Section 9

- problems 1(d) and 4 (dipole moment, etc.)
- Casimir effect

The matrix element in the QFT is

$$\langle p, s | H_I | p', s' \rangle = \frac{q}{2m} \int d\vec{x}\; e^{i(p-p')\cdot x}\, \bar{u}_s(p) \left(p'_\mu + p_\mu + i\sigma_{\mu\nu}(p'^\nu - p^\nu) \right) u_{s'}(p')\, A^\mu(x)$$

We wish to extract the nonrelativistic limit,

$$p^\mu = \left(m + mv^2/2 + O(v^4),\; mv + O(v^3) \right)$$

Let's write out all the terms that can appear in the spinor contraction,

$$\bar{u}(p_0 + p'_0) u A^0 + \bar{u}(p_i + p'_i) u A^i + \bar{u}(i\sigma_{0i}(p'^i - p^i)) u A^0 + \bar{u}(i\sigma_{ij}(p'^j - p^j)) u A^j + \bar{u}(i\sigma_{i0}(p'^0 - p^0)) u A^i$$

- $O(v^0)$, reproduces qA^0 term (1(b))
- $O(v)$, reproduces $\vec{p}\cdot\vec{A} + \vec{A}\cdot\vec{p}$ term — magnetic force
- looks $O(v)$ but turns out to be negligible, $O(v^2)$
- $O(v)$, gives $\vec{\mu}\cdot\vec{B}$ magnetic dipole term (1(d))
- $O(v^2)$, negligible

The strategy to evaluate the dipole term is just to expand everything out.

$$\langle p,s|H_I|p',s'\rangle \supset \frac{q}{2m}\int d\vec{x}\; \underbrace{e^{i(p-p')\cdot x}\,(p^j-p'^j)}_{-i\partial^j\left(e^{i(p-p')\cdot x}\right)}\;\underbrace{\bar{u}^s(p)}_{\sqrt{m}\,(\xi_s^\dagger\;\xi_s^\dagger)}\;\underbrace{\frac{[\gamma^i,\gamma^j]}{2}}_{-i\epsilon^{ijk}\begin{pmatrix}\sigma^k & \\ & \sigma^k\end{pmatrix}}\;\underbrace{u_{s'}(p')}_{\sqrt{m}\begin{pmatrix}\xi_{s'}\\ \xi_{s'}\end{pmatrix}}\;A^i(x) \qquad \sigma^{\mu\nu}=\frac{i}{2}[\gamma^\mu,\gamma^\nu]$$

$+\,O(v)$ corrections (negligible)

by direct calculation

$$= q\,\epsilon^{ijk}\,\xi_s^\dagger\sigma^k\xi_{s'}\int d\vec{x}\;\partial^j\left(e^{i(p-p')\cdot x}\right)A^i$$

$$= q\,\epsilon^{ijk}\,\xi_s^\dagger\sigma^k\xi_{s'}\int d\vec{x}\;e^{i(p-p')\cdot x}\,\partial_j A^i \qquad \vec{B}=\nabla\times\vec{A},\quad B_k=\epsilon_{ijk}\partial_i A_j$$

$$= -q\,\xi_s^\dagger(\vec{\sigma}\cdot\vec{B})\,\xi_{s'}\;\delta(\vec{p}-\vec{p}')$$

$$\vec{S}=\frac{\vec{\sigma}}{2}$$

which must match $2m\langle\vec{p},s|(-\vec{\mu}\cdot\vec{B})|\vec{p}',s'\rangle = -gq'\,\xi_s^\dagger(\vec{S}\cdot\vec{B})\,\xi_{s'}\,\delta(\vec{p}-\vec{p}') \longrightarrow g=2.$

Next: what is the effect of $H_I \supset i\bar{\Psi}\sigma^{\mu\nu}\Psi F_{\mu\nu}$? We get contributions like:

$$\underbrace{\bar{u}\sigma^{ij}u\,F_{ij}}_{\text{gives }\vec{\sigma}\cdot\vec{B}\text{ by same logic}} + \underbrace{\bar{u}\sigma^{i0}u\,F_{i0}}_{\sigma^{i0}\,\propto\begin{pmatrix}\sigma^i & \\ & -\sigma^i\end{pmatrix}}$$

which causes a cancellation.

this term is negligible in the NR limit.

$\longrightarrow$ <u>shifts magnetic dipole moment.</u>

Next consider $H_I \supset \bar{\Psi}\sigma^{\mu\nu}\gamma^5\psi F_{\mu\nu}$. in Weyl rep, $\gamma^5 = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$ so it exchanges the roles of the terms!

$$\underbrace{\bar{u}\,\sigma^{ij}\gamma^5 u F_{ij}}_{\text{cancellation, negligible}} + \underbrace{\bar{u}\,\sigma^{i0}\gamma^5 u F_{i0}}_{\text{yield } \vec{\sigma}\cdot\vec{E} \,\longrightarrow\, \underline{\text{electric dipole moment}}}$$

e^- EDM : CP-violating new physics

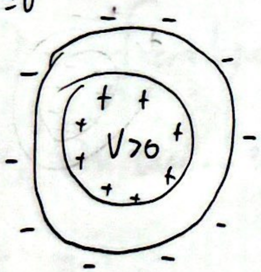
n EDM : should be there due to CP violation in strong sector, not observed!

Next consider $H_I \supset \bar{\Psi}\gamma^\mu\psi\,\partial^\nu F_{\mu\nu} = -\bar{\Psi}\gamma^\mu\Psi J_\mu$. This just replaces A_μ with $-J_\mu$, so leading effect is

$$H_I^{nr} \supset \rho(\vec{x})$$

which seems weird. Zero unless the particle is right on top of other charge! What classical EM setup could do this?

Spherical capacitor:



Zero external fields! sole effect is a higher voltage inside, so if very small, electrostatic interaction energy is $V\cdot\frac{4}{3}\pi r^3\cdot\rho(\vec{x})$ if centered at $\vec{x}$.

The charge of the particle is spread out — this is a charge radius.
- p's CR defines its size, had discrepancy
- e^-'s CR seems zero → "e^- is pointlike"

Finally, for $H_I \supset \bar{\Psi}\gamma^\mu\gamma^5\Psi J_\mu$ we get $H_I^{nr} \supset \vec{S}\cdot\vec{J}(\vec{x})$. shows up if particle carries current of a tiny toroidal solenoid — an anapole moment. These things don't show up in the multipole expansion b/c they're near field, not far field.

Recall the energy density of the vacuum for a massless real scalar field, in 1d,

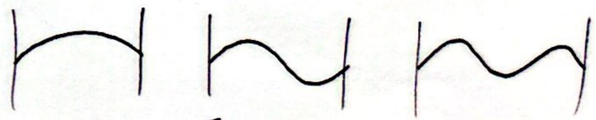
$$H = \int dk \, \frac{w_k}{2} \left(a_k^\dagger a_k + a_k a_k^\dagger \right)$$

$$= \underbrace{\int dk \, w_k a_k^\dagger a_k}_{\text{zero for vacuum}} + \underbrace{\int dk \, \frac{w_k}{2}}_{k = w_k} \underbrace{\delta(k-k)}_{\text{really means volume } V \text{ of space}}$$

$$\rho_0 = \frac{1}{2\pi} \int_0^\infty w \, dw = \infty ! \quad [\text{unobservable constant}]$$

We could ignore this, except that it's possible to change ρ_0 by adding other things! Imagine boundary conditions

$$\phi(0) = \phi(L) = 0 \quad [\text{analogue of conducting plates in EM}]$$



$$k_n = \frac{n\pi}{L} = w_n$$

Modes inside are discrete, with vacuum energy

$$H_0 = \sum_{n=0}^\infty \frac{w_n}{2} = \sum_{n=0}^\infty \frac{\pi}{2L} n \qquad \rho_0' = \frac{\pi}{2L^2} \sum_{n=0}^\infty n$$

By comparison, without the plates (letting $x = w/w_1$ in analogy with n)

$$\rho_0 = \frac{\pi}{2L^2} \int_0^\infty x \, dx \qquad F = -\frac{dE}{dL} = -(\rho_0 - \rho_0') = \frac{\pi}{2L^2} \left(\sum_{n=0}^\infty n - \int_0^\infty x \, dx \right).$$

Unfortunately, result is mathematically ill-defined.

Need to invoke 2 concepts:

- renormalization: focus on observables (in this case, take the difference)
- regularization: remove idealizations to make more things finite, and thus mathematically defined.

If $\phi(x)$ was height of string, extreme form of regularization is to remember string made of atoms → sufficiently high freqs. don't actually exist. But this is too hard to implement. Easier: for any physical barrier, sufficiently high ω waves don't see it → at high n, x the sum and integral must really be same. We can artificially implement this by taking

$$\sum_{n=0}^{\infty} n \to \sum_{n=0}^{\infty} n e^{-\epsilon n} = \frac{e^{-\epsilon}}{(1-e^{-\epsilon})^2} = \frac{1}{\epsilon^2} - \frac{1}{12}$$

$$\int_0^{\infty} x\,dx \to \int_0^{\infty} x e^{-\epsilon x}\,dx = \frac{1}{\epsilon^2}$$

dependence on cutoff param. ϵ cancels! for small ϵ, $F = -\frac{\pi}{24L^2}$ which matches expt.

Remarks:

- result is indep of cutoff function [Schwartz 15.3], matches analytic continuation
- sometimes sloppily expressed as "$1+2+3+\ldots = -\frac{1}{12}$"
- interpretation still controversial (Jaffe, 2005)