Section 8

- common issues in PS 6, PS 7
- amplitude calculations w/ fermions.

PS 6 Most people proved $\mathrm{tr}\,\gamma^\mu\gamma^\nu\gamma^\rho = 0$ wrong. Here's 2 valid proofs.

$$\begin{aligned}\mathrm{tr}\,\gamma^\mu\gamma^\nu\gamma^\rho &= \mathrm{tr}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^5\gamma^5 \\ &= \mathrm{tr}\,\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^5 \\ &= (-1)^3\,\mathrm{tr}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^5\gamma^5 \\ &= -\mathrm{tr}\,\gamma^\mu\gamma^\nu\gamma^\rho\end{aligned}$$

$$\begin{aligned}\mathrm{tr}\,\gamma^\mu\gamma^\nu\gamma^\rho &= \frac{1}{4}\,\mathrm{tr}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\sigma \\ &= \frac{1}{4}\,\mathrm{tr}\,\gamma_\sigma\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \\ &= -\frac{1}{2}\,\mathrm{tr}\,\gamma^\rho\gamma^\nu\gamma^\mu \\ &= -\frac{1}{2}\,\mathrm{tr}\,\gamma^\mu\gamma^\rho\gamma^\nu \qquad \text{we aren't done here! the order doesn't match!} \\ &= -\frac{1}{2}\,\underbrace{\mathrm{tr}\,\gamma^\mu\{\gamma^\rho,\gamma^\nu\}}_{\text{zero}} + \frac{1}{2}\,\gamma^\mu\gamma^\nu\gamma^\rho\end{aligned}$$

PS 7 How to check 3e? (what is $\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$)

- is a scalar under rotations
- is quadratic in $\vec{E}$, $\vec{B}$

$$\left.\right\} \alpha E^2 + \beta B^2 + \gamma \vec{E}\cdot\vec{B}$$

- flips sign under parity

$$\left\{\begin{array}{l} t\to t,\ \vec{r}\to-\vec{r},\ \partial_t\to\partial_t,\ \vec{\nabla}\to-\vec{\nabla} \\ \quad\text{implies} \\ \rho\to\rho,\ \vec{J}\to-\vec{J},\ A^0\to A^0,\ \vec{A}\to-\vec{A}\end{array}\right\} \begin{array}{l}\partial_0 A_1\,\partial_2 A_3 \to (-1)^3\partial_0 A_1\,\partial_2 A_3\text{, flips sign} \\ \vec{E}\to-\vec{E},\ \vec{B}\to\vec{B},\ \text{only } \vec{E}\cdot\vec{B} \text{ flips sign.}\end{array}$$

We know there are 24 terms, so must get $\pm 8\vec{E}\cdot\vec{B}$ (check sign by evaluating one).

$$\epsilon^{ijk}B^k = -F^{ij}$$
$$\underbrace{\epsilon^{ijk}\epsilon^{ijl}}_{\delta^{kl}}B^k = -F^{ij}\epsilon^{ijl}$$
$$B^l = -F^{ij}\epsilon^{ijl}$$

Okay, but how to do <u>cleanly</u> with indices? (with no casework)

$$\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = 4\epsilon^{0\nu\rho\sigma}F_{0\nu}F_{\rho\sigma}$$

$$= \sum_{\nu\rho\sigma} 4(-1)^3\epsilon^{0\nu\rho\sigma}F^{0\nu}F^{\rho\sigma}$$ need to do this to switch to Euclidean

$$= -4\epsilon^{ijk}F^{0i}F^{jk}$$

$$= 4E^i\epsilon^{ijk}F^{jk} = -8E^iB^i = -8\vec{E}\cdot\vec{B}.$$

$$\boxed{T_{\mu\nu} \sim \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} \qquad \delta g^{\mu\nu} = \epsilon g^{\mu\nu} \qquad \delta S \propto \epsilon T_{\mu\nu}g^{\mu\nu} = T^{\mu}{}_{\mu} = 0}$$

Significance of (3g), tracelessness $g_{\mu\nu}T^{\mu\nu} = 0$.

- arises from symmetry of EM under scale changes (dilations) $\rightarrow$ more generally conformal sym.
- implies no bending of starlight in scalar gravity, $\mathcal{L} \supset \phi\, g_{\mu\nu}T^{\mu\nu}$. preferred result is GR, $\mathcal{L} \supset h_{\mu\nu}T^{\mu\nu}$.

$$\boxed{J^\mu = x_\nu T^{\mu\nu}, \qquad 0 = \partial_\mu J^\mu = x_\nu \overbrace{\partial_\mu T^{\mu\nu}}^{0} + \overbrace{\partial_\mu x_\nu}^{g_{\mu\nu}} T^{\mu\nu} = T^{\mu}{}_{\mu}}$$

Many had wrong argument for (4a).

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2A_\mu A^\mu \longrightarrow \partial_\mu F^{\mu\nu} + m^2A^\nu = 0.$$

many rewrote as $(\partial^2 + m^2)A^\nu + \partial^\nu(\partial_\mu A^\mu) = 0$ but this does <u>not</u> show each term is zero.

instead take ∂_ν of eom to get $\underbrace{\cancel{\partial_\nu\partial_\mu F^{\mu\nu}}}_{0} + m^2\partial_\nu A^\nu = 0 \rightarrow \partial_\mu A^\mu = 0$. [many said this is a gauge choice, but there is no gauge sym!]

A disturbing Q: how do we know $m=0$ exactly for photon? Naive answer: b/c for any $m \neq 0$ get 3 polarizations instead of 2

But it's not that simple, since as $m \to 0$ the extra polarization decouples if J^ν conserved. → "there are no discontinuities in physics". all shades of grey.

$$A_\nu = \bar{A}_\nu + \frac{1}{m}\partial_\nu \phi \qquad \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\nu A^\nu - A^\nu J_\nu \to -\bar{A}^\nu J_\nu - \frac{1}{m}\partial^\nu \phi J_\nu$$

$$\downarrow \qquad\qquad \downarrow$$

indep. of ϕ $\qquad$ $\frac{1}{2}m^2 \bar{A}_\nu \bar{A}^\nu + m^2 \bar{A}_\nu \partial^\nu \phi + \frac{1}{2}(\partial_\nu \phi)(\partial^\nu \phi)$

As $m \to 0$ the mixing term goes away and we are left with

$$\mathcal{L} \supset \frac{1}{2}(\partial_\nu \phi)^2 - \frac{1}{m}\partial^\nu \phi J_\nu$$

So coupling to current blows up, unless current is conserved, in which case ϕ decouples!

Next let's show how to do (4c), starting from the answer to 4b,

$$\partial_\mu F^{\mu\nu} = \frac{\alpha}{2}\epsilon^{\mu\nu\rho}F_{\mu\rho} \qquad \text{want to get } (\partial^2 + \alpha^2)F_{\mu\nu} = 0.$$

Doesn't seem we can do much because most indices already contracted. But we can "remove" the ϵ's, by using

$$\epsilon^{\mu\nu\rho}\epsilon_{\mu\nu\sigma} = 2\delta^\rho_\sigma \qquad \epsilon^{\mu\nu\rho}\epsilon_{\mu\alpha\beta} = \delta^\nu_\alpha \delta^\rho_\beta - \delta^\nu_\beta \delta^\rho_\alpha$$

So the proof is... [strategy is to differentiate twice & use eom to remove extra derivatives]

$$\partial_\mu F^{\mu\nu} = \frac{\alpha}{2}\epsilon^{\nu\mu\rho}F_{\mu\rho}$$

contract with $\epsilon_{\nu\alpha\beta}$

$$\epsilon_{\nu\alpha\beta}\partial_\mu F^{\mu\nu} = \frac{\alpha}{2}(F_{\beta\alpha} - F_{\alpha\beta}) = \alpha F_{\beta\alpha}$$

apply ∂^β, use eom.

$$\epsilon_{\nu\alpha\beta}\partial^\beta\partial_\mu F^{\mu\nu} = \alpha\,\partial^\beta F_{\beta\alpha} = \frac{\alpha^2}{2}\epsilon_{\nu\alpha\mu}F^{\mu\nu}$$

contract with $\epsilon^{\alpha\delta\sigma}$

$$\partial^\delta\partial_\mu F^{\mu\sigma} - \partial^\sigma\partial_\mu F^{\mu\delta} = \alpha^2 F^{\sigma\delta}$$

apply ∂_δ

$$\partial^2(\partial_\mu F^{\mu\sigma}) = \alpha^2\partial_\delta F^{\sigma\delta}$$

rearrange, reindex.

$$(\partial^2+\alpha^2)(\partial_\mu F^{\mu\nu}) = 0$$

use eom

$$(\partial^2+\alpha^2)\left(\frac{\alpha}{2}\epsilon^{\nu\mu\rho}F_{\mu\rho}\right) = 0$$

contract with $\epsilon_{\nu\alpha\beta}$

$$(\partial^2+\alpha^2)F_{\alpha\beta} = 0.$$

In one sense, this is the trickiest derivation in the course. But there are only 2 things you can do (apply ∂ or ϵ) and we just do them in alternating order to make progress.

Let's get some practice with fermion amplitudes. We consider Yukawa theory,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 + \bar\psi(i\partial\!\!\!/ - m)\psi - g\phi\bar\psi\psi$$

Compute leading-order amplitude for $\psi\psi\to\psi\psi$, with

$$|i\rangle = |\psi^s(p)\,\psi^r(q)\rangle = \sqrt{4E_pE_q}\,a_p^{s\dagger}a_q^{r\dagger}|0\rangle$$

$$|f\rangle = |\psi^{r'}(q')\,\psi^{s'}(p')\rangle$$

The leading term is $O(g^2)$,

$$\frac{(-ig)^2}{2}\int d^4x\int d^4y\,\langle f|\,T\big(\phi(x)\bar\psi(x)\psi(x)\phi(y)\bar\psi(y)\psi(y)\big)\,|i\rangle.$$

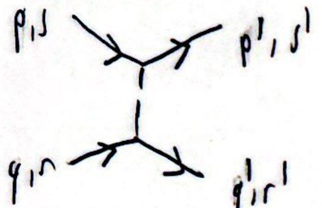
use Wick's theorem, get $D_F(x-y) : \bar\psi(x)\psi(x)\bar\psi(y)\psi(y):$

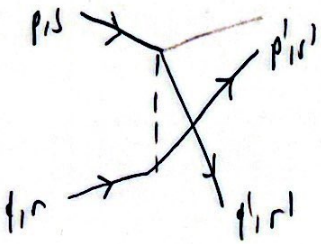
So far, this is identical to what we've seen in scalar Yukawa theory. What differs is that fermion fields are quantized with anticommutators, which means we pick up signs when we exchange their orders under normal ordering. Get

$$\frac{(-ig)^2}{2}\int d^4x\, d^4y\, D_F(x-y)\,\langle \psi^{r'}(q')\psi^{s'}(p')|\,\bar{\psi}_a(x)\,\bar{\psi}_b(y)\,|0\rangle \underbrace{\langle 0|\,\psi_a(x)\psi_b(y)\,|\psi^s(p)\psi^r(q)\rangle}_{e^{-ip\cdot y}u_b^s(p)\,e^{-iq\cdot x}u_a^r(q)\,-\,e^{-ip\cdot x}u_a^s(p)\,e^{-iq\cdot y}u_b^r(q)}(-1)$$

We get 4 terms of which there are two pairs, giving

$$\mathcal{M} = (-ig)^2\left(\frac{\bar{u}^{s'}(p')u^s(p)\,\bar{u}^{r'}(q')\cdot u^r(q)}{(p'-p)^2-\mu^2+i\epsilon} - \frac{\bar{u}^{s'}(p')u^r(q)\,\bar{u}^{r'}(q')u^s(p)}{(q'-p)^2-\mu^2+i\epsilon}\right)$$





New features: spinors for external fermions, sign for exchanging external fermions.