Section #7

- feedback on PS 6
- relativistic EM

{ be careful to write indices distinctly. η vs. γ, μ vs. ν
relabel indices often — tedious but necessary.

read official solns, they explain tricks that make later prob easier.

At this point, many objects have both Lorentz & spinor indices, which can be confusing.

$\mu\nu\rho\sigma$, from 0 to 3 — transforms by Λ, up vs. down matters

abcd, from 0 to 3 — transforms by $u(\Lambda)$, position irrelevant (but still summed in pairs)

$$(i\not\partial - m)\psi = 0 \qquad (i\not\partial_{ab} - m\mathbb{1}_{ab})\psi_b = 0 \qquad (i\gamma^\mu\partial_\mu - m)\psi = 0 \qquad (i\gamma^\mu_{ab}\partial_\mu - m\mathbb{1}_{ab})\psi_b = 0$$

Dirac eq. — spinor indices explicit — Lorentz indices explicit — all indices explicit

There's a tradeoff here. With all indices explicit, everything commutes with everything (γ^μ_{ab} is just a #), besides derivatives, but it's clunky. If some types of indices implicit, then we use the order to imply how indices are contracted, which means order matters! For example, in vector calculus

$$V_i\partial_i W_j = (\partial_i W_j)V_i$$ but without indices, must be written as $(\vec{V}\cdot\nabla)\vec{W}$

$A_{ij}B_{jk} = B_{jk}A_{ij}$ but can only be written as $(AB)_{ik}$

Conventional choice: Lorentz explicit, spinor implicit. (and in Yang-Mills there are color indices implicit too!)

Use antisymmetry/symmetry to save writing. Let $A^{\widehat{\mu\nu}} = A^{\mu\nu} - A^{\nu\mu}$. Then

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\eta^{\nu\rho}J^{\mu\sigma} \quad \text{(all 4 terms at once)} \qquad (J^{\mu\nu})_{\alpha\beta} = i\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} = i\,\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta}$$

1b: $$[J^{\mu\nu}, J^{\rho\sigma}]_{\alpha\delta} = -\eta^{\beta\gamma}\left(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta}\delta^{\rho}_{\gamma}\delta^{\sigma}_{\delta} - \delta^{\rho}_{\alpha}\delta^{\sigma}_{\beta}\delta^{\mu}_{\gamma}\delta^{\nu}_{\delta}\right)$$

$$= -\left(\delta^{\mu}_{\alpha}\eta^{\nu\rho}\delta^{\sigma}_{\delta} - \delta^{\rho}_{\alpha}\eta^{\sigma\mu}\delta^{\nu}_{\delta}\right)$$

$$= -\eta^{\nu\rho}\left(\delta^{\mu}_{\alpha}\delta^{\sigma}_{\delta} - \delta^{\mu}_{\delta}\delta^{\sigma}_{\alpha}\right)$$

$$= -\eta^{\nu\rho}\left(\delta^{\mu}_{\alpha}\delta^{\sigma}_{\delta}\right)$$

$$= i\eta^{\nu\rho}J^{\mu\sigma}$$

takes more thought, but much quicker.

$$\mathrm{tr}(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\gamma^{\delta}) = 0$$

$$\mathrm{tr}(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\gamma^{\delta}\gamma^{\alpha}) = 4\left(\eta^{\mu\nu}\eta^{\rho\sigma}\eta^{\delta\alpha} + 14 \text{ terms w/ alternating signs}\right)$$

What is the point of all trace identities? Consider

$$\mathrm{tr}(\gamma^0\gamma^3\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^2) \quad \text{is this nonzero?}$$

Distinct γ matrices anticommute, so this is

$$\pm\mathrm{tr}(\gamma^0\gamma^0\gamma^1\gamma^2\gamma^2\gamma^2\gamma^3\gamma^3) = \pm\mathrm{tr}(\gamma^1\gamma^2)$$

all γ's square to ± 1

We can always reduce to trace of at most 4 distinct γ matrices. 2(b) tells us all such traces vanish. Only get nonzero result if you have even # of each type of γ matrix!

Relativistic EM: To get massless photons, build Lagrangian from 4-vector A^μ,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\Big\downarrow \text{E-L} \qquad\qquad\qquad \Big\downarrow$$

$$\partial_\mu F^{\mu\nu} = 0 \qquad\qquad \epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\nu\rho} = 0.$$

2 of Maxwell's eqs. 2 of Maxwell's eqs.

why is this true?

$$\epsilon^{\mu\nu\rho\sigma}\left(\partial_\mu \partial_\nu A_\rho - \partial_\mu \partial_\rho A_\nu\right)$$

each term vanishes because

$$\epsilon^{\mu\nu\rho\sigma} \partial_\mu \partial_\nu A_\rho = -\epsilon^{\nu\mu\rho\sigma} \partial_\nu \partial_\mu A_\rho = -\epsilon^{\mu\nu\rho\sigma} \partial_\mu \partial_\nu A_\rho.$$

This Lagrangian has a gauge sym: action stays the same under $A_\mu \to A_\mu + \partial_\mu \alpha$ for function $\alpha(x)$. This implies that some subset of the E-L eqs. don't say anything $\to$ fewer E-L eqs. than fields $\to$ time evolution indeterminate

Trivial example: $L(q) = 0 \to q = \alpha(t)$

A bit less trivial: $L(q_1, q_2) = \frac{1}{2} m(\dot{q}_1 - \dot{q}_2)^2 + V(q_1 - q_2)$ is unchanged by $\begin{array}{l} q_1 \to q_1 + \alpha(t) \\ q_2 \to q_2 + \alpha(t). \end{array}$

There are formally two dof but only one physical dof. Can also see this in the Hamiltonian formulation. Phase space should contain only physical states. Indeed, here

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} = m(\dot{q}_1 - \dot{q}_2) \qquad p_2 = \frac{\partial L}{\partial \dot{q}_2} = m(\dot{q}_2 - \dot{q}_1) \longrightarrow p_1 + p_2 = 0.$$ the set of momenta is 1d, not 2d.

There's another weird thing that can happen:

$$L(q_1, q_2) = \frac{1}{2} m \dot{q}_1^2 + q_2 f(q_1)$$ (2nd way for Lagrangian d.o.f. to be unphysical)

no dep. on $\dot{q}_2 \to p_2 = 0$, E-L eq. for q_2 is $f(q_1) = 0$. q_2 is nondynamical, a Lagrange multiplier.

To describe a <u>massive</u> spin 1 particle, take

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu \to (\partial^2 + m^2) A^\mu = 0, \quad \partial_\mu A^\mu = 0.$$

There are no $\partial_0 A_0$ terms $\to A_0$ is a Lagrange multiplier which yields constraint $\partial_\mu A^\mu = 0$. There are $4 - 1 = 3$ dynamical d.o.f. corresponding to the 3 spin states of a spin 1 particle.

The massless photon has only 2 d.o.f. ($\to$ helicity ± 1) so we must remove one more, which is achieved by gauge symmetry.

This has profound implications! Gauge inv. of S implies $\mathcal{L} \supset -A_\mu J^\mu$ must have $\partial_\mu J^\mu = 0$, so photons couple to conserved charge. A similar story for gravity:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad \text{gauge sym. } h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

symmetric, 10 d.o.f. $-$ (4 nondynamical, 4 gauge) $= 2$ physical ($h = \pm 2$)

Coupling to gravity of form $\mathcal{L} \supset h_{\mu\nu} T^{\mu\nu}$ → for gauge inv. need $\partial_\mu T^{\mu\nu} = 0$. For a general theory, the only conserved 2-tensor is the stress-energy tensor → gravity couples to stress-energy.

These two are the only* possibilities for long-range forces, because:

(remarkably, Weinberg was able to prove these results without field theory, just considering soft emission amplitudes)

- must be mediated by massless particle (to be long-range)
- must be mediated by boson (to have classical limit) $\xrightarrow[\text{thm.}]{\text{Spin-stat.}}$ must have integer helicity
- CPT thm. implies helicities come in pairs. we get

$h = 0$

$h = \pm 1$ couples to J^μ w/ $\partial_\mu J^\mu = 0$ → photon

$h = \pm 2$ couples to $T^{\mu\nu}$ ⟶ graviton

$h = \pm 3$ nothing to couple to

...

Why not $h=0$? Because of renormalization: no way to keep it massless. But there is a sym. that keeps the photon and graviton massless: gauge symmetry! Field theory arguments therefore fix the structure of all observed macroscopic forces, & explain why we don't see more.

↳ and thus the everyday world