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Multiperiod interval-based stochastic dominance with application to dynamic portfolios

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We consider a multi-stage generalization of the *interval-based stochastic dominance* (ISD) principles introduced by Liu *et al.* [Interval-based stochastic dominance: Theoretical framework and application to portfolio choices. *Ann. Oper. Res.*, 2021, **307**, 329–361]. The ISD criterion was motivated specifically in a financial context to allow for contiguous integer SD orders on different portions of a portfolio return distribution against a benchmark distribution. A continuous spanning of SD conditions between *first*-, *second*-, and *third-order stochastic dominance* was introduced in that context, relying on a reference point. Here, by extending the partial order to random data processes, we apply ISD conditions to a multi-period portfolio selection problem and verify the modeling and computational implications of such an extension. Several theoretical and methodological issues arise in this case that motivate this contribution. The problem is formulated in scenario form as a multi-stage stochastic recourse program, and we study two possible generalizations of ISD principles in which we either enforce ISD constraints at each stage, independently from the scenario tree process evolution, or we do so conditionally along the scenario tree. We present a comprehensive set of computational results to show that, depending on the benchmark investment policy and the adopted ISD formulation, stochastic dominance conditions of first- or second-order can be enforced dynamically over a range of possible values of the reference point, and their solution carries a specific rationale. The computational constraints induced by the multistage ISD formulation are also emphasized and discussed in detail.

Keywords: Multi-stage stochastic programming; Interval stochastic dominance; Multi-period portfolio selection; Large-scale stochastic programming

JEL Classifications: C44, C61, C63, G11, G41

1. Introduction

Portfolio optimization has been at the center of finance theory for several decades. Since the seminal contributions by Harry Markowitz on mean-variance optimization in the '50s, many portfolio selection paradigms have been introduced over roughly six decades, as well documented in the review articles by Fabozzi *et al.* (2010), Kolm *et al.* (2014), Kim *et al.* (2018) and Cornuejols *et al.* (2018), with implications in decision theory, finance, computational methods, and more generally optimization approaches; see also Andriosopoulos *et al.* (2019). The adoption of single-period optimization models under alternative specifications of risk or acceptability measures, as well as risk-reward trade-off functions, is well documented; see Pflug and Römisch (2007), leading to a wide range of optimization results and computational approaches. In this research, we focus on a multi-period portfolio problem under an assumption of finite investment horizon and an underlying return process discrete in time and space. Multi-period portfolio models are typically motivated by the presence of market frictions and transaction costs in the problem formulation, as well as investment goals over the investment horizon, or by specific properties of the underlying return processes; see Dempster *et al.* (2003), Topaloglou *et al.* (2008), Guidolin and Timmermann (2007) and Mulvey *et al.* (2007). In this work, we consider a vector auto-regressive model for asset returns with joint dependence on two state variables for interest rates and inflation, as further discussed below.

Under an assumption of discrete uncertainty and underlying varying market conditions, closed-form solutions are seldom possible, and we resort to numerical techniques such as the popular stochastic dynamic programming approach, see Infanger (2011), or multistage stochastic programming, see

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Consigli *et al.* (2016). The former, specifically in the presence of Markov decision problems (MDP), leads to the definition of optimal investment policies over the planning horizon by backward induction. The latter, adopted in this research, leads to the definition of scenario-based optimal contingency plans and the specification of an *implementable, optimal here-and-now* (H&N) portfolio allocation to be revised in following stages. Multistage approaches have also been motivated by focusing on the value generated by recourse decisions in a multistage framework, see Dempster *et al.* (2003), Pflug *et al.* (2012), Barro *et al.* (2019) and Gomez *et al.* (2024). More generally, as discussed in previous contributions (Liu and Pan 2003, Consigli *et al.* 2015, Barro *et al.* 2019, 2022), the inclusion of recourse decisions within a multistage framework has positive effects in terms of risk control and performance enhancements relative to static approaches. An alternative to methods explicitly solving multi-period decision problems is provided by the solution of single-period portfolio problems with rebalancing as new market information is revealed, also resulting in dynamic strategies (Jung and Kim 2017). We return to this point later in this paper when discussing, in the computational section, a set of results benchmarking the two approaches.

A relevant motivation for formulating a portfolio problem based on SD principles, consistent with financial practice and traditional approaches in the fund management industry, comes from the distinction between *active* versus *passive* portfolio management, see Dempster *et al.* (2008). Contrary to the former, this latter formulation requires the introduction of a benchmark portfolio, and portfolio performance is analyzed in relative rather than absolute terms. A hybrid model, combining the relative portfolio selection criterion with a minimal return guarantee, is the popular *enhanced portfolio indexation* model discussed by Roman *et al.* (2013), Mansini *et al.* (2014) and Bruni *et al.* (2015). The adoption of such a relative approach, possibly under an enhancement condition, leads naturally to a problem formulation under stochastic dominance principles, and indeed this was advocated already by Ogryczak and Ruszczyński (1999) early on, anticipating a rich stream of contributions during the last two decades. *First-order stochastic dominance* (FSD) in a portfolio optimization context may be considered reflecting a benchmark performance enhancement objective, while *second-order stochastic dominance* (SSD) may be associated with an optimal portfolio replication problem, see Kopa and Post (2009) and Post and Kopa (2013).

The concept of *interval-based stochastic dominance* (ISD) was introduced in Liu *et al.* (2021) to allow in a classical static portfolio selection problem for *stochastic dominance* constraints other than FSD, SSD or *third-order stochastic dominance* (TSD), including those as specific sub-cases. The rationale for such generalization was, in extreme summary, related to the possibility of discriminating between contiguous SD partial orders with respect to a benchmark distribution. The definition of ISD-1 and ISD-2 criteria, spanning from FSD to SSD and from SSD to TSD, respectively, was based on the introduction of a reference point, as further clarified here below. This research builds essentially on two relevant streams of contributions related one to the introduction of non-integer SD criteria in *decision making under risk* and the second to the formulation of financial optimization problems with SD conditions. In this latter case the benchmark distribution may be exogenously generated by a given investment rule that the portfolio manager wishes to dominate. Such benchmark policy may be characterized outside the decision space of the investment problem or, as in this article, be so-called of *attainable* type, when the benchmark is constructed relying on the same decision space, or subspace, of the problem.

As for the foundations of non integer SD theory, from a decision-making perspective already in the 80s, Fishburn (1980) introduced fractional integration as a generalization of SD conditions to allow a continuum of stochastic dominance relations. Then Leshno and Levy (2002) proposed *almost stochastic dominance* (ASD) as a possible approach to relax FSD conditions, these latter easily resulting in hardly solvable decision problems. The same year Levy and Levy (2002), inspiring the ISD concept considered by Liu *et al.* (2021), proposed the concept of *prospect stochastic dominance* in accordance with the principles laid down in their popular contributions by Kahneman and Tversky (1974). In a financial context, Lizyayev and Ruszczyński (2012) applied ASD to a portfolio problem as a linear programming problem. More recently, specifically focusing on the preference domain spanned by FSD and SSD, Müller *et al.* (2017) extended utility theory results to account for a continuum of risk preferences. ASD analysis was then generalized by Tsetlin *et al.* (2015) through *generalized almost-SD* resulting in a more consistent set of principles and wider applicability.

The stream of contributions considering SD-based financial-type of problems in a static, one period framework, is even more long-dated and rich since the early work by Hanoch and Levy (1969) and Hadar and Russell (1969) where SD principles found their first applications in economics and finance. Then after we may recall in extreme summary, the early contributions by Bawa (1975), Ogryczak and Ruszczyński (1999) and Levy (2006) clarifying the relationship between expected utility theory and risk preferences in financial-type of problems. In these optimization problems, stochastic dominance constraints were introduced to ensure that the managed portfolios dominate benchmarks' returns. Dentcheva and Ruszczyński (2006) first considered a portfolio problem with SSD constraints, wherein risky asset returns were characterized by a joint discrete distribution. Luedtke (2008) proposed mixed-integer linear and linear programming formulations for FSD and SSD constraints, respectively, validated in a financial type of problem. Several authors extended the range of finance applications significantly with first examples of multiperiod SD modeling, primarily SSD-type under assumptions of interstage independence, see Post (2003), Kopa *et al.* (2018) and Kallio and Hardoroudi (2018) focusing on financial planning problems, Roman *et al.* (2013) on enhanced portfolio indexation problems, Yang *et al.* (2010) and Consigli *et al.* (2020) on classical asset-liability management problems and Kopa *et al.* (2018) and Moriggia *et al.* (2019) on pension fund management problems.

The multistage extension of SD and, for our purposes, ISD is complex from both a decision-theoretical and a methodological perspective. We tackle this problem by restricting the application domain to a portfolio selection problem in which a portfolio manager is assumed to maximize the expected value of her/his portfolio while enforcing ISD conditions with respect to alternative portfolio strategies.

1.1. *Motivation and contribution*

SSD approaches, well-established in decision theory as reflecting the preferences of risk-averse decision makers, have attracted significant interest in finance and may, in principle, be regarded as a reference decision model for relative portfolio selection, as shown, among others, by Fábián *et al.* (2011) and Roman *et al.* (2013).

In our early contribution (Liu *et al.* 2021), we provided an example in which two rather different portfolio distributions could not be distinguished relying only on FSD or SSD principles, as both dominate a benchmark according to SSD and neither one according to FSD. Furthermore, in that work, we verified that indeed FSD with respect to a benchmark can hardly be attained in practice, thus motivating a possible relaxation of constraints.

The introduction of a reference point over the support of the benchmark distribution helped in such case to discriminate between the two cases, leading respectively to ISD-1 and ISD-2 partial orders. Relying on the early statistical characterization of SD by Bawa (1975), Fishburn (1977) and introducing a reference point, say β, we generalized the concept of *partial semi-moments* (PSM) to enforce a continuum between the orders SD-k and SD-$(k+1)$ for an integer k.

In a multi-period formulation, we wish to enforce ISD principles at every decision stage of the problem. This amounts to introducing a dynamic reference function β_t that would help a decision maker discriminating between lower and upper tails of a benchmark distribution over time and, in a portfolio context in the case of ISD-1, identifying an investment strategy dominating a benchmark policy by FSD on the lower tail and by SSD on the upper tail of the distribution.

We rely on the diagram in figure 1 to summarize the relationship linking integer SD orders to ISD over one and over multiple periods.

Figure 1 clarifies a set of relevant extensions and provides a core motivation of this research study. On the top level we have the single period relationships involving ISD-k, that for $k = 1, 2$, spans from kth to $(k+1)$th order, studied in Liu *et al.* (2021). The reference point at the grounds of the ISD extension helps spanning the domain of the returns distributions generated by the optimal portfolio and the benchmark policy. As in the static, one period instance, an increasing β leads to a stronger kth SD order, while for β going to $-\infty$ we switch to the $(k+1)$th SD order. This article focuses on the ISD multiperiod extension. Such extension is summarized on the bottom part of figure 1. For a problem in scenario form two possible approaches can be adopted in a multistage model: either we impose the partial order in every subtree of the problem, thus conditionally on the scenario tree evolution, or we just consider the distributions at the given stages over the planning horizon. We refer to the first case as McISD-k and to the latter as MISD-k. Under any of the two approaches, the multi-period extension aims at enforcing the partial order with respect to a benchmark investment strategy over the planning horizon of the problem. We claim that by doing so the first stage results would benefit relative to a single period approach with rebalancing. We show in the computational evidence that indeed, under several input ISD-based feasibility conditions, through the β update, the optimal strategies would lead to an increasingly effective policy in terms of strategy benchmarking witnessed by a consistently decreasing ISD order over the stages of the problem.

The key motivations of this contribution can then be summarized as follows:

- Given an evolving benchmark distribution, the proposed dynamic extension helps tackling genuine multi-period asset as well as asset-liability management problems, in which we require the enforcement of ISD conditions with respect to distributions induced, say, by solvency requirements as in Consigli *et al.* (2018) or by alternative investment policies, see Consigli *et al.* (2020).
- From a methodological perspective, we analyse the implications of two possible approaches to multistage ISD and associated integer SD, for a stochastic program in scenario form. Either we require ISD-feasibility with respect to the benchmark in every sub-tree of the *multistage stochastic recourse problem* or just with respect to the distribution defined at every stage. We show that conditional SD implies stage-wise SD, while the converse does not hold.
- More generally, through the multistage extension we remain consistent with prior evidence of in- and out-of-sample superior performance and risk control effectiveness of multi-period relative to one-period myopic investment strategies, see Dempster *et al.* (2003), Mulvey (2004) and Consigli *et al.* (2016), Mei *et al.* (2022). In the context of this article such extension, particularly in presence of unsolvable FSD instances, does exploit

$$
\begin{tikzcd}
\mathrm{SD}\text{-}k & & \mathrm{ISD}\text{-}k \arrow[ll, "\beta=+\infty"'] \arrow[rr, "\beta=-\infty"] & & \mathrm{SD}\text{-}(k+1) \\
\mathrm{MSD}\text{-}k \arrow[u, "T=1"] & \mathrm{MISD}\text{-}k \arrow[l, "\beta_t=+\infty"'] \arrow[ur, "T=1"] \arrow[rr, leftrightarrow, "\beta_t\to\beta_n", "\beta_t\leftarrow\beta_n"'] & & \mathrm{McISD}\text{-}k \arrow[ul, "T=1"'] \arrow[r, "\beta_n=-\infty"] & \mathrm{MSD}\text{-}(k+1) \arrow[u, "T=1"']
\end{tikzcd}
$$

Figure 1. Relationship between integer SD orders to ISD over one period and with multiple periods.

the relaxation of FSD principles allowed by the ISD-paradigm.

This article aims then at the following contributions:

- The mathematical formulation of a dynamic instance of an ISD-constrained optimization problem, including multistage SD-conditions of the first (FSD) and second (SSD) order as sub-cases.
- The probabilistic foundation of ISD principles for generic random processes and their implications in a discrete financial optimization problem. Based on the introduction of a reference function with stage- or state-dependent updating, we analyse its relevance in defining optimal investment plans against alternative specification of benchmark policies.
- A careful characterization of the feasibility region for multiperiod ISD-based optimization problems with underlying scenario tree processes, whose financial and computational implications are evaluated in a case study.
- The evaluation of the computational feasibility of extremely large-scale problems resulting from large scenario trees even over a relatively limited number of stages. We present results collected on a *high-performance computing* (HPC) platform.

The article is organized as follows: in section 2 we motivate the multi-stage extension of ISD as a partial order between stochastic processes and clarify its implications in presence of discrete data tree processes. In this section we also clarify the distinctive features of two possible specifications of ISD-based feasibility conditions. In section 3 we go more in detail in the mathematical characterization of scenario-based ISD conditions for multistage problems. In section 4 we introduce the mathematical formulation of a portfolio selection problem based on multi-stage ISD principles and present the statistical model and scenario generation approach supporting the problem formulation. Finally, in section 5 we present an extended set of computational results to validate the introduced approach and analyze its implications with an application to the US equity market.

2. Multiperiod interval stochastic dominance

In this section, we first review the definition of traditional integer-order SD and interval SD; then extend ISD to the dynamic case. The relationships in figure 1 will then be clear and rigorously established. Under a discrete, scenario-based formulation, the multiperiod extension requires a discussion on the rules adopted on the reference point updating at every stage as well as a parsimonious definition of the ISD constraints associated with different model specifications.

Given two random variables X and Y defined in a probability space $(\Omega, \mathcal{F}, P)$, with Ω the sample space, $\mathcal{F}$ the σ-algebra and P the probability measure, when dealing with canonical SD orders, we say that X dominates Y with order k: $X \succeq_k Y$, if $F_k(X,\eta) \leq F_k(Y,\eta)$ for every $\eta \in \mathbb{R}$, where $F_k(X,\eta) = \frac{1}{(k-1)!}\mathbb{E}((\eta - X)_+^{k-1})$, $k \geq 1$. We are primarily interested to the cases $k = 1, 2, 3$.

The set of X stochastically dominating Y is defined as $\mathcal{A}_k(Y) = \{X \in L^{k-1}(\Omega, \mathcal{F}, P; \mathbb{R}) \mid X \succeq_k Y\}$, for $k = 1, 2, 3$.

The concept of interval stochastic dominance follows when introducing the reference point β.

Definition 1 Interval SD (Liu *et al.* 2021) Given two random variables X and Y, we say that X stochastically dominates Y in the kth-based interval if, for given $\beta \in \overline{\mathbb{R}}$, we have

$$\begin{cases} F_k(X;\eta) \leq F_k(Y;\eta), \quad \forall\, \eta \leq \beta, & \text{(1-1)} \\ F_{k+1}(X;\eta) \leq F_{k+1}(Y;\eta), \quad \forall\, \eta \geq \beta. & \text{(1-2)} \end{cases} \tag{1}$$

For $k = 1, 2$ we denote this dominance order by ISD-k for, respectively *first* and *second* order ISD and write $X \succeq_{(k,\beta)} Y$. We allow β to take values over $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\}$. For $k = 1, 2$, we define the feasible set of X ISD-k dominating Y as $\mathcal{A}_{k,\beta}(Y) = \{X \in L^{k-1}(\Omega, \mathcal{F}, P; \overline{\mathbb{R}}) \mid X \succeq_{(k,\beta)} Y\}$.

Associated with the definition of ISD is the semi-moment:

$$F_{(k,\beta)}(X,\eta) = \frac{1}{(k-1)!}\mathbb{E}((\eta - X)_+^{k-1})\chi_{(-\infty,\beta]}(\eta) + \frac{1}{k!}\mathbb{E}((\eta - X)_+^{k})\chi_{[\beta,\infty)}(\eta), \tag{2}$$

of a random variable X, with $\eta \in \mathbb{R}$. Then we have:

Proposition 1 *Let X and Y be random variables, $X \succeq_{(k,\beta)} Y$ if and only if*

$$F_{(k,\beta)}(X,\eta) \leq F_{(k,\beta)}(Y,\eta),$$

for every $\eta \in \mathbb{R}$, as proven by Liu *et al.* (2021).

We present further specific properties of ISD useful in the economy of this research in appendix 1 with the necessary proofs.

The above properties help conveying the financial rationale of the proposed criterion in which a portfolio manager may wish to enforce a strict dominance, say first order, on the lower portion of the benchmark distribution and a weaker one, say second order, above β. The criterion may also be understood as providing a relaxation of a strong integer based dominance order, in case of infeasibility of the associated portfolio optimization problem.

A multi-period extension of ISD principles calls for the introduction of random sequences and the definition of a partial order that remains consistent over time. From a financial perspective, the multi-period framework allows the derivation of an investment policy that dominates stochastically a given benchmark strategy over time. Let the time set be discrete and finite: $\mathcal{T}_0 := \{t_0, t_1, t_2, .., t_n\}$ where typically $t_0 = 0$, current time and $t_n = T$, the end of the investment horizon. At points we may need to consider $\mathcal{T} = \mathcal{T}_0 - \{t_0\}$. We assume now a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ generated by a random process, say ω_t, defined as a discrete *tree process* which embodies the overall uncertainty underlying the decision problem. We require the set of asset returns $r: (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ to have finite pth moments, i.e. $\int_\Omega |r(\omega)|^p \, d\mathbb{P}(\omega) < \infty$, for $p \geq 1$, $r(\omega) \in \mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ and $X_t(\omega)$ be the resulting portfolio value at time t. Specific to this model, this space will also include the *benchmark investment value*, say $Y_t(\omega)$ that the decision maker intends to dominate. In a dynamic set-up, $\mathcal{F}_t$ is the filtration at time $t \in \mathcal{T}_0$: we have $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_T$ that provides the evolution of the information set available to the decision maker, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$ is the σ-algebra at the root node. According to equation (1), the decision-maker might hold different levels of ambiguous risk attitude above or below the reference point β. Thus, it is natural to consider that the risk attitude of the decision-maker may vary according to the variation of the reference point, denoted by β_t in stage t.

Under these assumptions, the multistage stochastic dominance condition can be specified as follows.

Definition 2 MISD-k Let X_t and Y_t be two random processes defined over $t \in \mathcal{T}$ and β_t a finite countable sequence of reference points. Taking $s < t$, we say that X_t ISD-dominates Y_t with order (k, β_t), and write $X_t \succeq_{(k,\beta_t)|\mathcal{F}_s} Y_t$, conditional on $\mathcal{F}_s$, if $F_{(k,\beta_t)}(X_t, \eta; \mathcal{F}_s) \le F_{(k,\beta_t)}(Y_t, \eta; \mathcal{F}_s)$ a.e. for $\eta \in \mathcal{L}^p(\Omega, \mathcal{F}_s, P; \mathbb{R})$.

Here:

$$F_{(k,\beta_t)}(X_t, \eta, \mathcal{F}_s) = \frac{1}{(k-1)!}\mathbb{E}((\eta - X_t)_+^{k-1} \mid \mathcal{F}_s)\chi_{(-\infty,\beta_t]}(\eta) + \frac{1}{k!}\mathbb{E}((\eta - X_t)_+^{k} \mid \mathcal{F}_s)\chi_{[\beta_t,\infty)}(\eta). \quad (3)$$

We consider two kinds of dynamics of the reference point. The first type is stage-dependent, meaning β_t might differ at different stages according to a given criterion, resulting in an updating of the reference point known since the beginning. The information process related to the reference point is in this case assumed to be known since the first stage. The second type of dynamics for β is state-dependent, i.e. the value β_t depends on the value of historical realizations of $X_1, X_2, \dots, X_{t-1}$ and $Y_1, Y_2, \dots, Y_{t-1}$: β_t is $\mathcal{F}_{t-1}$-measurable, i.e. $\mathcal{F}_t$-predictable. Consider in particular the conditional and unconditional moments $\mathbb{E}((\eta - X_t)_+^k \mid \mathcal{F}_{t-1})$ and $\mathbb{E}((\eta - X_t)_+^k \mid \mathcal{F}_0)$, respectively. Then, we have that:

(1) X_t dominates *conditionally* Y_t over $\mathcal{T}$ with order (k, β_t) if $X_t \succeq_{(k,\beta_t \mid \mathcal{F}_t)} Y_t$, with respect to the conditional distributions. Or, that
(2) X_t dominates *stage-wise* Y_t over $\mathcal{T}$ with order (k, β_t) if $X_t \succeq_{(k,\beta_t \mid \mathcal{F}_0)} Y_t$.

Remark 1 β updating: Some comments are due in relation to the definition of the reference function β_t, whose updating will drive the decision process over the planning horizon. The decision maker, under general assumptions, may very well focus in specific problems on a given quantile of the distribution and leave β_t to adapt accordingly. A practical approach is to fix a quantile of the benchmark return distribution Y_t (resp. the distribution of benchmark Y_t conditional on $\mathcal{F}_{t-1}$) in each stage (resp. each node) of the problem. Let in particular q_β denote the quantile of the distribution, say $q_\beta = 0.5$ for the median to discriminate between contiguous SD orders. Then in every stage, depending on the benchmark distribution, the reference point will be updated. In the ISD case, below the β quantile of Y, to be denoted by q_β, we consider the stronger kth-order SD to describe the dominance relation while above q_β we use the weaker $(k+1)$th-order SD. In the stage-wise extension, the ISD condition between X_t and Y_t applies to the quantiles of the distributions of Y_t, while in the conditional extension of MISD, we rely on the filtration and the sub-trees distributions of Y_t conditional on a historical path till stage $t-1$, represented by $\mathcal{F}_{t-1}$.

In a financial application, under ISD-1, we assume that the portfolio manager has a primary interest to determine a portfolio strategy able to stochastically dominate over time a benchmark policy to the first order only on the lower domain of the distribution, while SSD would be sufficient above β_t. Now a β coefficient expressed in relative terms, say $\beta = -1\%$ monthly return would call for FSD of X_n over Y_n for negative returns below -1% and SSD above. Let such $\beta = -1\%$ correspond to $q_\beta = 0.25$ the first quartile of the benchmark distribution. As the number of nodes increases stage by stage, the MISD-1 criterion would always adapt requiring FSD for returns below the first quartile and SSD above. The effectiveness of such updating will depend on the height of the tree at that stage, thus on the adopted sampling scheme. Over the return domain of the portfolio and benchmark policies, as β increases, theoretically to ∞, MISD-1 would converge to the multistage FSD while for $\beta \to -\infty$ we would fall into the SSD case. This reflects a special case in Definition 2 when we take the reference function β_t to be a constant at ∞, resulting in this case in the multi-stage stochastic dominance order defined as MSD-k by $X_t \succeq_{(k,\infty)} Y_t$ and denoted by $X_t \succeq_{(k)} Y_t$. In the *conditional* case, we would have $\{X_t \mid \mathcal{F}_{t-1} \succeq_{(k,\infty)} Y_t \mid \mathcal{F}_{t-1}\}$ and write $\{X_t \mid \mathcal{F}_{t-1}\} \succeq_{(k)} \{Y_t \mid \mathcal{F}_{t-1}\}$. We specialize this discussion to the case of scenario tree processes in the next section.

3. Stochastic orders over discrete tree processes

We consider a scenario tree structure for the filtration generated by the portfolio process X_t and the benchmark process Y_t, as illustrated in figure 2.

Information is assumed to evolve according to a non recombining scenario tree as in figure 2. We label nodes in the tree as $n \in \mathcal{N}_t$ at time stage t, where every n has a unique ancestor node $n-$ and for $t \le T-1$ there exists a non-empty set of children nodes $n+ \in \mathcal{N}_{t+1}$ whose cardinality is denoted by $\#n+$. For every leaf node $n \in \mathcal{N}_T$ and random process ω a scenario is a path $\omega_n, \omega_{n-}, \omega_{n--}, \dots, \omega_{n_0}$, where n_0 is the root node. For every node n, we denote with t_n the time associated with that node. Each node carries a probability of occurrence given by p_n such that $\sum_{n \in \mathcal{N}_T} p_n = 1$ and for every non terminal node $p_n = \sum_{m \in n+} p_m$, $\forall n \in \mathcal{N}_t$, $t \le T-1$. On each node n of stage t, the values of X_t and Y_t are X_n and Y_n.

When dealing with stochastic dominance problems, the algebra of the tree process requires a specific notation to rule first- or second-order ISD conditions along the tree. For stage-wise ISD, the reference points β_t are *stage-dependent*, i.e. they are real numbers in $\mathbb{R}$. While for conditional ISD, the reference points are scenario-dependent and predictable: we

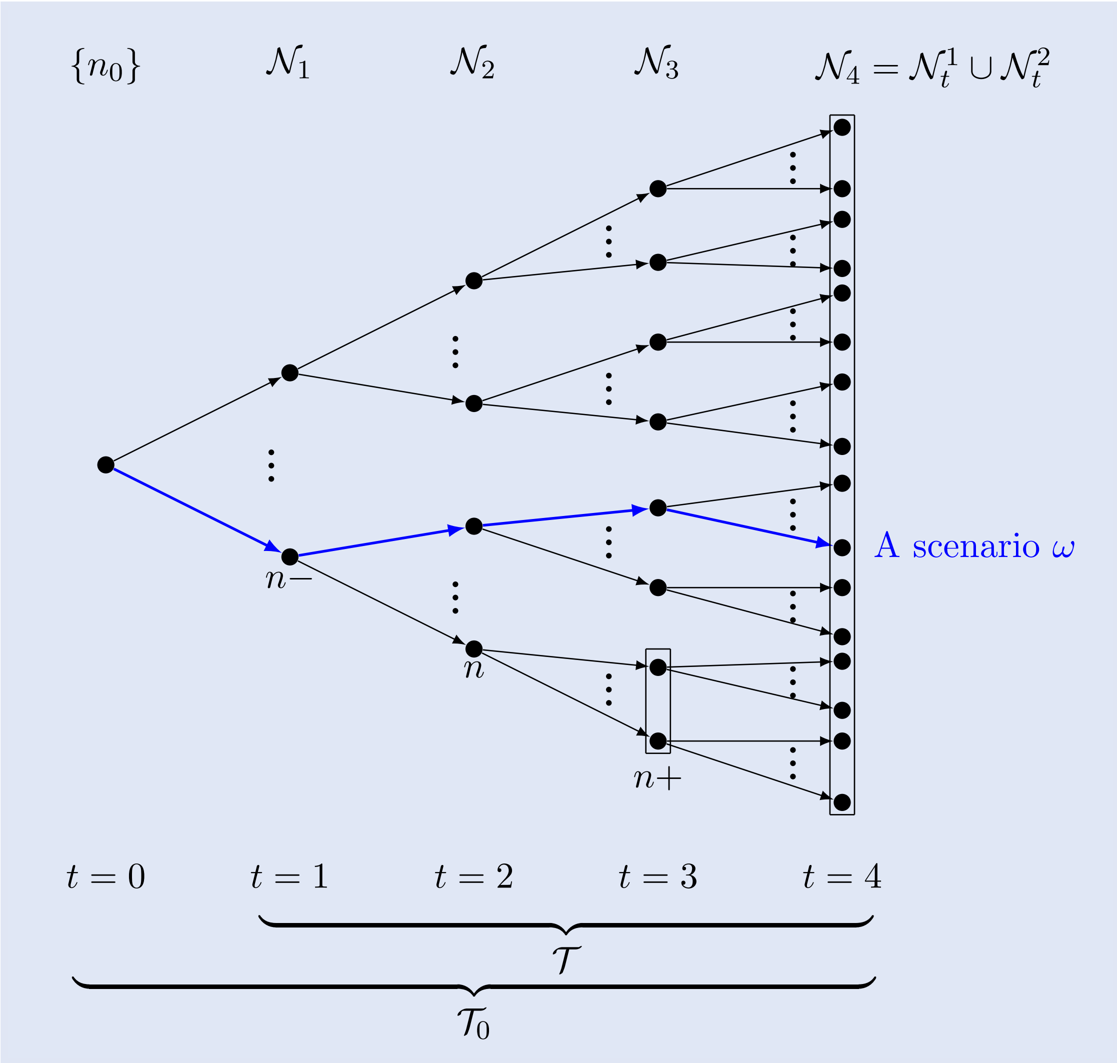

Figure 2. A 4-stage scenario tree.

consider β_n for conditional ISD case as the reference point defined on node $n \in \mathcal{N}_t$ and stage $t = 1, \ldots, T-1$. Then, in this case, for each non-leaf node n, we estimate β_n as the q_β-quantile of the values $\{Y_n, n \in n+\}$ only on the children nodes. Table 1 specifies the scenario tree conventions adopted throughout the paper.

We consider next a selected set of results that help highlighting relevant implications of the decision and modelling frameworks laid down above. We show in particular that McISD-k, for $k = 1, 2$ implies stage-wise MISD-k. From a financial perspective the MISD approach, by focusing on the distributions generated at every stage relies on a richer set of realizations, likely resulting into a more effective control of the tail investment risk.

We specialize Definition 2 to tree processes here next.

Definition 3 Let $\{X_n\}$ and $\{Y_n\}$ be two stochastic data tree processes and $\beta = \{\beta_t\} \in \mathbb{R}^{\mathcal{T}}$ be a set of stage-dependent benchmarks: then X_n stochastically dominates Y_n stage-wise with order (k, β), shortly denoted as $X_n \succeq_{(k,\beta)} Y_n$, if at any stage $t \in \mathcal{T}$, we have $X_{\mathcal{N}_t} \succeq_{(k,\beta_t)} Y_{\mathcal{N}_t}$, i.e. $F_{(1,\beta_t)}(X_{\mathcal{N}_t}, \eta) \leq F_{(1,\beta_t)}(Y_{\mathcal{N}_t}, \eta)$ for every $\eta \leq \beta_t$. Here, $X_{\mathcal{N}_t}$ and $Y_{\mathcal{N}_t}$ represent realizations of $\{X_n\}$ and $\{Y_n\}$, respectively, in stage t, with probability determined by the tree process evolution.

We distinguish from Definition 3, the case of a scenario-dependent reference function:

Definition 4 Let now $\beta = \{\beta_n\} \in \mathbb{R}^{\mathcal{N}}$ be a sequence of scenario-dependent reference points. Then we state that X_n conditionally dominates Y_n with order (k, β), shortly $X_n \succeq_{\{(k,\beta)|\mathcal{F}\}} Y_n$, if for any $n \in \mathcal{N}_t$, $t \in \mathcal{T} - \{T\}$, we have a conditional dominance of $X_{n+} \succeq_{(k,\beta_n)} Y_{n+}$, i.e. $F_{(k,\beta_n)}(X_{n+}, \eta) \leq F_{(k,\beta_n)}(Y_{n+}, \eta)$ for every $\eta \leq \beta_n$. Here, X_{n+} and Y_{n+} represent two random variables taking values in the children nodes of n, and the probability associated with each sample corresponds to the conditional probabilities.

Under the given definitions, it is easily proven the following result.

Corollary 1 Conditional implies Stagewise *For $n \in \mathcal{N}$, let X_n and Y_n be two random tree processes and $\beta \in \mathbb{R}^{\mathcal{T}}$ a reference vector. If $X_n \succeq_{\{(k,\beta)|\mathcal{F}\}} Y_n$ conditionally on $\mathcal{F}$, then $X_n \succeq_{(k,\beta)} Y_n$.*

Proof (Sketch) The key point is to divide the overall distribution of X_n (Y_n resp.) in each stage $t \in \mathcal{T}$ into many partitions $\mathcal{N}_t = \bigcup_{n \in \mathcal{N}_{t-1}} n+$. MISD imposes a set of moment constraints

Table 1. Scenario tree notation.

Time and stage notation	
$\mathcal{T}_0$	Time set: $\mathcal{T}_0 := \{0, 1, 2, .., T\}$ are the stages and T is the investment horizon
$\mathcal{T}$	Positive time set: $\mathcal{T} := \{1, 2, .., T\}$
Probability space	
$(\Omega, \mathcal{F}, \mathbb{P})$	Probability space with sample space Ω, σ-algebra $\mathcal{F}$ and measure $\mathbb{P}$
$\mathcal{F}_t$	Information set, σ-algebra at time $t \in \mathcal{T}$
Scenario tree convention	
n	Node n whose time occurrence is denoted by t_n when this is requested
n_0	Root node
$n-$	Parent of node n
$n+$	Set of children of node n
$\mathcal{N}_t$	Set of nodes at stage $t \in \mathcal{T}$
$\mathcal{N}_T$	Set of all leaf nodes (same number to all scenarios)
p_n	unconditional probability of node n
$p_{m,n}$	Conditional probability to reach node n from node m
X_n	Portfolio value in node n
Y_n	Benchmark value in node n
$X_{\mathcal{N}_t}$	Portfolio values across all node in $\mathcal{N}_t$
$Y_{\mathcal{N}_t}$	Benchmark values across all nodes in $\mathcal{N}_t$
X_{n+}	Portfolio values in children nodes $n+$
Y_{n+}	Benchmark values in children nodes $n+$
Stage-wise ISD	
β_t	Reference point at stage t
$\mathcal{N}_{[t]} = \{n_{[1]} \le \cdots \le n_{[\#\mathcal{T}]}\}$	Sorted nodes in stage $t \in \mathcal{T}$
$(\mathcal{N}^1_{[t]}, \mathcal{N}^2_{[t]})$	Partition of $\mathcal{N}_t$ induced by β in stage $t \in \mathcal{T}$
l_t	Reference integer label at stage $t \in \mathcal{T}$
Conditional ISD	
β_n	Reference point in node n
$\mathcal{N}_{[n+]} = \{e_{[1]} \le \cdots \le e_{[\#n+]}\}$	Sorted nodes in the subtree from $n \in \mathcal{N}_t$
$(\mathcal{N}^1_{[n+]}, \mathcal{N}^2_{[n+]})$	Partition of $n+$ induced by β at the node $n \in \mathcal{N}_t$, $t \in \mathcal{T}$
n^*_n	Reference integer defined in $n \in \mathcal{N}_t$ associated to the partition of $n+$

Table 2. Tree reformulation of ISD-1 and ISD-2.

Model	MISD-1	MISD-2	McISD-1	McISD-2
reference point β	Stage-dependent β_t, $t \in \mathcal{T}$		Scenario-dependent β_n, $n \in \mathcal{N}_t$, $t \in \mathcal{T}$	
Left of β	FSD	SSD	FSD	SSD
reformulation	MILP (4)–(7)	LP (9a)	MILP (11a)–(11f)	LP (12a)–(12d)
Right of β	SSD	TSD	SSD	TSD
reformulation	LP (8a)	QCLP (10a)	LP (11g)–(11l)	QCLP (12e)–(12m)

on the overall stage distribution, while McISD on the partitions induced by the current set of sub-trees. By applying the law of total probability to the distribution term to those moment constraints, we have the desired relationship. ■

Furthermore, as in a one stage problem stronger SD conditions imply weaker ones, as stated in the following lemma.

LEMMA 1 *Let X_n and Y_n be random tree processes with finite horizon. Then, for $k = 1, 2$*

(i) *If $X_n \succeq_k Y_n$ and $X_n \succeq_{\{k|\mathcal{F}\}} Y_n$ then $X_n \succeq_{(k,\beta)} Y_n$ and $X_n \succeq_{\{(k,\beta)|\mathcal{F}\}}$ respectively, for every reference function β.*

(ii) *If $X_n \succeq_{(k,\beta)} Y_n$ and $X_n \succeq_{\{(k,\beta)|\mathcal{F}\}} Y_n$ for some reference function β then $X_n \succeq_{(k+1)} Y_n$ and $X_n \succeq_{\{(k+1)|\mathcal{F}\}} Y_n$.*

Lemma 1 is a direct application to a stochastic discrete tree process of Lemma A.4 in appendix 2, established for general stochastic processes. Corollary 1 and Lemma 1 have relevant practical implications and will be verified numerically in section 5.

One may refer to table 2 for a summary of the tree reformulation of the MISD/McISD of order 1 or 2, focusing on the domains partition determined by the reference point β.

Table 2 conveys the characterization of the feasibility region under the different modeling options and anticipates the resulting formulations of the optimization problems as mixed-integer linear programs (MILP), linear programs (LP) and quadratically constrained linear programs (QCLP). The next sections 3.1 and 3.2 go in detail into the specification of the full set of equations adopted under the above assumptions.

3.1. Scenario-based formulation of MISD constraints

Based on the tree process in section 3, we can apply the discrete reformulation of static first-order ISD proposed in Liu *et al.* (2021) on each stage of the scenario tree, leading to a reformulation of MISD.

Consider for every t the sorted benchmark series $Y_{n_{[1]}} \leq Y_{n_{[2]}} \leq \cdots \leq Y_{n_{[\#\mathcal{N}_t]}}$ leading to the labeling set denoted by $\mathcal{N}_{[t]} = \{n_{[1]} \leq \cdots \leq n_{[\#\mathcal{N}_t]}\}$. Then, for every $\beta_t \in \mathbb{R}$ there exists $1 \leq l_t < \#\mathcal{N}_t$ such that $Y_{n_{[l_t]}} \leq \beta_t < Y_{n_{[l_t+1]}}$, for nodal values contiguous to l_t. We call l_t the reference integer nodal label on stage $t \in \mathcal{T}$.

We have, as a result, the partition $\mathcal{N}_{[t]} = \mathcal{N}^1_{[t]} \cup \mathcal{N}^2_{[t]}$ with $\mathcal{N}^1_{[t]} = \{n_{[1]} \leq \cdots \leq n_{[l_t]}\}$ and $\mathcal{N}^2_{[t]} = \{n_{[l_t+1]} \leq \cdots \leq n_{[\#\mathcal{T}]}\}$. Such partition, following Luedtke (2008) and Liu *et al.* (2021), is induced by the benchmark values $\{Y_n\}$, $n \in \mathcal{N}_t$. In detail, we have

3.1.1. MISD-1 feasibility. We require $X_n \succeq_1 Y_n$ for nodes in $\mathcal{N}^1_t$, and $X_n \succeq_2 Y_n$ for nodes in $\mathcal{N}^2_t$. On the left-hand side of l_t, we adopt a binary variable

$$c_{d,n} \in \{0,1\}, \quad d \in \mathcal{N}^1_t, n \in \mathcal{N}_t, \quad t \in \mathcal{T} \tag{4}$$

to specify whether the value of X_n is greater or equal to Y_d, for all $d \in \mathcal{N}^1_t - \{n_{[1]}\}$, $n \in \mathcal{N}_t$. Then MISD-1 requires:

$$X_n + Mc_{d,n} \geq Y_d, \quad d \in \mathcal{N}^1_t - \{n_{[1]}\}, \quad n \in \mathcal{N}_t, t \in \mathcal{T}, \tag{5a}$$

$$\sum_{n \in \mathcal{N}_t} q_n c_{d,n} \leq \sum_{s \in \mathcal{N}^1_t,\ Y_s < Y_d} p_s, \quad d \in \mathcal{N}^1_t \cup \{n_{[l_t+1]}\}, \quad t \in \mathcal{T}. \tag{5b}$$

Here, $M >> 0$ is a sufficiently large number such that constraint (5a) guarantees $c_{d,n} = 1$ when $X_n < Y_d$, while constraint (5b) restricts the number of samples when $X_n < Y_d$. And $p_d = P(Y_t = Y_d)$ is the relative frequency of the value Y_d in a sample $Y_{n_{[1]}} \leq Y_{n_{[2]}} \leq \cdots \leq Y_{n_{[\#\mathcal{N}_t]}}$ to be specified exogenously.

A set of constraints to rule MISD-1 in the neighbourhood of the reference point β_t is needed in the form of a mixed integer constraint:

$$X_n + Mc_{n_{[l_t+1]},n} \geq \beta_t, \quad n \in \mathcal{N}_t, \quad t \in \mathcal{T}, \tag{6a}$$

$$\sum_{n \in \mathcal{N}_t} q_n c_{n_{[1]},n} \leq P(Y_t \leq \beta_t), \quad t \in \mathcal{T}, \tag{6b}$$

$$X_n - M(1 - c_{n_{[1]},n}) \leq \beta_t, \quad n \in \mathcal{N}_t, \quad t \in \mathcal{T}. \tag{6c}$$

Moreover, we have restrictions on the lower bound for X_n induced by the smallest sampled value of the benchmark:

$$X_n \geq Y_{n_{[1]}}, \quad n \in \mathcal{N}_t, \quad t \in \mathcal{T}. \tag{7}$$

Above the reference point β_t we require an SSD relationship for any node in $\mathcal{N}^2_t$. Following Dentcheva and Ruszczynski (2003) and Liu *et al.* (2021), MISD-1 implies:

$$Y_d - X_n \leq a_{d,n}, \quad d \in \mathcal{N}^2_t, \quad n \in \mathcal{N}_t, \quad t \in \mathcal{T}, \tag{8a}$$

$$\sum_{n \in \mathcal{N}_t} q_n a_{d,n} \leq \sum_{d_1 \in \mathcal{N}_t} p_{d_1}(Y_d - Y_{d_1})_+, \quad d \in \mathcal{N}^2_t, \quad t \in \mathcal{T}, \tag{8b}$$

$$\beta_t - X_n \leq a_{n_{[l_t]},n}, \quad n \in \mathcal{N}_t, \quad t \in \mathcal{T}, \tag{8c}$$

$$\sum_{n \in \mathcal{N}_t} q_n a_{n_{[l_t]},n} \leq \sum_{d \in \mathcal{N}_t} p_d(\beta_t - Y_d)_+, \tag{8d}$$

$$a_{d,n} \geq 0, \quad d \in \{n_{[l_t]}\} \cup \mathcal{N}^2_t, \quad n \in \mathcal{N}_t, \quad t \in \mathcal{T}. \tag{8e}$$

We can conclude that, given the tree processes $\{X_n\}$ and $\{Y_n\}$, for $n \in \mathcal{N}_t$, $t \in \mathcal{T}_0$ and a series of deterministic reference points $\beta_t \in \mathbb{R}$, we can reformulate the stage-wise ISD constraint of first-order between X_n and Y_n as (4), (5a)–(5b), (6a)–(6c), (7), (8a).

3.1.2. MISD-2 feasibility. In a similar fashion, we propose the following reformulation of the stage-wise second-order ISD constraints between $\{X_n\}$ and $\{Y_n\}$, discriminating in each stage between SSD conditions left of β_t and TSD conditions on its right.

On the left-hand side of l_t, we have the SSD constraints similar to (8a):

$$X_n \geq Y_{n_{[1]}}, \quad n \in \mathcal{N}_t, \quad t \in \mathcal{T}, \tag{9a}$$

$$Y_d - X_n \leq a_{d,n}, \quad d \in \mathcal{N}_t, \quad n \in \mathcal{N}_t, \quad t \in \mathcal{T}, \tag{9b}$$

$$\sum_{n \in \mathcal{N}_t} q_n a_{d,n} \leq \sum_{d_1 \in \mathcal{N}_t} p_{d_1}(Y_d - Y_{d_1})_+, \quad d \in \mathcal{N}^1_t, \quad t \in \mathcal{T}, \tag{9c}$$

$$a_{d,n} \geq 0, \quad d \in \mathcal{N}_t, \quad n \in \mathcal{N}_t, \quad t \in \mathcal{T}. \tag{9d}$$

Here, the auxiliary variable $a_{d,n}$ takes values of $a_{d,n} = (Y_d - X_n)_+$ at optimality.

On the right-hand side of l_t, we introduce a linear-quadratic constraint system to guarantee a partial TSD relationship between $\{X_n\}$ and $\{Y_n\}$:

$$\beta_t - X_n \leq \theta_n, \quad n \in \mathcal{N}_t, \quad t \in \mathcal{T}, \tag{10a}$$

$$\sum_{n \in \mathcal{N}_t} q_n \theta_n \leq \sum_{d \in \mathcal{N}_t} p_d(\beta_t - Y_d)_+, \quad t \in \mathcal{T}, \tag{10b}$$

$$\sum_{n \in \mathcal{N}_t} q_n a^2_{d,n} \leq \sum_{d_1 \in \mathcal{N}_t} p_{d_1}(Y_d - Y_{d_1})^2_+, \quad d \in \mathcal{N}^2_t, \quad t \in \mathcal{T}, \tag{10c}$$

$$\sum_{n \in \mathcal{N}_t} q_n \theta^2_n \leq \sum_{d \in \mathcal{N}_t} p_d(\beta_t - Y_d)^2_+, \tag{10d}$$

$$\theta_n \geq 0, \quad n \in \mathcal{N}_t, \quad t \in \mathcal{T}, \tag{10e}$$

$$\sum_{n \in \mathcal{N}_t} q_n X_n \geq \sum_{d \in \mathcal{N}_t} p_d Y_d, \quad t \in \mathcal{T}, \tag{10f}$$

$$X_{n_1} - X_{n_2} \leq c_{n_1,n_2}, \quad n_1, n_2 \in \mathcal{N}_t, \quad t \in \mathcal{T}, \tag{10g}$$

$$\sum_{n_1 \in \mathcal{N}_t} q_{n_1} c_{n,n_1} \leq \sum_{d \in \mathcal{N}_t} p_d a_{d,n} - \sum_{d \in \mathcal{N}_t} p_d Y_d + X_n, \quad n \in \mathcal{N}_t, \quad t \in \mathcal{T} \tag{10h}$$

$$\beta_t - X_n \leq \lambda_n, \quad n \in \mathcal{N}_t, \quad t \in \mathcal{T}, \tag{10i}$$

$$\sum_{n \in \mathcal{N}_t} q_n \lambda_n \leq \sum_{d \in \mathcal{N}_t} p_d(\beta_t - Y_d)_+, \quad t \in \mathcal{T}, \tag{10j}$$

$$c_{n_1,n_2} \geq 0, \quad n_1, n_2 \in \mathcal{N}_t, \quad t \in \mathcal{T}, \tag{10k}$$

$$\lambda_n \geq 0, \quad n \in \mathcal{N}_t, \quad t \in \mathcal{T}. \tag{10l}$$

Here, (10a)–(10e) characterize the dominance relationship $F_3(X, \eta) \le F_3(Y, \eta)$ for $\eta \in \{Y_d \mid d \in \mathcal{N}_t^2\} \cup \{\beta_t\}$. The eq.s (10a)–(10e), however, are not sufficient to enforce the ISD-3 condition. For we consider the constraints (10f)–(10l) to guarantee the inequality $F_3(X, \eta) \le F_3(Y, \eta)$ for $\eta \ge \beta_t$ by checking the derivative of the quadratic polynomials. At the optimum the auxiliary variables satisfy $e_{d,n} = (X_n - Y_d)_+$.

REMARK 2 Under either ISD-1 or ISD-2 feasibility conditions the definition in input of the reference points β_t in each stage implies the enforcement of the appropriate stochastic dominance principles left or right of a quantile q_β of the return distributions. A constant reference point β over the planning horizon would lead to a stage-dependent quantile to be evaluated ex-post relying on the problem solution. On the contrary, a time-varying reference function may lead to a relatively stable control in the tail of the distributions.

3.2. *Scenario-based formulation of McISD constraints*

We consider the reformulation of conditional ISD constraints under the scenario tree framework. The key difference from the stage-wise formulation is that now ISD-k conditions, for $k = 1, 2$, are enforced on every 2-stage sub-tree with associated conditional probabilities. The dominance comparisons, represented by c are computed conditional on each non-leaf node $d \in \mathcal{N}_t, t \in \mathcal{T} - \{T\}$.

In the conditional case, the reference point characterizing the ISD formulation applies to every 2-stage subtree as the random process evolves. The decision maker is in this case assumed to seek an optimal contingency plan along the tree dominating the benchmark *locally*. We show below that, indeed the solution of the resulting constrained stochastic program, when feasible, would imply stage-wise ISD of order 1 or 2, while the converse is not necessarily true. Here, for every node n, excluding the leaf nodes, we shall find the reference points for the distributions specified on the children nodes in the current sub-tree. Denote by $\mathcal{N}_{[n+]} = \{e_{[1]} \le \cdots \le e_{[\#n+]}\}$ the labeling scheme adopted to identify the set of children nodes of $n \in \mathcal{N}_t$. Now, as above, there exists a nodal dependent n_n^* such that $\mathcal{N}_{[n+]}^1 = \{e_{[1]}, \ldots, e_{[n_n^*]}\}$ and $\mathcal{N}_{[n+]}^2 = \{e_{[n_n^*+1]}, \ldots, e_{[\#n+]}\}$.

We have $\mathcal{N}_{[n+]} = \mathcal{N}_{[n+]}^1 \cup \mathcal{N}_{[n+]}^2$ as the partition of $\mathcal{N}_{n+}$ induced by β_n from $n \in \mathcal{N}_t,\ t \in \mathcal{T}$.

3.2.1. McISD-1 feasibility.

By applying the reformulation method in (4)–(8e), to each non-leaf node, we have that McISD-1 constraints between $\{X_n\}$ and $\{Y_n\}$ with respect to the scenario-dependent reference points $\beta_n \in \mathbb{R}$, $n \in \mathcal{N}_t$, $t = 1, \ldots, T-1$, can be reformulated as:

$$X_n + Mc_{e,n} \ge Y_e, \quad e \in \mathcal{N}_{[d+]}^1 - \{e_{[1]}\},$$
$$n \in d+,\ d \in \mathcal{N}_t,\ t \in \mathcal{T} - \{T\}, \tag{11a}$$

$$\sum_{n \in d+} q_n c_{e,n} \le \sum_{s \in \mathcal{N}_{[d+]}^1, Y_n < Y_e} p_s, \quad e \in \mathcal{N}_{[d+]}^1 - \{e_{[n_d^*+1]}\}$$
$$d \in \mathcal{N}_t,\ t \in \mathcal{T} - \{T\}, \tag{11b}$$

$$X_n + Mc_{e_{[n_d^*+1]},n} \ge \beta_d, \quad n \in d+,\ d \in \mathcal{N}_t,\ t \in \mathcal{T} - \{T\}, \tag{11c}$$

$$\sum_{n \in d+} q_n c_{e_{[1]},n} \le P(Y_t \le \beta_d \mid Y_{t-1} = Y_d),$$
$$d \in \mathcal{N}_t,\ t \in \mathcal{T} - \{T\}, \tag{11d}$$

$$X_n - M(1 - c_{e_{[1]},n}) \le \beta_d,$$
$$n \in d+,\ d \in \mathcal{N}_t,\ t \in \mathcal{T} - \{T\}, \tag{11e}$$

$$X_n \ge Y_{e_{[1]}}, \quad n \in d+,\ t \in \mathcal{T} - \{T\}, \tag{11f}$$

$$Y_e - X_n \le a_{e,n},\ e \in \mathcal{N}_{[d+]}^2,$$
$$n \in d+,\ d \in \mathcal{N}_t,\ t \in \mathcal{T} - \{T\}, \tag{11g}$$

$$\sum_{n \in d+} q_n a_{e,n} \le \sum_{e_1 \in d+} p_{e_1}(Y_e - Y_{e_1})_+,$$
$$e \in \mathcal{N}_{[d+]}^2,\ d \in \mathcal{N}_t,\ t \in \mathcal{T} - \{T\}, \tag{11h}$$

$$\beta_d - X_n \le a_{e_{[n_d^*]},n}, \quad n \in d+,\ d \in \mathcal{N}_t,\ t \in \mathcal{T} - \{T\}, \tag{11i}$$

$$\sum_{n \in d+} q_n a_{e_{[n_d^*]},n} \le \sum_{e \in d+} p_e(\beta_d - Y_e)_+,$$
$$d \in \mathcal{N}_t,\ t \in \mathcal{T} - \{T\}, \tag{11j}$$

$$c_{e,n} \in \{0, 1\}, \quad e \in \mathcal{N}_{[d+]}^1,\ n \in d+,\ d \in \mathcal{N}_t,\ t \in \mathcal{T} - \{T\}, \tag{11k}$$

$$a_{e,n} \ge 0, \quad e \in \{e_{[n_d^*]}\} \cup \mathcal{N}_{[d+]}^2,\ n \in d+,\ t \in \mathcal{T} - \{T\}. \tag{11l}$$

Similar to MISD-1 (4)–(8a), the left of β part of McISD-1 has a MILP reformulation shown in (11a)–(11f), while the right of β part has a LP reformulation shown in (11g)–(11l). The only difference from MISD-1 is that McISD asks a group of FSD constraints (11b) and (11d) and group of SSD constraints (11h) and (11j) at each non-leaf node, where MISD requires only one group of FSD and SSD constraints at each stage. Thus, we shall take account of all children nodes $d+$ of each non-leaf node d in (11b), (11d), (11h), (11j). In contrast, MISD takes into account all nodes $\mathcal{N}_t$ for each constraints in (5b), (6b), (8b), (8d).

3.2.2. McISD-2 feasibility.

We can reformulate conditional ISD constraints of the second-order between two random tree processes $\{X_n\}$ and $\{Y_n\}$ with respect to a series of scenario-dependent reference points $\beta_n \in \mathbb{R}$, $n \in \mathcal{N}_t$, $t = 1, \ldots, T-1$, as a linear-quadratic constrained system:

$$X_n \ge Y_{e_{[1]}}, \quad e, n \in d+,\ d \in \mathcal{N}_t,\ t \in \mathcal{T} - \{T\}, \tag{12a}$$

$$Y_e - X_n \le a_{e,n}, \quad e, n \in d+,\ d \in \mathcal{N}_t,\ t \in \mathcal{T} - \{T\}, \tag{12b}$$

$$\sum_{n \in d+} q_n a_{e,n} \le \sum_{e_1 \in d+} p_{e_1}(Y_e - Y_{e_1})_+,$$
$$e \in \mathcal{N}_{[d+]}^1,\ d \in \mathcal{N}_t,\ t \in \mathcal{T} - \{T\}, \tag{12c}$$

$$a_{e,n} \ge 0, \quad e \in d+,\ n \in d+,\ d \in \mathcal{N}_t,\ t \in \mathcal{T} - \{T\}, \tag{12d}$$

$$\beta_d - X_n \le \theta_n, \quad n \in d+,\ d \in \mathcal{N}_t,\ t \in \mathcal{T} - \{T\}, \tag{12e}$$

$$\sum_{n \in d+} q_n \theta_n \le \sum_{e \in d+} p_e(\beta_d - Y_e)_+,$$

$$d \in \mathcal{N}_t, \;\; t \in \mathcal{T} - \{T\}, \tag{12f}$$

$$\sum_{n \in d+} q_n a_{e,n}^2 \le \sum_{e_1 \in d+} p_{e_1} (Y_e - Y_{e_1})_+^2,$$

$$e \in \mathcal{N}_{[d+]}^2 \;\; d \in \mathcal{N}_t, \;\; t \in \mathcal{T} - \{T\}, \tag{12g}$$

$$\sum_{n \in d+} q_n \theta_n^2 \le \sum_{e \in d+} p_e (\beta_d - Y_e)_+^2, \quad d \in \mathcal{N}_t, \;\; t \in \mathcal{T} - \{T\}, \tag{12h}$$

$$\theta_n \ge 0, \quad n \in d+, \;\; d \in \mathcal{N}_t, \;\; t \in \mathcal{T} - \{T\}, \tag{12i}$$

$$\sum_{n \in d+} q_n X_n \ge \sum_{e \in d+} p_e Y_e, \quad d \in \mathcal{N}_t, \;\; t \in \mathcal{T} - \{T\}, \tag{12j}$$

$$X_{n_1} - X_{n_2} \le c_{n_1,n_2}, \quad n_1, n_2 \in d+, \;\; d \in \mathcal{N}_t, \;\; t \in \mathcal{T} - \{T\}, \tag{12k}$$

$$\sum_{n_1 \in d+} q_{n_1} c_{n,n_1} \le \sum_{e \in d+} p_e a_{e,n} - \sum_{e \in d+} p_e Y_e + X_n,$$

$$n \in d+, \;\; d \in \mathcal{N}_t, \;\; t \in \mathcal{T} - \{T\}, \tag{12l}$$

$$c_{n_1,n_2} \ge 0, \quad n_1, n_2 \in d+, \;\; d \in \mathcal{N}_t, \;\; t \in \mathcal{T} - \{T\}. \tag{12m}$$

Here, (12a)–(12d) characterize an LP reformulation of SSD left of β part, and (12e)–(12m) gives a quadratically constrained reformulation of TSD at right of β part.

Reformulations (11a) and (12a) clarify the methodological implications of imposing the ISD-k partial order in every subtree of the stochastic program.

4. Multistage portfolio selection with dynamic interval SD

We formulate a multi-period portfolio problem based on the ISD principles discussed in sections 2 and 3. As explained in Liu *et al.* (2021) this approach, specifically in the case of ISD-1 problems, is consistent with an investor following a bi-criteria decision rule: as expected utility maximizer, with concave and increasing utility function on the SSD domain, and as requiring a tight risk control on the FSD domain.

We assume $i = 1, 2, \dots, I$ risky assets and one risk-free asset for the cash account and an investor seeking an optimal investment strategy over the horizon $\mathcal{T}$ Given an initial endowment and an exogenous benchmark strategy, the investor seeks the maximization of the expected terminal wealth while requiring over time the ISD conditions to be enforced.

The investment strategy evolves in each stage always assuming the cash return $r_{0,t}$ known at the beginning of a stage and the asset returns $\mathbf{r}_t = [r_{1,t}, \dots, r_{I,t}]^\top$, as the return of the benchmark $\{Y_t\}$, realized only at the end of a stage. The portfolio manager aims at deriving an optimal *non anticipative* strategy stochastically dominating over time the benchmark strategy with order ISD-$k.\beta$.

At the initial stage, $\mathbf{u}_0, \mathbf{b}_0, \mathbf{s}_0$ define the unique decision *under full uncertainty*, also referred to as *here-and-now* or *implementable* root node decision. After the root node, the portfolio allocations and selling or buying decisions $\mathbf{u}_t, \mathbf{b}_t, \mathbf{s}_t$ define the optimal controls. For $n \in \mathcal{N}_t$ nodal decisions are denoted by $\mathbf{u}_n, \mathbf{b}_n, \mathbf{s}_n, u_{0,n}$ for $t = 0, 1, \dots, T-1$. No portfolio rebalancing is allowed on leaf nodes, at the end of the planning horizon, in stage T for $n \in \mathcal{N}_T$.

We have the following portfolio revision equations for $t \in \mathcal{T} - \{T\}$:

$$\mathbf{u}_0 = \bar{\mathbf{u}} + \mathbf{b}_0 - \mathbf{s}_0, \quad n = n_0, \;\; t = 0 \tag{13}$$

$$\mathbf{u}_n = \mathbf{r}_n^\top \mathbf{u}_{n-} + \mathbf{b}_n - \mathbf{s}_n, \quad n \in \mathcal{N}_t, \;\; t \in \mathcal{T} - \{T\}, \tag{14}$$

where $\mathbf{r}_n^\top \mathbf{u}_{n-} = [r_{1,n} u_{1,n-}, \dots, r_{I,n} u_{I,n-}]^\top = \sum_{i=1}^I r_{i,n} u_{i,n-}$ in (14), clarifies the rebalancing process, based first on the assessment of the current portfolio gross return r_n and only after on buying or selling decisions. Equation (13) defines the first stage, root-node re-balancing equation. Equation (14) refers to portfolio revisions over the planning horizon up to the beginning of the last stage. In the given notation the return vector $\mathbf{r}_n$ is a gross return, specified as the ratio between two end-of-period prices. We assume that the investor cannot collect external funds and thus the investment strategy is fully self-financed. When rebalancing however she/he will face transaction costs c_b and c_s for buying and selling decisions, respectively. Based on an initial cash surplus $\bar{u}_0$, the following equations will then define the cash balance evolution at the root node and then over the investment horizon:

$$u_{0,0} = \bar{u}_0 - (1 + c_b)\|\mathbf{b}_0\|_1 + (1 - c_s)\|\mathbf{s}_0\|_1,$$

$$u_{0,n} = r_{0,n-} u_{0,n-} - (1 + c_b)\|\mathbf{b}_n\|_1 + (1 - c_s)\|\mathbf{s}_n\|_1,$$

$$n \in \mathcal{N}_t, \;\; t \in \mathcal{T}.$$

For $n \in \mathcal{N}_t$, stage $t = 1, \dots, T$, we specify the portfolio value process or wealth process, as:

$$X_n = \mathbf{r}_n^\top \mathbf{u}_{n-} + r_{0,n-} u_{0,n-}, \quad n \in \mathcal{N}_t, \;\; t \in \mathcal{T}.$$

The portfolio manager is assumed to maximize the expected wealth at the end of the planning horizon under the ISD constraints. The optimization problem with a MISD constraint reads:

$$\max_{\mathbf{u},\mathbf{b},\mathbf{s}} \quad \sum_{n \in \mathcal{N}_T} p_n X_n \tag{15a}$$

$$\text{s.t.} \quad \mathbf{u}_0 = \bar{\mathbf{u}} + \mathbf{b}_0 - \mathbf{s}_0, \tag{15b}$$

$$\mathbf{u}_n = \operatorname{diag}(\mathbf{r}_n)\mathbf{u}_{n-} + \mathbf{b}_n - \mathbf{s}_n, \quad n \in \mathcal{N}_t, \;\; t \in \mathcal{T} - \{T\}, \tag{15c}$$

$$u_{0,0} = \bar{u}_0 - (1 + c_b)\|\mathbf{b}_0\|_1 + (1 - c_s)\|\mathbf{s}_0\|_1, \tag{15d}$$

$$u_{0,n} = r_{0,n-} u_{0,n-} - (1 + c_b)\|\mathbf{b}_n\|_1 + (1 - c_s)\|\mathbf{s}_n\|_1,$$

$$n \in \mathcal{N}_t, \;\; t \in \mathcal{T} - \{T\}, \tag{15e}$$

$$X_n = \mathbf{r}_n^\top \mathbf{u}_{n-} + r_{0,n-} u_{0,n-}, \quad n \in \mathcal{N}_t, \;\; t \in \mathcal{T}, \tag{15f}$$

$$X_{\mathcal{N}_t} \succeq_{(k,\beta_t)} Y_{\mathcal{N}_t}, \quad t = 0, 1, \dots, T-1, \tag{15g}$$

$$\mathbf{u}_n \ge 0, \quad \mathbf{b}_n \ge 0, \quad \mathbf{s}_n \ge 0, \quad u_{0,n} \ge 0, \quad n \in \mathcal{N}_t,$$

$$t = 0, 1, \dots, T-1. \tag{15h}$$

Constraints (15h) are the non negativity constraints restricting portfolio evolution. Constraints (15g) impose the stagewise ISD conditions $\succeq_{(k,\beta|\mathcal{F})}$ on the optimal strategy under $\mathcal{F} = \mathcal{F}_0$ for given β_t, $t = 0, 1, \dots, T-1$.

Following the feasibility condition (15g), for the McISD-k problem formulation we have

$$\max_{\mathbf{u},\mathbf{b},\mathbf{s}} \sum_{n\in\mathcal{N}_T} p_n X_n \tag{16a}$$

$$\text{s.t.} \quad X_{n+} \succeq_{(k,\beta_n|n)} Y_{n+}, \quad n \in \mathcal{N}_t, \ \ t = 0, 1, \ldots, T-1, \tag{16b}$$

$$(15\text{b})\text{–}(15\text{f}), (15\text{h}) \tag{16c}$$

Here, X_{n+} and Y_{n+} induce the discrete distribution across all children nodes of n, and the probability associated with each sample corresponds to the conditional branching probability along the scenario tree. The constraint (16b) entails the conditional MISD $\succeq_{\{(k,\beta\,|\,\mathcal{F})\}}$ for $\mathcal{F} = \mathcal{F}_n$.

Under any specification of the ISD conditions, the model development requires the generation of the full set of random coefficients characterizing problem (15a) and, specifically in our setting, the definition of the benchmark process to be stochastically dominated. For this purpose We introduce here next the statistical model and in the following sections 4.2 and 4.3 clarify how that would lead to the construction of a scenario tree for the assets and the benchmarks.

4.1. Asset return model

We consider a statistical model based on an assumption of common dependence on the US inflation to be denoted by π and on the 3-month risk-free Treasury bill interest rate r_0: we denote the associated processes by π_t and $r_{0,t}$. Net monthly returns of risky assets are computed, for $i \in \mathcal{I}$ as $r_{i,t} = \frac{v_{i,t}}{v_{i,t-1}} - 1$ where $v_{i,t}$ refers to the *exchange-traded-fund* (ETF) value at time t. We assume an auto-regressive model of order 1 for π_t and $r_{0,t}$ with correlated residuals:

$$\pi_t = \alpha_0 + \alpha_1 \pi_{t-1} + \alpha_2 r_{0,t-1} + e_t^{\pi} \tag{17a}$$

$$r_{0,t} = \beta_0 + \beta_1 \pi_{t-1} + \beta_2 r_{0,t-1} + e_t^{r}. \tag{17b}$$

Asset returns are then assumed to depend jointly on those two state variables and we test for statistical lagged 1 dependence between asset returns, according to the following model:

$$r_{i,t} = \lambda_{0,i} + \lambda_{1,i}\pi_t + \lambda_{2,i} r_{0,t} + \sum_{j\in I} \theta_{j,i} r_{j,t-1} + e_t^i, \qquad i = 1, 2, \ldots, I. \tag{18}$$

Further to assets' co-dependence in the mean function, we assume correlated residuals, resulting into a vector autoregressive model (18). We perform OLS estimation on this model specification and then apply a canonical scenario generation algorithm. In a financial context, we enforce arbitrage-free conditions along the tree following the established results by Klaassen (2002) and Geyer *et al.* (2010). The full set of statistical estimates, the algorithm for scenario generation and it's outputs are presented in appendix 4. A set of out-of-sample results is presented in section 5, based on estimation updates from 12-year data histories and input in the scenario tree algorithm to formulate and solve multistage stochastic recourse problems always out-of-sample and under uncertainty. The procedure is repeated to span an extended period with quarterly steps to validate the adopted methodology.

4.2. Scenario generation

The state-of-the-art on scenario generation is continuously evolving, see Consigli *et al.* (2016), Barro *et al.* (2022) and Kaut (2021).

Scenario generation is performed for given input tree structure by first simulating the tree processes for the inflation and the 3-month interest rate and then by deriving all assets' return scenarios according to models (17a) and (18) and the algorithm 4. The benchmark scenarios are then derived according to the model specified in section 4.3.

We assume an asset universe characterized by equity and fixed income ETFs to span alternative investors' risk profiles.

Scenario tree generation is based on a Monte Carlo sampling approach to derive scenario paths of the assets' return vector process from the root to the leaf nodes, according to a *branching sequence* specified as $\{N_1 - N_2 - \cdots - N_T\}$ and resulting into a symmetric tree with N_t to denote the branching degree common to all nodes in stage t, leading to $\Pi_{t=1}^{T} N_t$ scenarios at the end of the planning horizon. We present the algorithm for generating an arbitrage-free scenario tree in appendix 4. Here next we limit ourselves to provide evidence of the terminal cdf's generated for every asset in table 3.

We report here next the statistical evidence and the set of terminal distributions for the asset universe adopted in the case study in section 5, shown in table 3.

For statistical validation we present in table 4 the monthly mean return and standard deviations together with their ratio (Sharpe ratio), to convey the convergence, sufficient for our purposes, of the asset returns distributions to the historical first and second moments. We generate a very rich scenario tree of 24500 scenarios and branching $[50 - 10 - 7 - 7]$ with the purpose of validating the implemented scenario algorithm. A data set of monthly returns from January 2002 to June 2023 is considered for model estimation. We can find that the asset

Table 3. Bond and equity ETFs adopted in the computations. SHY, VCSH and VCIT are fixed-income constant maturity ETFs.

US bond (short)	iShares 1-3 Year Treasury Bond ETF	SHY
US Corporate bond (short)	Vanguard Short-Term Corporate Bond Index Fund	VCSH
US Corporate bond (middle)	Vanguard Medium-Term Corporate Bond Index Fund	VCIT
US stock (Utilities)	Utilities Sector SPDR ETF	XLU
US stock (Health)	Health Care Sector SPDR ETF	XLV
US stock (Energy)	Energy Sector SPDR ETF	XLE

Note: XLU, XLV and XLE are equity sector indices from the SP500 index pool.

Table 4. Monthly asset returns descriptive statistics mean μ, standard deviation σ and Sharpe ratios.

	Original data			Scenario output		
	μ	σ	SR	μ	σ	SR
SHY	0.000036	0.002896	0.012318	0.000033	0.002824	0.011619
VCSH	0.000204	0.006584	0.030942	0.000203	0.006541	0.030965
VCIT	0.000765	0.015378	0.049749	0.000964	0.014425	0.066814
XLU	0.006640	0.038831	0.170985	0.006502	0.038072	0.170783
XLV	0.010508	0.038548	0.272592	0.010057	0.037000	0.271805
XLE	0.005992	0.078634	0.076202	0.006502	0.078644	0.082677

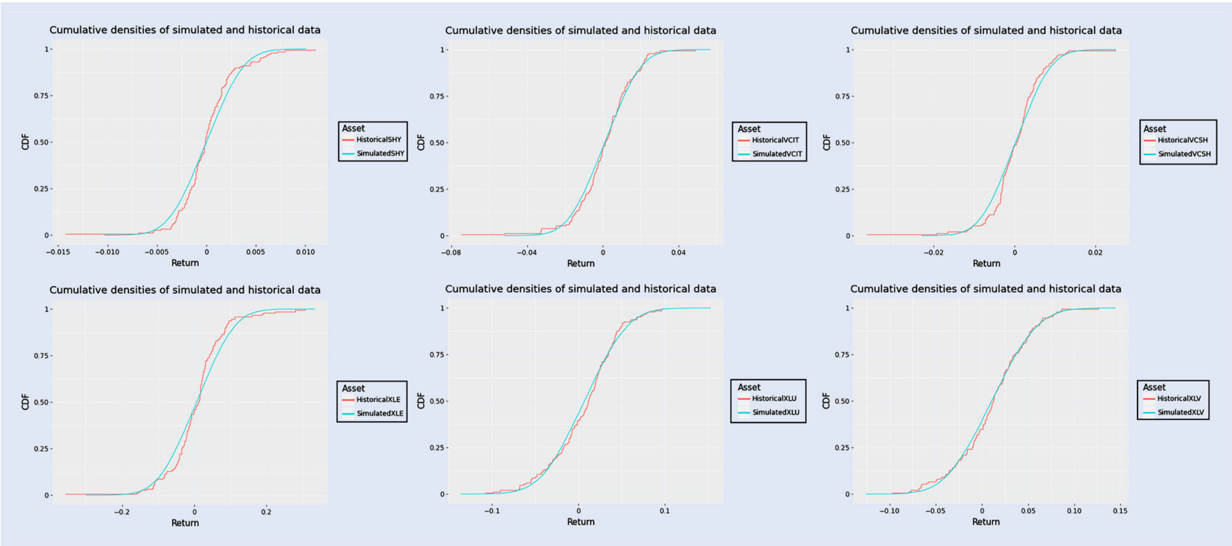

Figure 3. Scenario-based (green line) and historical (red line) CDFs at the end of the investment horizon. Left to right and top to bottom, the ETFs: SHY, VCSH, VCIT, XLU, XLV, XLE.

statistics from scenario generation and original data sample match sufficiently well from table 4 and figure 3.

We consider those 6 assets to derive three possible benchmark strategies in the next section.

We show in the numerical results in section 5 that, given the feasibility conditions specified for MISD or McISD problems, a trade-off issue arises: on one side we need a sufficient set of realizations of the portfolio and the benchmark processes along the scenario tree to derive consistent evidence on the ISD partial order in every stage. On the other an increasing number of scenarios and stages would lead to a *curse of dimensionality* problem and numerical instability, thus jeopardizing the solution of the problem. We refer shortly to the former as *MISD requirement* and to the latter as *SP requirement*. We see in the computational section that indeed such trade-off problem is particularly relevant for MISD instances.

4.3. Benchmark policy

We consider in the numerical results three possible benchmark strategies constructed relying on the assets in table 3. A benchmark investment strategy may, in general but needs not, belong to the set of attainable investment policies supported by the specification of the asset universe, see Moriggia *et al.* (2019) and Liu *et al.* (2021). In this study we verify the model performance against (1) an equally weighted portfolio based on all assets, (2) on a *fixed income portfolio* based on the three bond ETFs, and (3) against an *equity portfolio*, based on the three equity ETFs. The portfolio manager seeks the maximization of the expected terminal portfolio value under the constraint to stochastically dominate the benchmark return distributions in every stage for a given ISD-$k.\beta$ order.

Once the asset returns' scenarios have been generated, the following model is considered to generate the benchmark returns is considered, for $t \in \mathcal{T}$, $n \in \mathcal{N}_t$, $j = \{1, 2, 3\}$:

$$y_n^j = w^j \sum_{i \in \mathcal{I}_1} \frac{r_{i,n}}{\#\mathcal{I}_1} + (1 - w^j) \sum_{i \in \mathcal{I}_2} \frac{r_{i,n}}{\#\mathcal{I}_2} \tag{19}$$

where $\mathcal{I}_1$ includes all equity assets and $\mathcal{I}_2$ includes all fixed-income assets. The model is validated against the three benchmarks: equally weighted y_n^1 based on $w^1 = 0.5$, equal weights but including only fixed income assets: y_n^2 with $w^2 = 1$, and finally y_n^3 with equal weights but only equity for $w^3 = 0$.

The benchmark portfolio, instrumental to derive the ISD conditions will follow, for $n \in \mathcal{N}_t$, $t \in \mathcal{T}$, with $Y_0 = \bar{u}$ the same initial portfolio value of the optimization problem:

$$Y_n^j = Y_{n-}^j (1 + y_n^j). \tag{20}$$

Depending on the benchmark returns, for $j = 1, 2, 3$ we can generate different probability distributions, that, for given β, the portfolio manager intends to dominate under the stage-wise or conditional ISD formulations: $X_n \succeq_{\{(k,\beta) \mid \mathcal{F}\}} Y_n^j$ as postulated in (15g).

We present in figure 4 the return distributions generated at the end of every stage by a unit investment either in all assets

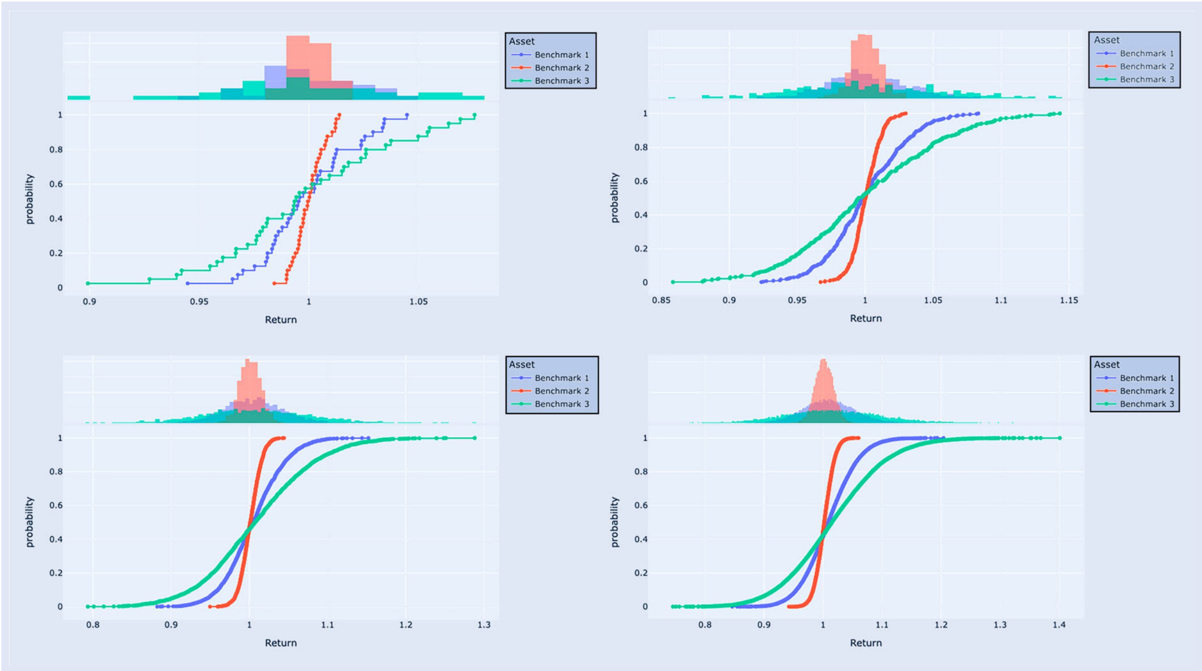

Figure 4. Comparative CDFs, branchings (40, 10, 7, 7) benchmarks 1, 2 and 3 CDFs in each stage left to right, top to bottom: stages 1, 2, 3 and 4.

Table 5. Simulated mean μ_j, standard deviation σ_j and Sharpe ratios SR_j of benchmark strategies $j = 1{,}2$ and 3 over a 1 year horizon with quarterly stages.

stage	μ_1	σ_1	SR_1	μ_2	σ_2	SR_2	μ_3	σ_3	SR_3
1	0.01081	0.02225	0.27004	0.00320	0.00907	0.35279	0.01842	0.04004	0.46011
2	0.01439	0.03270	0.24409	0.00438	0.01318	0.33240	0.02443	0.05899	0.41417
3	0.01539	0.04001	0.21072	0.00494	0.01614	0.30651	0.02585	0.07306	0.35382
4	0.01840	0.04662	0.21465	0.00591	0.01862	0.31777	0.03094	0.08575	0.36088

(benchmark 1, equally weighted) or in bonds only (benchmark 2), or in equity only (benchmark 3), based on a rich scenario tree with branching $[40-10-7-7]$ resulting into 19600 scenarios at the end of the year. We summarise in table 5 a set of statistics associated with the distributions in figure 4.

The three distributions in figure 4 are considered in the largest test problem, based on 19600 scenarios discussed in section 5. As mentioned above, those represent three attainable benchmarks taken as representative of a conservative (benchmark 2), a balanced (benchmark 1) or a risky (benchmark 3) approach to portfolio management.

4.4. Implementation details

We summarize the implementation and methodological steps leading to the solution of a multistage ISD-constrained portfolio optimization problem. This develops along three phases:

(i) *Inputs and scenario generation*: the input information includes the asset universe and the benchmark specification, the number of stages and their time distribution, the tree structure, the statistical estimates, and the initial conditions for the return processes and the benchmark. Scenarios were generated locally with the Python open-source language. Alternatively, this step can rely on MATLAB or R environments.

(ii) Based on the selection of the $k.q_\beta$ order and the MISD rather than the McISD constraints specification, the problem formulation was based on Python, and the model generation on the Gurobi 10.0 tool. For a given set of scenarios and problem specifications, this tool generates a standard to be submitted to a solution algorithm. This phase, typically time-intensive, was already performed on the *high-performance computing* (HPC) platform described next.

(iii) *Solution and output analysis*: in our application, all problems were formulated and solved on an HPC cluster with an Intel(R) Xeon(R) Gold 6230R CPU @ 2.10 GHz, featuring 52 physical cores and 52 logical processors. Gurobi v10.0.0 was adopted to solve the stochastic programs with automatic selection of the solution algorithm: mixed integer linear programming, linear programming, or quadratically constrained programming for ISD-2 problems. All results were then collected and analyzed within the main Python programming environment.

5. Numerical results

We present in this section an extended set of results to analyse the computational and financial implications induced by a multistage SD formulation with all the details discussed so far. In particular, we intend to analyse:

- The computational cost induced by an increasing number of scenarios over a 1-year, 4-stage problem under either one of the two possible multistage ISD formulations.
- The evidence collected primarily for comparative purposes from a representative case problem under different MISD-k and McISD-k specifications. Specifically related to the root node optimal portfolio allocation, the optimal objective, portfolio diversification and the stochastic dominance orders.
- A comprehensive set of in- and out-of-sample results for large scale problems spanning the period from January 2021 to December 2023. The collected evidence helps supporting the adoption of a multistage framework and validating the proposed interval-based SD approach. Specific attention is devoted to assess the advantages of a multiperiod relative to a single period setting and differentiate an ISD-1 from an SSD multistage instances.

The optimization problem (15a) is formulated with an asset universe defined in section 4.2 based on bond and equity ETFs: the acronyms SHY, VCSH and VCIT are for fixed-income constant maturity ETFs, while XLU, XLV and XLE correspond to equity sector indices from the SP500 index pool. We provide numerical results for the 3 attainable benchmark strategies specified in equation (19), to define a balanced, a conservative and a risky benchmark, respectively. The investment horizon is 1 year with 4 quarterly stages for the multistage problems, we present also results for single stage problems with 3 month planning horizon. In this latter case, adopting for comparative purposes, exactly the same scenario set considered in the first of the multistage problem.

We adopt the following convention to characterize a problem instance: $\mathcal{L}(S_g, k.q_\beta, m, j, T, t_0)$ with: S_g to specify a scenario tree or the number of scenarios as explained next, k for the ISD order jointly with q_β for the quantile associated with the reference point, $m = 1$, 2 for MISD and McISD, respectively, $j = 1$, 2, 3 for the adopted benchmark: 1 for the equally weighted portfolio, 2 for the fixed-income-only and 3 for the equity-only benchmarks. $T = \{0.25, 1\}$ for the planning horizon, where 0.25 is a quarter of a year for single period problems, while 1 is the reference 1 year multistage horizon and, finally, t_0 refers to the timing of the root node decision, always in the face of uncertainty.We remind that a $k.q_\beta$ label would imply that the kth SD order is enforced on the $1 - q_\beta$ benchmark return domain and the $(k + 1)$th order on the remaining domain partition identified by the q_β quantile.

The following scenario trees are supported in the case-study with results presented either in this section 5 or in appendix: $\{S_1, S_2, S_3, S_4, S_5, S_6\} = \{1296, 1512, 1764, 5880, 7840, 19600\}$ scenarios with associated branching structures $[N_1 - N_2 - N_3 - N_4] = [6 - 6 - 6 - 6]$, $[7 - 6 - 6 - 6]$, $[7 - 7 - 6 - 6]$, $[15 - 8 - 7 - 7]$, $[20 - 8 - 7 - 7]$ and $[40 - 10 - 7 - 7]$, respectively.

The notation $\mathcal{L}(S_g^*, k.q_\beta^*, m^*, j^*, T^*, t_0^*)$ defines the set of test-problems spanning all the settings above or selected subsets. Notice that the S_1 scenario tree, based on an asset universe of 6 assets, represent the minimal branching structure complying with the arbitrage-free requirements for scenario generation, discussed in section 4.2.

The data-set considered in this study spans the January 2009-December 2023 period: in particular for one year planning horizon, say from January to December 2021, we rely for statistical estimation on monthly data over the preceding 12 years. A unit initial wealth is assumed in every problem instance and we span the 2021–2023 period with quarterly updated moving windows, all relying on 12 years of historical data for statistical estimation and 1 year out-of-sample scenarios.

We present in section 5.4 solution evidence associated with 12 problem instances spanning from January 2021 to December 2023, Qj to denote the jth quarter of every year, with $t_0 = \{2021Q1, 2021Q2, 2021Q3, 2021Q4, 2022Q1, \ldots, 2023Q4\}$. In- and out-of-sample results are considered. In that section we do also present the results associated with single period models with rebalancing at the end of every quarter.

5.1. Problem dimension and solution evidence

All the problems are solved on the HPC cluster with a Gurobi interface for solution. Either MISD or McISD formulation leads in case of $k = 1$ to mixed-integer stochastic linear programs solved by Gurobi's branch and bound methods. SSD constrained problems are linear programs that are solved by Gurobi's primal or dual simplex methods. Finally, the ISD-2 case is a quadratically constrained problem solved by Gurobi's barrier method.

We clarify in great detail in appendix 3 the coefficients specification and variables leading to a relevant difference in size and computational feasibility of problems formulated under McISD rather than enforcing stage-wise MISD conditions. Here, we verify directly those implications on a set of test-problems in table 6.

The evidence in table 6 allows several remarks.

- The most striking evidence comes from the problem dimension and solution times of multistage conditional versus stage-wise ISD formulations. McISD problems are all solved to optimality and in seconds.
- MISD formulations are computationally far more challenging than the McISD formulations. We recall that SD conditions established in the latter imply the same conditions on the former, from Lemma 1.
- Multiperiod SSD problems are all solved under conditional or stagewise ISD with relatively fast solution times. We show in the following sections that indeed the optimal investment strategies and optimal values change depending on the adopted formulation.
- Despite the employment of the HPC platform we see that we were unable to solve any MISD problem with stochastic dominance order above 2.0, even for the 1296 scenario problem. In this case due to the large dimension and the quadratic constraints the risk of numerical instability in presence

Table 6. Numerical evidence of multi-stage ISD problems from ISD-1.75 to ISD-2.25 of increasing dimension.

	MISD-1.75			MISD-2.0			MISD-2.25		
No scenarios	1296	1512	1764	1296	1512	1764	1296	1512	1764
Branching	6^{*4}	$7*6^{*3}$	$7^{*2}*6^{*2}$	6^{*4}	$7*6^{*3}$	$7^{*2}*6^{*2}$	6^{*4}	$7*6^{*3}$	$7^{*2}*6^{*2}$
Problem class	MILP	MILP	MILP	LP	LP	LP	QCP	QCP	QCP
No rows	1760265	2389563	3245019	1755591	2384112	3238665	3487873	4741028	6445605
No columns	1760257	2389553	3244995	3486307	4739201	6443463	3486307	4739201	6443463
Nonzeroes	15661921	21295543	28959619	15619963	21246592	28902541	44514074	61079907	83147422
Binary vars	433458	589701	802396	0	0	0	0	0	0
Quadratic vars	0	0	0	0	0	0	433458	589701	802445
Solution CPU (secs)	8192.7672	24481.0462	64365.16	2442.46	2817.57	3630.64	∞	∞	∞
	McISD-1.75			McISD-2.0			McISD-2.25		
No scenarios	1296	1512	1764	1296	1512	1764	1296	1512	1764
Branching	6^{*4}	$7*6^{*3}$	$7^{*2}*6^{*2}$	6^{*4}	$7*6^{*3}$	$7^{*2}*6^{*2}$	6^{*4}	$7*6^{*3}$	$7^{*2}*6^{*2}$
Problem class	MILP	MILP	MILP	LP	LP	LP	QCP	QCP	QCP
No rows	42495	49583	57871	1755591	2384112	3238665	52078	60770	70962
No columns	41977	48977	57153	38869	45351	52925	49747	58049	67779
Non-zeroes	197401	230359	269041	155443	181408	211963	266036	310495	362904
Binary vars	4662	5439	6342	0	0	0	0	0	0
Quadratic vars	0	0	0	0	0	0	777	906	1053
Solution CPU (secs)	26.0955	29.3431	32.45	10.49	11.37	12.81	15.23	18.58	21.32

of singularities in the coefficient matrix is high. The stage-wise formulation of multistage SD problems appears viable only for first-order ISD, $k = 1$. See the evidence below.

In section 5.2 we go more in detail on the set of testproblems $\mathcal{L}(S_1, k.q^*_\beta, m^*, 2, 1, 2022Q1)$ for which we have been able to collect a full range of comparative evidences, from $k.q_\beta = 1.0$ to 2.25.

Based on the evidence in table 6, strongly supporting the conditional ISD problem formulation, we further enriched the tree structure to generate the following very large-scale problems, that we see all solved even for the largest scenario tree S_6 under the selected ISD conditions, namely for $q_\beta = \{1.75, 2.0, 2.25\}$. We provide more details on the solution of the largest problem in section 5.3 and appendix 6.

The three problem instances with 5880, 7840 and 19600 scenarios were the largest ones solved in this project. We see that the McISD-1.75 problem with 19600 scenarios was solved in roughly 50 minutes. Much faster solution times for the SSD-constrained LP problems and QCLP problems.

Sections 5.2 and 5.3 present evidence on just one instance of the smallest and the largest problems, with two main qualitative objectives: first to provide numerical and graphical evidence of the problem solutions and, second, to analyse the financial implications of the models.

The financial perspective is captured by analysing the optimal portfolios associated with different instances and the terminal returns distributions evaluated in-sample over the 2022-2023 period. From a methodological viewpoint, we consider to which extent different ISD problems' formulations affect the stochastic dominance conditions over time.

5.2. Spanning multistage SD partial orders

We analyse the evidence emerging from the solution of the following problem instances: $\mathcal{L}(S_1, k.q_\beta = \{2.25, 2.0, 1.8995, 1.75, 1.5, 1.25, 1.0\}, m = \{1, 2\}, 2, 1, t_0 = 2022Q1)$ in which we focus on SD conditions with respect to the conservative benchmark strategy. The aim of this section is primarily of a comparative nature to collect evidence on the problem solution under MISD or McISD constraints. We show that indeed the stage-wise formulation leads to a relaxation of conditional ISD feasibility conditions. We verify:

- The optimal root node portfolios and objective values for different ISD orders and under MISD versus McISD formulations.
- The evolution of the dominance order over each stage under the conditional or stage-wise approaches for selected problems. This analysis is performed graphically in figure 5 and we provide evidence in appendix 5.

Table 8 includes column-wise the type of problem, then 7 columns with the here and now optimal allocation and on the right the optimal value, the Hirschman-Herfindahl index (HHI) and the stochastic dominance order resulting, after the problem solution, when comparing the portfolio returns distributions and the benchmark distribution at the end of every stage.

We stop at ISD-2.25 for two reasons, as mentioned before: lack of solution of MISD instances and, as for the McISD formulation, almost identical first stage optimal portfolios above that ISD order. The collected evidence in table 8 is sufficient to derive several interesting remarks:

(i) The MISD formulation and feasibility conditions relax those induced by McISD: indeed for $k.q_\beta \geq 2.25$ the lack of solution of MISD quadratic problems is caused by the problem size and numerical instability, see table 6. On the contrary, for $k.q_\beta \leq 1.75$, McISD instances turn out to be infeasible: it is sufficient that any subtree of the problem causes infeasibility to have lack of convergence to an optimum. We see that such problem does not emerge in case of MISD formulations.

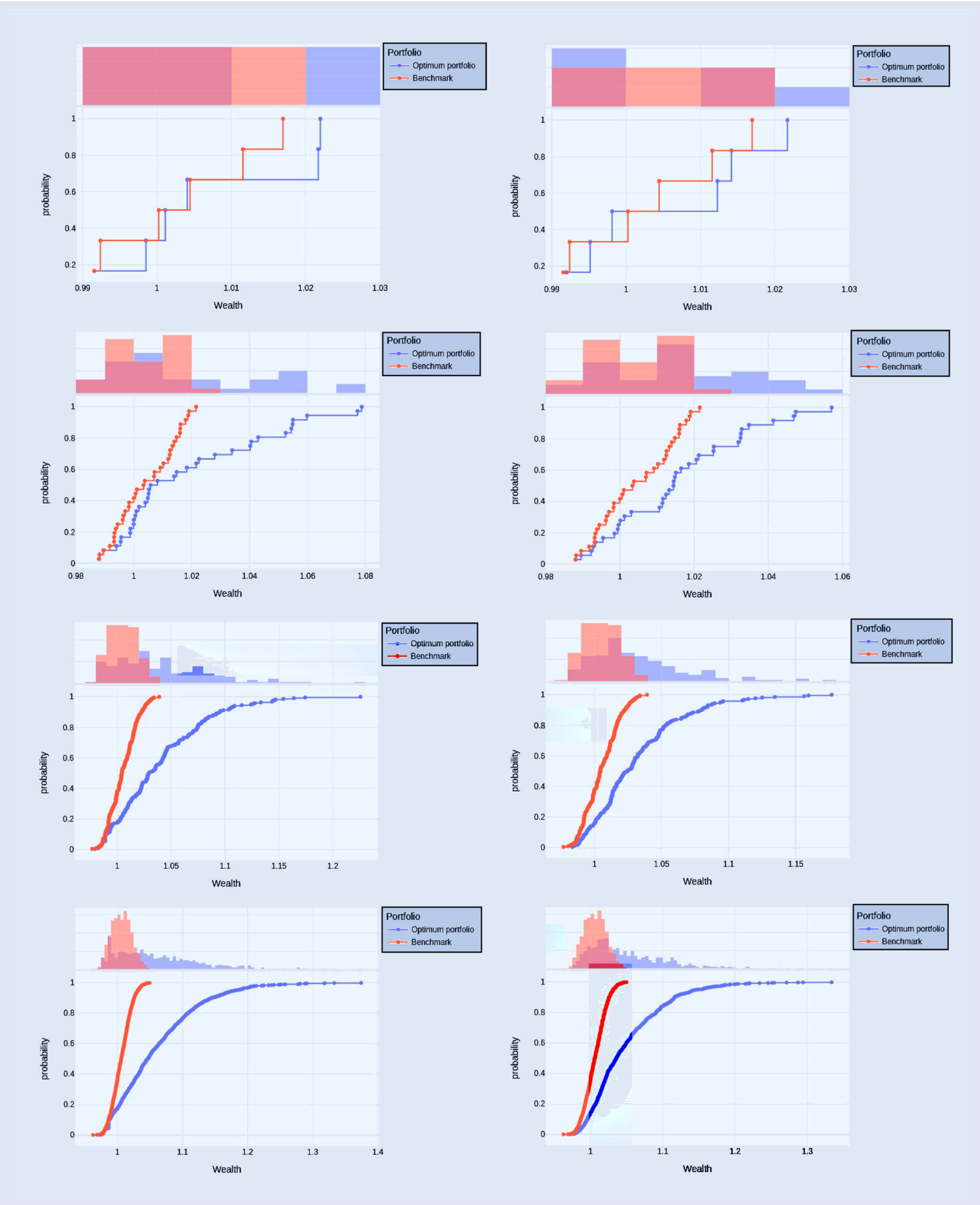

Figure 5. MISD (left) and McISD (right) 1.8995 CDFs every stage in 6^4 scenarios problem. Benchmark 2.

(ii) When focusing only on the two problems jointly solved, we see that indeed the MISD optimal values are greater than the McISD values (this is a consistent evidence, see below and in appendix), there are minor variations on the optimal root node allocation and they induce, ex-post similar ISD conditions relative to benchmark 1.

(iii) Looking at solved MISD-1 problems, we see that portfolio diversification increases and the optimal value decreases as the ISD order decreases, thus leading to increasingly diversified portfolios for $k.q_\beta = \{1.5, 1.25, 1.0\}$. Based on this evidence, we claim that interval-based SD does indeed span between FSD and SSD: an evidence we did already establish for the static portfolio problem in Liu *et al.* (2021).

(iv) The ISD evidence in the last column is particularly relevant. It should be analysed in strict relationship with the problem instance in the first column and jointly with the CDFs plotted in figure 5. Namely ex-post the input ISD order is always met and already at the second stage indeed, the dominance over the benchmark distribution is of the first order. An evidence confirmed below in section 5.3. Relative to the adopted benchmark strategy, we can infer here that the optimal portfolios dominate to the first order a strongly risk-averse portfolio strategy.

We present in figure 5 the CDF's in each stage of the problem associated with the strongest ISD order solved under both formulations, this is $\mathcal{L}(S_1, 1.8985, m^*, 2, 1, 2022Q1)$. The plots are self-explanatory and we just recall that we are here considering only benchmark strategy 2 and we show on the left column the CDFs from the MISD case and on the right column from the McISD.

An evidence emerging in this case and then confirmed essentially in all our experiments, is that given the problem instance and the partition of the domain induced by the reference point β, then the ISD order systematically converges to FSD at the last stage and very often even before. Furthermore we see that the conservative strategy is relatively easy to dominate.

In the following section we provide more robust financial evidence based on a much richer characterization of ISD under the conditional approach, from now on to be regarded as the reference model. We present further evidence on the solution of these test-problems in appendix 5.

5.3. Optimal 1-year benchmark-based McISD strategies

Following the evidence in section 5.2, we concentrate now on the results collected on a set of large-scale test problems, whose SD feasibility is expressed in conditional form. The focus is primarily on the results associated with the solution of a very large instance, leading to sufficiently robust evidence and the benchmarking of single- against multi-period formulations. We refer to table 7 for the associated problem dimensions and solution times.

Table 9, left to right, displays the problem instance, the root node decisions, the optimal objective values, the portfolios' diversification properties (HHI) of those solutions and the evidence in terms of stochastic dominance with respect to the benchmarks. The information on the rightmost column includes the ISD orders at the end of the first, second, third and forth stages for given problem instance in terms of quantiles $k.q_\beta$. We focus on the most distinctive cases $k.q_\beta = \{2.25, 2.0, 1.75, 1.5\}$, the latter to verify also if an increasing number of scenarios ease the ISD feasibility conditions as the SD order becomes more stringent. We can see that problem $k.q_\beta = 1.75$ is now solved while 1.5 still faces infeasibility problems. An increasing branching at the root node helps an accurate calibration of the stochastic dominance already in the first stage. Notice that, based on a unit initial capital, the Opt-Value column displays the expected wealth at the end the year for the multistage instances and at the end of the first quarter for single period problems. These latter are identified by the acronym ISD in the first column.

Table 9 allows the comparison of the root node decisions and solution evidence on the largest problem instance across the three benchmarks for the multistage instances as well as the single stage problems. Key to the comparison between single and multistage problems is the adoption of the very same scenario set in the common first stage and based on 40 nodes.

We highlight the following evidence from table 9:

- Taking also table 8 into account, we can see that for McISDs 2.0 and 2.25 the optimal root node decisions are very close one another. The same is true for the single period problems with, however a much lower diversification. Under the stronger 1.75 partial order all models show greater diversification and a lower expected terminal portfolio value.
- The optimal portfolios adapt to the benchmarks and benchmark 1 does actually provides the highest diversification. In the case of the single period problem against the risky benchmark 3 we see that the optimal portfolios are fully concentrated.

We can anticipate that the evidence collected for these problem instances are confirmed once we update the information and derive solutions over a more extended time span.

We present here next in figure 6 the graphical evidence resulting from the solution of problems $\mathcal{L}(19600, 1.75, 2, j^*, 1, 2023Q1)$. From top to bottom we show for every stage $t = 1, 2, 3, 4$, the CDF's generated by the solution of the three problems: left column against benchmark 1, central for benchmark 2 and right against benchmark 3. We can see that already in the third stage the portfolio strategy is actually leading to first order SD with respect to the benchmarks. The evidence in the second column can be compared with the second column in figure 5 for the same problem based however on a smaller scenario set.

Table 7. Large scale stochastic programs with McISD feasibility conditions.

	McISD-1.75			McISD-2.0			McISD-2.25		
No scenarios	5880	7840	19600	5880	7840	19600	5880	7840	19600
Branching	$15\cdot 8\cdot 7^2$	$20\cdot 8\cdot 7^2$	$40\cdot 10\cdot 7^2$	$15\cdot 8\cdot 7^2$	$20\cdot 8\cdot 7^2$	$40\cdot 10\cdot 7^2$	$15\cdot 8\cdot 7^2$	$20\cdot 8\cdot 7^2$	$40\cdot 10\cdot 7^2$
Problem class	MILP	MILP	MILP	LP	LP	LP	QCLP	QCLP	QCLP
Solver	barrier	barrier	barrier	D simplex	D simplex	D simplex	barrier	barrier	barrier
No rows	196149	261629	655329	172656	230306	577086	245350	327330	811199
No columns	192199	256359	642059	178489	238079	596379	233569	311619	781619
Nonzeroes	936481	1167271	3135751	751396	1002751	2519071	1687463	1737638	4368078
Binary vars	20595	27480	69240	0	0	0	0	0	0
Quadratic vars	0	0	0	0	0	0	2930	3906	9771
Average CPU time (sec's)	335.25	472.71	3021.34	35.28	43.42	55.45	84.74	91.38	157.56

Table 8. Root portfolios and diversification of ISD models with a 6^4 scenario tree and Benchmark 2.

		SHY	VCSH	VCIT	XLU	XLV	XLE	Risk-free	OptValue	HHI	ISD-$k.q_\beta$
ISD-2.25	MISD				unsolvable						
	McISD	0	0	0.7036	0	0.2963	0	0	1.0496	0.5829	(1.648,1.0,1.0,1.0)
ISD-2.0	MISD	0	0	0.5103	0	0.4896	0	0	1.0601	0.5002	(1.437,1.0,1.0,1.0)
	McISD	0	0	0.7036	0	0.2963	0	0	1.0496	0.5829	(1.648,1.0,1.0,1.0)
ISD-1.8985	MISD	0	0	0.5103	0	0.4896	0	0	1.0595	0.500	(1.437,1.0,1.0,1.0)
	McISD	0	0	0.7049	0	0.2937	0.0012	0	1.0488	0.5833	(1.648,1.0,1.0,1.0)
ISD-1.75	MISD	0	0	0.5103	0	0.4896	0	0	1.0584	0.5002	(1.437,1.0,1.0,1.0)
	McISD				infeasible						
ISD-1.5	MISD	0.0140	0	0.4729	0.0096	0.4681	0.0350	0	1.0566	0.4444	(1.429,1.0,1.0,1.0)
	McISD				infeasible						
ISD-1.25	MISD	0.1924	0	0.3657	0.0186	0.4150	0.0081	0	1.0561	0.3434	(1.0,1.0,1.0,1.0)
	McISD				infeasible						
ISD-1.0	MISD	0.1662	0	0.3948	0.0108	0.4246	0.0035	0	1.0555	0.3639	(1.0,1.0,1.0,1.0)
	McISD				infeasible						

Table 9. $\mathcal{L}(S_6, q_\beta, 2, j^*, T^*, 2022\text{Q1}), q_\beta = \{1.5, 1.75, 2.0, 2.25\}, j = \{1, 2, 3\}, T = \{0.25, 1\}$ solution evidence, optimal root-node portfolios.

Model-Benchmark	SHY	VCSH	VCIT	XLU	XLV	XLE	Risk-free	OptValue	HHI
Benchmark 1									
McISD-2.25	0.15554	0	0.29190	0.13330	0.19611	0.15399	0.06913	1.054	0.194
McISD-2.0	0.16753	0	0.28914	0.13529	0.19594	0.15418	0.05790	1.054	0.195
McISD-1.75	0.09819	0.10780	0.24405	0.13945	0.18176	0.15092	0.07779	1.053	0.162
McISD-1.5				infeasible					
ISD-2.25	0.2324	0	0	0.273	0.4946	0	0	1.0122	0.3732
ISD-2.0	0.2324	0	0	0.273	0.4946	0	0	1.0122	0.3732
ISD-1.75	0	0	0.3455	0.2314	0.4221	0.001	0	1.0117	0.3511
Benchmark 2									
McISD-2.25	0.37818	0.26528	0.34803	0	0.0085	0	0	1.039	0.334
McISD-2.0	0.36713	0.28096	0.34390	0	0.00800	0	0	1.039	0.332
McISD-1.75	0.33333	0.33333	0.33333	0	0	0	0	1.038	0.333
McISD-1.5				infeasible					
ISD-2.25	0.6864	0	0	0.0893	0.2243	0	0	1.0049	0.5294
ISD-2.0	0.6864	0	0	0.0893	0.2243	0	0	1.0049	0.5294
ISD-1.75	0.1199	0	0.0299	0.125	0.1086	0.0169	0.5996	1.0041	0.4025
Benchmark 3									
McISD-2.25	0	0	0	0.33340	0.33348	0.33310	0	1.05	0.333
McISD-2.0	0	0	0	0.33333	0.33333	0.33333	0	1.051	0.333
McISD-1.75	0	0	0	0.33333	0.33333	0.33333	0	1.0486	0.333
McISD-1.5				infeasible					
ISD-2.25	0	0	0	0	1	0	0	1.0162	1
ISD-2.0	0	0	0	0	1	0	0	1.0162	1
ISD-1.75	0	0	0	0	1	0	0	1.0162	1

It is worth noticing that indeed in the case of the riskiest benchmark 3 the optimal strategy since the first stage replicates perfectly the equity portfolio. As for benchmarks 1 and 2 we see that the benchmark distributions are initially well replicated and then after the managed portfolios increasingly dominate the benchmark distributions.

In the following subsection we will restrict the analysis to the most representative benchmark strategy 1 and focus specifically on the ISD 1.75 and 2.0 partial orders to provide further evidence on the multistage versus single stage formulations.

5.4. Consistency evidence: single versus multistage models

This section analyses the evidence from the solution of a sequence of problems relying on the moving windows described early in this section. We restrict the analysis to SSD and McISD-1.75 problems. We consider the following instances: $\mathcal{L}(S_6, k.q^*_\beta, 2, 1, T^*, t^*_0)$, for $k.q_\beta = \{1.75, 2.0\}$, $T = \{0.25, 1\}$ and $t_0 = \{2021\text{Q1}, \ldots, 2023\text{Q4}\}$. We aim at verifying the consistency of the results collected for the single instances in previous sections and the financial implications of the multi-period extension. In section 5.6 we do also analyse the out-of-sample evidence associated with the two formulations.

We consider here below the evolution of the optimal root node portfolios in figure 7 and two types of information collected by solving the same type of problem repeatedly over the 3 years:

- In table 10 we show the annual in-sample performances of the single against the 4 stage solutions, when assuming that the single period problems are solved with updated input information, in sequence over the given subsequent year. In the last 2 columns we compare the end of the year compounded returns generated when solving the 4single

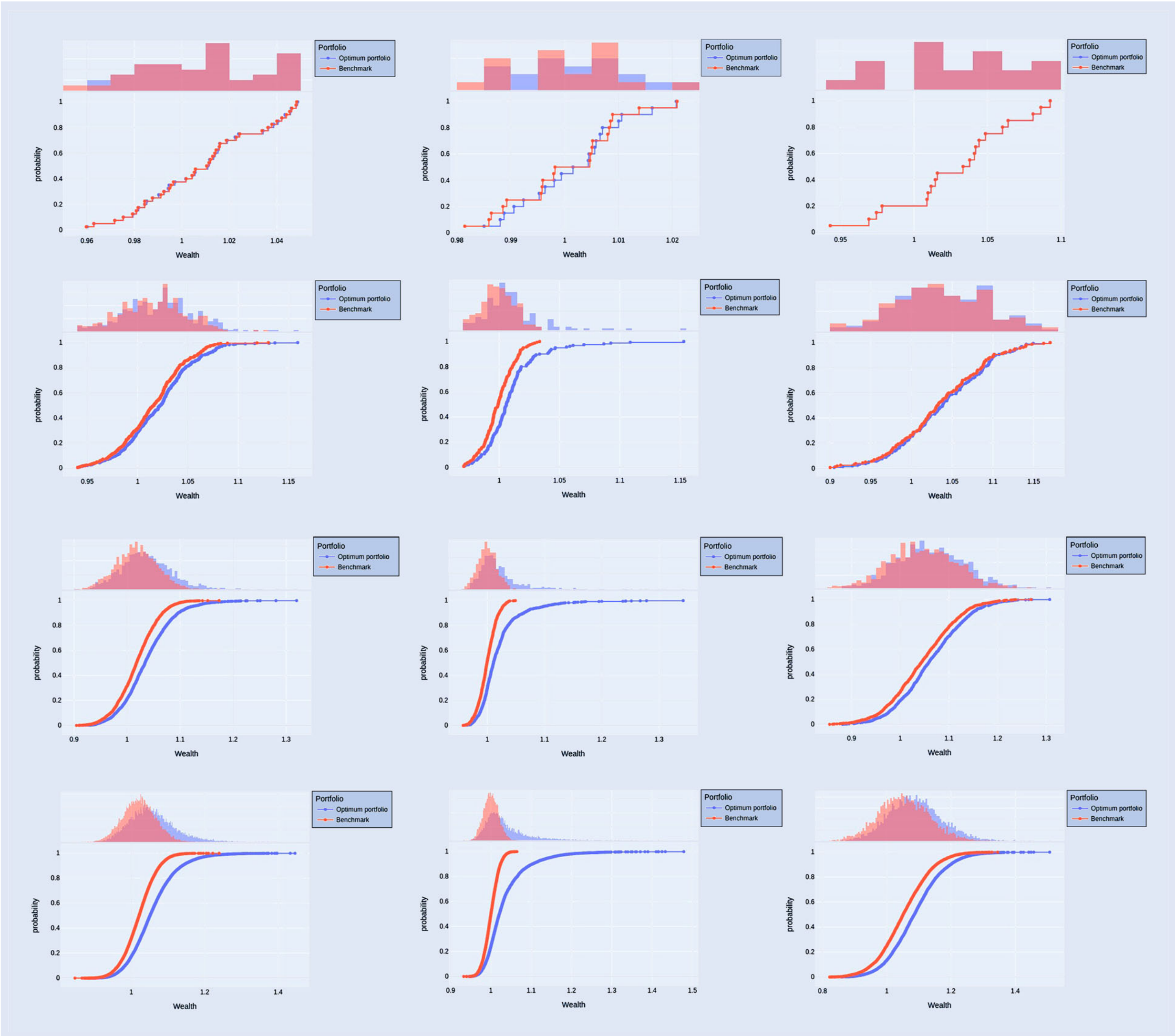

Figure 6. McISD-1.75: 19600 scenario problem. Benchmarks 1 (left CDF's), 2 (central CDF's) and 3 (right CDF's) stochastic dominance evidence after solution.

stage problems in each consecutive quarter. The last row, for instance compares the expected wealth at the end of 2023 for $t_0 = 1.1.2023$ with the expected wealth that would be generated when solving 4 single period problems and compounding. The evidence is limited to ISD-1.75 and SSD problems on the grounds of the rarely distinguishable solutions above 2.0 dominance as reported in tables 11, 9 and 8.

- Table 11 displays instead quarterly statistics associated with the sequence of single and multistage solutions, including evidence on the average. In the upper section from the solution of 12 single period instances and on the lower section the average quarterly statistics associated with the multistage solutions, for the two stochastic dominance orders, 1.75 and 2.0.

It is important to underline that market data and statistics leading to new scenarios for the single period problems are all updated as uncertainty reveals and the problems always solved out-of-sample.

Consider the optimal root node portfolios in figure 7. On the upper part the portfolios from the multistage solutions and below those from single period solutions.

We highlight:

- The relative stability of the optimal root portfolios in the multi- relative to the single-stage. Neither model includes transaction costs nor investment bounds and the underlying scenarios agree perfectly in the first quarter.
- Column-wise the optimal portfolios differ and single-period solutions vary significantly across subsequent quarters.
- Top to bottom the optimal portfolios under ISD-1.75 differ from the SSD portfolios. We see next that in-sample the latter do indeed generate higher expected returns. The information in table 11 supports the distinction between the two models. See also in appendix 6 the portfolio strategies associated with the two models along selected median scenarios.

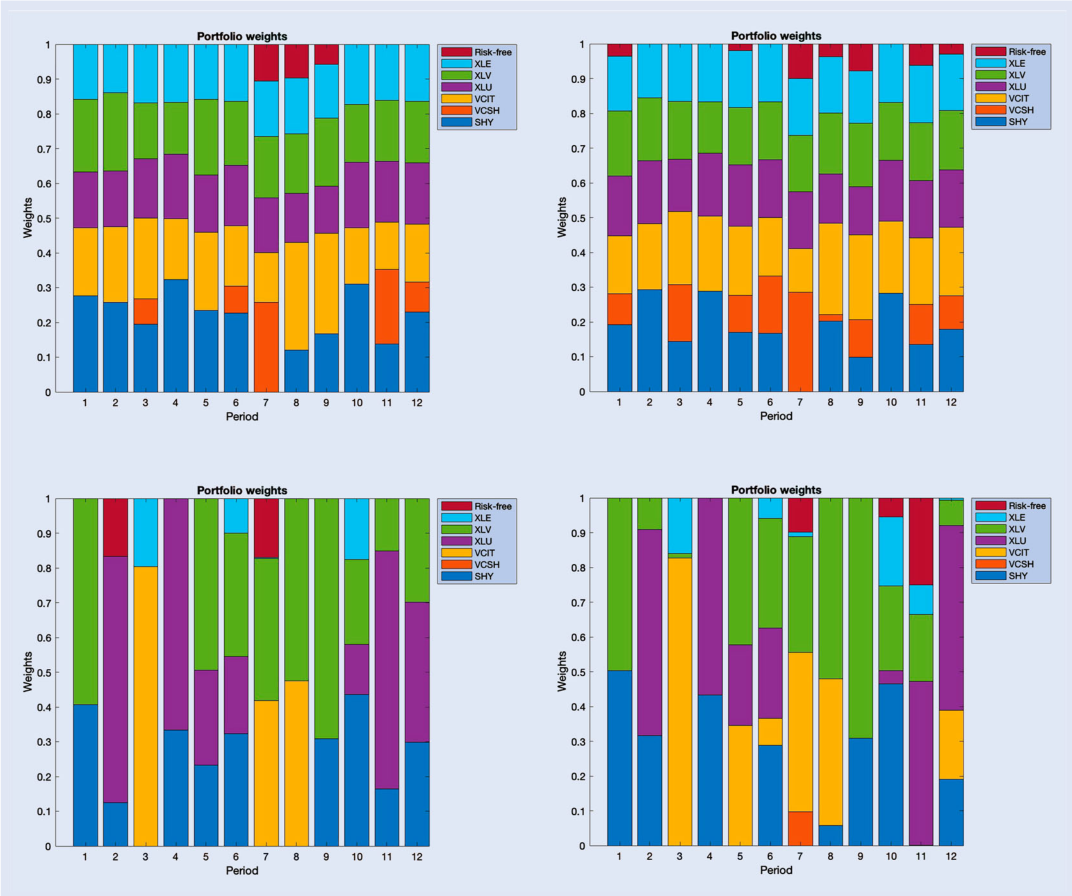

Figure 7. Optimal root-node portfolios January 2021–December 2023: top-down: McISD-2.0 left and McISD 1.75 right, ISD 2.0 left and ISD 1.75 right. Benchmark 1.

We compare in table 10 the expected wealth after 12 months generated by single or multiperiod models, starting with unit capital invested at the beginning of the months indicated in column 1. The single period solutions are compounded every quarter to be comparable with the multistage solutions. Results are shown in column **Total return** for the sequence of single period problems and in the last column for the multistage solutions. In the upper section of the table for ISD-1.75 problems and in the lower part for SSD problems. Without going in great detail we see that the multistage solutions consistently outperform on overlapping years the outcome of a sequence of single stage solutions and that indeed the terminal wealths generated by SSD solutions overperform in-sample the 1.75 solutions consistently. We provide a final set of evidence in table 11.

The following statistics are presented for $k.q_\beta = \{1.75, 2.0, 2.25\}$, always based on optimal solutions. In this case, these are all comparable quarterly statistics averaged over the 12 solutions, line by line: the expected quarterly wealth, the standard deviation at the horizon, the associated Sharpe ratio, the Conditional Value-at-Risk, the Herfindahl index for portfolio diversification and in the last line the average ISD order estimated after the solution. On the top section for single period and at the bottom for multistage problems. For these latter problems we show quarterly averages computed first with respect to the stages and then across the 12 problems. We can then confirm that the multistage instances guarantee a tighter risk control, higher risk-adjusted return, as shown by the Sharpe ratios, and a far more effective dominance order with respect to the first benchmark than the single period instances.

We summarize in the next section the evidence collected through sections 5.1 to 5.4.

5.5. *Summary evidence: McISD-based decision making*

We summarize a set of relevant methodological and financial evidence from the presented results so far:

- Related to the multistage extension: from the set of problems we have solved the multistage model provides better in- and out-of-sample evidence. The evidence collected from 2021 to 2023 on the largest scenario tree shows also that one period optimal strategies are period-by-period significantly more volatile and portfolios less diversified than in the multistage case.
- When increasing the number of scenarios we have seen that the MISD formulation becomes unsolvable while when reducing the ISD orders we couldn't solve problems formulated with conditional ISD below order 1.75.
- The conditional ISD model, however, is able to handle very large instances with several thousands scenarios, leading to effective replication and outperformance of the benchmarks considered in the

Table 10. Expected end-of-year wealth: multistage versus compounded single stage solutions. ISD 1.75 and 2.0, benchmark 1.

12 months starting	Q1	Q2	Q3	Q4	**Total return**	
			ISD-1.75			**McISD 1.75**
Jan 2021	1.0042	1.0046	1.0048	1.006	**1.0197**	**1.0512**
Apr 2021	1.0046	1.0048	1.006	1.0117	**1.0274**	**1.0487**
Jul 2021	1.0048	1.006	1.0117	1.008	**1.03084**	**1.05364**
Oct 2021	1.006	1.0117	1.008	1.0017	**1.02765**	**1.04889**
Jan 2022	1.0117	1.008	1.0017	1.0046	**1.026226**	**1.05724**
Apr 2022	1.008	1.0017	1.0046	1.0087	**1.023183**	**1.05841**
Jul 2022	1.0017	1.0046	1.0087	1.0073	**1.02247**	**1.05326**
Oct 2022	1.0046	1.0087	1.0073	1.01078	**1.03176**	**1.05525**
Jan 2023	1.0087	1.0073	1.01078	1.0139	**1.04131**	**1.05329**
			SSD-2.00			**McISD-2.0**
Jan 2021	1.0049	1.0048	1.0048	1.007	**1.0216**	**1.05232**
Apr 2021	1.0048	1.0048	1.007	1.0122	**1.02909**	**1.04991**
Jul 2021	1.0048	1.007	1.0122	1.0082	**1.03257**	**1.05441**
Oct 2021	1.007	1.0122	1.0082	1.002	**1.0297**	**1.0496**
Jan 2022	1.0122	1.0082	1.002	1.0048	**1.02745**	**1.05898**
Apr 2022	1.0082	1.002	1.0048	1.0087	**1.0239**	**1.05914**
Jul 2022	1.002	1.0048	1.0087	1.0078	**1.0235**	**1.05392**
Oct 2022	1.0048	1.0087	1.0078	1.0131	**1.03483**	**1.05316**
Jan 2023	1.0087	1.0078	1.0131	1.0143	**1.04461**	**1.0541**

Table 11. January 2021 to December 2023 statistics, benchmark 1, average over 12 instances.

One-stage problem	ISD-1.75	ISD-2	ISD-2.25
$\mathrm{E}(W_1)$	1.0072	1.0077	1.0077
$\sigma(W_1)$	0.0218	0.0234	0.0234
$\mathrm{SR}(W_1)$	0.3220	0.3237	0.3237
$\mathrm{CVaR}_{0.95}(W_1)$	− 0.0339	− 0.0357	− 0.0357
$\mathrm{HHI}(W_1)$	0.4305	0.4642	0.4642
$\mathrm{ISD}{-}k.q_\beta(W_1)$	1.3719	1.5475	1.5475
4-stage problem: average statistics	McISD-1.75	McISD-2	McISD-2.25
$\mathrm{E}(\bar{W}_4)$	1.0134	1.0137	1.0137
$\sigma(\bar{W}_4)$	0.0164	0.0167	0.0167
$\mathrm{SR}(\bar{W}_4)$	0.8174	0.8193	0.8195
$\mathrm{CVaR}_{0.95}(\bar{W}_4)$	− 0.0144	− 0.0145	− 0.0145
$\mathrm{HHI}(\bar{W}_4)$	0.1809	0.1963	0.1960
$\mathrm{ISD}{-}k.q_\beta(W_1, W_2, W_3, W_4)$	1.1888	1.2721	1.2808

study, with FSD always attained at the end of the planning horizon.

- As the ISD order decreases, feasibility conditions become more stringent and improve portfolio diversification systematically. When comparing in the multiperiod case SSD and McISD-1.75 results, optimal strategies differ to a sufficient extent to motivate the search for stronger ISD criteria. On the smallest S_1 scenario set we could span effectively from FSD to SSD, including the solution of the FSD problem.

The choice of a 1-year horizon with quarterly policy revision is relevant and consistent with financial practice: the derivation of a strategic asset allocation over such planning horizon is common in the fund management industry, where indeed benchmark portfolios are adopted to verify the funds' relative performance. To date we had rather limited evidence on the effectiveness of optimal multistage SSD portfolios, which are here instead carefully evaluated and would provide a viable modeling framework. McISD-1.75 models would differ by requiring FSD over the lower quartile and allowing for SSD on the remaining support, likely leading to a better downside risk control, consistently with the theoretical results. In practice the definition of a benchmark portfolio needs not be based, as in this study, on assets included in the portfolio space. It will then be the portfolio manager, who based on statistical and financial analysis will first identify the most appropriate investment universe. The adoption of an McISD approach will be viable after selecting a subset of strategic asset classes relevant for the problem. A critical issue is associated with the generation of arbitrage-free scenarios, forcing the branching degree at every node to be greater than or equal to the set of assets. Furthermore, the definition of the most appropriate ISD criterion is relevant and will typically rely on market analysis and a subjective specification of a quantile return necessary to characterize the strategy. We show in table 7 that without necessarily considering S_6, the instances with $S_5 = 7840$ scenarios are solved in minutes. The definition of a minimal ISD-1 order able to guarantee the problem feasibility will typically require a preliminary simulation study and it is problem dependent.

The following decision process can then be envisaged in practice: definition of benchmark and asset universe, selection of a relevant subset of assets, statistical characterization of β and q_β for $k = 1, 2$, definition of the investment horizon, the stages and tree process. Simulation study for β calibration and optimization for SSD and the strongest attainable ISD-1 order. Output analysis and validation after solution as discussed in the next section.

5.6. Out-of-sample results

This final section presents the evidence collected when back-testing on market data the optimal strategies generated by the solutions to the largest scenario problems. We take every optimal root node portfolio in figure 7 and assess their performance after applying the ETF values expressed by the market at the end of every quarter. For ease of exposition we consider in table 12 the evolution every half year from January 2021 (1 capital invested) to December 2023. We distinguish again between single period and 4-stage problems and span the three ISD conditions already discussed above. In the last 5 columns, we present the quarterly statistics associated with each problem instance.

After the volatile Covid-19 year, it is common knowledge that 2021 and 2022 have been extremely positive for equity markets with a significant outflow from fixed-income markets penalized by increasing interest rates globally and pushed by the unprecedented expansion of public deficits. The evidence in table 12 reflects such trend where we see on the bottom line that an investment of 100 fully in equity would have led to a capital of 149 after 2 years with a limited loss during 2023. The evidence on the top part of the table is of interest and related to the presented models. The first section is devoted to the optimal single period portfolios and we see that due to the volatile composition of those portfolios, the ex-post outcome is pretty poor. The multistage solutions are instead effective in replicating the $1/N$ benchmark and actually the selected ISD order makes very little difference. We refer to the evidence in figure 7 to clarify the relationship between optimal McISD-2.0 and 1.75 portfolios. It is rather interesting that the in-sample performances in table 10 turn out pretty accurate out-of-sample.

6. Conclusions

This article provides a comprehensive treatment of the methodological and computational implications induced by an extension to a multi-period setting of the one period ISD framework introduced by Liu *et al.* (2021). The portfolio optimization model studied in that early contribution is here extended to several stages.

We have considered two possible approaches to enforce ISD: conditionally along the tree, thus in every 2-stage subtree embedded in the given tree structure or stage-wise, thus considering the benchmark nodal realizations in every stage of the tree. Several relevant evidences emerge form the analysis and if, on one hand, they can be claimed as primary

Table 12. Back testing results over 12 quarters from January 2021 to December 2023, benchmark strategy 1.

		W_0	$W_{0.5}$	W_1	$W_{1.5}$	W_2	$W_{2.5}$	W_3	Mean (%)	Std (%)	SR	Max (%)	Min (%)
$T = 1$	$\text{ISD}_{2.25}$	100	100.74	107.29	101.66	104.36	100.56	98.06	−0.06	4.79	−0.01	7.80	−7.28
$T = 1$	ISD_2	100	100.74	107.29	101.66	104.36	100.56	98.06	−0.06	4.79	−0.01	7.80	−7.28
$T = 1$	$\text{ISD}_{1.75}$	100	101.31	106.76	99.06	101.47	98.04	99.43	0.04	4.40	0.01	7.35	−4.64
$T = 4$	$\text{ISD}_{2.25}$	100	108.30	112.59	111.24	115.39	112.81	114.73	1.21	3.70	0.33	7.34	−4.80
$T = 4$	ISD_2	100	108.30	112.60	111.27	115.43	112.83	114.79	1.22	3.70	0.33	7.33	−4.80
$T = 4$	$\text{ISD}_{1.75}$	100	108.00	112.13	111.20	115.44	112.83	114.80	1.22	3.69	0.33	7.34	−4.78
benchmark	(1/N strategy)	100	108.15	112.30	111.45	115.51	112.78	114.84	1.22	3.75	0.33	7.47	−4.78
	60-40 equity-bond	100	110.02	115.45	116.14	121.72	118.07	120.15	1.63	4.39	0.37	8.73	−5.07
	40-60 equity-bond	100	106.30	109.22	106.88	109.54	107.65	109.68	0.82	3.18	0.26	6.22	−4.49
	0-100 equity-bond	100	99.01	97.39	89.84	87.90	88.63	90.52	−0.80	2.42	−0.33	3.75	−4.57
	100-0 equity-bond	100	117.60	128.56	136.24	149.06	140.78	142.90	3.25	7.19	0.45	13.75	−6.22

contributions of this work, on the other, they highlight some of its limitations and possible ways forward:

- The enforcement of stochastic dominance principles between random processes in a discrete setting, required several theoretical results previously established for integer SD-principles (Dentcheva and Ruszczynski 2003) and only recently leading to computational evidence in a multi-period setting (Kopa *et al.* 2018, Moriggia *et al.* 2019). In a discrete scenario-based model, the feasibility regions associated with two possible approaches, have been fully characterized and their financial implications studied in a multistage stochastic program. We have reported that an increasing number of scenarios with yet a limited number of stages may lead to numerical instability and possibly to lack of convergence with quadratically constrained problems above MISD-2.25 and on the other hand stronger McISD conditions below order 2 generate infeasibility problems.
- The resulting stage-wise and conditional ISD orders have been shown the former to define a relaxation of the feasibility conditions imposed by the latter: such relaxation, despite the adoption of a high-performance-computing platform, however comes at a very high computational cost, making the conditional approach computationally far more viable. Yet, under the McISD approach we were unable to solve large problems with an ISD order below 1.75.
- Through extended computations, we have shown that in a dynamic model, under either the MISD or McISD formulations, most initial ISD input conditions led almost always to FSD at the end of the 4-stage horizon. This evidence has been shown in the comparative study to be robust against all benchmark strategies.
- When solvable, the advantages of adopting a multistage rather than a single stage framework have been assessed in- and out-of-sample. In the multistage model, under the given assumptions of attainable benchmarks and a limited number of stages, the SSD approach appears sufficient to attain most of the benefits induced by the multi-period model.
- The above points suggest that surely the McISD approach should represent the standard from a methodological perspective and also that indeed in a multiperiod setting it is necessary to first investigate an appropriate ISD-1.q_β order on the benchmark distribution and then rely on a stage-dependent β updating to enhance the likelihood of convergence.
- The solution of a very large-scale problem with 19600 over a 1-year planning horizon helped showing an effective dynamic dominance over the stages and the desirable dependence of the optimal portfolio strategies on the adopted benchmark: from a financial perspective the more volatile and risky benchmark led, under the ISD constraints, to an accurate portfolio replication while the more conservative benchmark led to an optimal strategy with a clear upside and a significant improvement of the average performance. More complex benchmarks strategies and an asset universe not necessarily based on the same set of assets can be considered under the proposed modeling framework.

The main focus of this contribution is on the methodological implications of a dynamic extension of ISD principles and a comprehensive analysis of two possible approaches when applied to a multiperiod portfolio problem. The MISD approach has several advantages from the perspective of accurate risk control relative to a benchmark and weaker feasibility requirements, but faces a strong curse of dimensionality when increasing the number of scenarios and/or stages of the problem. The McISD approach has been shown to be far less computationally expensive and able under a sufficiently rich scenario tree to enforce McISD-1 conditions over each stage. The financial application finally helped testing the methodology in a classical financial optimization framework and emphasising the ability of the proposed approach to dominate in-sample three different benchmark distributions.

Future research will focus on sampling methods for ex-ante reference point calibration and the definition of a minimal feasible McISD order. In parallel, we will tackle the study of a decomposition method to speed up solution times for very large-scale McISD problems, possibly resulting from both a more extended asset universe and increasing number of stages. Finally, we aim at extending the study of SD-based portfolio efficiency, well established in the single period setting for integer SD criteria, see Fábián *et al.* (2011), Roman *et al.* (2013), Hodder *et al.* (2015) and Fang and Post (2017), to the ISD paradigm first in a single stage and then in a multistage framework.

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No potential conflict of interest was reported by the author(s).

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Appendices

Appendix 1. Some properties of ISD

In this appendix, we review some properties of static ISD.

LEMMA A.1 *Let θ and β two reference points such that $\theta \leq \beta$. If $X \succeq_{(k,\beta)} Y$ then $X \succeq_{(k,\theta)} Y$.*

Proof Note that $F_{k+1}(X,\eta) = \int_{-\infty}^{\eta} F_k(X,\xi)d\xi$ and

$$F_{(k,\beta)}(X,\eta) = F_k(X,\eta)\chi_{(-\infty,\beta]}(\eta) + F_{k+1}(X,\eta)\chi_{[\beta,\infty)}(\eta).$$

Assume that $\theta \leq \beta$.
If $\eta \leq \theta$ then $\eta \leq \beta$ and

$$F_{(k,\beta)}(X,\eta) = F_k(X,\eta)\chi_{(-\infty,\beta]} = F_k(X,\eta)\chi_{(-\infty,\theta]} = F_{(k,\theta)}(X,\eta)$$

If $\eta \geq \beta$ then $\eta \geq \theta$ and

$$\begin{aligned} F_{(k,\beta)}(X,\eta) &= F_k(X,\eta)\chi_{[\beta,\infty)}(\eta) = F_k(X,\eta)\chi_{[\theta,\infty)}(\eta) \\ &= F_{(k,\theta)}(X,\eta) \end{aligned}$$

If $X \succeq_{(k,\beta)} Y$ then $F_{(k,\beta)}(X,\eta) \leq F_{(k,\beta)}(Y,\eta)$ for every $\eta \in \mathbb{R}$.

- For $\eta \leq \theta$ we have $F_{(k,\theta)}(X,\eta) = F_{(k,\beta)}(X,\eta) \leq F_{(k,\beta)}(Y,\eta) = F_{(k,\theta)}(Y,\eta)$.
- If $\theta \leq \eta \leq \beta$ then $F_k(X,\eta) = F_{(k,\beta)}(X,\eta) \leq F_{(k,\beta)}(Y,\eta) = F_k(Y,\eta)$ then $F_{(k,\theta)}(X,\eta) = F_{k+1}(X,\eta) \leq F_{k+1}(Y,\eta) = F_{(k,\theta)}(Y,\eta)$.
- If $\eta \geq \beta$ then $F_{(k,\theta)}(X,\eta) = F_{(k,\beta)}(X,\eta) \leq F_{(k,\beta)}(Y,\eta) = F_{(k,\theta)}(Y,\eta)$.

Therefore, $F_{(k,\theta)}(X,\eta) \leq F_{(k,\theta)}(Y,\eta)$ for every $\eta \in \mathbb{R}$. Thus, $X \succeq_{(k,\theta)} Y$ ■

LEMMA A.2 *Let X and Y be random variables. Then,*

(i) *If $X \succeq_{(k)} Y$ then $X \succeq_{(k,\beta)} Y$ for every $\beta \in \overline{\mathbb{R}}$.*
(ii) *If $X \succeq_{(k,\beta)} Y$ for some $\beta \in \overline{\mathbb{R}}$ then $X \succeq_{(k+1)} Y$.*

Proof (i) Assume that $X \succeq_{(k)} Y$ and $\beta \in \mathbb{R}$ then $F_k(X,\eta) \leq F_k(Y,\eta)$ for every $\eta \in \mathbb{R}$. In particular, for $\eta \leq \beta$ we have $F_k(X,\eta) \leq F_k(Y,\eta)$ and for every $\eta \in \mathbb{R}$, $F_{k+1}(X,\eta) \leq F_{k+1}(Y,\eta)$. Therefore, $F_{(k,\theta)}(X,\eta) \leq F_{(k,\theta)}(Y,\eta)$ for every $\eta \in \mathbb{R}$. Thus, $X \succeq_{(k,\beta)} Y$.

(ii) If $X \succeq_{(k,\beta)} Y$ for some $\beta \in \mathbb{R}$ then $F_{(k,\theta)}(X,\eta) \leq F_{(k,\theta)}(Y,\eta)$ for every $\eta \in \mathbb{R}$. If $\eta \leq \beta$ then $F_k(X,\eta) = F_{(k,\beta)}(X,\eta) \leq F_{(k,\beta)}(Y,\eta) = F_k(Y,\eta)$ and $F_{k+1}(X,\eta) \leq F_{k+1}(Y,\eta)$. Since $F_{k+1}(X,\eta) = F_{(k,\beta)}(X,\eta) \leq F_{(k,\beta)}(Y,\eta) = F_{k+1}(Y,\eta)$ for $\eta \geq \beta$ we have $X \succeq_{(k+1)} Y$.

■

THEOREM A.1 *We have the relations*

$$\succeq_{(k)} = \bigcap_{\beta \in \overline{\mathbb{R}}} \succeq_{(k,\beta)} = \succeq_{(k,\infty)}$$

and

$$\succeq_{(k+1)} = \succeq_{(k,-\infty)} \supset \bigcup_{\beta \in \overline{\mathbb{R}}} \succeq_{(k,\beta)}$$

Proof Note that $\lim_{\beta\to\infty} F_{(k,\beta)}(X,\eta) = F_k(X,\eta)$ and $\lim_{\beta\to-\infty} F_{(k,\beta)}(X,\eta) = F_{k+1}(X,\eta)$ for every $\eta \in \mathbb{R}$. Then we have the inclusion $\bigcap_{\beta\in\overline{\mathbb{R}}} \succeq_{(k,\beta)} \subset \succeq_{(k)} \subset \succeq_{(k,\infty)}$. By Lemmas A.1 and A.2 we have $\bigcap_{\beta\in\overline{\mathbb{R}}} \succeq_{(k,\beta)} \supset \succeq_{(k)} \supset \succeq_{(k,\infty)}$, $\succeq_{(k+1)} = \succeq_{(k,-\infty)}$ and $\succeq_{(k+1)} \supset \bigcup_{\beta\in\overline{\mathbb{R}}} \succeq_{(k,\beta)}$. ■

COROLLARY A.1 *SD-k is a special case of ISD-k.*

Proof In fact, by Theorem A.1 we can consider SD-k as ISD-k for the reference point $\beta = \infty$. ■

Appendix 2. Some properties of conditional ISD

For reference functions θ and β we regard the pointwise order $\theta \leq \beta$: in the stagewise case $\theta_t \leq \beta_t$ for every $t \in \mathcal{T}$ and in the conditional case $\theta(s) \leq \beta(s)$ for $s \in S_t$, $t \in \mathcal{T}$.

LEMMA A.3 *Let X_t and Y_t stochastic processes over $t \in \mathcal{T}$ and θ, β reference functions such that $\theta \leq \beta$.*

(i) *If $X_t \succeq_{(k,\beta_t)\,|\,\mathcal{F}_0} Y_t$ then $X_t \succeq_{(k,\theta_t)\,|\,\mathcal{F}_0} Y_t$.*
(ii) *If $X_t \succeq_{(k,\beta_t\,|\,\mathcal{F}_t)} Y_t$ then $X_t \succeq_{(k,\theta_t\,|\,\mathcal{F}_t)} Y_t$.*

LEMMA A.4 *Let X_t and Y_t stochastic processes over $t \in \mathcal{T}$. Then,*

(i) *If $X_t \succeq_{(k)} Y_t$ and $\{X_t \mid \mathcal{F}_{t-1}\} \succeq_{(k)} \{Y_t \mid \mathcal{F}_{t-1}\}$ then $X_t \succeq_{(k,\beta_t)\,|\,\mathcal{F}_0} Y_t$ and $X_t \succeq_{(k,\beta_t\,|\,\mathcal{F}_t)} Y_t$, respectively, for every reference function β.*
(ii) *If $X_t \succeq_{(k,\beta_t)\,|\,\mathcal{F}_0} Y_t$ and $X_t \succeq_{(k,\beta_t\,|\,\mathcal{F}_t)} Y_t$ for some reference function β then $X_t \succeq_{(k)} Y_t$ and $\{X_t \mid \mathcal{F}_{t-1}\} \succeq_{(k)} \{Y_t \mid \mathcal{F}_{t-1}\}$.*

The proof of Lemmas A.3 and A.4 are straightforward because we use the definitions of MISD-k (respectively, McISD-k) and MSD-k to apply Lemmas A.1 and A.2 in appendix 1 at each stage (respectively at each scenario) and then we conclude the proof by using the definitions of MISD-k (respectively, McISD-k) and MSD-k.

Appendix 3. Alternative ISD formulations and computations

Tables A2 and A1 clarify, however, that for a given number of assets, the two formulations carry very different computational implications.

As the number of stages and scenarios increase the resulting stochastic programming instance suffers easily from *curse of dimensionality*. We face in this case a relevant trade-off problem, since a sufficiently rich branching degree is needed to enforce and assess the stochastic dominance relationships with a negative effect on the computational sustainability of the instance sent to the solver. To ease such constraint we have resumed to an *high performance computing* (HPC) platform as further discussed in section 5.

In order to visually demonstrate and compare the scale of different dynamic ISD models more intuitively, we consider a symmetric scenario tree, i.e. the branching degrees are equal across all

Table A1. Size of technology matrix, binary variables and quadratic terms in different multi-stage ISD models.

	MISD-1.0	MISD-1.q_β
No. rows	$(m+1)\sum_{t=1}^{T}\prod_{j=1}^{t}N_j + m\left(2\prod_{j=1}^{T}N_j+1\right)+1+\sum_{t=1}^{T}\left(\left(\prod_{j=1}^{t}N_j\right)^2+\prod_{j=1}^{t}N_j-1\right)$	$(m+1)\sum_{t=1}^{T}\prod_{j=1}^{t}N_j + m\left(2\prod_{j=1}^{T}N_j+1\right)+1+\sum_{t=1}^{T}\left(\left(\prod_{j=1}^{t}N_j\right)^2+4\prod_{j=1}^{t}N_j+2\right)$
No. columns	$(3m+1)\left(1+\sum_{t=1}^{T}\prod_{j=1}^{t}N_j\right)+\sum_{t=1}^{T}\left(\prod_{j=1}^{t}N_j\right)^2$	$(3m+1)\left(1+\sum_{t=1}^{T}\prod_{j=1}^{t}N_j\right)+\sum_{t=1}^{T}\left(\prod_{j=1}^{t}N_j+2\right)\left(\prod_{j=1}^{t}N_j\right)$
Binary variables	$\sum_{t=1}^{T}\left(\prod_{j=1}^{T}N_j\right)^2$	$\sum_{t=1}^{T}(l_t+1)\left(\prod_{j=1}^{t}N_j\right)$
Quadratic terms	0	0
	McISD-1.0	**McISD-1.q_β**
No. rows	$(m+1)\sum_{t=1}^{T}\prod_{j=1}^{t}N_j + m\left(2\prod_{j=1}^{T}N_j+1\right)+1+\sum_{t=1}^{T}\prod_{j=1}^{t-1}N_j\,(N_t^2+N_t-1)$	$(m+1)\sum_{t=1}^{T}\prod_{j=1}^{t}N_j + m\left(2\prod_{j=1}^{T}N_j+1\right)+1+\sum_{t=1}^{T}\prod_{j=1}^{t-1}N_j\,(N_t^2+4N_t+2)$
No. columns	$(3m+1)\left(1+\sum_{t=1}^{T}\prod_{j=1}^{t}N_j\right)+\sum_{t=1}^{T}N_t\prod_{j=1}^{t}N_j$	$(3m+1)\left(1+\sum_{t=1}^{T}\prod_{j=1}^{t}N_j\right)+\sum_{t=1}^{T}(N_t+2)\prod_{j=1}^{t}N_j$
Binary variables	$\sum_{t=1}^{T}N_t\prod_{j=1}^{t}N_j$	$\sum_{t=1}^{T}\sum_{r=1}^{\prod_{j=1}^{t-1}N_j}(l_{t,r}+1)N_t$
Quadratic terms	0	0
	MISD-2.0	**MISD-2.q_β**
No. rows	$(m+1)\sum_{t=1}^{T}\prod_{j=1}^{t}N_j + m\left(2\prod_{j=1}^{T}N_j+1\right)+1+\sum_{t=1}^{T}\left(\left(\prod_{j=1}^{t}N_j\right)^2+\prod_{j=1}^{t}N_j-1\right)$	$(m+1)\sum_{t=1}^{T}\prod_{j=1}^{t}N_j + m\left(2\prod_{j=1}^{T}N_j+1\right)+1+\sum_{t=1}^{T}\left(2\left(\prod_{j=1}^{t}N_j+1\right)^2+1\right)$
No. columns	$(3m+1)\left(1+\sum_{t=1}^{T}\prod_{j=1}^{t}N_j\right)+\sum_{t=1}^{T}\left(\prod_{j=1}^{t}N_j\right)^2$	$(3m+1)\left(1+\sum_{t=1}^{T}\prod_{j=1}^{t}N_j\right)+\sum_{t=1}^{T}\left(2\prod_{j=1}^{t}N_j+1\right)\left(\prod_{j=1}^{t}N_j\right)$
Binary variables	0	0
Quadratic terms	0	$\sum_{t=1}^{T}\left(\prod_{j=1}^{t}N_j-l_t+1\right)\left(\prod_{j=1}^{t}N_j\right)$
	McISD-2.0	**McISD-2.q_β**
No. rows	$(m+1)\sum_{t=1}^{T}\prod_{j=1}^{t}N_j + m\left(2\prod_{j=1}^{T}N_j+1\right)+1+\sum_{t=1}^{T}\prod_{j=1}^{t-1}N_j\,(N_t^2+N_t-1)$	$(m+1)\sum_{t=1}^{T}\prod_{j=1}^{t}N_j + m\left(2\prod_{j=1}^{T}N_j+1\right)+1+\sum_{t=1}^{T}\prod_{j=1}^{t-1}N_j\,(2(N_t+1)^2+1)$
No. columns	$(3m+1)\left(1+\sum_{t=1}^{T}\prod_{j=1}^{t}N_j\right)+\sum_{t=1}^{T}N_t\prod_{j=1}^{t}N_j$	$(3m+1)\left(1+\sum_{t=1}^{T}\prod_{j=1}^{t}N_j\right)+\sum_{t=1}^{T}\sum_{r=1}^{\prod_{j=1}^{t}N_j}(2N_t+1)$
Binary variables	0	0
Quadratic terms	0	$\sum_{t=1}^{T}\sum_{r=1}^{\prod_{j=1}^{t-1}N_j}(N_t-l_{t,r}+1)$

Table A2. Technology matrix, binary variables and quadratic terms for multi-stage ISD models under a [5-5-5-5] scenario tree and 6 assets.

	MISD-1.0	MISD-1.2	MISD-1.5	MISD-1.8	MISD-2.0	MISD-2.2	MISD-2.5	MISD-2.8
No. rows	420,643	422,995	422,995	422,995	420,643	423,779	423,779	423,779
No. columns	421,739	382,064	382,064	382,064	421,739	791,557	795,447	799,337
Binary variables	406,900	325,520	203,450	81,380	0	0	0	0
Quadratic terms	0	0	0	0	0	11153	7263	3373
	McISD-1.0	McISD-1.2	McISD-1.5	McISD-1.8	McISD-2.0	McISD-2.8	McISD-2.5	McISD-2.2
No. rows	17,491	19,363	19,363	19,363	17,491	26,071	27,007	28,879
No. columns	15,619	15,931	15,931	15,931	15,619	378,457	379,237	380,797
Binary variables	780	624	390	156	0	0	0	0
Quadratic terms	0	0	0	0	0	42,900	42,900	42,900

$n \in N_t$ in stage t. We can then determine the size of the technology matrix associated with different reformulations for first-order MISD, see equations (4)–(8e), second-order MISD, equations (9a)–(10a) as well as first-, see equations (11a) and second-order McISD from (12a). Meanwhile, we account for the binary variables as well as the quadratic terms in the reformulations. The estimated matrix dimensions are shown in table A1.

From table A1, we can observe that,

- MISD versus McISD of order 1 or 2: we can notice the increase by a factor of 1 of the number of rows and the number of columns in the former relative to the latter: $\sum_{t=1}^{T}((\prod_{j=1}^{t} N_j)^2 + \prod_{j=1}^{t} N_j - 1)$ versus $\sum_{t=1}^{T} \prod_{j=1}^{t-1} N_j(N_t^2 + N_t - 1)$ for the rows and $\sum_{t=1}^{T}(\prod_{j=1}^{t} N_j)^2$ versus $\sum_{t=1}^{T} N_t \prod_{j=1}^{t} N_j$ for the columns.
- Note that the number of binary variables is a quadratic polynomial in the variables l_t and $\prod_{j=1}^{t} N_j$ for $t = 1, \ldots, T$ in the MISD-1.β. Meanwhile in the McISD-1.β case then it is a quadratic polynomial in the variables $l_{t,r}$ and N_t for $r = 1, \ldots, \prod_{j=1}^{t-1} N_j$ and $t = 1, \ldots, T$. Furthermore, the number of binary variables in the conditional case is bound above by the polynomial $\sum_{t=1}^{T} \prod_{j=1}^{t} N_j N_t$, since $l_{t,r} \le N_t - 1$ for $r = 1, \ldots, \prod_{j=1}^{t-1} N_j$ and $t = 1, \ldots, T$. On the other hand, the number of binary variables in the stagewise case is bound above by the polynomial $\sum_{t=1}^{T}(\prod_{j=1}^{t} N_j)^2$, since $l_t \le \prod_{j=1}^{t} N_j - 1$ for $t = 1, \ldots, T$.

A stage-wise formulation leads to a quadratic polynomial in the variables $\prod_{j=1}^{t} N_j$, $t = 1, \ldots, T$ with leading term $\sum_{t=1}^{T}(\prod_{j=1}^{t} N_j)^2$ resulting in extremely large instances for sufficiently large scenario trees.On the contrary, the conditional approach leads to a quadratic polynomial in the variables $\prod_{j=1}^{t-1} N_j$, $\prod_{j=1}^{t} N_j$ and N_t, $t = 1, \ldots, T$ with leading term $\sum_{t=1}^{T} N_t \prod_{j=1}^{t} N_j$. In particular, the ratio between the leading terms of stagewise and conditional is $\prod_{j=1}^{t-1} N_j$ which implies that, relative to the conditional approach, the stagewise approach leads to an exponential increase in dimension of the associated feasibility conditions.

Appendix 4. Scenario generation

We shall consider a model based on inflation and a 3-month interest rate. Notice that monthly gross returns are computed as $r_{i,t} = \frac{v_{i,t}}{v_{i,t-1}} - 1$ over 1 month (and can be negative or positive). This is different from the interest rate which I call $r_{0,t}$ that can be quarterly, is generally positive and we consider also monthly frequency.

We describe the model by the following system of equations:

$$\begin{pmatrix} \pi_t \\ r_{0,t} \end{pmatrix} = \beta_0 + \beta_1 \begin{pmatrix} \pi_{t-1} \\ r_{0,t-1} \end{pmatrix} + C_0 \begin{pmatrix} e_t^{\pi} \\ e_t^{r} \end{pmatrix}, \tag{A1a}$$

$$r_{i,t} = \lambda_{0,i} + \lambda_{1,i}\pi_t + \sum_{j=0,1,\ldots,i-1} \theta_{j,i} r_{j,t} + \theta_{i,i} r_{i,t-1} + \sum_{j=0,1,\ldots,i} C_{i,j} e_t^{j}, \quad i = 1, 2, \ldots, 6, \ t \in \mathcal{T}, \tag{A1b}$$

where the matrix C is the Cholesky factor of the correlation matrix of the residuals. The estimation of the parameters was carried out by using linear regression. The coefficients of the model are in table A3.

When considering the sequence of testproblems from January 2021 to, the last one starting on, October 2023, statistical estimates are updated quarterly maintaining a data history of the past 12 years to preserve the non-anticipativity assumption and generate genuine out-of-sample results.

The correlation matrix is given by:

$$Corr = \begin{pmatrix} 1.0000 & & & & & & & \\ 0.1556 & 1.0000 & & & & & & \\ 0.0570 & -0.0005 & 1.0000 & & & & & \\ -0.1764 & -0.1536 & -0.0010 & 1.0000 & & & & \\ 0.1003 & -0.0007 & 0.0002 & -0.0019 & 1.0000 & & & \\ 0.1230 & 0.0200 & 0.0022 & -0.0002 & 0.0010 & 1.0000 & & \\ 0.0134 & -0.1056 & -0.0014 & -0.0026 & 0.0025 & -0.0277 & 1.0000 & \\ -0.0449 & 0.0222 & 0.0048 & -0.0145 & 0.0205 & -0.0220 & -0.0201 & 1.0000 \end{pmatrix}$$

Table A3. OLS estimates, monthly data December 01 2009–June 01 2022 for GDP inflation, 3-month interest rate TMIR and risky assets SHY, VCSH, VCIT, XLU, XLV, and XLE.

	Constant	π_{t-1}	$r_{0,t-1}$							R^2	F coeff	p-value	e_t^*
π_t	0.0008	1.0108	− 0.106							0.969	2.42E+03	0	$N(0, 0.0000128)$
	constant	π_{t-1}	$r_{0,t-1}$										
$r_{0,t}$	− 0.00081	0.0408	1.0078							0.9749	2.97E+03	0	$N(0, 0.00000246)$
	constant	π_t	$r_{0,t}$	$r_{1,t-1}$									
$r_{1,t}$	0.000578	− 0.05298	0.07233	0.03415						0.1175	6.7492	0.00026	$N(0, 0.0000093)$
	constant	π_t	$r_{0,t}$	$r_{1,t}$	$r_{2,t-1}$								
$r_{2,t}$	0.000872	− 0.0435	0.05219	1.17861	− 0.0659					0.3186	17.646	0	$N(0, 0.000038)$
	constant	π_t	$r_{0,t}$	$r_{1,t}$	$r_{2,t}$	$r_{3,t-1}$							
$r_{3,t}$	− 0.00041	0.0495	− 0.04588	0.32841	2.176	0.029263				0.951	323.38	0	$N(0, 0.0000257)$
	constant	π_t	$r_{0,t}$	$r_{1,t}$	$r_{2,t}$	$r_{3,t}$	$r_{4,t-1}$						
$r_{4,t}$	0.00612	0.1601	− 0.4146	− 2.0476	0.3321	1.1829	− 0.2238			0.2891	10.101	2.34E-09	$N(0, 0.00119)$
	constant	π_t	$r_{0,t}$	$r_{1,t}$	$r_{2,t}$	$r_{3,t}$	$r_{4,t}$	$r_{5,t-1}$					
$r_{5,t}$	0.0137	− 0.09625	− 0.201	− 3.8057	3.4635	− 0.6956	0.2983	− 0.186		0.301	9.0902	2.58E-09	$N(0, 0.00114)$
	constant	π_t	$r_{0,t}$	$r_{1,t}$	$r_{2,t}$	$r_{3,t}$	$r_{4,t}$	$r_{5,t}$	$r_{6,t-1}$				
$r_{6,t}$	− 0.0063	0.5707	− 0.8375	− 11.976	10.7226	− 2.2231	− 0.123	0.5992	− 0.1309	0.451	15.1061	0	$N(0, 0.0039)$

Note: R^2 is the coefficient of determination of the regression, the F coefficient defines the significance of the adopted model specification, the p-value is the probability of the Jarque–Bera test to validate the hypothesis of normality of the residuals, whose distributions are in the last column.

Appendix 5. $\mathcal{L}(1296, k.q_\beta = \{1.8985, 2.0\}, m^*, 2, 1, 2022Q1)$ ISD dominance over 4 stages, benchmark 2

Algorithm 1 Generation of scenario tree for risky assets SHY, VCSH, VCIT, XLU, XLV and XLE

Require: Set data history of risky assets and compute the benchmark.
Input: Set the root nodal return r_0 at time $t = 0$.
Specify scenario tree structure: Set tree branching structure $b = (b_1, b_2, \dots, b_T)$, resulting into $\#\mathcal{N}_T = \sum_{t=0}^{T} \prod_{j=1}^{t} b_j$ scenarios in stage T.
for $t \in \mathcal{T}$, $t < T$ **do**
  for $n \in \mathcal{N}_t$ **do**
    Step 1: Generate a random vector $y \in \mathbb{R}^8$ with normal distribution $N(0, I_8)$.
    Step 2:

$$\begin{pmatrix} \pi_n \\ r_{0,n} \end{pmatrix} = \beta_0 + \beta_1 \begin{pmatrix} \pi_{n-} \\ r_{0,n-} \end{pmatrix} + C_0 \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

    for $j = 3$ to 8 **do**

$$r_{n,j} = \lambda_{0,j} + \pi_{n,j}\lambda_{1,j} + r_{0,n,j}\lambda_{2,t} + \theta_{j,1:(j-1)} r_{n,1:(j-1)} + \theta_{j,j} r_{n-,j} + C_{1,j,1:j} y_{3:j}$$

    end for
    Check for absence of arbitrage:
    Define $h = 0$
    while True **do**
      Step 1: Consider the matrix $a = (r_{l,j})_{3 \le j \le 8, l \in n+}$.
      Step 2: Solve the optimization problem

$$\min 0 \tag{A2}$$

$$\text{s.t.} \quad w > 0, \quad w \in \mathbb{R}^m \tag{A3}$$

$$aw = 1_m \tag{A4}$$

      if (A2) is unfeasible **then** $h = 0$.
      else
        $h = 1$
      end if
    end while
  end for
end for **Output:** $((r_{n,j})_{3 \le j \le 8})_{n \in \mathcal{N}}$.

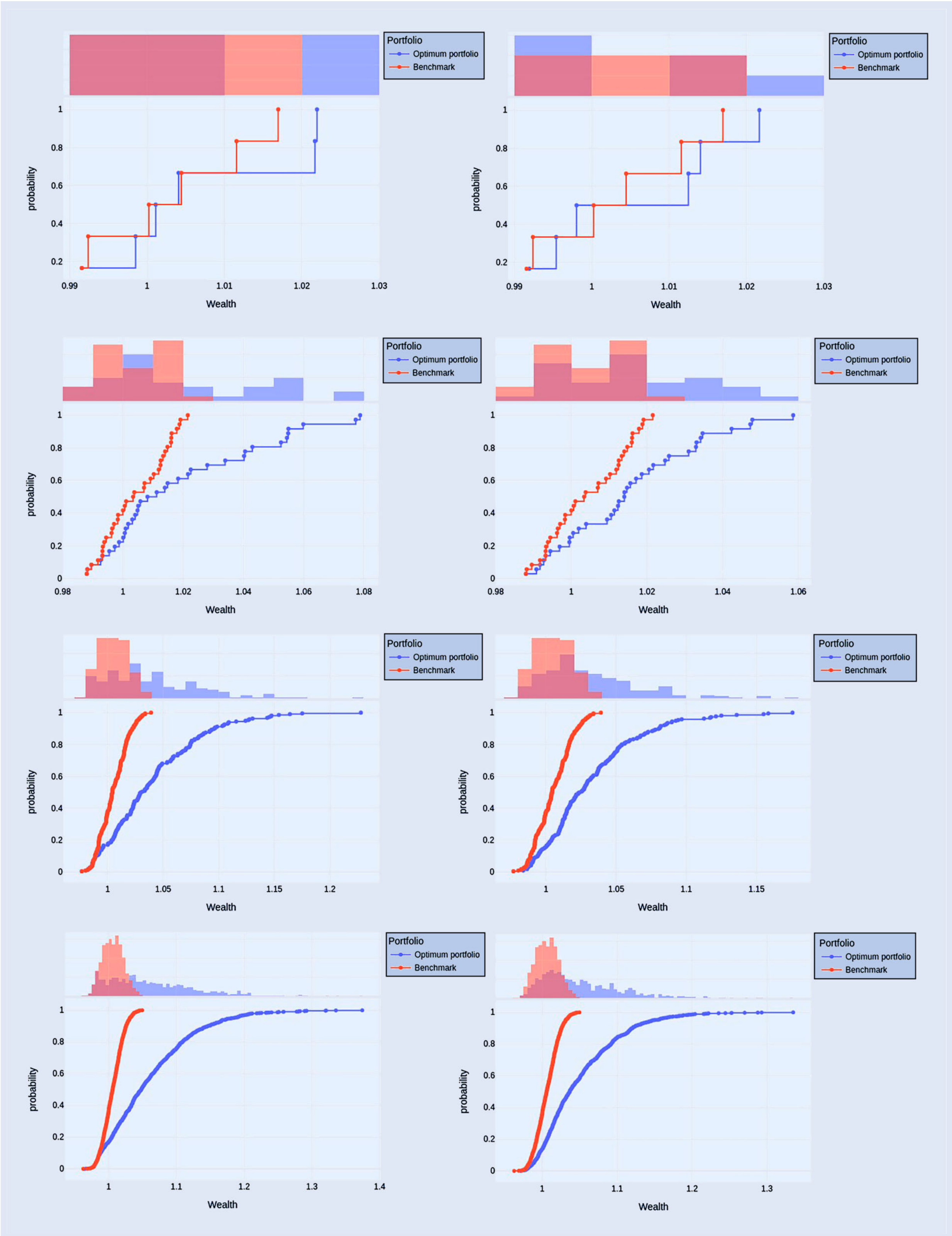

Figure A1. MISD (left) and McISD (right) 2.0 CDFs every stage in 6^4 scenarios problem.

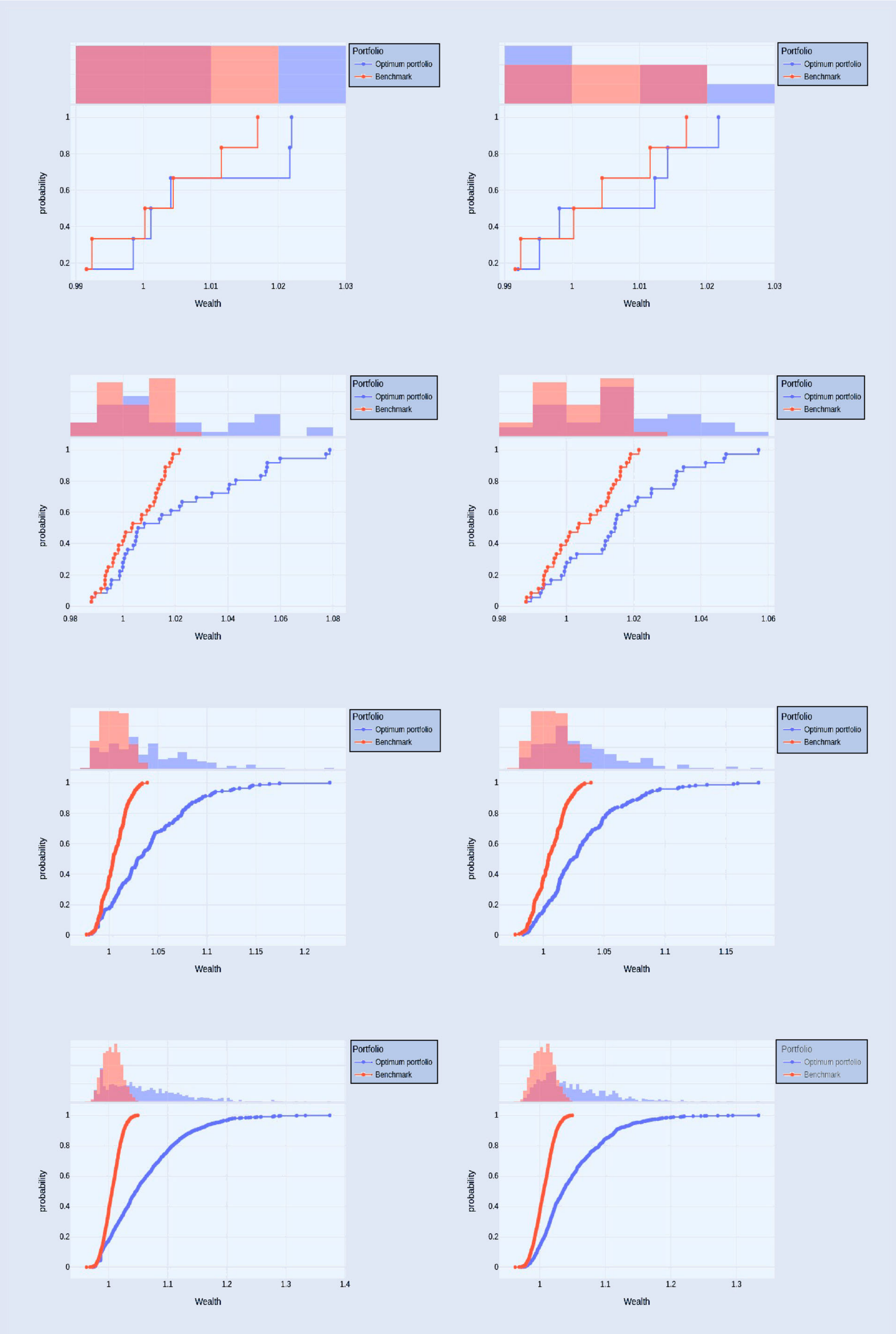

Figure A2. MISD (left) and McISD (right) 1.8985, CDFs every stage in 6^4 scenarios problem.

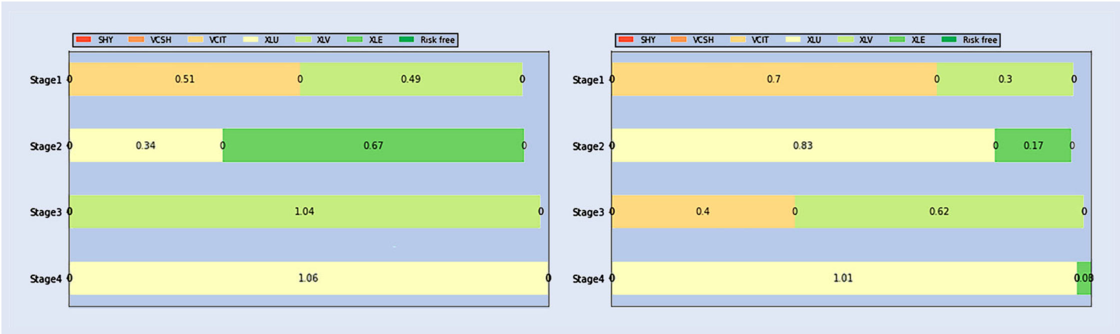

Figure A3. Optimal portfolio strategy along the median wealth scenario [6*4] tree. MISD 2.0 left and McISD 2.0 right.

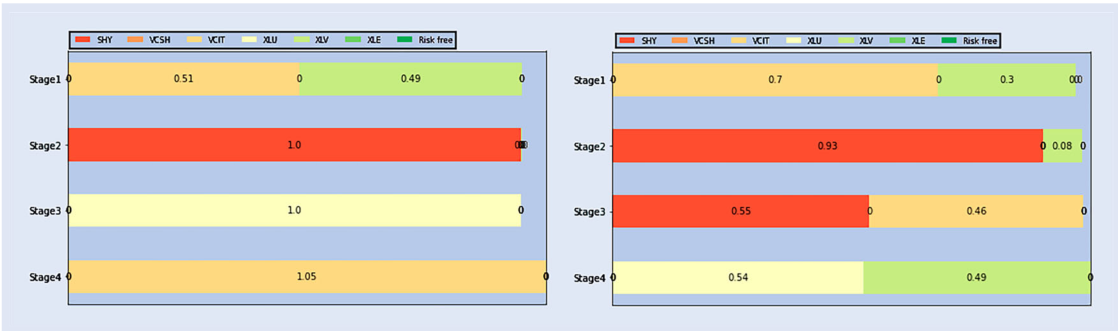

Figure A4. Optimal portfolio strategy along the median wealth scenario [6*4] tree. MISD-1.8985 left and McISD-1.8985 right.

Appendix 6. In-sample results: $\mathcal{L}(S_6, k.q_\beta = \{1.75, 2.0\}, j = 1, T = 1, t_0^*)$ optimal portfolio strategies along a median scenario, Jan 2021–Dec 2023

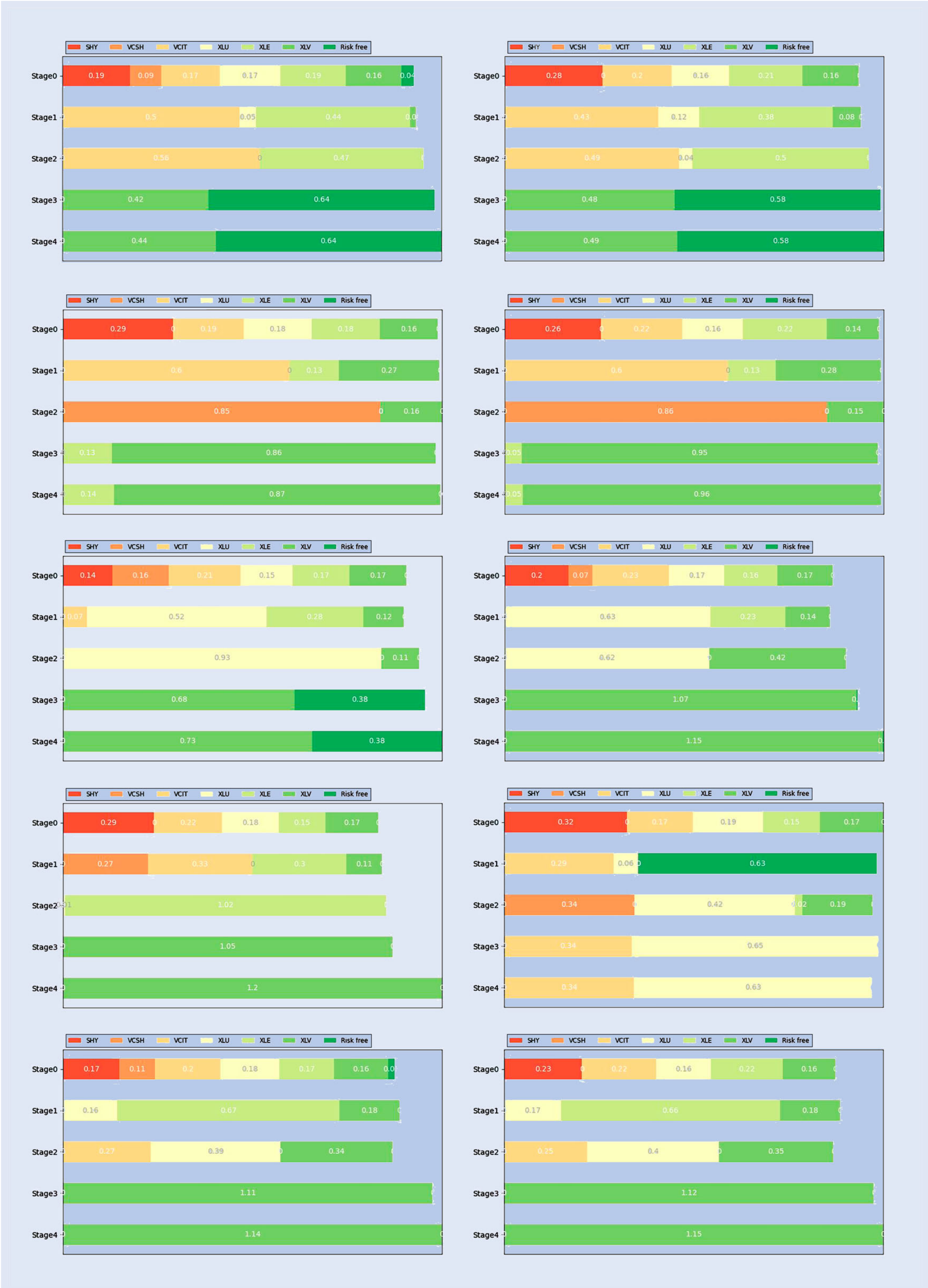

Figure A5. $40 * 10 * 7^{*2}$, McISD-1.75 versus 2.0 (left to right), benchmark 1, median scenario optimal portfolios.

Figure A6. $40 * 10 * 7^2$, McISD-1.75 versus 2.0 (left to right), benchmark 1, median scenario optimal portfolios.

Figure A7. $40 * 10 * 7^2$, McISD-1.75 versus 2.0 (left to right), benchmark 1, median scenario optimal portfolios.