Section 6.

- PS 5 review - virtual particles
- all about rep. theory.

2a: how to do final $\int d\Omega$?

2c: normalization of vertex.

$$\mathcal{L}_{int} = -\lambda \psi^* \psi \psi^* \psi$$

leads to structure like

$$\ldots (\text{stuff}) \quad \psi^* \psi^* \psi \psi \quad (\text{stuff}) \ldots$$

which has $2 \times 2 = 4$ choices. so

 $= -4i\lambda$.

generally: $\mathcal{L}_{int} = \dfrac{\phi_1^n}{n!} \dfrac{\phi_2^m}{m!} \ldots$

produces vertex with <u>no</u> numeric factor.

$$\Gamma = \frac{1}{2M} \int \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1} \frac{d^3\vec{p}_2}{(2\pi)^3 2E_2} |-ig|^2 (2\pi)^4 \delta^{(4)}(k-p_1-p_2)$$

$$= \frac{g^2}{32\pi^2 M} \int \frac{d^3\vec{p}_1 \, d^3\vec{p}_2}{E_1 E_2} \delta^{(4)}(k-p_1-p_2) \qquad k^\mu = (M, \vec{0})$$

$$= \frac{g^2}{32\pi^2 M} \int \frac{d^3\vec{p}_1}{E_1^2} \delta(M-2E_1) \qquad d^3\vec{p}_1 \to |\vec{p}_1|^2 d|\vec{p}_1| \, d\Omega$$

$$= \frac{g^2}{32\pi^2 M} \int_0^\infty d|\vec{p}_1| \frac{|\vec{p}_1|^2}{E_1^2} \delta(M-2E_1) \underbrace{\int d\Omega}_{4\pi}$$

trick: $E_1^2 = |\vec{p}_1|^2 + m^2$

$E_1 dE_1 = |\vec{p}_1| d|\vec{p}_1|$

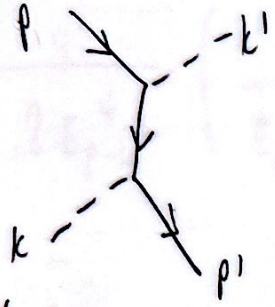
$$= \frac{g^2}{8\pi M} \int_m^\infty dE_1 \frac{|\vec{p}_1|}{E_1} \delta(M-2E_1)$$

note: $\int d\Omega$ is 2π if identical particles

$$= \frac{g^2}{16\pi M} \sqrt{1-(2m/M)^2}$$

1b: many mixed up t and u, see soln.

A conceptual question: what's going on here?



$\psi\phi \to \psi\phi$

is the intermediate ψ matter or antimatter? $\qquad iM \sim (-ig)^2 \dfrac{i}{(p-k')^2 - m^2 + i\epsilon}$

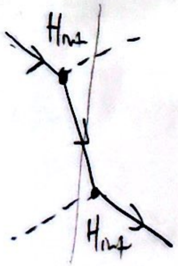
it is a "virtual" particle — what does that mean?

according to undergrad QM, the matrix element looks like

$$M \sim \sum_n \frac{\langle f|H_{int}|n\rangle\langle n|H_{int}|i\rangle}{E - E_n}$$ (from TDPT + Fermi's golden rule)

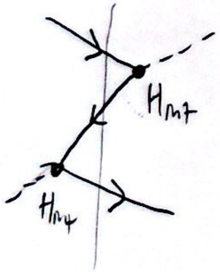
expec. values of H_0.

there are 2 possibilities for $|n\rangle$:



$|n\rangle = |\vec{k}, \vec{k}', \psi \text{ with } \vec{p} - \vec{k}'\rangle$ gives $\dfrac{g^2/2E^\psi_{p-k}}{E^\psi_{\vec{p}} + E^\phi_{\vec{k}} - (E^\phi_{\vec{k}} + E^\phi_{\vec{k}'} + E^\psi_{\vec{p}-\vec{k}'})} = \cancel{\dfrac{g^2}{\sqrt{|\vec{p}|^2+m^2} + \sqrt{|\vec{k}'|^2+m^2} - \sqrt{(\vec{p}-\vec{k})^2+m^2}}}$

$\neq$



$|n\rangle = |\psi \text{ with } \vec{p}, \psi \text{ with } \vec{p}', \bar{\psi} \text{ with } \vec{p} - \vec{k}'\rangle$ $\quad \dfrac{g^2/2E^\psi_{p-k}}{E^\psi_{\vec{p}'} + E^\phi_{\vec{k}'} - (E^\psi_{\vec{p}} + E^\psi_{\vec{p}'} + E^\psi_{\vec{p}-\vec{k}'})} = \cancel{\dfrac{g^2}{\sqrt{|\vec{k}|^2+m^2} - \sqrt{|\vec{k}'|^2+m^2} - \sqrt{(\vec{p}-\vec{k})^2+m^2}}}$

$\neq \cancel{g^2}$

the sum is

$$\frac{g^2}{2E_{p-k}^4}\left(\frac{1}{E_p - E_{k'} - E_{p-k'}} + \frac{1}{E_{k'} - E_p - E_{p-k'}}\right) = +g^2 \frac{1}{-(E_{p-k'})^2 + (E_p - E_{k'})^2}$$

$$= -g^2 \frac{1}{(p^0 - k'^0)^2 - |\vec{p} - \vec{k'}|^2 - m^2}$$ which matches the Feynman propagator.

virtuality is artifact of description (using $T\exp(-i\int H dt)$, Feynman prop.)

keeps stuff LI and accounts for multiple diagrams at once.

perturbation theory is artificial!

- E is always conserved
- state always evolves smoothly

but... there is amp to be in these intermediate states.

another Q: people say a static source is surrounded by a cloud of zero frequency virtual photons. what?

$H = \frac{p^2}{2m} + \frac{1}{2}kx^2 + Fx$ is in a coherent state.

it's $|0\rangle + \alpha|1\rangle + \dots$ but the $|1\rangle$ doesn't oscillate. might say: zero freq. virtual excitation.

Same with QFT!

More representation theory

Let's review rotations. What are they? Simple answer: linear trans. that preserve lengths of vectors,

$$\vec{v} \to R\vec{v} \qquad (R\vec{v})\cdot(R\vec{v}) = \vec{v}\cdot\vec{v} \quad \to R^T R = I.$$

But vectors aren't the only things! A scalar f doesn't change, $f \to f$. And an outer product of 2 vectors has

$$M = \vec{v}\,\vec{w}^T \to R\vec{v}\,\vec{w}^T R^T = RMR^T.$$

More generally, a tensor (rank 2) has to transform this way,

$$\underset{\text{3d rep.}}{v_i \to R_{ij} v_j} \qquad \underset{\text{9d rep.}}{M_{ij} \to R_{ik} M_{kl} R_{jl}} \qquad \underset{\text{1d rep.}}{f \to f}$$

The 9d rep. is reducible: symmetric stays symmetric ($S \to RSR^T$, $S^T \to RSR^T$), antisym, and identity. Can write

$$M_{ij} = \underset{\text{5d rep}}{S_{ij}} + \underset{\text{3d rep}}{A_{ij}} + \underset{\text{1d rep.}}{f I_{ij}}$$

5d rep	3d rep
moment of inertia	torque
stress tensor	magnetic field

more generally, 7d, 9d, 11d, ...

there are also two <u>invariant</u> objects: δ_{ij} and ϵ_{ijk}. these are particular tensors whose components stay the same under R.

<u>everything</u> can be built up from the "fundamental" 3d rep and these tensors.

ex: $\delta_{ij} v_i w_j$ is scalar $\quad \epsilon_{ijk} v_i w_j$ is vector $(\vec{v}\times\vec{w})_k$

That is the reason we spend so much time studying vectors, $\cdot$, and $\times$. But in QM we see an even more "fundamental" representation, the spin ½ particle!

$$|\psi\rangle = \psi_\uparrow |\uparrow\rangle + \psi_\downarrow |\downarrow\rangle \qquad \psi_a \to \underbrace{S(R)_{ab}}_{\text{unitary } 2\times 2 \text{ matrix}} \psi_b$$ is a 2d rep., call it 2.

The global picture of reps. is...

dim	classical	quantum	
1	scalar	spin 0	"s orbs"
2	—	spin ½	
3	vector	spin 1	"p orbs"
4	—	spin 3/2	
5	traceless sym. 2-tensor	spin 2	"d orbs"
6	—	spin 5/2	
7	traceless sym 3-tensor	spin 3	"f orbs"

It's all the same math!

vector $\otimes$ vector = traceless sym 2 $\oplus$ vector $\oplus$ scalar

spin 1 $\otimes$ spin 1 = spin 2 $\oplus$ spin 1 $\oplus$ spin 0

$$3 \times 3 = 5 + 3 + 1$$

But now we see the spinor ψ_a is the <u>true</u> fundamental object. So we should build everything out of it.

[why not earlier? b/c spinor rep. is <u>complex</u> & <u>projective</u>]

For example, product of 2 spinors: spin ½ $\otimes$ spin ½ = spin 1 + spin 0. How to extract them?

spin 0: $\epsilon_{ab} \psi_a \chi_b$ spin 1 = vector: $V_i = (\sigma_i)_{ab} \psi^*_a \chi_b = \langle \psi | \sigma_i | \chi \rangle$

only invariant for spinors

Clebsch-Gordan coeffs. for extracting 3 from $2\otimes 2$.

acted on by J^i with $[J^i, J^j] = i\epsilon^{ijk} J^k$

I

The story for Lorentz is a step more complex...

$$[J^i, J^j] = i\epsilon^{ijk} J^k$$
$$[J^i, K^j] = i\epsilon^{ijk} K^k \qquad J^i_\pm = \frac{J^i \pm i K^i}{2} \rightarrow \begin{array}{l} [J^i_+, J^i_-] = 0 \\ [J^i_\pm, J^j_\pm] = i\epsilon^{ijk} J^k_\pm \end{array}$$
$$[K^i, K^j] = -i\epsilon^{ijk} J^k$$

Lorentz algebra is just two copies of the rotation algebra!

Classification of reps. by how they transform under each of the $J_\pm$:

"spin" \ "spin"	0	$\frac{1}{2}$	1
0	Lorentz scalar (1d)	LH Weyl spinor (2d)	anti-self dual antisym 2-tensor (3d)
$\frac{1}{2}$	RH Weyl spinor (2d)	4-vector (4d)	LH Weyl Rarita-Schwinger (6d)
1	self-dual (3d) antisym 2-tensor	RH Weyl Rarita-Schwinger (6d)	traceless sym. 2-tensor (9d)

example: how does $V^\mu W^\nu$ decompose?

$$\left(\tfrac{1}{2}, \tfrac{1}{2}\right) \otimes \left(\tfrac{1}{2}, \tfrac{1}{2}\right) = (0,0) + (0,1) + (1,0) + (1,1)$$

scalar $\eta_{\mu\nu} V^\mu W^\nu$

tensor $\frac{V^\mu W^\nu + V^\nu W^\mu}{2} - \frac{\eta^{\mu\nu}\eta_{\mu\nu} V^\mu W^\nu}{4}$

antisym. tensor $\frac{V^\mu W^\nu - V^\nu W^\mu}{2} = A^{\mu\nu}$.

further decompose as $A^{\mu\nu} = \bar{A}^{\mu\nu} + \tilde{A}^{\mu\nu}$

$$\bar{A}^{\mu\nu} = -\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{A}^{\rho\sigma}$$
$$\tilde{A}^{\mu\nu} = +\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\tilde{A}^{\rho\sigma}.$$

invariant tensors if we stick with proper representations are $\eta_{\mu\nu}, \epsilon_{\mu\nu\rho\sigma}$

everything you know fits in here. but where is the Dirac spinor?!

it's a reducible rep, $\left(\tfrac{1}{2}, 0\right) + \left(0, \tfrac{1}{2}\right)$, motivated by parity

the Weyl spinors are the real fundamentals. (very useful in theory) the product of two Dirac spinors:

$$\left(\left(\tfrac{1}{2},0\right)\oplus\left(0,\tfrac{1}{2}\right)\right)\otimes\left(\left(\tfrac{1}{2},0\right)\oplus\left(0,\tfrac{1}{2}\right)\right) = (0,0) + \overbrace{(1,0) + (0,1)}^{\text{antisym. tensor}} + (0,0) + \overbrace{\left(\tfrac{1}{2},\tfrac{1}{2}\right) + \left(\tfrac{1}{2},\tfrac{1}{2}\right)}^{\text{two 4-vectors}}$$

$$\bar{\psi}\psi \qquad \bar{\psi}[\gamma^\mu,\gamma^\nu]\psi \qquad \bar{\psi}\gamma^5\psi \qquad \bar{\psi}\gamma^\mu\psi \qquad \bar{\psi}\gamma^\mu\gamma^5\psi$$

the γ matrizes are fundamentally Clebsch-Gordan coeffs for extracting the 4-vector part of the product.