Section 3

Common issues on PS 1!

$\vec{v} = \vec{w}$ unless $\vec{v}$ is parallel to the $\hat{z}$ axis, in which case $\vec{v} = -\vec{w}$

$\vec{v} \cdot \vec{w} = v_x w_x + v_y w_y - v_z w_z$

$\vec{\tau} = \vec{r} \cdot \vec{p}$

$x(t) = x(0) + \int_0^t v(t)\, dt$

$\frac{dx(t)}{dt} = v(t) + \int_0^t a(t)\, dt + \int_0^t v(t)$?!

(Don't use sources — use your brain. We have original problems.)

$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

vs. $(\partial_x E_y - \partial_y E_x)\hat{z} + \ldots = -\frac{\partial B_z}{\partial t}\hat{z} - \ldots$

$Q\phi - \phi Q = [Q, \phi] = \phi$

$\phi^\dagger Q - Q^\dagger \phi^\dagger = \phi^\dagger$

$A^\mu = B^\mu$ for $\mu = 0$

$A^\mu = -B^\mu$ for $\mu = i$ — spurious signs

$V = A^\mu B^\mu$ — mismatched index pos.

$V^\mu = A^\mu B_\mu$ — extra index

$V^\mu = A^\mu B_\mu C^\mu$ — repeated index

to avoid confusion, rename as needed!

$A^0 B_0 + A^1 B_1 + A^2 B_2 + A^3 B_3$ — use compact notation.

it is not true that $a(\vec{k}) = a^\dagger(-\vec{k})$

problem 2: you can save a lot of time!

problem 1: it's not e^{ivk_1}, it's $e^{i \tanh^{-1}(v) k_1}$.

$f(\vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^3} f(\vec{k}) e^{+i\vec{k}\cdot\vec{x}}$

$f(\vec{k}) = \int d^3\vec{x}\, f(\vec{x}) e^{-i\vec{k}\cdot\vec{x}}$

what is true is that if a function $f(\vec{x})$ is real,

$f(\vec{k}) = f(-\vec{k})^*$

if $O(\vec{x})$ is Hermitian,

$O(\vec{k}) = O(-\vec{k})^\dagger$

but:

$\phi(\vec{k}) = \frac{a^\dagger(-\vec{k}) + a(+\vec{k})}{\sqrt{2\omega_k}}$

Why are commutators so important? Consider 1D QM, states $|\psi\rangle$. Physical content given by action of <u>unitary</u> ops. like

trans. $T(a) = e^{ipa}$ (p, H Hermitian) trans sym. <u>means</u> $T(a)U(t)|\psi\rangle = U(t)T(a)|\psi\rangle$ for any $|\psi\rangle$

time trans. $U(t) = e^{-iHt}$

↳ generators are physical observables

For infinitesimal $\epsilon = a = t$, get

$$(1 + ip\epsilon)(1 - iH\epsilon)|\psi\rangle = (1 - iH\epsilon)(1 + ip\epsilon)|\psi\rangle \longrightarrow [p, H]|\psi\rangle = 0 \rightarrow [p, H] = 0.$$

Commutation of the generators implies finite syms and also means:

① let $H|\psi\rangle = E|\psi\rangle$ and $|\psi'\rangle = T(a)|\psi\rangle \rightarrow H|\psi'\rangle = HT(a)|\psi\rangle = T(a)H|\psi\rangle = E|\psi'\rangle$

translations don't change energy. (another way to say we have sym.)

② let $p|\psi\rangle = p_0|\psi\rangle$ and $|\psi'\rangle = U(t)|\psi\rangle \rightarrow p|\psi'\rangle = p_0|\psi'\rangle$

time evolution doesn't change $p \rightarrow p$ conserved (Noether in Hamiltonian)

③ p and H have simultaneous eigenvectors → can be known simultaneously, measurements commute.

What about nontrivial commutators? Simplest new one is x, and p generates translations in $x \rightarrow p = -i\partial_x \rightarrow [x, p] = i$.

Turn it around: if $[x, p] = i$ then $p = -i\partial_x$ (Stone – von Neumann).

General point: in QM,

physical ops = unitary, can compose and invert = Lie group.

physical obs. = Hermitian, have commutation relations = Lie algebra

how they specifically act on $\mathcal{H}$ = rep. theory.

BCH: comm. relations of generators tell us mult. struc. of group. so Lie algebra determines much of Lie group structure.

Simplest example of rep. theory: (NR rotation, no metric signs)

→ derived in PS 2 through classical considerations

$[J_i, J_j] = i\epsilon_{ijk} J_k \qquad R(\hat{n}, \theta) = e^{-i\theta \hat{n}\cdot\vec{J}}$ comm. rel. not specific to QM, general structure.

$J_i = 0$ trivial rep., no spin

$J_i = \frac{\sigma_i}{2}$ spin $1/2$

$[Q, \phi] = \phi$, ϕ lowers Q by 1

→ $[H, a^+] = \omega a^+$ a^+ raises H by ω.

general: let $J_\pm = J_1 \pm i J_2 \to [J_3, J_\pm] = \pm J_\pm \to J_+$ raises J_3 eig. by 1.

if $J_3|m\rangle = m|m\rangle$ and $J_+|j\rangle = 0$ then $J_-|-j\rangle = 0$ → states are $m = -j, -j+1, \ldots, j-1, j$.

not just for QM states, also works for vectors, tensors...

vector:
$|m=1\rangle \leftrightarrow \hat{x} + i\hat{y}$
$|m=0\rangle \leftrightarrow \hat{z}$
$|m=-1\rangle \leftrightarrow \hat{x} - i\hat{y}$

3d rep. has to be spin 1

rank 2 tensor T_{ij} has 9 components

= traceless sym. part (5d, spin 2 rep.)
antisym. part (3d, spin 1 rep.)
trace (1d, spin 0 rep.)

...

Wigner's classification:

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$
$$[J_i, k_j] = i\epsilon_{ijk} k_k$$
$$[k_i, k_j] = -i\epsilon_{ijk} J_k$$
$$[J_i, P_j] = i\epsilon_{ijk} P_k$$

$$[J_i, H] = [P_i, H] = 0$$
$$[k_i, P_j] = -iH\delta_{ij}$$
$$[k_i, H] = -iP_i$$

① simul. diag. $\vec{P}, H$ to get $|p^\mu, \sigma\rangle$.

② syms. act as expected: $e^{i\theta J_3}|p^\mu, \sigma\rangle = |(\Lambda p)^\mu, \sigma'\rangle$ rotation

$e^{i \tanh v\, k_1}|p^\mu, \sigma\rangle = |(\Lambda p)^\mu, \sigma'\rangle$ boost

③ what happens to σ? easiest to consider ops. that fix p^μ.

- if $p^2 = m^2$, can LT to $p^\mu = (m, 0, 0, 0)$. only ops. that fix are $J_i \to$ spin reps.
- if $p^2 = 0$, can LT to $p^\mu = (w, 0, 0, w)$. ops. that fix are

$$R = J_z \qquad T_1 = J_x + k_y \qquad T_2 = J_y - k_x$$

$$[R, T_\pm] = \pm T_\pm.$$

define states $R|h\rangle = h|h\rangle$.

then we must have $T_\pm|h\rangle = \rho|h \pm 1\rangle$.

the general rep. has all integer or half-integer h.

usually take $\rho = 0$, only one h.

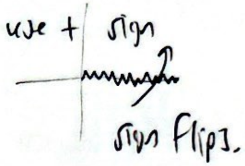
CPT sym: if h, have $-h$

e.g. photon has $h = \pm 1$

graviton has $h = \pm 2$.

"photon is spin 1" wrong!

Branch cuts: $\sqrt{z}$ not consistently defined! if $z = re^{i\theta}$,

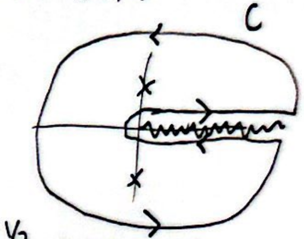


$$\sqrt{z} = \pm|\sqrt{r}|\, e^{i\theta/2}$$

use $|r^{1/3}|\, e^{i\theta/3}$

A contour integral with a branch cut!

$$I = \int_0^\infty \frac{x^{1/3}\, dx}{1 + x^2}$$



$$\int_C \frac{z^{1/3}\, dz}{1 + z^2} = \int_C \frac{dz}{2i}\left(\frac{z^{1/3}}{z - i} - \frac{z^{1/3}}{z + i}\right)$$
$$= (2\pi i)\frac{1}{2i}\left(i^{1/3} - (-i)^{1/3}\right) = \left(\frac{\sqrt{3} - i}{2}\right)\pi$$
$$= I - e^{2\pi i/3} I \quad \to I = \pi/\sqrt{3}.$$