A Bit About Hilbert Spaces

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October 29, 2016

Inner Product Space (or "Pre-Hilbert" Spaces)

An **inner product space** (over reals) is a vector space $\mathcal{V}$ and an **inner product**, which is a mapping

$$\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathbf{R}$$

that has the following properties $\forall x, y, z \in \mathcal{V}$ and $a, b \in \mathbf{R}$:

- Symmetry: $\langle x, y \rangle = \langle y, x \rangle$
- Linearity: $\langle ax + by, z \rangle = a \langle x, z \rangle + b \langle y, z \rangle$
- Postive-definiteness: $\langle x, x \rangle \geqslant 0$ and $\langle x, x \rangle = 0 \iff x = 0$.

Norm from Inner Product

For an inner product space, we define a norm as

$$\|x\| = \sqrt{\langle x, x\rangle}.$$

Example

$\mathbf{R}^d$ with standard Euclidean inner product is an inner product space:

$$\langle x, y\rangle := x^T y \qquad \forall x, y \in \mathbf{R}^d.$$

Norm is

$$\|x\| = \sqrt{x^T y}.$$

What norms can we get from an inner product?

Theorem (Parallelogram Law)

A norm $\|v\|$ can be generated by an inner product on $\mathcal{V}$ iff $\forall x, y \in \mathcal{V}$

$$2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2,$$

and if it can, the inner product is given by the ***polarization identity***

$$\langle x, y\rangle = \frac{\|x\|^2 + \|y\|^2 - \|x-y\|^2}{2}.$$

Example

ℓ_1 norm on $\mathbf{R}^d$ is NOT generated by an inner product. [Exercise]

Is ℓ_2 norm on $\mathbf{R}^d$ generated by an inner product?

Pythagorean Theroem

Definition

Two vectors are **orthogonal** if $\langle x, y\rangle = 0$. We denote this by $x \perp y$.

Definition

x is orthogonal to a set S, i.e. $x \perp S$, if $x \perp s$ for all $x \in S$.

Theorem (Pythagorean Theorem)

If $x \perp y$, *then* $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

Proof.

We have

$$\begin{aligned} \|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \\ &= \|x\|^2 + \|y\|^2. \end{aligned}$$

Projection onto a Plane (Rough Definition)

- Choose some $x \in \mathcal{V}$.
- Let M be a subspace of inner product space $\mathcal{V}$.
- Then m_0 is the **projection of** x **onto** M**,**
  - if $m_0 \in M$ and is the closest point to x in M.
- In math: For all $m \in M$,

$$\|x - m_0\| \leqslant \|x - m\|.$$

Hilbert Space

- Projections exist for all finite-dimensional inner product spaces.
- We want to allow infinite-dimensional spaces.
- Need an extra condition called **completeness**.
- A space is **complete** if all Cauchy sequences in the space converge.

Definition

A **Hilbert space** is a complete inner product space.

Example

Any finite dimensional inner product space is a Hilbert space.

The Projection Theorem

Theorem (Classical Projection Theorem)

- *$\mathcal{H}$ a Hilbert space*
- *M a closed subspace of $\mathcal{H}$*
- *For any $x \in \mathcal{H}$, there **exists a unique** $m_0 \in M$ for which*

$$\|x - m_0\| \leqslant \|x - m\| \; \forall m \in M.$$

- *This m_0 is called the **[orthogonal] projection of x onto M.***
- *Furthermore, $m_0 \in M$ is the projection of x onto M iff*

$$x - m_0 \perp M.$$

Projection Reduces Norm

Theorem

Let M be a closed subspace of $\mathcal{H}$. For any $x \in \mathcal{H}$, let $m_0 = Proj_M x$ be the projection of x onto M. Then

$$\|m_0\| \leqslant \|x\|,$$

with equality only when $m_0 = x$.

Proof.

$$\begin{aligned}
\|x\|^2 &= \|m_0 + (x - m_0)\|^2 \text{ (note: } x - m_0 \perp m_0) \\
&= \|m_0\|^2 + \|x - m_0\|^2 \text{ by Pythagorean theorem} \\
\|m_0\|^2 &= \|x\|^2 - \|x - m_0\|^2
\end{aligned}$$

If $\|x - m_0\|^2 = 0$, then $x = m_0$, by definition of norm. □