$$= (\lambda-1)(\lambda-3)$$

$\checkmark$

$\boxed{\text{The Eigenvalues are } \lambda = 1 \text{ or } \lambda = 3.}$

7b) The larger eigenvalue is $\lambda = 3$

Eigenspace of $\lambda = 3$ is E_3

$$E_3 = N\left(\begin{bmatrix} \lambda-2 & -1 \\ -1 & \lambda-2 \end{bmatrix}\right)$$

$$= N\left(\begin{bmatrix} 3-2 & -1 \\ -1 & 3-2 \end{bmatrix}\right) = N\left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)$$

The Eigenvectors in E_3 is...

$$= \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \xrightarrow[R_2 = R_2 + R_1]{} \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$-v_1 + 1 = 0$
$-v_1 = -1$
$v_1 = 1$

$v_2 = 1$

$$\boxed{\begin{array}{c} \text{Eigenvector of } E_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ E_3 = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \end{array}}$$ ✓

7c) verify that eigenvector is an eigenvector of Matrix M

Start with $A \cdot \vec{v}$ where A is the Matrix M and $\vec{v}$ = eigenvector ✓

$$A \cdot \vec{v} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+1 \\ 1+2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \longrightarrow 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$, yes this is the same vector as the eigenvector, just scaled by 3!

This means the eigenvector is an eigenvector of the matrix M, and the scalar, 3, is the eigenvalue. ✓ ←

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⑧ $3x_1 + 5x_2 = 59$
$7x_1 + 2x_2 = 99$ $\longrightarrow \left[\begin{array}{cc|c} 3 & 5 & 59 \\ 7 & 2 & 99 \end{array}\right]$

$\text{Det} = 3 \cdot 2 - 7 \cdot 5$
$= 6 - 35 = -29$
one solution!

$$\left[\begin{array}{cc|c} 3 & 5 & 59 \\ 7 & 2 & 99 \end{array}\right] \xrightarrow[R_1 = R_1/3]{} \left[\begin{array}{cc|c} 1 & 5/3 & 59/3 \\ 7 & 2 & 99 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 5/3 & 59/3 \\ 0 & -\frac{29}{3} & -\frac{116}{3} \end{array}\right]$$

$R_2 = R_2 - 7R_1$
$2 - 7(5/3) = -\frac{29}{3}$
$99 - 7(59/3) = -\frac{116}{3}$

$$\begin{bmatrix} 1 & 5/3 & 59/3 \\ 0 & -\frac{29}{3} & -\frac{116}{3} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 5/3 & 59/3 \\ 0 & 1 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 13 \\ 0 & 1 & 4 \end{bmatrix}$$

$R_2 = R_2 \cdot \left(-\frac{3}{29}\right)$
$-\frac{29}{3} \cdot -\frac{3}{29} = 1$
$-\frac{116}{3} \cdot -\frac{3}{29} = 4$

$R_1 = R_1 - \frac{5}{3}R_2$
$5/3 - 5/3(1) = 0$
$59/3 - \frac{5}{3}(4) = \frac{39}{3} =$

$$\left[\begin{array}{cc|c} 1 & 0 & 13 \\ 0 & 1 & 4 \end{array}\right] \longrightarrow \boxed{\begin{array}{l} x_1 = 13 \\ x_2 = 4 \end{array}}$$ ✓

(9) $9x_1 + 9x_2 - 7x_3 = 6$

$3x_1 - 2x_2 - 9x_3 = 3$

$6x_1 + 0x_2 + 1x_3 = -10$

$\text{Det} = 9(-2\cdot 1 - 0\cdot(-9)) - 9(3\cdot 1 - 6(-9)) + (-7)$ $(3\cdot 0 - 6(-2))$

$= -18 - 513 - 84 = -615$

Because the determinant is not 0, there is a solution!

1)
$$\left[\begin{array}{ccc|c} 9 & 9 & -7 & 6 \\ 3 & -2 & -9 & 3 \\ 6 & 0 & 1 & -10 \end{array}\right] \xrightarrow[R_1 = R_1/9]{} $$

2)
$$\begin{bmatrix} 1 & 1 & -7/9 & 6/9 \\ 3 & -2 & -9 & 3 \\ 6 & 0 & 1 & -10 \end{bmatrix} \longrightarrow$$

$R_2 = R_2 - 3R_1$

3)
$$\begin{bmatrix} 1 & 1 & -7/9 & 6/9 \\ 0 & -5 & -\frac{20}{3} & 1 \\ 6 & 0 & 1 & -10 \end{bmatrix}$$

$3 - 3(1) = 0$

$-2 - 3(1) = -5$

$-9 - 3(-7/9) = -\frac{20}{3} = -\frac{27}{3} + \frac{7}{3}$

$3 - 3(6/9) = 1 = 3 - \frac{3}{1}\cdot\frac{6}{9} = 3 - 2$

4)
$$\left[\begin{array}{ccc|c} 1 & 1 & -7/9 & 6/9 \\ 0 & -5 & -\frac{20}{3} & 1 \\ 0 & -6 & \frac{17}{3} & -14 \end{array}\right] \longrightarrow$$

$R_3 = R_3 - 6R_1$

$6 - 6(1) = 0$

$0 - 6(1) = -6$

$1 + 6(+7/9) = \frac{17}{3} = \frac{3}{3} + \frac{6\cdot 7}{9} = \frac{3+14}{3}$

$-10 - 6(6/9) = -14 = -10 - \frac{6\cdot 6}{9} = -10 - 4$

(5)
$$\begin{bmatrix} 1 & 1 & -7/9 & 6/9 \\ 0 & 1 & 4/3 & -1/5 \\ 0 & -6 & \frac{17}{3} & -14 \end{bmatrix} \rightarrow$$

$R_2 = R_2/-5$

$+\frac{20}{3}\left(+\frac{1}{5}\right)$ $= \frac{4}{3}$

6)
$$\begin{bmatrix} 1 & 0 & -\frac{19}{9} & 13/15 \\ 0 & 1 & 4/3 & -1/5 \\ 0 & -6 & \frac{17}{3} & -14 \end{bmatrix}$$

$R_1 = R_1 - R_2$

$-7/9 - 4/3 = -\frac{7}{9} - \frac{12}{9} = -\frac{19}{9}$

$\frac{2}{3} + \frac{1}{5} = \frac{2\cdot 5 + 1\cdot 3}{15} = \frac{13}{15}$

7) $R_3 = R_3 + 6R_2$

$$\begin{bmatrix} 1 & 0 & -\frac{19}{9} & 13/15 \\ 0 & 1 & 4/3 & -1/5 \\ 0 & 0 & \frac{41}{3} & -\frac{76}{5} \end{bmatrix} \longrightarrow$$

$-6 + 6(1) = 0$

$\frac{17}{3} + 6(\frac{24}{}4/3) = \frac{41}{3}$

$-14 + 6(-1/5) = \frac{-76}{5} = \frac{-70}{5} - \frac{6}{5}$

$R_3 = R_3 \cdot \frac{3}{41}$

8)
$$\begin{bmatrix} 1 & 0 & -19/9 & 13/15 \\ 0 & 1 & 4/3 & -1/5 \\ 0 & 0 & 1 & \frac{-228}{205} \end{bmatrix} \longrightarrow$$

$\frac{-76}{5}\cdot\frac{3}{41} = \frac{-228}{205}$

9) $R_1 = R_1 + 19/9 R_3$

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{911}{615} \\ 0 & 1 & 4/3 & -1/5 \\ 0 & 0 & 1 & \frac{-228}{205} \end{bmatrix}$$

$-\frac{19}{9} + \frac{19}{9}(1) = 0$

$13/15 + \frac{19}{9}\left(\frac{-228}{205}\right)$

$= \frac{123}{123}\cdot\frac{13}{15} - \frac{4332}{1845}$

$= \frac{1599 - 4332}{1845}$

$= \frac{-2733}{1845} = \frac{-911}{615}$

10) $R_2 = R_2 - \frac{4}{3}R_3$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{-911}{615} \\ 0 & 1 & 0 & \frac{263}{205} \\ 0 & 0 & 1 & \frac{-228}{205} \end{bmatrix}$$

$= 4/3 - 4/3(1) = 0$

$= -1/5 - 4/3\left(\frac{-228}{205}\right)$

$\frac{(123)}{(123)}\frac{-1}{5} + \frac{912}{615}$

$= \frac{-123 + 912}{615} = \frac{789}{615} \div 3 = \frac{263}{205}$

Answer:

$$\boxed{\begin{aligned} x_1 &= -\frac{911}{615} \\ x_2 &= \frac{263}{205} \\ x_3 &= \frac{-228}{205} \end{aligned}}$$

✓

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$11x_1 + 3x_2 + 9x_3 = 237$
$5x_1 + 4x_2 + 2x_3 = 101$
$37x_1 + 18x_2 + 24x_3 = -777$

$\rightarrow \left[\begin{array}{ccc|c} 11 & 3 & 9 & 237 \\ 5 & 4 & 2 & 101 \\ 37 & 18 & 24 & -777 \end{array}\right]$

$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$

$\text{Det} = 11(\overset{96}{24\cdot 4} - \overset{-36}{18\cdot 2}) - 3(\overset{120}{24\cdot 5} - \overset{74}{37\cdot 2}) + 9(\overset{90}{18\cdot 5} - \overset{148}{37\cdot 4})$

$= 11(60) - 3(46) + 9(-58)$

$= 660 - 138 - 522 = 660 - 660 = 0$

Because the determinant = 0, this means that this could have infinite # of solutions or cannot be solved.

1) $R_1 = R_1/11$

$\left[\begin{array}{cccc} 1 & 3/11 & 9/11 & 237/11 \\ 0 & 29/11 & \frac{-23}{11} & -74/11 \\ 37 & 18 & 24 & -777 \end{array}\right] \rightarrow$

2) $R_3 = R_3 - 37R_1$

$\left[\begin{array}{cccc} 1 & 3/11 & 9/11 & 237/11 \\ 0 & 29/11 & -23/11 & -74/11 \\ 0 & 87/11 & \frac{-69}{11} & -\frac{17316}{11} \end{array}\right] \rightarrow$

$37 - 37(1) = 0$
$18 - 37(3/11) = \frac{87}{11} = \frac{198-111}{11}$
$24 - 37(9/11) = \frac{264-333}{11} = \frac{-69}{11}$
$-777 - 37\left(\frac{237}{11}\right) = \frac{-8547-8769}{11}$

3) $R_2 = R_2 \cdot 11/29$

$\left[\begin{array}{cccc} 1 & 3/11 & 9/11 & 237/11 \\ 0 & 1 & \frac{-23}{29} & \frac{-74}{29} \\ 0 & \frac{87}{11} & \frac{-69}{11} & -\frac{17316}{11} \end{array}\right]$

$\frac{-23}{\cancel{11}} \cdot \frac{\cancel{11}}{29} =$
$-74/\cancel{11} \cdot \frac{\cancel{11}}{29}$

4) $R_1 = R_1 - 3/11\, R_2$

$\left[\begin{array}{cccc} 1 & 0 & 30/29 & \frac{645}{29} \\ 0 & 1 & \frac{-23}{29} & -74/29 \\ 0 & 87/11 & \frac{-69}{11} & \frac{-17316}{11} \end{array}\right] \rightarrow$

$3/11 - 3/11(1) = 0$
$\frac{29}{29} \cdot 9/11 - 3/11\left(\frac{-23}{29}\right) = \frac{261+69}{319} = \frac{330}{319}$
$\frac{29}{29} \cdot \frac{237}{11} - 3/11\left(\frac{-74}{29}\right) = \frac{237\cdot 29 + 3\cdot 74}{319}$

5) $R_3 = R_3 - 87/11\, R_2$

$\left[\begin{array}{cccc} 1 & 0 & 30/29 & 645/29 \\ 0 & 1 & \frac{-23}{29} & \frac{-74}{29} \\ 0 & 0 & 0 & \frac{-1554}{1} \end{array}\right] \rightarrow$

$\frac{87}{11} - \frac{87}{11}(1) = 0$
$\frac{-69}{11} - \frac{87}{11}\left(\frac{-23}{29}\right) =$
$= \frac{-69\cdot 29 + 87\cdot 23}{319} =$
$= \frac{-2001+2001}{319} = 0$

$= \frac{-17316}{11} - \frac{87}{11}\left(\frac{-74}{29}\right)$

$\frac{-502164 + 6438}{319} = \frac{-495726}{319}$

$= -1554$ ✓

6) $R_1 = R_1 - \frac{645}{29} R_3$
$R_2 = R_2 + 74/29\, R_3$

$\left[\begin{array}{cccc} 1 & 0 & \frac{30}{29} & 0 \\ 0 & 1 & \frac{-23}{29} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

Because the last row is $[0\ 0\ 0 \mid 1]$, this means that there is no solution b/c we imply $0 = 1$, which is not true.
Therefore, the system of equations has NO solution!
It can't be solved. ♥