Section #2.

- Dirac deltas
- contour integrals
- the big picture.
- answer pset Q's

$$\begin{tikzcd}
\text{expt.} \arrow[r] \arrow[dr] & \text{syms.} \arrow[d, "\text{rep.}"] & & & & & \\
& \text{particles} \arrow[r, "\text{rep.}"] & \text{fields} \arrow[r, "\text{EFT}"] & \mathcal{L} \arrow[r, "\text{c.q.}"] & \text{quantum theory} \arrow[r, "\text{Feynman}"] \arrow[d, bend right, "\text{Wick, Dyson}"'] & \mathcal{M} \arrow[r, "\text{int.}"] & \text{observables} \\
& & & & \text{correlation fncs.} \arrow[ur, bend right, "\text{LSZ}"'] & &
\end{tikzcd}$$

(particle physics, condensed matter, nuclear physics, quantum optics, cosmology)

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\text{QFT} \arrow[r, shift left, "\text{classical field limit}"] \arrow[d, shift left, "\text{particle limit}"] & \text{CFT} \arrow[l, shift left, "\text{canonical quantization}"] \arrow[d, shift left, "\text{action at distance}"] \\
\text{particle QM} \arrow[u, shift left, "\text{2nd quant.}"] \arrow[r, "\text{classical limit}"', "\text{canonical quantization}" below] & \text{CM} \arrow[u, shift left, "\text{continuum description}"]
\end{tikzcd}$$

(acoustics, hydrodynamics, electromagnetism, most of engineering, Earth science)

(atomic physics, chemistry)

(everyday physics, planetary & galactic dynamics)

QFT is the subject that unifies particle QM & classical field theory.

all quantum fields exhibit this wave-particle duality. in one limit, quantum EM field becomes classical EM field; in another, few discrete photons.

QFT is the common language of huge part of physics!

most accurate calculational tool ever, and contains all physics you've ever learned.

Φ expt.

$$\begin{tikzcd}
\text{expt.} \arrow[r] \arrow[dr] & \text{basic syms.} \arrow[d, "\text{rep. theory}"] & & & & \\
& \text{types of particles} \arrow[r, "\text{rep.}", "\text{theory}"'] & \text{types of fields} \arrow[r, "\text{EFT}"] & \text{Lagrangian} \arrow[r, "\text{canon.}", "\text{quant.}"'] & \text{interacting Hamiltonian} \arrow[r, "\text{Feynman}", "\text{rules}"'] & \text{scattering amplitudes}
\end{tikzcd}$$

the "steps": $\text{interacting Hamiltonian} \xrightarrow[\text{theorem}]{\text{Wick's}} \text{correlation functions} \xrightarrow[\text{reduction}]{\text{LSZ}} \text{scattering amplitudes} \xrightarrow[\text{int.}]{\text{phase space}}$ cross sections, decay rates etc.

Delta functions:

$$V_1 + iV_2 = (u+iv,\ -v+iu) = (f,\ if)$$

defined by $\displaystyle\int_{-\infty}^{\infty} dx\, f(x)\,\delta(x) = f(0)$

implies: $\displaystyle\int dx\, f(x)\,\delta(x-a) = f(a)$ $\qquad \displaystyle\int d^3x\, f(\vec{x})\,\delta^{(3)}(\vec{x}) = f(\vec{0})$

$$\int dx\, f(x)\,\delta(cx) = \frac{1}{|c|} f(0) \xrightarrow[\text{generally}]{\text{more}} \int dx\, f(x)\,\delta(g(x)) = \sum_{\substack{x_n \\ g(x_n)=0}} \frac{\delta(x_n)}{|g'(x_n)|}$$

<u>Ex</u>: what is $\int dx\, f(x)\,\delta(x^2-1)$? Solns $x = \pm 1$ where derivative is $\pm 2 \rightarrow \frac{1}{2}(f(1)+f(-1))$

<u>bonus</u>:

$$\delta^{(3)}(\vec{x}) = (?)\,\delta(r)$$

derivatives: $\displaystyle\int dx\, f(x)\,\delta'(x) = -\int dx\, f'(x)\,\delta(x) = -f'(0)$

<u>Contour integrals</u>: $\int_C f(z)\,dz$. let's consider components: $f = u + iv$, $dz = dx + i\,dy$.

$$\int_C f(z)\,dz = \int_C (u\,dx - v\,dy) + i(v\,dx + u\,dy)$$

$$f'(z) \equiv \lim_{\epsilon \to 0} \frac{f(z+\epsilon a) - f(z)}{\epsilon a} \rightarrow \frac{f(z+\epsilon)-f(z)}{\epsilon} = \frac{f(z+i\epsilon)-f(z)}{i\epsilon}$$

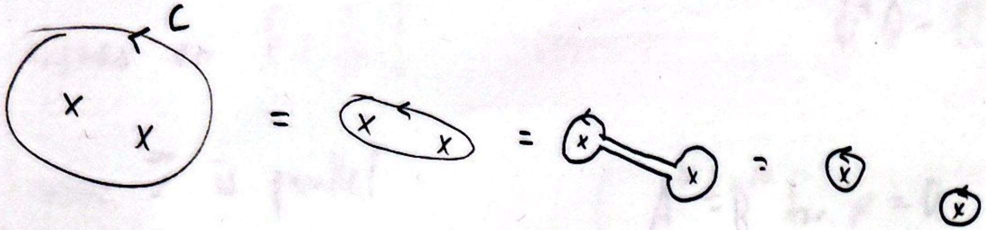
(for any a) $\qquad \left(\dfrac{\partial f}{\partial x} = \dfrac{1}{i}\dfrac{\partial f}{\partial y}\right) \qquad \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} \qquad -i\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial y}$

$$\rightarrow \underbrace{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}_{\substack{(v, +u) = V_2 \\ \text{is curl-free.}}} \qquad \underbrace{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}_{\substack{(u,-v) \text{ is curl-free.} \\ \| \\ V_1}}$$

$$\int_C f(z)\,dz = \int_C (\vec{V}_1 + i\vec{V}_2)\cdot d\vec{r}$$

is independent of path.

So we can always deform closed integration contours to zero, unless there are singularities



just need to know how to evaluate a small loop around each singularity

$$\oint \frac{dz}{z} = \oint_0^{2\pi} \frac{d(re^{i\theta})}{re^{i\theta}} = \int_0^{2\pi} d\theta \, \frac{ie^{i\theta}}{e^{i\theta}} = 2\pi i.$$

$$\oint \frac{dz}{z^2} = \int_0^{2\pi} d\theta \, \frac{ie^{i\theta}}{re^{2i\theta}} = \frac{i}{r}\int_0^{2\pi} d\theta \, e^{-i\theta} = 0. \quad \text{(and so on)}$$

for simple singularities, looking like $1/z^n$, only $1/z$ terms ("poles") do anything. So in general we have

$$\int_C f(z)\,dz = \sum_i 2\pi i \,\mathrm{Res}(z_i) \quad \text{where } f \text{ looks like } \frac{\mathrm{Res}(z_i)}{z-z_i} \text{ near } z=z_i.$$

Ex: what is $\displaystyle\int_C \frac{f(z)\,dz}{z^2-1}$ where C is a big ccw circle, and f is not singular?

$$\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{1}{2}\left(-\frac{1}{z+1} + \frac{1}{z-1}\right) \rightarrow 2\pi i\left(\frac{1}{2}f(1) - \frac{1}{2}f(-1)\right).$$

challenging example: handling "essential" singularity.

$$\oint dz \, e^{iaz} e^{ib/z} = ?$$

note: wild oscillations, growth, and damping as $z \to 0$!