

ROBERT SEDGEWICK | KEVIN WAYNE

<http://algs4.cs.princeton.edu>

2.2 MERGESORT

- ▶ *mergesort*
- ▶ *bottom-up mergesort*
- ▶ *sorting complexity*
- ▶ *comparators*
- ▶ *stability*

Two classic sorting algorithms: mergesort and quicksort

Critical components in the world's computational infrastructure.

- Full scientific understanding of their properties has enabled us to develop them into practical system sorts.
- Quicksort honored as one of top 10 algorithms of 20th century in science and engineering.

Mergesort. [this lecture]



...

Quicksort. [next lecture]



...

Algorithms

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2.2 MERGESORT

- ▶ ***mergesort***
- ▶ ***bottom-up mergesort***
- ▶ ***sorting complexity***
- ▶ ***comparators***
- ▶ ***stability***

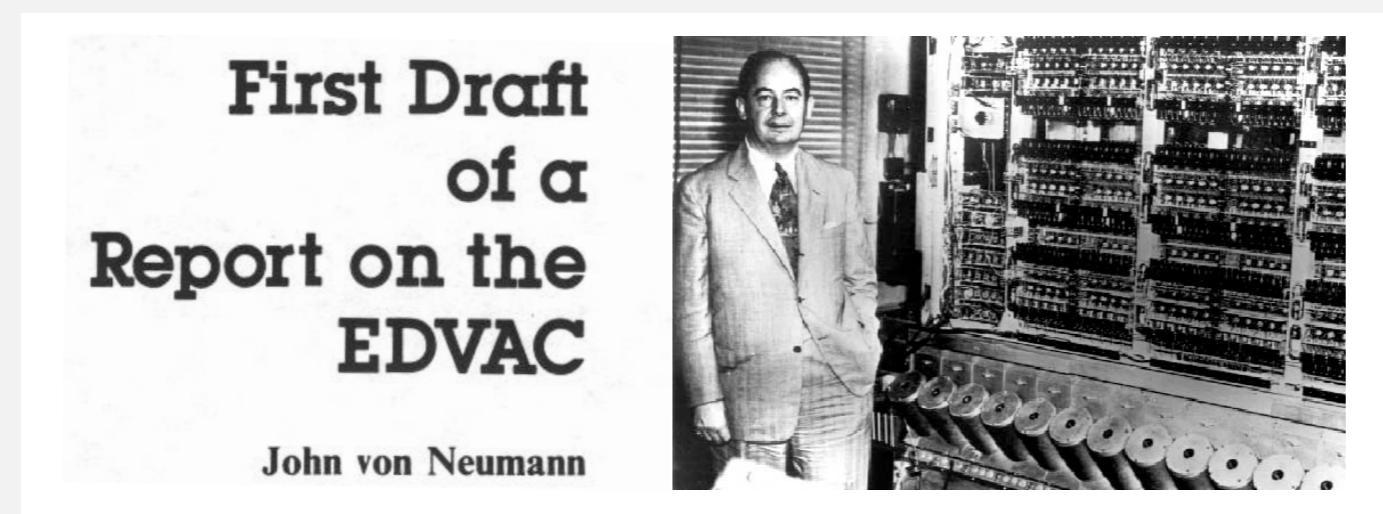
Mergesort

Basic plan.

- Divide array into two halves.
- Recursively sort each half.
- Merge two halves.

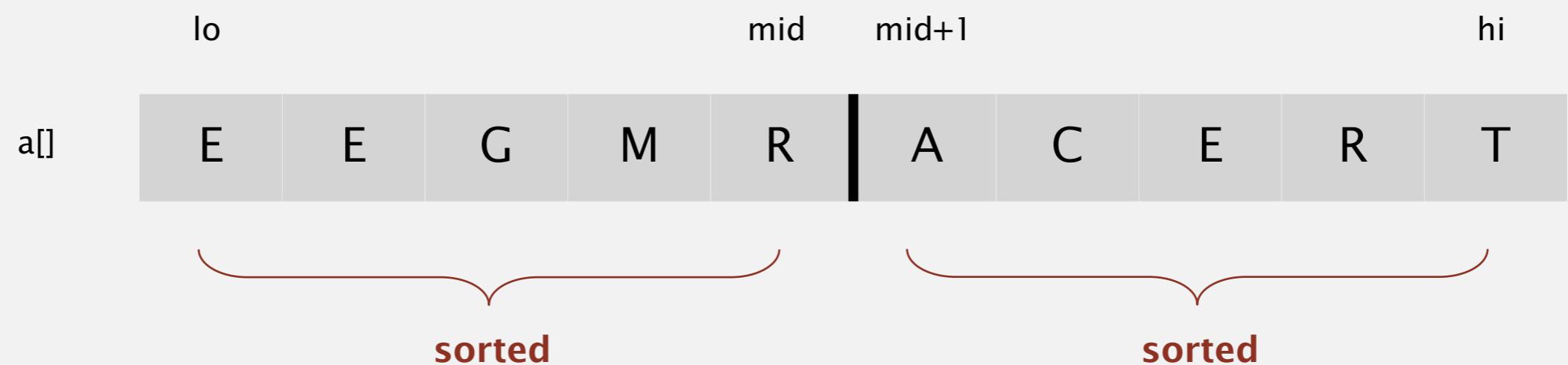
input	M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
sort left half	E	E	G	M	O	R	R	S	T	E	X	A	M	P	L	E
sort right half	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
merge results	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

Mergesort overview



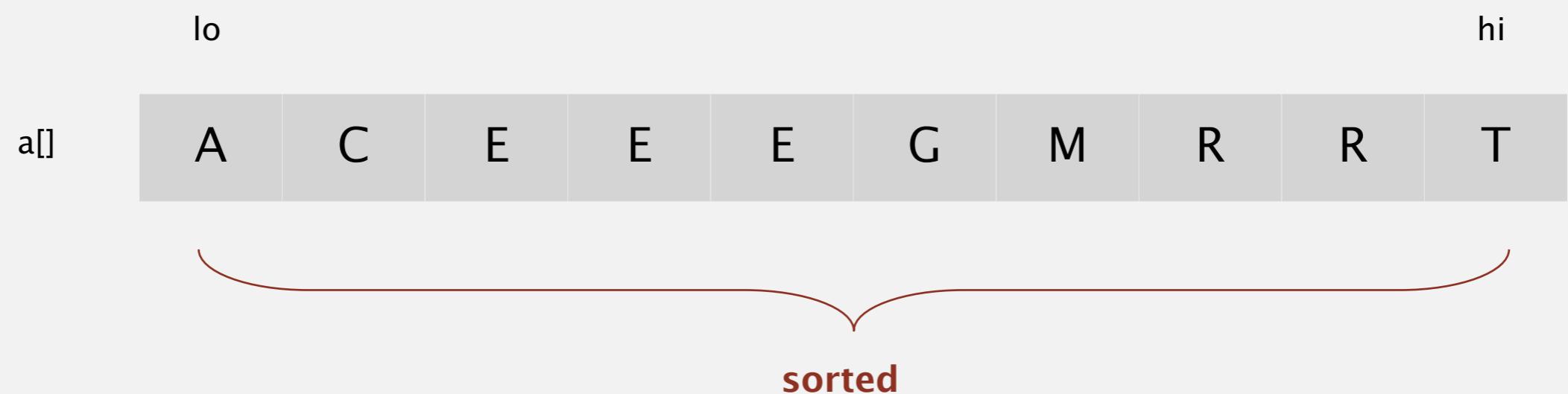
Abstract in-place merge demo

Goal. Given two sorted subarrays $a[lo]$ to $a[mid]$ and $a[mid+1]$ to $a[hi]$, replace with sorted subarray $a[lo]$ to $a[hi]$.



Abstract in-place merge demo

Goal. Given two sorted subarrays $a[lo]$ to $a[mid]$ and $a[mid+1]$ to $a[hi]$, replace with sorted subarray $a[lo]$ to $a[hi]$.



Merging: Java implementation

```
private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid, int hi)
{
    for (int k = lo; k <= hi; k++)
        aux[k] = a[k]; copy

    int i = lo, j = mid+1;
    for (int k = lo; k <= hi; k++)
    {
        if      (i > mid)          a[k] = aux[j++]; merge
        else if (j > hi)          a[k] = aux[i++];
        else if (less(aux[j], aux[i])) a[k] = aux[j++];
        else                      a[k] = aux[i++];
    }
}
```

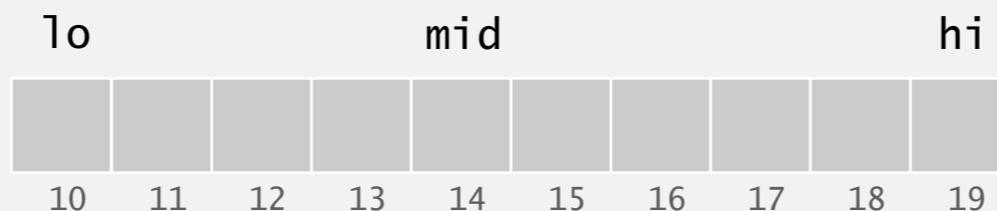


Mergesort: Java implementation

```
public class Merge
{
    private static void merge(...)
    { /* as before */ }

    private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
    {
        if (hi <= lo) return;
        int mid = lo + (hi - lo) / 2;
        sort(a, aux, lo, mid);
        sort(a, aux, mid+1, hi);
        merge(a, aux, lo, mid, hi);
    }

    public static void sort(Comparable[] a)
    {
        Comparable[] aux = new Comparable[a.length];
        sort(a, aux, 0, a.length - 1);
    }
}
```



Mergesort: trace

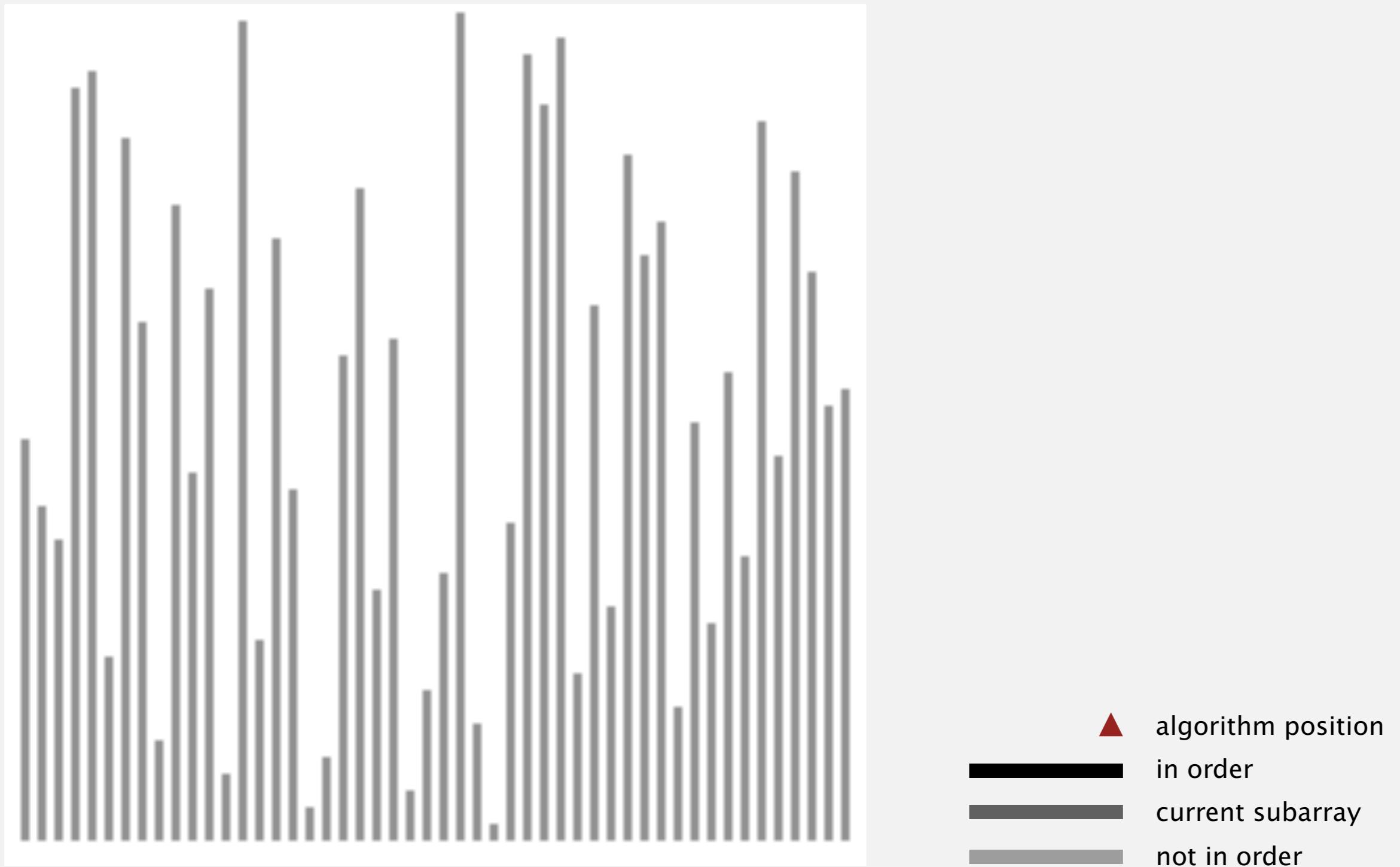
	a[]																
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	
to	M	E	R	G	E	S	0	R	T	E	X	A	M	P	L	E	
hi	E	M	R	G	E	S	0	R	T	E	X	A	M	P	L	E	
merge(a, aux, 0, 0, 1)	E	M	G	R	E	S	0	R	T	E	X	A	M	P	L	E	
merge(a, aux, 2, 2, 3)	E	G	M	R	E	S	0	R	T	E	X	A	M	P	L	E	
merge(a, aux, 0, 1, 3)	E	G	M	R	E	S	0	R	T	E	X	A	M	P	L	E	
merge(a, aux, 4, 4, 5)	E	G	M	R	E	S	0	R	T	E	X	A	M	P	L	E	
merge(a, aux, 6, 6, 7)	E	G	M	R	E	S	0	R	T	E	X	A	M	P	L	E	
merge(a, aux, 4, 5, 7)	E	G	M	R	E	O	R	S	T	E	X	A	M	P	L	E	
merge(a, aux, 0, 3, 7)	E	E	G	M	O	R	R	S	T	E	X	A	M	P	L	E	
merge(a, aux, 8, 8, 9)	E	E	G	M	O	R	R	S	E	T	X	A	M	P	L	E	
merge(a, aux, 10, 10, 11)	E	E	G	M	O	R	R	S	E	T	A	X	M	P	L	E	
merge(a, aux, 8, 9, 11)	E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E	
merge(a, aux, 12, 12, 13)	E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E	
merge(a, aux, 14, 14, 15)	E	E	G	M	O	R	R	S	A	E	T	X	M	P	E	L	
merge(a, aux, 12, 13, 15)	E	E	G	M	O	R	R	S	A	E	T	X	E	L	M	P	
merge(a, aux, 8, 11, 15)	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X	
merge(a, aux, 0, 7, 15)	A	E	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X



result after recursive call

Mergesort: animation

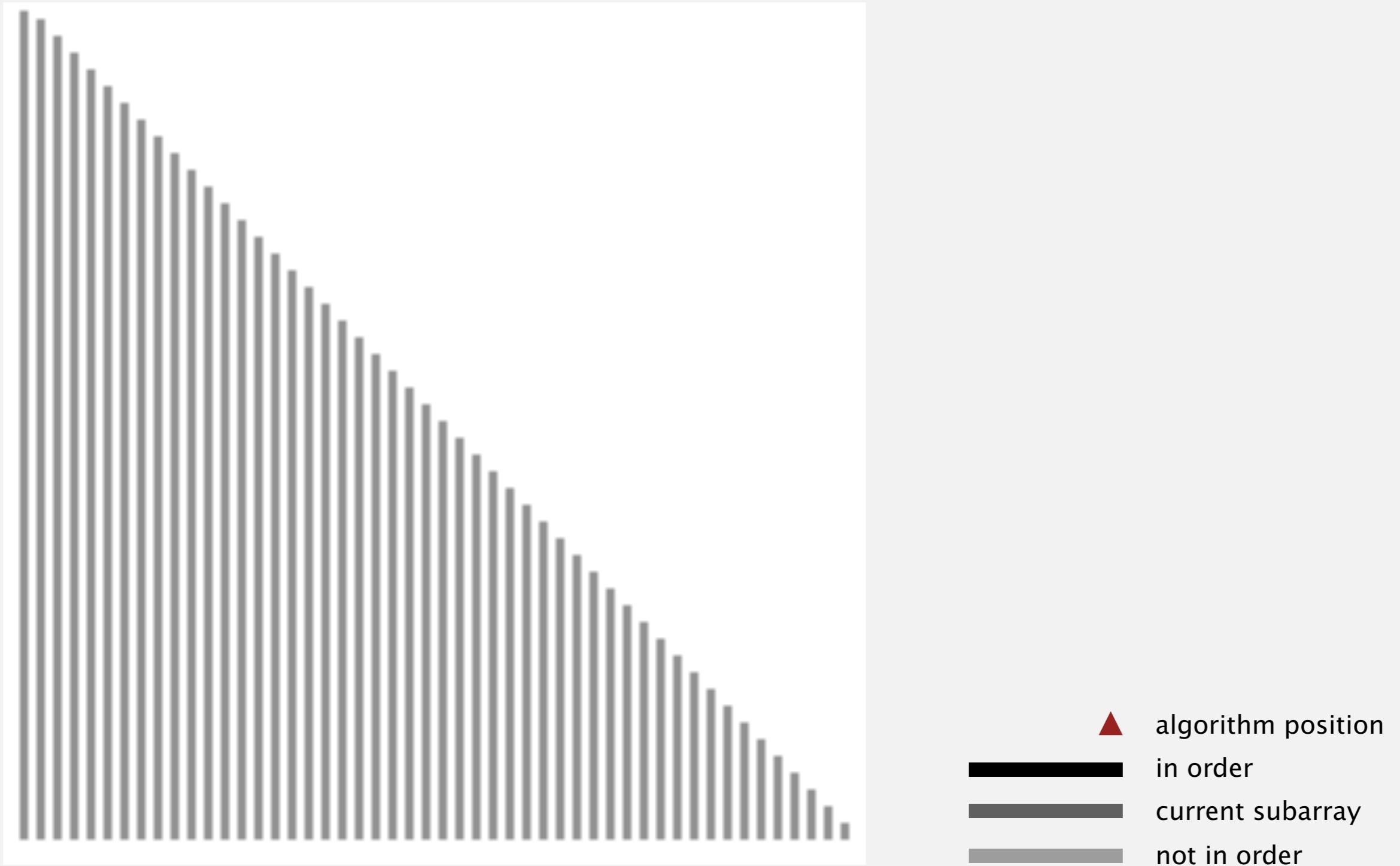
50 random items



<http://www.sorting-algorithms.com/merge-sort>

Mergesort: animation

50 reverse-sorted items



<http://www.sorting-algorithms.com/merge-sort>

Mergesort: empirical analysis

Running time estimates:

- Laptop executes 10^8 compares/second.
- Supercomputer executes 10^{12} compares/second.

	insertion sort (N^2)			mergesort ($N \log N$)		
computer	thousand	million	billion	thousand	million	billion
home	instant	2.8 hours	317 years	instant	1 second	18 min
super	instant	1 second	1 week	instant	instant	instant

Bottom line. Good algorithms are better than supercomputers.

Mergesort: number of compares

Proposition. Mergesort uses $\leq N \lg N$ compares to sort an array of length N .

Pf sketch. The number of compares $C(N)$ to mergesort an array of length N satisfies the recurrence:

$$C(N) \leq C(\lceil N/2 \rceil) + C(\lfloor N/2 \rfloor) + N \quad \text{for } N > 1, \text{ with } C(1) = 0.$$

↑
left half ↑ right half ↑ merge

We solve the recurrence when N is a power of 2:

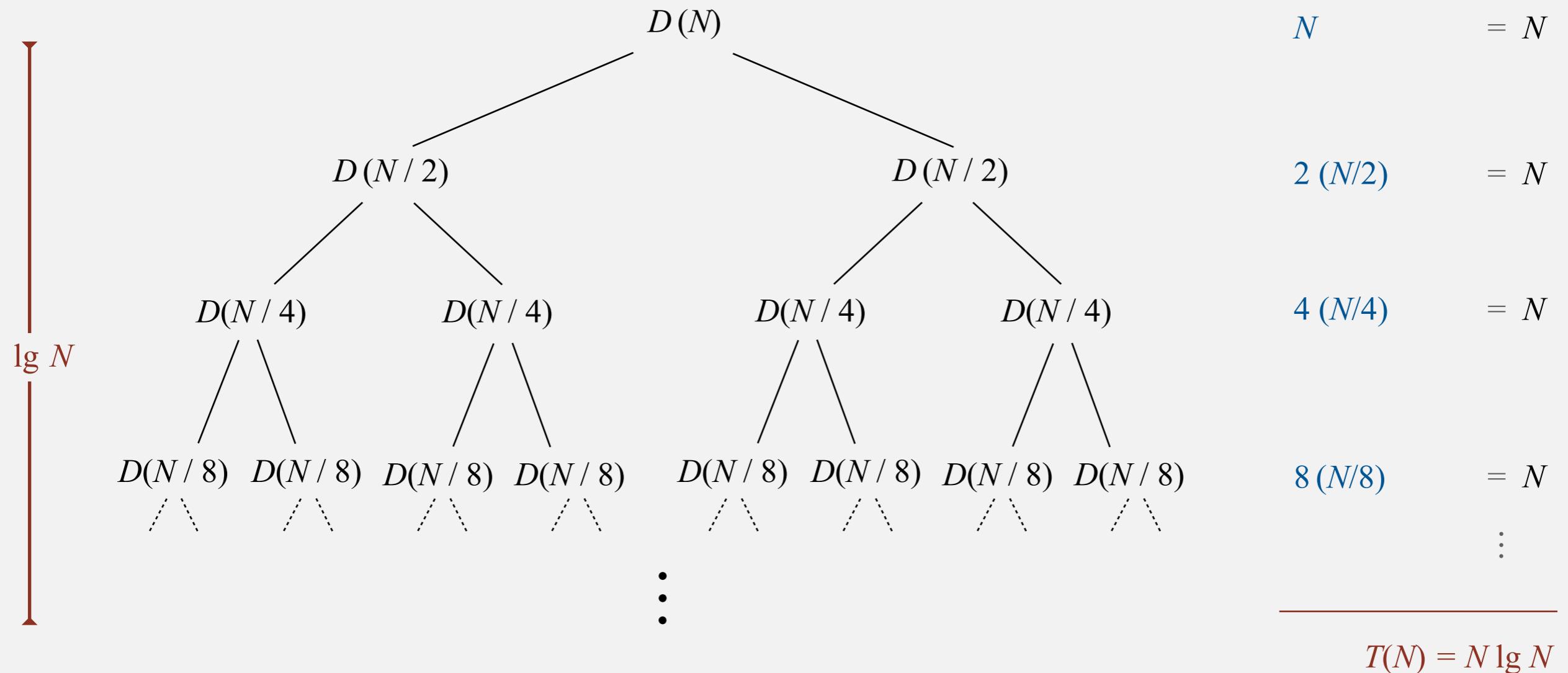
← result holds for all N
(analysis cleaner in this case)

$$D(N) = 2 D(N/2) + N, \text{ for } N > 1, \text{ with } D(1) = 0.$$

Divide-and-conquer recurrence: proof by picture

Proposition. If $D(N)$ satisfies $D(N) = 2 D(N/2) + N$ for $N > 1$, with $D(1) = 0$, then $D(N) = N \lg N$.

Pf 1. [assuming N is a power of 2]



$$T(N) = N \lg N$$

Divide-and-conquer recurrence: proof by induction

Proposition. If $D(N)$ satisfies $D(N) = 2 D(N/2) + N$ for $N > 1$, with $D(1) = 0$, then $D(N) = N \lg N$.

Pf 2. [assuming N is a power of 2]

- Base case: $N = 1$.
- Inductive hypothesis: $D(N) = N \lg N$.
- Goal: show that $D(2N) = (2N) \lg (2N)$.

$$\begin{aligned} D(2N) &= 2 D(N) + 2N && \text{given} \\ &= 2 N \lg N + 2N && \text{inductive hypothesis} \\ &= 2 N (\lg (2N) - 1) + 2N && \text{algebra} \\ &= 2 N \lg (2N) && \text{QED} \end{aligned}$$

Mergesort: number of array accesses

Proposition. Mergesort uses $\leq 6N \lg N$ array accesses to sort an array of length N .

Pf sketch. The number of array accesses $A(N)$ satisfies the recurrence:

$$A(N) \leq A(\lceil N/2 \rceil) + A(\lfloor N/2 \rfloor) + 6N \text{ for } N > 1, \text{ with } A(1) = 0.$$

Key point. Any algorithm with the following structure takes $N \log N$ time:

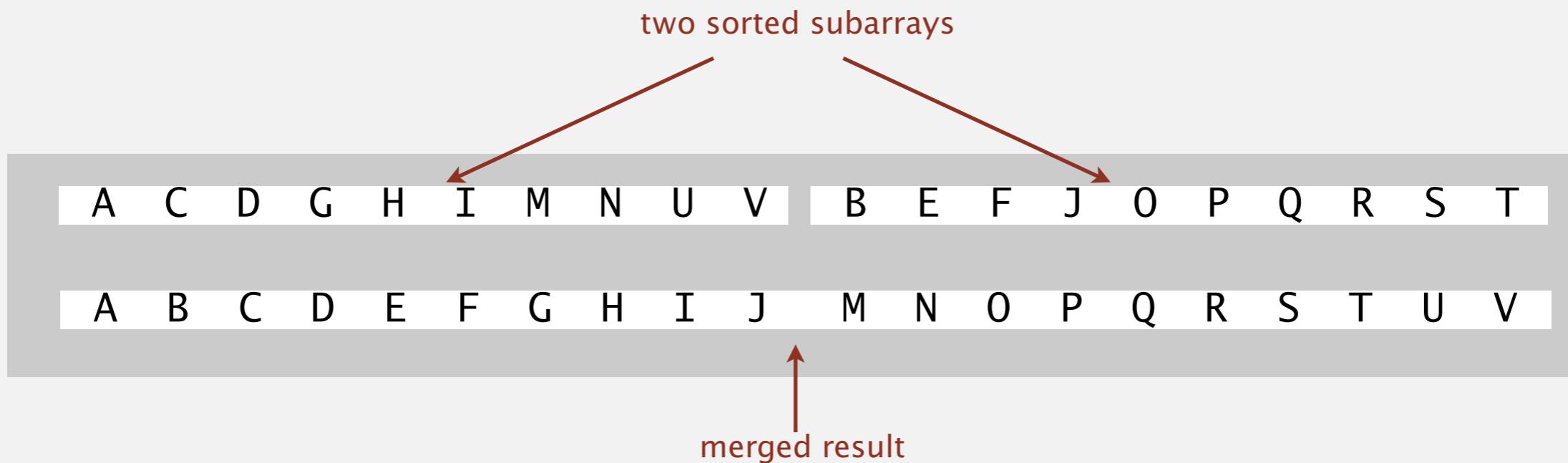
```
public static void linearithmic(int N)
{
    if (N == 0) return;
    linearithmic(N/2); ← solve two problems
    linearithmic(N/2); ← of half the size
    linear(N); ← do a linear amount of work
}
```

Notable examples. FFT, hidden-line removal, Kendall-tau distance, ...

Mergesort analysis: memory

Proposition. Mergesort uses extra space proportional to N .

Pf. The array `aux[]` needs to be of length N for the last merge.



Def. A sorting algorithm is **in-place** if it uses $\leq c \log N$ extra memory.

Ex. Insertion sort, selection sort, shellsort.

Challenge 1 (not hard). Use `aux[]` array of length $\sim \frac{1}{2} N$ instead of N .

Challenge 2 (very hard). In-place merge. [Kronrod 1969]

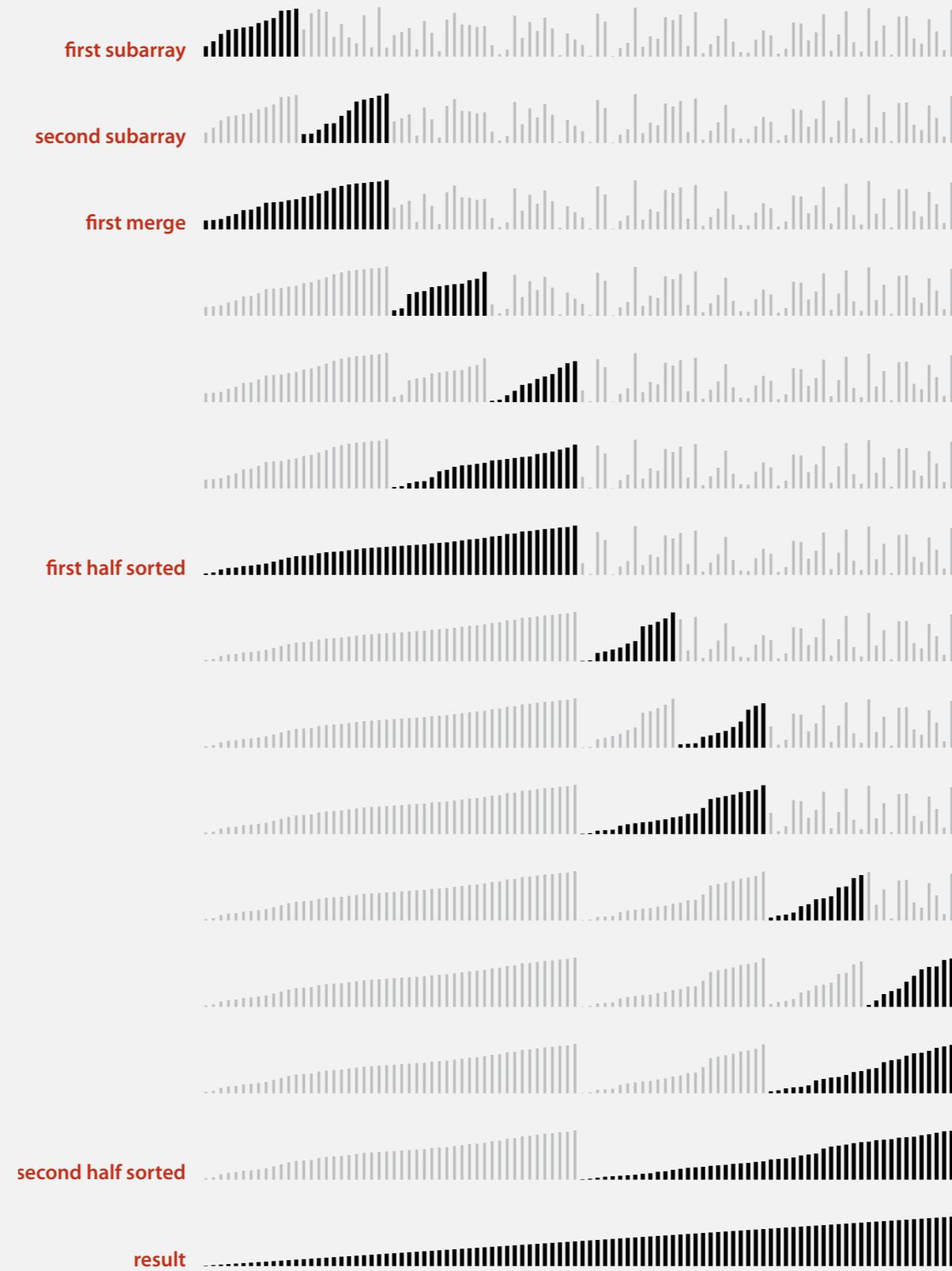
Mergesort: practical improvements

Use insertion sort for small subarrays.

- Mergesort has too much overhead for tiny subarrays.
- Cutoff to insertion sort for ≈ 10 items.

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
    if (hi <= lo + CUTOFF - 1)
    {
        Insertion.sort(a, lo, hi);
        return;
    }
    int mid = lo + (hi - lo) / 2;
    sort (a, aux, lo, mid);
    sort (a, aux, mid+1, hi);
    merge(a, aux, lo, mid, hi);
}
```

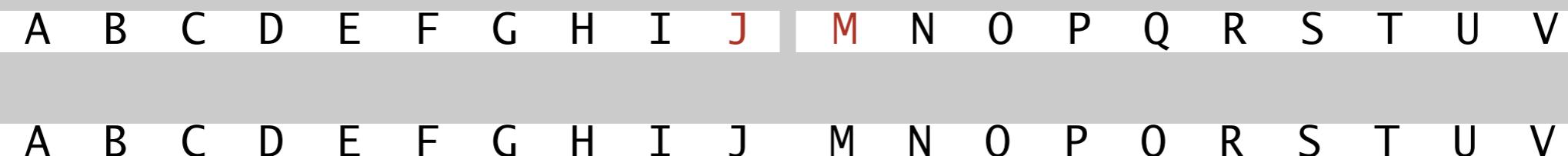
Mergesort with cutoff to insertion sort: visualization



Mergesort: practical improvements

Stop if already sorted.

- Is largest item in first half \leq smallest item in second half?
- Helps for partially-ordered arrays.



```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
    if (hi <= lo) return;
    int mid = lo + (hi - lo) / 2;
    sort(a, aux, lo, mid);
    sort(a, aux, mid+1, hi);
    if (!less(a[mid+1], a[mid])) return;
    merge(a, aux, lo, mid, hi);
}
```

Mergesort: practical improvements

Eliminate the copy to the auxiliary array. Save time (but not space) by switching the role of the input and auxiliary array in each recursive call.

```
private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid, int hi)
{
    int i = lo, j = mid+1;
    for (int k = lo; k <= hi; k++)
    {
        if (i > mid)          aux[k] = a[j++];
        else if (j > hi)       aux[k] = a[i++];
        else if (less(a[j], a[i])) aux[k] = a[j++]; ← merge from a[] to aux[]
        else                   aux[k] = a[i++];
    }
}

private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
    if (hi <= lo) return;
    int mid = lo + (hi - lo) / 2;
    sort (aux, a, lo, mid);           ↑
    sort (aux, a, mid+1, hi);
    merge(a, aux, lo, mid, hi);
}
```

switch roles of aux[] and a[]

assumes aux[] is initialize to a[] once,
before recursive calls

Java 6 system sort

Basic algorithm for sorting objects = mergesort.

- Cutoff to insertion sort = 7.
- Stop-if-already-sorted test.
- Eliminate-the-copy-to-the-auxiliary-array trick.

Arrays.sort(a)



<http://www.java2s.com/Open-Source/Java/6.0-JDK-Modules/j2me/java/util/Arrays.java.html>

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2.2 MERGESORT

- ▶ *mergesort*
- ▶ ***bottom-up mergesort***
- ▶ *sorting complexity*
- ▶ *comparators*
- ▶ *stability*

Bottom-up mergesort

Basic plan.

- Pass through array, merging subarrays of size 1.
- Repeat for subarrays of size 2, 4, 8,

	a[i]																
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	
sz = 1	M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E	
merge(a, aux, 0, 0, 1)	E	M	R	G	E	S	O	R	T	E	X	A	M	P	L	E	
merge(a, aux, 2, 2, 3)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E	
merge(a, aux, 4, 4, 5)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E	
merge(a, aux, 6, 6, 7)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E	
merge(a, aux, 8, 8, 9)	E	M	G	R	E	S	O	R	E	T	X	A	M	P	L	E	
merge(a, aux, 10, 10, 11)	E	M	G	R	E	S	O	R	E	T	A	X	M	P	L	E	
merge(a, aux, 12, 12, 13)	E	M	G	R	E	S	O	R	E	T	A	X	M	P	L	E	
merge(a, aux, 14, 14, 15)	E	M	G	R	E	S	O	R	E	T	A	X	M	P	E	L	
sz = 2	E	G	M	R	E	S	O	R	E	T	A	X	M	P	E	L	
merge(a, aux, 0, 1, 3)	E	G	M	R	E	O	R	S	E	T	A	X	M	P	E	L	
merge(a, aux, 4, 5, 7)	E	G	M	R	E	O	R	S	A	E	T	X	M	P	E	L	
merge(a, aux, 8, 9, 11)	E	G	M	R	E	O	R	S	A	E	T	X	M	P	E	L	
merge(a, aux, 12, 13, 15)	E	G	M	R	E	O	R	S	A	E	T	X	E	L	M	P	
sz = 4	E	E	G	M	O	R	R	S	A	E	T	X	E	L	M	P	
merge(a, aux, 0, 3, 7)	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X	
merge(a, aux, 8, 11, 15)	A	E	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X
sz = 8																	
merge(a, aux, 0, 7, 15)																	

Bottom-up mergesort: Java implementation

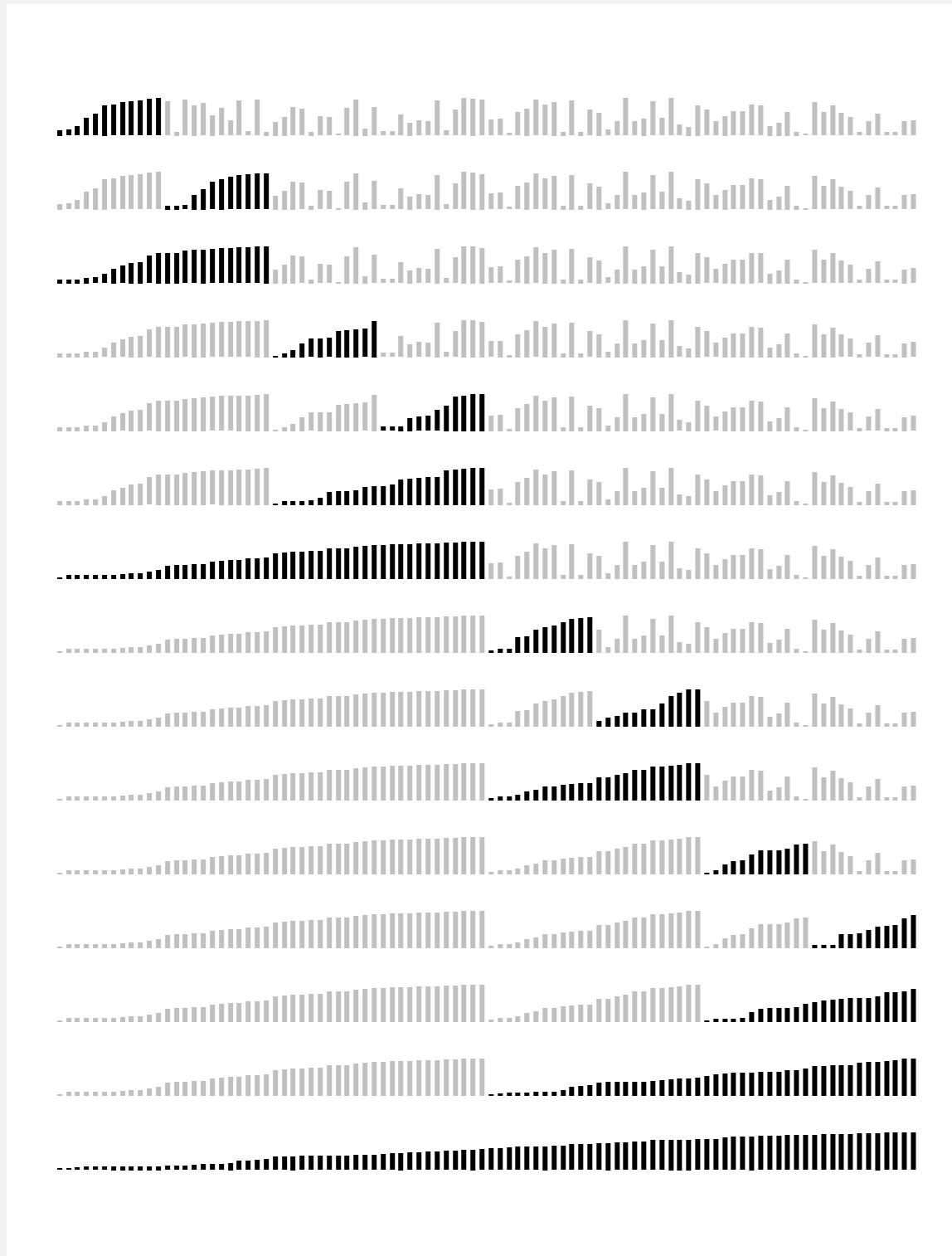
```
public class MergeBU
{
    private static void merge(...)
    { /* as before */ }

    public static void sort(Comparable[] a)
    {
        int N = a.length;
        Comparable[] aux = new Comparable[N];
        for (int sz = 1; sz < N; sz = sz+sz)
            for (int lo = 0; lo < N-sz; lo += sz+sz)
                merge(a, aux, lo, lo+sz-1, Math.min(lo+sz+sz-1, N-1));
    }
}
```

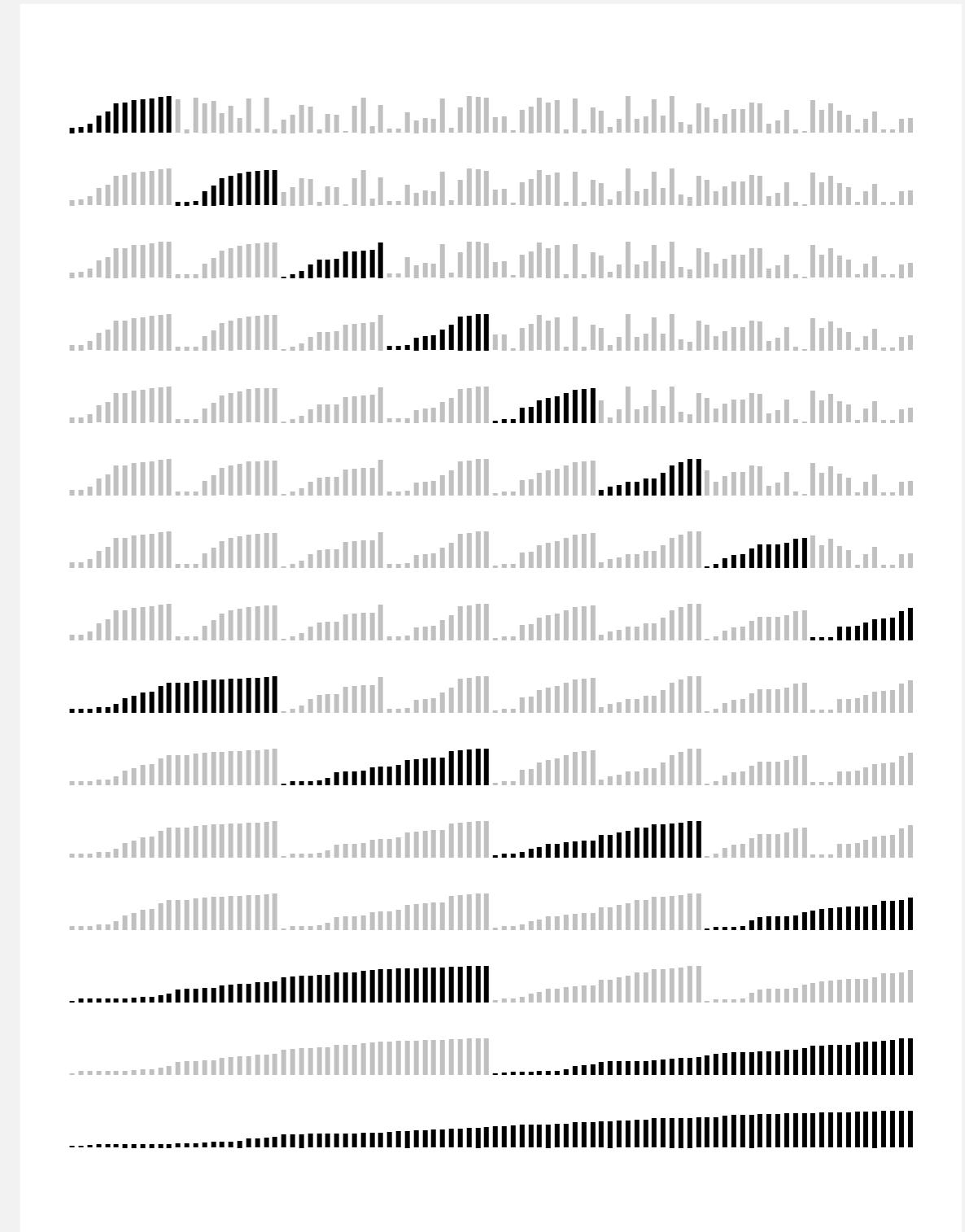
but about 10% slower than recursive,
top-down mergesort on typical systems

Bottom line. Simple and non-recursive version of mergesort.

Mergesort: visualizations



top-down mergesort (cutoff = 12)



bottom-up mergesort (cutoff = 12)

Natural mergesort

Idea. Exploit pre-existing order by identifying naturally-occurring runs.

input

1	5	10	16	3	4	23	9	13	2	7	8	12	14
---	---	----	----	---	---	----	---	----	---	---	---	----	----

first run

1	5	10	16	3	4	23	9	13	2	7	8	12	14
---	---	----	----	---	---	----	---	----	---	---	---	----	----

second run

1	5	10	16	3	4	23	9	13	2	7	8	12	14
---	---	----	----	---	---	----	---	----	---	---	---	----	----

merge two runs

1	3	4	5	10	16	23	9	13	2	7	8	12	14
---	---	---	---	----	----	----	---	----	---	---	---	----	----

Tradeoff. Fewer passes vs. extra compares per pass to identify runs.

Timsort

- Natural mergesort.
- Use binary insertion sort to make initial runs (if needed).
- A few more clever optimizations.



Tim Peters

Intro

This describes an adaptive, stable, natural mergesort, modestly called timsort (hey, I earned it <wink>). It has supernatural performance on many kinds of partially ordered arrays (less than $\lg(N!)$ comparisons needed, and as few as $N-1$), yet as fast as Python's previous highly tuned samplesort hybrid on random arrays.

In a nutshell, the main routine marches over the array once, left to right, alternately identifying the next run, then merging it into the previous runs "intelligently". Everything else is complication for speed, and some hard-won measure of memory efficiency.

...

Consequence. Linear time on many arrays with pre-existing order.

Now widely used. Python, Java 7, GNU Octave, Android,

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Complexity of sorting

Computational complexity. Framework to study efficiency of algorithms for solving a particular problem X .

Model of computation. Allowable operations.

Cost model. Operation count(s).

Upper bound. Cost guarantee provided by **some** algorithm for X .

Lower bound. Proven limit on cost guarantee of **all** algorithms for X .

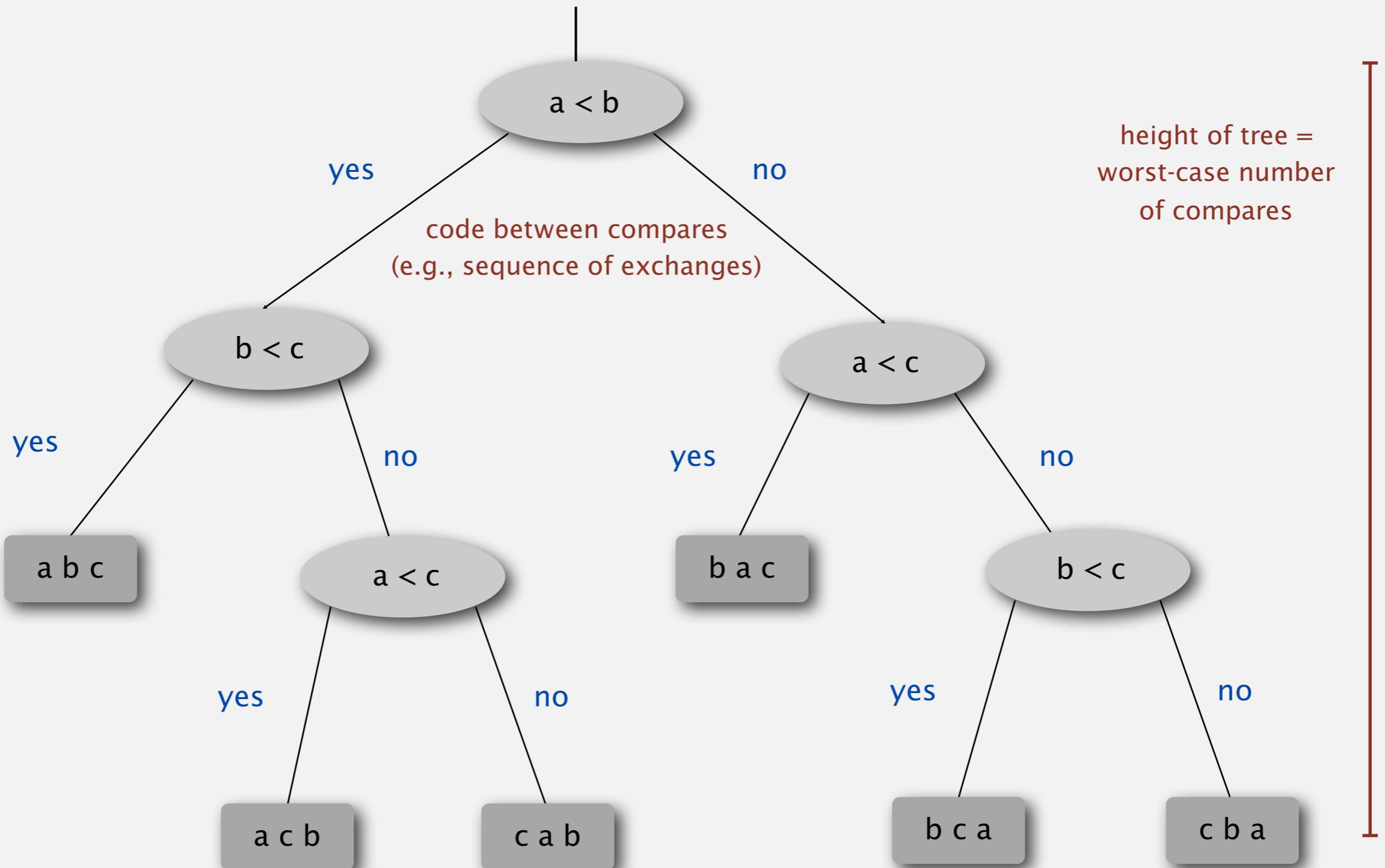
Optimal algorithm. Algorithm with best possible cost guarantee for X .

lower bound ~ upper bound

Example: sorting.

- Model of computation: decision tree. ← can access information only through compares
(e.g., Java Comparable framework)
- Cost model: # compares.
- Upper bound: $\sim N \lg N$ from mergesort.
- Lower bound:
- Optimal algorithm:

Decision tree (for 3 distinct keys a, b, and c)



height of tree =
worst-case number
of compares

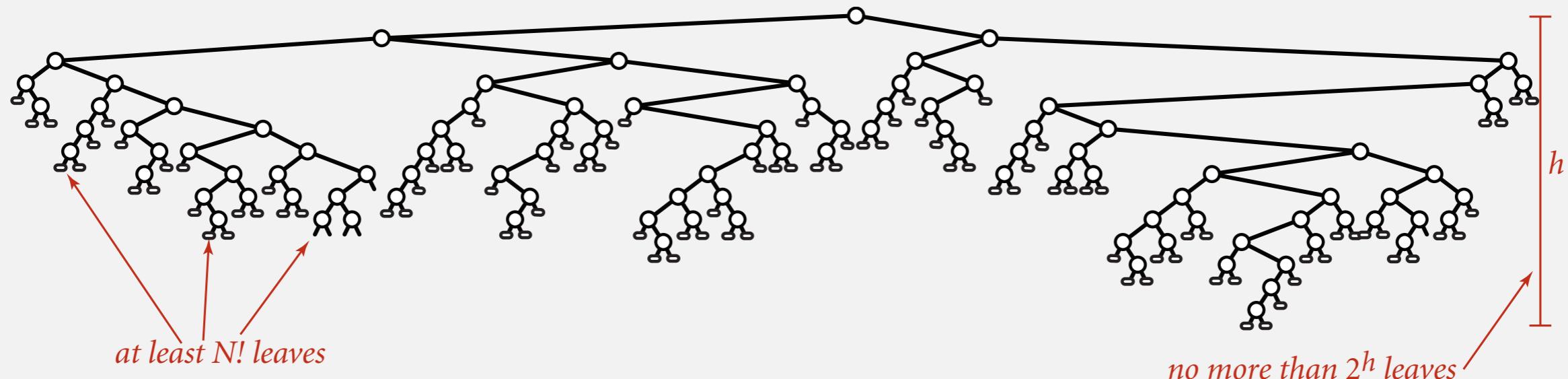
each leaf corresponds to one (and only one) ordering;
(at least) one leaf for each possible ordering

Compare-based lower bound for sorting

Proposition. Any compare-based sorting algorithm must use at least $\lg(N!) \sim N \lg N$ compares in the worst-case.

Pf.

- Assume array consists of N distinct values a_1 through a_N .
- Worst case dictated by **height h** of decision tree.
- Binary tree of height h has at most 2^h leaves.
- $N!$ different orderings \Rightarrow at least $N!$ leaves.



Compare-based lower bound for sorting

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Pf.

- Assume array consists of N distinct values a_1 through a_N .
- Worst case dictated by **height h** of decision tree.
- Binary tree of height h has at most 2^h leaves.
- $N!$ different orderings \Rightarrow at least $N!$ leaves.

$$\begin{aligned} 2^h &\geq \# \text{leaves} \geq N! \\ \Rightarrow h &\geq \lg(N!) \sim N \lg N \end{aligned}$$



Stirling's formula

Complexity of sorting

Model of computation. Allowable operations.

Cost model. Operation count(s).

Upper bound. Cost guarantee provided by some algorithm for X .

Lower bound. Proven limit on cost guarantee of all algorithms for X .

Optimal algorithm. Algorithm with best possible cost guarantee for X .

Example: sorting.

- Model of computation: decision tree.
- Cost model: # compares.
- Upper bound: $\sim N \lg N$ from mergesort.
- Lower bound: $\sim N \lg N$.
- Optimal algorithm = mergesort.

First goal of algorithm design: optimal algorithms.

Complexity results in context

Compares? Mergesort **is** optimal with respect to number compares.

Space? Mergesort **is not** optimal with respect to space usage.



Lessons. Use theory as a guide.

Ex. Design sorting algorithm that guarantees $\frac{1}{2} N \lg N$ compares?

Ex. Design sorting algorithm that is both time- and space-optimal?

Complexity results in context (continued)

Lower bound may not hold if the algorithm can take advantage of:

- The initial order of the input.

Ex: insert sort requires only a linear number of compares on partially-sorted arrays.

- The distribution of key values.

Ex: 3-way quicksort requires only a linear number of compares on arrays with a constant number of distinct keys. [stay tuned]

- The representation of the keys.

Ex: radix sort requires no key compares — it accesses the data via character/digit compares.

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Sort countries by gold medals

NOC	Gold	Silver	Bronze	Total
United States (USA)	46	29	29	104
China (CHN)§	38	28	22	88
Great Britain (GBR)*	29	17	19	65
Russia (RUS)§	24	25	32	81
South Korea (KOR)	13	8	7	28
Germany (GER)	11	19	14	44
France (FRA)	11	11	12	34
Italy (ITA)	8	9	11	28
Hungary (HUN)§	8	4	6	18
Australia (AUS)	7	16	12	35

Sort countries by total medals

NOC	Gold	Silver	Bronze	Total
United States (USA)	46	29	29	104
China (CHN)§	38	28	22	88
Russia (RUS)§	24	25	32	81
Great Britain (GBR)*	29	17	19	65
Germany (GER)	11	19	14	44
Japan (JPN)	7	14	17	38
Australia (AUS)	7	16	12	35
France (FRA)	11	11	12	34
South Korea (KOR)	13	8	7	28
Italy (ITA)	8	9	11	28

Sort music library by artist

	Name	Artist	Time	Album
12	<input checked="" type="checkbox"/> Let It Be	The Beatles	4:03	Let It Be
13	<input checked="" type="checkbox"/> Take My Breath Away	BERLIN	4:13	Top Gun - Soundtrack
14	<input checked="" type="checkbox"/> Circle Of Friends	Better Than Ezra	3:27	Empire Records
15	<input checked="" type="checkbox"/> Dancing With Myself	Billy Idol	4:43	Don't Stop
16	<input checked="" type="checkbox"/> Rebel Yell	Billy Idol	4:49	Rebel Yell
17	<input checked="" type="checkbox"/> Piano Man	Billy Joel	5:36	Greatest Hits Vol. 1
18	<input checked="" type="checkbox"/> Pressure	Billy Joel	3:16	Greatest Hits, Vol. II (1978 - 1985) (Disc 2)
19	<input checked="" type="checkbox"/> The Longest Time	Billy Joel	3:36	Greatest Hits, Vol. II (1978 - 1985) (Disc 2)
20	<input checked="" type="checkbox"/> Atomic	Blondie	3:50	Atomic: The Very Best Of Blondie
21	<input checked="" type="checkbox"/> Sunday Girl	Blondie	3:15	Atomic: The Very Best Of Blondie
22	<input checked="" type="checkbox"/> Call Me	Blondie	3:33	Atomic: The Very Best Of Blondie
23	<input checked="" type="checkbox"/> Dreaming	Blondie	3:06	Atomic: The Very Best Of Blondie
24	<input checked="" type="checkbox"/> Hurricane	Bob Dylan	8:32	Desire
25	<input checked="" type="checkbox"/> The Times They Are A-Changin'	Bob Dylan	3:17	Greatest Hits
26	<input checked="" type="checkbox"/> Livin' On A Prayer	Bon Jovi	4:11	Cross Road
27	<input checked="" type="checkbox"/> Beds Of Roses	Bon Jovi	6:35	Cross Road
28	<input checked="" type="checkbox"/> Runaway	Bon Jovi	3:53	Cross Road
29	<input checked="" type="checkbox"/> Rasputin (Extended Mix)	Boney M	5:50	Greatest Hits
30	<input checked="" type="checkbox"/> Have You Ever Seen The Rain	Bonnie Tyler	4:10	Faster Than The Speed Of Night
31	<input checked="" type="checkbox"/> Total Eclipse Of The Heart	Bonnie Tyler	7:02	Faster Than The Speed Of Night
32	<input checked="" type="checkbox"/> Straight From The Heart	Bonnie Tyler	3:41	Faster Than The Speed Of Night
33	<input checked="" type="checkbox"/> Holding Out For A Hero	Bonny Tyler	5:49	Meat Loaf And Friends
34	<input checked="" type="checkbox"/> Dancing In The Dark	Bruce Springsteen	4:05	Born In The U.S.A.
35	<input checked="" type="checkbox"/> Thunder Road	Bruce Springsteen	4:51	Born To Run
36	<input checked="" type="checkbox"/> Born To Run	Bruce Springsteen	4:30	Born To Run
37	<input checked="" type="checkbox"/> Jungleland	Bruce Springsteen	9:34	Born To Run
38	<input checked="" type="checkbox"/> Turn! Turn! Turn! (To Everything)	The Byrds	3:57	Forrest Gump The Soundtrack (Disc 2)

Sort music library by song name

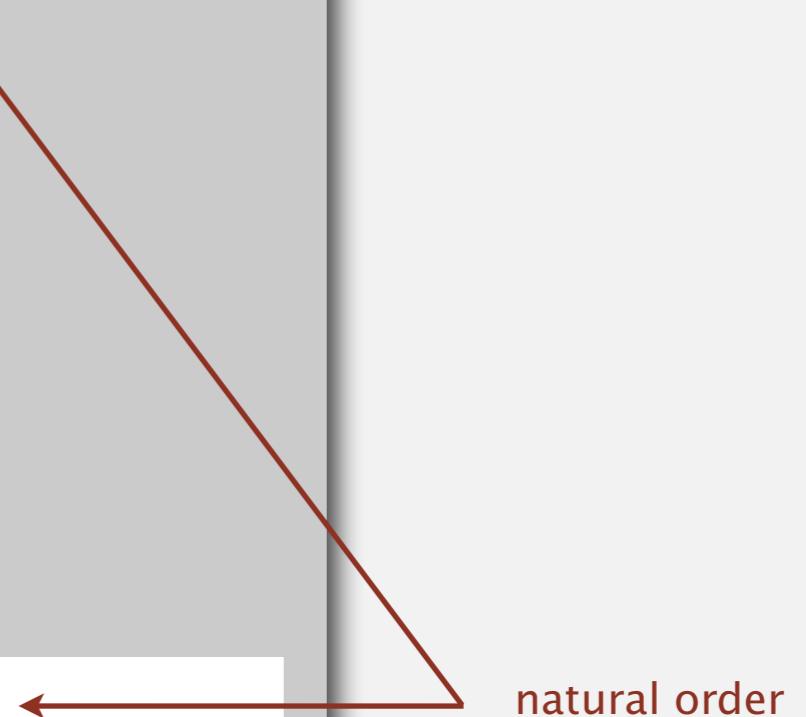
	Name	Artist	Time	Album
1	<input checked="" type="checkbox"/> Alive	Pearl Jam	5:41	Ten
2	<input checked="" type="checkbox"/> All Over The World	Pixies	5:27	Bossanova
3	<input checked="" type="checkbox"/> All Through The Night	Cyndi Lauper	4:30	She's So Unusual
4	<input checked="" type="checkbox"/> Allison Road	Gin Blossoms	3:19	New Miserable Experience
5	<input checked="" type="checkbox"/> Ama, Ama, Ama Y Ensancha El ...	Extremoduro	2:34	Deltoya (1992)
6	<input checked="" type="checkbox"/> And We Danced	Hooters	3:50	Nervous Night
7	<input checked="" type="checkbox"/> As I Lay Me Down	Sophie B. Hawkins	4:09	Whaler
8	<input checked="" type="checkbox"/> Atomic	Blondie	3:50	Atomic: The Very Best Of Blondie
9	<input checked="" type="checkbox"/> Automatic Lover	Jay-Jay Johanson	4:19	Antenna
10	<input checked="" type="checkbox"/> Baba O'Riley	The Who	5:01	Who's Better, Who's Best
11	<input checked="" type="checkbox"/> Beautiful Life	Ace Of Base	3:40	The Bridge
12	<input checked="" type="checkbox"/> Beds Of Roses	Bon Jovi	6:35	Cross Road
13	<input checked="" type="checkbox"/> Black	Pearl Jam	5:44	Ten
14	<input checked="" type="checkbox"/> Bleed American	Jimmy Eat World	3:04	Bleed American
15	<input checked="" type="checkbox"/> Borderline	Madonna	4:00	The Immaculate Collection
16	<input checked="" type="checkbox"/> Born To Run	Bruce Springsteen	4:30	Born To Run
17	<input checked="" type="checkbox"/> Both Sides Of The Story	Phil Collins	6:43	Both Sides
18	<input checked="" type="checkbox"/> Bouncing Around The Room	Phish	4:09	A Live One (Disc 1)
19	<input checked="" type="checkbox"/> Boys Don't Cry	The Cure	2:35	Staring At The Sea: The Singles 1979–1985
20	<input checked="" type="checkbox"/> Brat	Green Day	1:43	Insomniac
21	<input checked="" type="checkbox"/> Breakdown	Deerheart	3:40	Deerheart
22	<input checked="" type="checkbox"/> Bring Me To Life (Kevin Roen Mix)	Evanescence Vs. Pa...	9:48	
23	<input checked="" type="checkbox"/> Californication	Red Hot Chili Pepp...	1:40	
24	<input checked="" type="checkbox"/> Call Me	Blondie	3:33	Atomic: The Very Best Of Blondie
25	<input checked="" type="checkbox"/> Can't Get You Out Of My Head	Kylie Minogue	3:50	Fever
26	<input checked="" type="checkbox"/> Celebration	Kool & The Gang	3:45	Time Life Music Sounds Of The Seventies – C
27	<input checked="" type="checkbox"/> Chakna Chakna	Surjit Bindrakhia	5:11	Bombay Dreams

Comparable interface: review

Comparable interface: sort using a type's **natural order**.

```
public class Date implements Comparable<Date>
{
    private final int month, day, year;

    public Date(int m, int d, int y)
    {
        month = m;
        day   = d;
        year  = y;
    }
    ...
    public int compareTo(Date that)
    {
        if (this.year < that.year) return -1;
        if (this.year > that.year) return +1;
        if (this.month < that.month) return -1;
        if (this.month > that.month) return +1;
        if (this.day   < that.day)  return -1;
        if (this.day   > that.day)  return +1;
        return 0;
    }
}
```



natural order

Comparator interface

Comparator interface: sort using an **alternate order**.

```
public interface Comparator<Key>  
    int compare(Key v, Key w)      compare keys v and w
```

Required property. Must be a **total order**.

string order	example
natural order	Now is the time
case insensitive	is Now the time
Spanish language	café cafetero cuarto churro nube ñoño
British phone book	McKinley Mackintosh

pre-1994 order for
digraphs ch and ll and rr



Comparator interface: system sort

To use with Java system sort:

- Create Comparator object.
- Pass as second argument to `Arrays.sort()`.

```
String[] a;           uses natural order
...
Arrays.sort(a);
...
Arrays.sort(a, String.CASE_INSENSITIVE_ORDER);
...
Arrays.sort(a, Collator.getInstance(new Locale("es")));
...
Arrays.sort(a, new BritishPhoneBookOrder());
...
```

The diagram consists of a light gray rectangular box containing Java code. Inside the box, two red arrows point from specific parts of the code to explanatory text above them. One arrow points from the first call to `Arrays.sort(a);` to the text "uses natural order". Another arrow points from the last call to `Arrays.sort(a, new BritishPhoneBookOrder());` to the text "uses alternate order defined by Comparator<String> object".

Bottom line. Decouples the definition of the data type from the definition of what it means to compare two objects of that type.

Comparator interface: using with our sorting libraries

To support comparators in our sort implementations:

- Use Object instead of Comparable.
- Pass Comparator to sort() and less() and use it in less().

insertion sort using a Comparator

```
public static void sort(Object[] a, Comparator comparator)
{
    int N = a.length;
    for (int i = 0; i < N; i++)
        for (int j = i; j > 0 && less(comparator, a[j], a[j-1]); j--)
            exch(a, j, j-1);
}

private static boolean less(Comparator c, Object v, Object w)
{ return c.compare(v, w) < 0; }

private static void exch(Object[] a, int i, int j)
{ Object swap = a[i]; a[i] = a[j]; a[j] = swap; }
```

Comparator interface: implementing

To implement a comparator:

- Define a (nested) class that implements the Comparator interface.
- Implement the compare() method.

```
public class Student
{
    private final String name;
    private final int section;
    ...

    public static class ByName implements Comparator<Student>
    {
        public int compare(Student v, Student w)
        {   return v.name.compareTo(w.name);   }
    }
}
```

```
public static class BySection implements Comparator<Student>
{
    public int compare(Student v, Student w)
    {   return v.section - w.section;   }
}
```

 this trick works here
since no danger of overflow

Comparator interface: implementing

To implement a comparator:

- Define a (nested) class that implements the Comparator interface.
- Implement the compare() method.

`Arrays.sort(a, new Student.ByName());`

Andrews	3	A	664-480-0023	097 Little
Battle	4	C	874-088-1212	121 Whitman
Chen	3	A	991-878-4944	308 Blair
Fox	3	A	884-232-5341	11 Dickinson
Furia	1	A	766-093-9873	101 Brown
Gazsi	4	B	766-093-9873	101 Brown
Kanaga	3	B	898-122-9643	22 Brown
Rohde	2	A	232-343-5555	343 Forbes

`Arrays.sort(a, new Student.BySection());`

Furia	1	A	766-093-9873	101 Brown
Rohde	2	A	232-343-5555	343 Forbes
Andrews	3	A	664-480-0023	097 Little
Chen	3	A	991-878-4944	308 Blair
Fox	3	A	884-232-5341	11 Dickinson
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Battle	4	C	874-088-1212	121 Whitman
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Algorithms

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2.2 MERGESORT

- ▶ *mergesort*
- ▶ *bottom-up mergesort*
- ▶ *sorting complexity*
- ▶ *comparators*
- ▶ ***stability***

Stability

A typical application. First, sort by name; **then** sort by section.

`Selection.sort(a, new Student.ByName());`

Andrews	3	A	664-480-0023	097 Little
Battle	4	C	874-088-1212	121 Whitman
Chen	3	A	991-878-4944	308 Blair
Fox	3	A	884-232-5341	11 Dickinson
Furia	1	A	766-093-9873	101 Brown
Gazsi	4	B	766-093-9873	101 Brown
Kanaga	3	B	898-122-9643	22 Brown
Rohde	2	A	232-343-5555	343 Forbes

`Selection.sort(a, new Student.BySection());`

Furia	1	A	766-093-9873	101 Brown
Rohde	2	A	232-343-5555	343 Forbes
Chen	3	A	991-878-4944	308 Blair
Fox	3	A	884-232-5341	11 Dickinson
Andrews	3	A	664-480-0023	097 Little
Kanaga	3	B	898-122-9643	22 Brown
Gazsi	4	B	766-093-9873	101 Brown
Battle	4	C	874-088-1212	121 Whitman

@#%&@! Students in section 3 no longer sorted by name.

A **stable** sort preserves the relative order of items with equal keys.

Stability

Q. Which sorts are stable?

A. Need to check algorithm (and implementation).

sorted by time	sorted by location (not stable)	sorted by location (stable)
Chicago 09:00:00	Chicago 09:25:52	Chicago 09:00:00
Phoenix 09:00:03	Chicago 09:03:13	Chicago 09:00:59
Houston 09:00:13	Chicago 09:21:05	Chicago 09:03:13
Chicago 09:00:59	Chicago 09:19:46	Chicago 09:19:32
Houston 09:01:10	Chicago 09:19:32	Chicago 09:19:46
Chicago 09:03:13	Chicago 09:00:00	Chicago 09:21:05
Seattle 09:10:11	Chicago 09:35:21	Chicago 09:25:52
Seattle 09:10:25	Chicago 09:00:59	Chicago 09:35:21
Phoenix 09:14:25	Houston 09:01:10	Houston 09:00:13
Chicago 09:19:32	Houston 09:00:13	Houston 09:01:10
Chicago 09:19:46	Phoenix 09:37:44	Phoenix 09:00:03
Chicago 09:21:05	Phoenix 09:00:03	Phoenix 09:14:25
Seattle 09:22:43	Phoenix 09:14:25	Phoenix 09:37:44
Seattle 09:22:54	Seattle 09:10:25	Seattle 09:10:11
Chicago 09:25:52	Seattle 09:36:14	Seattle 09:10:25
Chicago 09:35:21	Seattle 09:22:43	Seattle 09:22:43
Seattle 09:36:14	Seattle 09:10:11	Seattle 09:22:54
Phoenix 09:37:44	Seattle 09:22:54	Seattle 09:36:14

Stability: insertion sort

Proposition. Insertion sort is **stable**.

```
public class Insertion
{
    public static void sort(Comparable[] a)
    {
        int N = a.length;
        for (int i = 0; i < N; i++)
            for (int j = i; j > 0 && less(a[j], a[j-1]); j--)
                exch(a, j, j-1);
    }
}
```

i	j	0	1	2	3	4
0	0	B ₁	A ₁	A ₂	A ₃	B ₂
1	0	A ₁	B ₁	A ₂	A ₃	B ₂
2	1	A ₁	A ₂	B ₁	A ₃	B ₂
3	2	A ₁	A ₂	A ₃	B ₁	B ₂
4	4	A ₁	A ₂	A ₃	B ₁	B ₂
		A ₁	A ₂	A ₃	B ₁	B ₂

Pf. Equal items never move past each other.

Stability: selection sort

Proposition. Selection sort is **not stable**.

```
public class Selection
{
    public static void sort(Comparable[] a)
    {
        int N = a.length;
        for (int i = 0; i < N; i++)
        {
            int min = i;
            for (int j = i+1; j < N; j++)
                if (less(a[j], a[min]))
                    min = j;
            exch(a, i, min);
        }
    }
}
```

i	min	0	1	2
0	2	B ₁	B ₂	A
1	1	A	B ₂	B ₁
2	2	A	B ₂	B ₁
		A	B ₂	B ₁

Pf by counterexample. Long-distance exchange can move one equal item past another one.

Stability: shellsort

Proposition. Shellsort sort is **not stable**.

```
public class Shell
{
    public static void sort(Comparable[] a)
    {
        int N = a.length;
        int h = 1;
        while (h < N/3) h = 3*h + 1;
        while (h >= 1)
        {
            for (int i = h; i < N; i++)
            {
                for (int j = i; j > h && less(a[j], a[j-h]); j -= h)
                    exch(a, j, j-h);
            }
            h = h/3;
        }
    }
}
```

h	0	1	2	3	4
	B ₁	B ₂	B ₃	B ₄	A ₁
4	A ₁	B ₂	B ₃	B ₄	B ₁
1	A ₁	B ₂	B ₃	B ₄	B ₁
	A ₁	B ₂	B ₃	B ₄	B ₁

Pf by counterexample. Long-distance exchanges.

Stability: mergesort

Proposition. Mergesort is **stable**.

```
public class Merge
{
    private static void merge(...)
    { /* as before */ }

    private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
    {
        if (hi <= lo) return;
        int mid = lo + (hi - lo) / 2;
        sort(a, aux, lo, mid);
        sort(a, aux, mid+1, hi);
        merge(a, aux, lo, mid, hi);
    }

    public static void sort(Comparable[] a)
    { /* as before */ }
}
```

Pf. Suffices to verify that merge operation is stable.

Stability: mergesort

Proposition. Merge operation is **stable**.

```
private static void merge(...)  
{  
    for (int k = lo; k <= hi; k++)  
        aux[k] = a[k];  
  
    int i = lo, j = mid+1;  
    for (int k = lo; k <= hi; k++)  
    {  
        if (i > mid) a[k] = aux[j++];  
        else if (j > hi) a[k] = aux[i++];  
        else if (less(aux[j], aux[i])) a[k] = aux[j++];  
        else a[k] = aux[i++];  
    }  
}
```

0	1	2	3	4		5	6	7	8	9	10
A ₁	A ₂	A ₃	B	D		A ₄	A ₅	C	E	F	G

Pf. Takes from left subarray if equal keys.

Sorting summary

	inplace?	stable?	best	average	worst	remarks
selection	✓		$\frac{1}{2} N^2$	$\frac{1}{2} N^2$	$\frac{1}{2} N^2$	N exchanges
insertion	✓	✓	N	$\frac{1}{4} N^2$	$\frac{1}{2} N^2$	use for small N or partially ordered
shell	✓		$N \log_3 N$?	$c N^{3/2}$	tight code; subquadratic
merge		✓	$\frac{1}{2} N \lg N$	$N \lg N$	$N \lg N$	$N \log N$ guarantee; stable
timsort		✓	N	$N \lg N$	$N \lg N$	improves mergesort when preexisting order
?	✓	✓	N	$N \lg N$	$N \lg N$	holy sorting grail