MATH 108 PROOF PORTFOLIO

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1. A direct proof

Definition 1.1. *We say that m is divisible by n if there exists an $k \in \mathbb{Z}$ such that $m = nk$ for $m, n \in \mathbb{Z}$.*

Definition 1.2. *We say that a function is injective if $f(x_1) = f(x_2)$, then $x_1 = x_2$.*

Definition 1.3. *We say that a function $X \to Y$ is surjective if for all $y \in Y$, there exist a $x \in X$ such that f(x) = y.*

Proposition 1.4 (Homework 5, Problem 6a)**.** *(a) 5^{600} is not divisible by 3.*

Proof. We want to show that $5^{600} \neq$ 3j for some $j \in \mathbb{Z}$, by definition 1.1. This means that $5^{600} \neq 0 \pmod 3$. Notice that 5 mod 3 = -1. This indicates that $5^{600} \equiv -1^{600} \pmod 3$. Note that $-1^{2k} = 1$, where $k \in \mathbb{N}$. Let k = 300. Notice that $-1^{600} = -1^{2(300)} = -1^{2k} = 1$. Since $1 \neq 0 \pmod 3$, 5^{600} is not divisible by 3.

□

2. Proof using a contrapositive

Proposition 2.1 (Homework 3, Problem 5b)**.** *(b) Suppose $a \in \mathbb{Z}$. If a^2 is not divisible by 4, then a is odd.*

Proof. By contrapositive, it is equivalent to prove "if a is even, then a^2 is divisible by 4."

By the definition of even numbers, let a = 2k for some $k \in \mathbb{Z}$.

Then

$$(2k)^2 = 4k^2.$$

Since the intergers are closed under multiplication, $k^2 \in \mathbb{Z}$. Let p = k^2. Notice that $a^2 = 4p$, so 4 divides a, by definition 1.1. Thus, a^2 is divisible by 4, by proof of contrapositive.

□

3. Proof by contradiction

Proposition 3.1 (Homework 3, Problem 6)**.** *6. Use proof by contradiction for the following statement. Suppose $a, b, c \in \mathbb{Z}$. If $a^2 + b^2 = c^2$, then a or b is even.*

Proof. The contraditon of the statement above is if $a^2 + b^2 = c^2$ and a and b are odd.

By the definition of odd integers, a = 2k + 1 and b = 2m + 1 for some $k, m \in \mathbb{Z}$.

Then

$$\begin{aligned} c^2 &= (2k+1)^2 + (2m+1)^2 \\ &= 4k^2 + 4k + 1 + 4m^2 + 4m + 1 \\ &= 4k^2 + 4m^2 + 4k + 4m + 2 \\ &= 2(2k^2 + 2m^2 + 2k + 2m + 1). \end{aligned}$$

Let t $= 2k^2 + 2m^2 + 2k + 2m + 1$. Notice that c^2 is an even integer such that $c^2 =$ 2t. This means that c is an even integer. Let c = 2q for some $q \in \mathbb{Z}$. Then, $c^2 =$ (2q)(2q) $= 4q^2$. This implies that c^2 is divisible by 4, by definition 1.1. Note, however, that c^2 is also equal to 2t, which is not divisible by 4, by definition 1.2. This is a contradiction. Hence the original statement holds.

□

4. If and only if (equivalence) proof

Proposition 4.1 (Homework 5, Problem 2a)**.** *a) Given sets A, B and C.*
Show that $A \times (B \cup C) = (A \times B) \cup (A \times C)$

Proof. We need to show that $A \times (B \cup C) \subset (A \times B) \cup (A \times C)$ and
$(A \times B) \cup (A \times C) \subset A \times (B \cup C)$.

Proof:

Let $x \in (A \times B) \cup (A \times C)$.
This means that $x \in (A \times B)$ or $x \in (A \times C)$.
Let x $= (x_1, x_2)$.
So $x_1 \in A$ and $x_2 \in B$ or $x_1 \in A$ and $x_2 \in C$.
Since $x_1 \in A$ and x_2 is an element of either B or C, so $x \in A \times (B \cup C)$.
Thus, $A \times (B \cup C) \subset (A \times B) \cup (A \times C)$.

Let $(x1, x2) \in A \times (B \cup C)$.
Then, $x_1 \in A$ and $x_2 \in (B \cup C)$.
So, $x_1 \in A$ and $x_2 \in B$ or $x_2 \in C$.
We see that x_2 can be in either set B or C.
This means that $(x_1, x_2) \in A \times B$ or $(x_1, x_2) \in A \times C$.
Therefore, $(A \times B) \cup (A \times C) \subset A \times (B \cup C)$.

□

5. An induction (or strong induction) proof

Proposition 5.1 (Practice Problems, Problem 1a)**.**

$$\sum_{k=0}^{n}(2k+1) = (n+1)^2.$$

Proof. Proof by induction

For the base case, we consider n = 0. Note that

$$\sum_{k=0}^{0}(2k+1) = 2(0)+1 = 1$$

.
Also, note that

$$(n+1)^2 = (0+1)^2 = 1$$

.
Thus, the base case holds for n = 0.

Inductive Hypothesis: Assume that $\sum_{k=0}^{n}(2k+1) = (n+1)^2$ is true for some n≤j where $j \in \mathbb{Z}_{\geq 0}$. We want to prove that $\sum_{k=0}^{j+1}(2k+1) = (n+2)^2$.

Then

$$\sum_{k=0}^{j+1}(2k+1) = \sum_{k=0}^{j}(2k+1) + 2(j+1) + 1$$

$$= (k+1)^2 + 2(j+1) + 1 \qquad \text{by Inductive Hypothesis,}$$

$$= j^2 + 2j + 1 + 2j + 2 + 1$$

$$= j^2 + 4j + 4$$

$$= (j+2)(j+2) = (j+2)^2.$$

By induction, the statement holds for all n ≥ 0.

□

6. A PROOF INVOLVING SETS

Proposition 6.1 (Midterm, Problem 4)**.** *Suppose that A, B and C are sets. Prove that A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C).*

Proof. We need to show that $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$ and $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$.

Let $x \in A \cup (B \cap C)$.

So, $x \in A$ or $x \in (B \cap C)$.
Then, $x \in A$ or $x \in B$ and $x \in C$.
It follows that, $x \in (A \cup B)$ and $x \in (A \cup C)$.
Hence, $x \in (A \cup B) \cap (A \cup C)$.

Let $x \in (A \cup B) \cap (A \cup C)$.
Then, $x \in (A \cup B)$ and $x \in (A \cup C)$.
So, $x \in A$ or $x \in B$ and $x \in A$ or $x \in C$.
Note $x \in A$ or $x \in (B \cap C)$.
It is clear that $x \in A \cup (B \cap C)$.

Therefore, we see that $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$ and $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$.

$\square$

7. AN INJECTIVITY AND SURJECTIVITY OF A FUNCTION PROOF

Proposition 7.1 (Practice Problem Final, Problem 6c)**.** *(6c) f:* $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, f(x) = 7x + 4$.

Proof. Note that $7x_1 + 4 = 7x_2 + 4$. By subtracting 4 and then multiplying $\frac{1}{7}$ on each side, we have that $x_1 = x_2$, by definiton 1.2. Thus, this function is injective. Since there exist a $y \in R_{\geq 0}$ such that f(x) = y for all $x \in R_{\geq 0}$, so y = 7x + 4, by definition 1.3. This means that x = (y - 4)/7. This means that f(x) = f((y - 4)/7) = 7((y - 4)/7) + 4 = y. Thus, this function is also surjective.

$\square$