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# Optimal trading and competition with information in the price impact model

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Information drives trading and hence affects price dynamics. We study how an informed trader optimally trades, how multiple traders compete with each other, and how their competition affects price dynamics as well as their inventories in the linear price impact model framework. We formulate the informed trading problem as a stochastic optimal control problem and a stochastic differential game in different cases. By virtue of different methods, problems are solved explicitly in all cases. In the case of a single trader, the risk-averse informed trader trades less in the beginning and then speeds up to reduce exposure to the volatility of dynamic information. In addition, the risk-averse trader reveals information more slowly and thus contributes less to price efficiency. When multiple risk-neutral traders compete with information, regardless of the competition structure, they tend to trade more rapidly in the early stage to fill orders at a good price. Competition and the presence of a leader help promote price efficiency, and competition also leads to the empirically observed mean-reverting behavior of price dynamics and order flows. Furthermore, among risk-averse informed traders, inventory management triggers predatory trading and active information leakage during informed trading, and the positions of traders with the same level of risk aversion tend to be identical and mean-reverting to zero under multiple rounds of competition.

*Keywords*: Informed trading; Price impact; Risk aversion; Stochastic optimal control; Stochastic differential game

## 1. Introduction

In financial markets, information drives trading. Informed traders can use information to know the asset's value in advance and profit from their corresponding trades. Moreover, the impact of such activities on the market is always of great concern. In this paper, we study how an informed trader optimally trades, how multiple traders compete with each other, and how their competition affects price dynamics as well as their inventories.

The pioneering work on informed trading was given by Kyle (1985). Noise traders provide stochastic market fluctuations while a single risk-neutral informed trader and risk-neutral market makers engage in a game to give the equilibrium price, and static private information will be gradually incorporated into the price throughout the whole transaction. The continuous-time version of the model by Kyle (1985) was subsequently developed by Back (1992). In this model, uniform information revelation can also be obtained and more general cases can be studied. In addition, Holden and Subrahmanyam (1992) and Back *et al.* (2000) studied competition among multiple informed traders. Unlike the case of a single informed trader, competition renders information to be disclosed much more rapidly, or even instantaneously. These exceptional studies demonstrate how informed trading should be conducted and how such trading activities affect the market. Furthermore, the assumption about static information can be extended. In fact, during the trade, informed traders can still have continuous acquisition of information, i.e. informed trading and information acquisition can be carried out simultaneously. Back and Pedersen (1998) considered this type of dynamic information, and they found the optimal trading strategy should be adapted not only to the price but also to the signal. The concept of dynamic information is extremely important in modern financial markets and is widely used in algorithmic and high-frenquency trading (see for example Cartea and Jaimungal 2016, Foucault *et al.* 2016, Cartea *et al.* 2018, Lehalle and Neuman 2019, Sastry and Thompson 2019, Donnelly and Lorig 2020, Forde *et al.* 2022, Fouque *et al.* 2022, Neuman and Voß 2022, Micheli *et al.* 2023).

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Generally speaking, the efficient use of dynamic information can lead to substantial revenues, as well as the significant improvement of the original trading strategy.

It is worth mentioning that information used by traders can be either news related to fundamental values or order book data which reflects current market sentiment. These miscellaneous information helps predict future prices, but at the same time, conceals huge uncertainty. Therefore, risk management of dynamic information is highly imperative. The CARA utility was first considered by Holden and Subrahmanyam (1994) and Baruch (2002) and further studied by Guéant (2016), Sastry and Thompson (2019), and Barger and Donnelly (2021). However, the CARA utility makes it difficult to solve some complicated models. Forsyth *et al.* (2012) and Almgren (2012) considered quadratic variation as an alternative risk measure, which makes the model convenient to handle. More importantly, quadratic running and terminal penalties are systematically and thoroughly studied in Cartea *et al.* (2015) to descirbe the risks faced during the trade, and this standard framework is now quite prevalent in the research on algorithmic trading (see for example Cartea and Jaimungal 2016, Fouque *et al.* 2022, Neuman and Voß 2022, 2023).

In this paper, based on the empirical observations in Cartea *et al.* (2015), like Donnelly and Lorig (2020) we decide to focus on the price impact of orders instead of exploring the equilibrium pricing rules of market makers. The discrete price impact model was initially developed by Almgren and Chriss (2000) to tackle optimal liquidation problem. Orders placed by traders cause temporary and permanent price impact. On the one hand, the transaction price is not the current market price and will suffer an additional slippage. On the other hand, the transaction itself reveals trading intentions or information, leading to the market being permanently affected by the order. And thanks to the thorough research in Cartea *et al.* (2015), we can use quadratic running and terminal penalties to represent the risk in the continuous-time model, and hence derive the explicit trading strategy for the informed trader who is risk averse to dynamic information. Even the risk-neutral trader won't fully reveal the information due to the presence of temporary price impact. Moreover, the risk-averse trader initially trades cautiously and subsequently accelerates their trading to mitigate exposure to dynamic information volatility, which is consistent with the numerical result in Sastry and Thompson (2019) where the CARA utility is considered in discrete time.

In addition, high sensitivity to information and quick processing of information are distinctive characteristics of high-frequency traders (HFTs) (see Brogaard *et al.* 2014, 2019). When HFTs engage in informed trading, competition is inevitable because the dynamic information they acquire is almost identical. More importantly, such competition can significantly affect market microstructure and therefore deserves special attention. Intertwined price impact was used by Carlin *et al.* (2007) to describe how traders affect each other when they execute orders, and in the same way, we can also study how multiple informed traders compete for liquidity in the market. Nash equilibrium is a key concept in a non-cooperative game. By solving Nash equilibria between traders, specific trading patterns such as predatory trading and liquidity provision can be studied (see Carlin *et al.* 2007, Schied and Zhang 2017, Luo and Schied 2019, Voß 2022, Neuman and Voß 2023, Micheli *et al.* 2023). In this paper, we would like to study the more general case. In reality, large institutional traders can open huge positions by virtue of their capital advantage, and HFTs can place orders further ahead to seize trading opportunities by virtue of their latency advantage. Thus, with identical information, they can always gain the majority of the profits, while retail traders can only follow them to get the remaining little profits. As a result, inspired by the ideas from DeMiguel and Xu (2009), Huang *et al.* (2019), Colombo and Labrecciosa (2019), and Fujii and Takahashi (2022), we decide to study the multiple-leader Stackelberg-Nash equilibrium of informed trading. We find the solution to the associated HJB equations and derive explicit closed-loop strategies for risk-neutral traders in the presence of temporary price impact. When temporary price impact approaches zero, we recover the results in Holden and Subrahmanyam (1992) and Back *et al.* (2000) that there is no equilibrium and information is revealed immediately. Furthermore, competition accelerates the incorporation of information into the market and the presence of a leader helps promote price efficiency. The price and the net order flow can both be approximately considered to be mean-reverting, which matches the assumptions in Carcano *et al.* (2005), Zhang and Zhang (2008), Bergault *et al.* (2022), Bechler and Ludkovski (2015), and Ma *et al.* (2020) respectively. For traders who are risk averse or need inventory management, unfortunately we are unable to obtain explicit closed-loop strategies. However, we can give explicit open-loop deterministic strategies in a Nash equilibrum for the sake of subsequent analysis. Inventory management triggers predatory trading and active information leakage during informed trading, and when traders have the same level of risk aversion, their positions tend to be identical and mean-reverting to zero under multiple rounds of competition.

Note that in this paper, for the sake of simplicity and tractability of the model, like many other papers we don't consider execution uncertainty, which has been theoretically and empirically studied by Cheng *et al.* (2017) and Carmona and Leal (2023), respectively. In fact, on the one hand, the use of limit orders forces traders to endure execution uncertainty and adverse selection. On the other hand, actively adding execution uncertainty allows to hide trading intentions and avoid the detection of other traders (see Yang and Zhu 2020, Sağlam 2020). Fortunately, execution uncertainty doesn't dramatically affect our model and results, so all of our analysis is meaningful.

This paper is structured as follows. Section 2 introduces dynamic information and formulates informed trading in the linear price impact model framework. Section 3 provides the feedback solution in the case of a single risk-averse trader. Section 4 provides closed-loop strategies in the case of multiple risk-neutral traders and analyzes price efficiency under different competition structures. Section 5 provides open-loop deterministic strategies in the case of risk-averse traders and illustrates trading patterns arising from inventory management. Section 6 concludes the paper.

## 2. Model setup

We assume that all information acquisition and informed trading occur within the time interval $[0, T]$. The interval is short and the risk-free rate is deemed zero. $N(N \geq 1)$ informed traders engage in informed trading with dynamic information while noise traders and distressed traders contribute to short-term imbalances between supply and demand resulting in stochastic price fluctuations. Informed traders can get some information about the asset's value, and thus they can trade ahead of other market participants without this knowledge to make profits.

The 'long-run' value, denoted by $Z$, is defined as the real value of the asset at time $T$. Informed traders have access to dynamic information, which enables them to have a knowledge of $Z$, and this knowledge will become increasingly precise over time. We follow Foucault *et al.* (2016) and suppose that informed traders' view of the asset's value $P_t$ follows

$$\mathrm{d}P_t = \sigma_1 \mathrm{d}B_t, \tag{1}$$

where $\sigma_1$ is a positive constant and $B_t$ is a Brownian motion. $\{\mathcal{F}_t^P\}_{0 \leq t \leq T}$ is used to denote the filtration generated by $P_t$. Here we make a more general assumption. We suppose that one part of the information about $Z$ can be fully determined by $P_T$, yet another part can not be known in advance even by informed traders. Informed traders acknowledge that dynamic information isn't perfectly accurate, and there may be a discrepancy between their view of the value $P_T$ and the real value $Z$. As a result, we assume

$$Z = P_T + \varepsilon, \tag{2}$$

where $\varepsilon \sim N(0, \sigma_2^2)$. $\sigma_2$ is a positive constant and $\varepsilon$ is independent of $\{\mathcal{F}_t^P\}_{0 \leq t \leq T}$.

Informed traders trade continuously in the market. $v_t^i$ and $Q_t^i$ is used to denote the trading rate and position of the $i$th informed trader at time $t$ respectively. A positive $v_t^i$ means buying, and a negative $v_t^i$ means selling. $Q_t^i$ follows

$$\mathrm{d}Q_t^i = v_t^i \,\mathrm{d}t. \tag{3}$$

Noise traders and distressed traders cause short-term imbalances between supply and demand, resulting in stochastic price fluctuations. Without the intervention of informed traders, price dynamics can be seen as a Brownian motion with no drift term. Orders from informed traders cause permanent and temporary price impact. The affected price process $S_t$ is driven by

$$\mathrm{d}S_t = \gamma \sum_{i=1}^{N} v_t^i \,\mathrm{d}t + \sigma_3 \,\mathrm{d}W_t, \tag{4}$$

where $\gamma$ and $\sigma_3$ are positive constants and $W_t$ is a Brownian motion independent of $B_t$.† $\{\mathcal{F}_t^S\}_{0 \leq t \leq T}$, the filtration generated by $S_t$, is independent of $\varepsilon$. The transaction price is given by

$$\widetilde{S}_t = S_t + \eta \sum_{i=1}^{N} v_t^i, \tag{5}$$

where $\eta$ is a positive constant. The $i$th informed trader's cash $X_t^i$ hence follows

$$\mathrm{d}X_t^i = -\widetilde{S}_t v_t^i \,\mathrm{d}t. \tag{6}$$

The market price $S_t$ can be seen as the view of the expected asset's value by other market participants without dynamic information at time $t$. Market participants acknowledge the presence of informed traders in the market and will adjust their view of the asset's value based on order flow. Previous studies on informed trading focus on equilibrium pricing rules proposed by Kyle (1985), while we consider that liquidity providers may not know when informed trading starts and ends. Therefore, like Donnelly and Lorig (2020) we decide to use price impact to describe the noisy relationship between order flow and price dynamics. The linear price impact model was first introduced by Almgren and Chriss (2000) and its rationale has been elaborated earlier in Carlin *et al.* (2007). The orders placed by informed traders change the market's view of the fair price and gradually reveal the information they have, causing permanent price impact on the market. The linear relationship between order flow and price move in (4), on the one hand, is supported by empirical observations in Cartea *et al.* (2015), and on the other hand, can guarantee no dynamic arbitrage (see Gatheral 2010, Guéant 2016). Actually, linear permanent price impact is highly similar with linear pricing rules in Kyle (1985). However, additional temporary price impact facilitates us to consider transaction costs. Temporary price impact arises from LOB dynamics. If a large amount of liquidity is needed in a short period, informed traders are forced to walk the LOB to execute their market orders, making the transaction price worse than the current market price. As the relationship between the trading rate and such slippage is noisy, a linear function is already a good approximation (see Cont *et al.* 2014, Frei and Westray 2015), although a power law function might be a better description (see Almgren 2003, Cartea *et al.* 2015). In addition, when there are many informed traders competing for liquidity in the market, their orders tend to interact with each other, resulting in intertwined temporary price impact in (5) (see Carlin *et al.* 2007, Schied and Zhang 2017, Voß 2022). What's more, replacing Arithmetic Brownian Motion by Geometric Brownian Motion (see Gatheral and Schied 2011, Forsyth *et al.* 2012) and other extensions (see Donnelly 2022 and references therein) have been extensively studied, but here we choose the simple price structure assumption in (4) and (5) to produce tractable problems and finally result in explicit expressions for solutions in the following sections.

Informed traders have access to both dynamic information and the current price. Therefore, they have the filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$, where $\mathcal{F}_t = \sigma(\mathcal{F}_t^P, \mathcal{F}_t^S)$. $\varepsilon$ is independent of $\{\mathcal{F}_t\}_{0 \leq t \leq T}$. $\mathcal{M}$ is used to denote the set of real-valued $\{\mathcal{F}_t\}_{0 \leq t \leq T}$-adapted processes. The set of admissible controls

† The correlation can also be considered. However, the resulting strategy does not change. For simplicity we opt to assume that the two Brownian motions are independent.

for informed traders is defined as

$$\mathcal{A} = \left\{ v \in \mathcal{M} : \mathbb{E} \int_0^T |v_t|^2 \, \mathrm{d}t < +\infty \right\}.$$

The $i$th informed trader has the initial cash $X_0^i$ and the initial position $Q_0^i$. No terminal condition is enforced on $Q_t^i$, and $Q_T^i$ depends on the strategy $v^i$. With a trading strategy $v^i \in \mathcal{A}$, the $i$th informed trader will end up with the cash $X_T^i$ as well as the position $Q_T^i$. For the $i$th informed trader, the position $Q_T^i$ is valued at the "long-run" value $Z$ instead of the market price $S_T$, and the terminal wealth $\Pi^i$ is hence given by

$$\Pi^i = X_T^i + Z Q_T^i. \tag{7}$$

We introduce another process $\Lambda_t = P_t - S_t$ and have

$$\mathrm{d}\Lambda_t = -\gamma \sum_{i=1}^N v_t^i \, \mathrm{d}t + \sigma_1 \, \mathrm{d}B_t - \sigma_3 \, \mathrm{d}W_t. \tag{8}$$

$\Lambda_t$ represents the difference in the perception of the expected asset's value by informed traders and other market participants. At time 0, informed traders observe the initial state $\Lambda_0$, which is an arbitrary constant that measures the impact of initial information on the expected value of the asset. $\Lambda_0 = 0$ means that initial information has no impact and informed traders have the same view of the expected asset's value as other market participants do at time 0. With straightforward computations, the terminal wealth $\Pi^i$ in (7) can by rewritten as

$$\Pi^i = \int_0^T \left( -\eta v_t^i \sum_{j=1}^N v_t^j + \Lambda_t v_t^i \right) \mathrm{d}t + \int_0^T \sigma_1 Q_t^i \, \mathrm{d}B_t + \varepsilon Q_T^i + (X_0^i + P_0 Q_0^i) \tag{9}$$

*Proof*

$$\begin{aligned}
\Pi^i &= (X_T^i - X_0^i) + (P_T Q_T^i - P_0 Q_0^i) + (Z - P_T) Q_T^i \\
&\quad + (X_0^i + P_0 Q_0^i) \\
&= \int_0^T \mathrm{d}X_t^i + \int_0^T P_t \, \mathrm{d}Q_t^i + \int_0^T Q_t^i \, \mathrm{d}P_t + (Z - P_T) Q_T^i \\
&\quad + (X_0^i + P_0 Q_0^i) \\
&= \int_0^T \left[ -\left( S_t + \eta \sum_{j=1}^N v_t^j \right) v_t^i + P_t v_t^i \right] \mathrm{d}t \\
&\quad + \int_0^T \sigma_1 Q_t^i \, \mathrm{d}B_t + \varepsilon Q_T^i + (X_0^i + P_0 Q_0^i)
\end{aligned}$$

With $\Lambda_t = P_t - S_t$ we get (9). ■

From the first term of (9) we can see that the final wealth of the informed trader depends not only on his own trading strategy, but also on the trading strategies of others. The second term represents the uncertainty caused by dynamic information during the trading horizon. According to Cheng *et al.* (2017) and Mariani and Fatone (2022), the Itô integration is well defined and has zero expectation due to $v^i \in \mathcal{A}$ (even when we add execution uncertainty). The third term represents the uncertainty from information inaccuracy $\varepsilon$. Obviously it has zero expectation because $\varepsilon$ and $Q_T^i$ are independent. The fourth term represents the expected initial wealth, which can't be changed by the trading strategy. In the following sections we will formulate different specific problems based on (9) for different scenarios.

## 3. The case of a single risk-averse trader

In this section, we consider the case where $N = 1$, i.e. a single trader in the market engages in informed trading. Without the impact of other informed traders, the terminal wealth (9) is given by

$$\Pi = \int_0^T [-\eta (v_t)^2 + \Lambda_t v_t] \, \mathrm{d}t + \int_0^T \sigma_1 Q_t \, \mathrm{d}B_t + \varepsilon Q_T + (X_0 + P_0 Q_0). \tag{10}$$

We also have

$$\mathbb{E}[\Pi] = \mathbb{E} \left[ \int_0^T [-\eta (v_t)^2 + \Lambda_t v_t] \, \mathrm{d}t \right] + (X_0 + P_0 Q_0).$$

We follow Cartea *et al.* (2015) to allow the informed trader to be risk averse by considering running and terminal penalties. For the second term in (10), the quadratic variation

$$QV = \mathbb{E} \int_0^T \sigma_1^2 (Q_t)^2 \, \mathrm{d}t$$

is used to measure the risk from dynamic information during the trading horizon. For the third term in (10), as $\varepsilon$ and $Q_T^i$ are independent, we use variance

$$Var = \sigma_2^2 \mathbb{E}[(Q_T)^2]$$

to measure the risk from information inaccuracy $\varepsilon$. The informed trader is committed to maximizing the utility function

$$\mathbb{E}[\Pi] - \lambda QV - \frac{\beta}{2} Var,$$

where $\lambda$ and $\beta$, two positive constants, are used to measure the aversion to two kinds of risk respectively. The problem of informed trading hence reduces to a stochastic control problem

$$\sup_{v \in \mathcal{A}} \mathbb{E} \left[ -\frac{\beta}{2} \sigma_2^2 (Q_T)^2 + \int_0^T [-\eta (v_t)^2 + \Lambda_t v_t - \lambda \sigma_1^2 (Q_t)^2] \, \mathrm{d}t \right]. \tag{11}$$

$X_0 + P_0 Q_0$ is omitted because it has no impact on our problem. $\mathcal{A}_t$ is the version of $\mathcal{A}$ restricted to $[t, T]$. $\mathbb{E}_t[\cdot]$ denotes the conditional expectation $\mathbb{E}[\cdot|\Lambda_t = \Lambda, Q_t = Q]$. We define the value function

$$V(t,\Lambda,Q) = \sup_{v\in\mathcal{A}_t} \mathbb{E}_t\left[-\frac{\beta}{2}\sigma_2^2(Q_T)^2 + \int_t^T [-\eta(v_u)^2 + \Lambda_u v_u - \lambda\sigma_1^2(Q_u)^2]\,\mathrm{d}u\right].$$

By dynamic programming principle, we obtain the HJB equation

$$V_t - \lambda\sigma_1^2 Q^2 + \frac{1}{2}(\sigma_1^2+\sigma_3^2)V_{\Lambda\Lambda} + \sup_{v\in\mathbb{R}}\left\{-\eta v^2 + (\Lambda + V_Q - \gamma V_\Lambda)v\right\} = 0 \qquad (12)$$

with the terminal condition

$$V(T,\Lambda,Q) = -\frac{\beta}{2}\sigma_2^2 Q^2.$$

We can obtain the optimal trading strategy by solving the HJB equation (12), and the result is shown in the following proposition.

PROPOSITION 3.1 *The optimal trading strategy $v^*$, solution of* (11)*, has an explicit feedback expression:*

$$v_t^* = \frac{\alpha(\Lambda_t+\gamma Q_t)}{\delta\sinh\big(\alpha(T-t)\big) + 2\eta\alpha\cosh\big(\alpha(T-t)\big)} - \alpha Q_t\cdot\frac{\delta\cosh\big(\alpha(T-t)\big)+2\eta\alpha\sinh\big(\alpha(T-t)\big)}{\delta\sinh\big(\alpha(T-t)\big)+2\eta\alpha\cosh\big(\alpha(T-t)\big)} \qquad (13)$$

*where $\delta = \gamma + \beta\sigma_2^2$ and $\alpha = \sqrt{\frac{\lambda\sigma_1^2}{\eta}}$.*

*Proof* See appendix. ■

The resulting trading strategy depends on the aversion to dynamic information $\lambda$ and the aversion to information inaccuracy $\beta$. It's natural to consider the risk-neutral case as follows.

COROLLARY 3.2 *When the informed trader is risk neutral to dynamic information ($\lambda = 0$) or when the information is static ($\sigma_1 = 0$), the optimal trading strategy reduces to*

$$v_t^* = \frac{\Lambda_t - \beta\sigma_2^2 Q_t}{\delta(T-t)+2\eta} \qquad (14)$$

*What's more, if the informed trader is fully risk neutral, i.e. $\lambda = 0$ and $\beta = 0$, the optimal trading strategy reduces to*

$$v_t^* = \frac{\Lambda_t}{\gamma(T-t)+2\eta} \qquad (15)$$

*Proof* When $\lambda = 0$, by solving the ODE we have $h(t) = \frac{2\eta\delta}{\delta(T-t)+2\eta}$ and $f(t) = \frac{2\eta}{\delta(T-t)+2\eta}$. With $v^* = \frac{f\Lambda+(\gamma f-h)Q}{2\eta}$, we get (14). Further letting $\beta = 0$, we obtain (15). ■

In figure 1, we show the trajectories of $\mathbb{E}[\Lambda_t]$ affected by the trading strategy (13). According to the dynamics of the difference in the perception $\Lambda_t$ in (8), we can analyze the trading patterns of the risk-averse informed trader by studying the trend of $\mathbb{E}[\Lambda_t]$. We take away the impact of the initial position by setting $Q_0 = 0$. The two panels illustrate the impact of two different risk aversion coefficients respectively. A large $\lambda$ means that the trader wants to reduce exposure to the volatility of dynamic information. In the early stage of trading, dynamic information is not precise enough, so the trader decreases his trades to keep his asset safe. In the later stage of trading, dynamic information is precise enough, so the trader speeds up his trading to gain profit. As a result, the trader doesn't trade uniformly, but instead initially trades cautiously and subsequently accelerates, and this pattern becomes more pronounced with more aversion to dynamic information (a larger $\lambda$). This result is consistent with the numerical result obtained in Sastry and Thompson (2019), although Sastry and Thompson (2019) used the CARA utility in the discrete model while we use the running penalty and provide the explicit solution in the continuous model. When $\lambda = 0$, the trader tends to trade uniformly to minimize transaction costs, but the trading rate depends on the aversion to information inaccuracy $\beta$. With a large $\beta$, the trader will keep a light position at the end of the trade to make his terminal wealth safe. Therefore, a larger $\beta$ will cause a smaller trading rate. Furthermore, given risk aversion $\lambda$ and $\beta$, the risk-averse trader trades more cautiously when the information is more volatile and more inaccurate (large $\sigma_1$ and $\sigma_2$).

As shown in the resulting strategies in (13), (14) and (15), the informed trader always trades in the same direction with $\Lambda_t$ unless the position is excessively heavy, and doesn't care about the position in the risk-neutral case. Dynamic information allows the informed trader to have a more accurate view of the asset's value relative to other market participants, resulting in the difference between $P_t$ and $S_t$. As a result, The speed with which $\Lambda_t$ moves towards zero indicates the speed with which the informed trader reveals the information to the market, and this speed can be used to measure price efficiency. From figure 1, we conclude that the risk-averse trader reveals information more slowly, and thus contributes less to price efficiency. However, as also stated in Barger and Donnelly (2021), even a risk-neutral informed trader will not fully reveal information at time $T$ due to transaction costs (see blue lines in figure 1). Actually, with the dynamics of the difference in the perception $\Lambda_t$ in (8) and the strategy in (15), we have

$$\mathrm{d}\Lambda_t = \frac{-\gamma\Lambda_t}{\gamma(T-t)+2\eta}\,\mathrm{d}t + \sigma_1\,\mathrm{d}B_t - \sigma_3\,\mathrm{d}W_t, \qquad (16)$$

which yields

$$\Lambda_t = \left(1 - \frac{\gamma t}{\gamma T + 2\eta}\right)\Lambda_0 + \int_0^t \frac{\gamma(T-t)+2\eta}{\gamma(T-u)+2\eta} \times (\sigma_1\,\mathrm{d}B_u - \sigma_3\,\mathrm{d}W_u). \qquad (17)$$

And further we obtain

$$v_t^* = \frac{\Lambda_0}{\gamma T + 2\eta} + \int_0^t \frac{\sigma_1\,\mathrm{d}B_u - \sigma_3\,\mathrm{d}W_u}{\gamma(T-u)+2\eta}.$$

The strategy is similar to the adaptive TWAP strategy proposed in Cheng *et al.* (2017). However, the difference is that

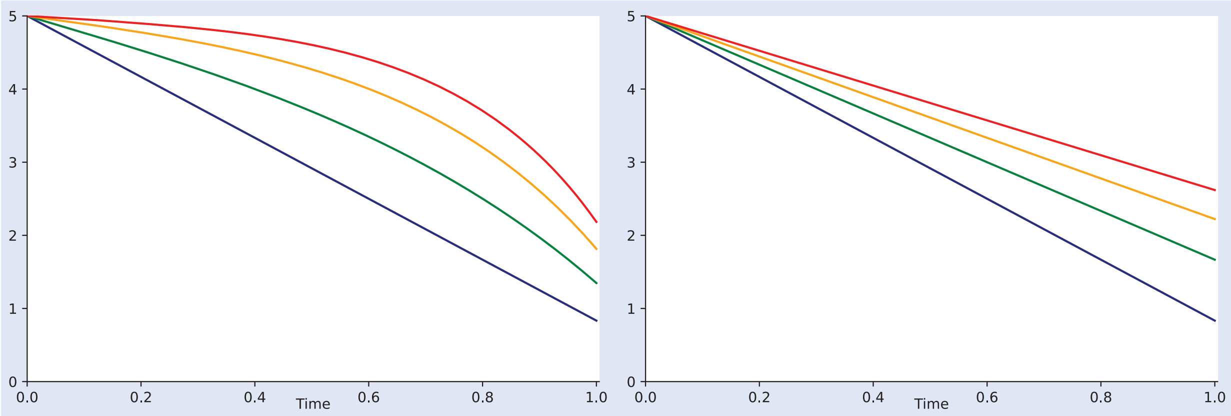


Figure 1. Sample trajectories of $\mathbb{E}[\Lambda_t]$ affected by the trading strategy (13). The parameters are $\Lambda_0 = 5$, $Q_0 = 0$, $T = 1$, $\gamma = 10$, $\eta = 1$. On the left, $\beta\sigma_2^2 = 0$, and blue, green, orange, red correspond to $\lambda\sigma_1^2 = 0, 3, 8, 15$ respectively. On the right, $\lambda\sigma_1^2 = 0$, and blue, green, orange, red correspond to $\beta\sigma_2^2 = 0, 3, 6, 9$ respectively.

our trading is driven by information rather than by position, and our strategy is adapted to price and information rather than position. In the case of no transaction cost ($\eta \to 0$), with (17) we have $\Lambda_T = 0$, which recovers the result in Kyle (1985) and Back (1992) that all information is revealed at time $T$. However, in the presence of transaction costs (a positive $\eta$), this result no longer holds. As

$$\mathbb{E}[\Lambda_T] = \frac{2\eta}{\gamma T + 2\eta}\Lambda_0,$$

the trader will not fully reveal the information at the end of the trade.

## 4. The case of multiple risk-neutral traders

In this section, we consider the case when multiple informed traders compete with the same dynamic information $P_t$. Inspired by the ideas from DeMiguel and Xu (2009), Huang *et al.* (2019), Colombo and Labrecciosa (2019), and Fujii and Takahashi (2022), we study a more general result: the multiple-leader Stackelberg-Nash equilibrium. And Nash equilibrium (no leader) and one-leader Stackelberg-Nash equilibrium can be seen as two special cases of our result. To derive explicit closed-loop strategies, we assume that these informed traders are all risk neutral. Such assumption not only makes the model solvable but also helps us better understand price efficiency under different competition structures.

We divide informed traders in the market into two levels: the leaders and the followers. Players in the same level can compete equally with each other, resulting in a Nash equilibrium. Leaders have priority over followers in developing trading strategies, so that traders between the two levels reach a Stackelberg equilibrium. According to the definition in Colombo and Labrecciosa (2019), we first give the concept of multiple-leader Stackelberg-Nash equilibrium.

Definition 1 Admissible trading rates of all leaders and followers form a multiple-leader Stackelberg-Nash equilibrium if

- given the trading rates of all leaders and other followers, no follower has an incentive to deviate from the equilibrium because the current trading rate already maximizes the corresponding goal.
- given the trading rates of other leaders and followers' optimal reactions to leaders, no leader has an incentive to deviate from the equilibrium because the current trading rate already maximizes the corresponding goal.

We assume that there are a total of $M + N$ informed traders in the market, including $M$ leaders and $N$ followers. We still use the model developed in section 2. For the sake of comprehension, the leaders' trading rates are still denoted by $v_t^i$ ($i = 1, 2, \ldots, M$), but the followers' trading rates are replaced by $\xi_t^i$ ($i = 1, 2, \ldots, N$). As all players are risk-neutral, they are devoted to maximizing their own expected revenue $\mathbb{E}[\Pi^i] - (X_0^i + P_0 Q_0^i)$. The problem faced by the $i$th leader is

$$\sup_{v^i \in \mathcal{A}} \mathbb{E}\left[\int_0^T \left[-\eta v_t^i \left(\sum_{j=1}^M v_t^j + \sum_{j=1}^N \xi_t^j\right) + \Lambda_t v_t^i\right] \mathrm{d}t\right], \quad (18)$$

and the problem faced by the $i$th follower is

$$\sup_{\xi^i \in \mathcal{A}} \mathbb{E}\left[\int_0^T \left[-\eta \xi_t^i \left(\sum_{j=1}^M v_t^j + \sum_{j=1}^N \xi_t^j\right) + \Lambda_t \xi_t^i\right] \mathrm{d}t\right]. \quad (19)$$

Risk-neutral traders don't care about their position. Therefore, we use $\mathbb{E}_t[\cdot]$ to denote the conditional expectation $\mathbb{E}[\cdot|\Lambda_t = \Lambda]$, and then we can define the $i$th leader's value function

$$V^i(t, \Lambda) = \sup_{v^i \in \mathcal{A}_t} \mathbb{E}_t\left[\int_t^T \left[-\eta v_u^i \left(\sum_{j=1}^M v_u^j + \sum_{j=1}^N \xi_u^j\right) + \Lambda_u v_u^i\right] \mathrm{d}u\right], \quad i = 1, 2, \ldots, M$$

and the $i$th follower's value function

$$U^i(t,\Lambda) = \sup_{\xi^i\in\mathcal{A}_t} \mathbb{E}_t\left[\int_t^T \left[-\eta\xi_u^i\left(\sum_{j=1}^M v_u^j + \sum_{j=1}^N \xi_u^j\right) + \Lambda_u\xi_u^i\right]\mathrm{d}u\right], \quad i=1,2,\ldots,N.$$

By dynamic programming principle, we obtain the HJB equations

$$V_t^i + \frac{1}{2}\left(\sigma_1^2+\sigma_3^2\right)V_{\Lambda\Lambda}^i + \sup_{v^i\in\mathbb{R}}\left\{-(\gamma V_\Lambda^i + \eta v^i)\left(\sum_{j=1}^M v^j + \sum_{j=1}^N \xi^j\right) + \Lambda v^i\right\} = 0, \quad i=1,2,\ldots,M \tag{20}$$

$$U_t^i + \frac{1}{2}(\sigma_1^2+\sigma_3^2)U_{\Lambda\Lambda}^i + \sup_{\xi^i\in\mathbb{R}}\left\{-(\gamma U_\Lambda^i + \eta\xi^i)\left(\sum_{j=1}^M v^j + \sum_{j=1}^N \xi^j\right) + \Lambda\xi^i\right\} = 0, \quad i=1,2,\ldots,N \tag{21}$$

with the terminal condition

$$V^i(T,\Lambda) = 0\ (i=1,2,\ldots,M) \quad \text{and} \quad U^i(T,\Lambda) = 0\ (i=1,2,\ldots,N).$$

By solving HJB equations (20) and (21), we derive the equilibrium strategies and the value functions in closed form, which are shown in the following proposition.

Proposition 4.1 *Consider $M$ leaders and $N$ followers that choose an optimal feedback control to solve the problems in* (18) *and* (19). *There exists a closed-loop multiple-leader Stackelberg-Nash equilibrium. The closed-loop strategy $v^i$ for the ith leader and $\xi^i$ for the ith follower have an explicit expression:*

$$\begin{aligned} v_t^{i,*} &= v_t^* = (N+1)\xi_t^*, \quad i=1,2,\ldots,M \\ \xi_t^{i,*} &= \xi_t^* = \frac{1}{\eta}\cdot\frac{MN+M+N-1}{(MN+M+N)^2-\varphi(t)}\Lambda_t, \quad i=1,2,\ldots,N \end{aligned} \tag{22}$$

*where* $\varphi(t) = e^{-\frac{MN+M+N-1}{MN+M+N+1}\frac{2\gamma}{\eta}(T-t)} \in (0,1]$. *The value function $V^i(t,\Lambda)$ for the ith leader and $U^i(t,\Lambda)$ for the ith follower, which solve* (20) *and* (21)*, have an explicit expression:*

$$\begin{aligned} V^i(t,\Lambda) &= V(t,\Lambda) = (N+1)U(t,\Lambda), \quad i=1,2,\ldots,M \\ U^i(t,\Lambda) &= U(t,\Lambda) = a(t)\Lambda^2 + b(t), \quad i=1,2,\ldots,N \end{aligned}$$

*where*

$$\begin{cases} a(t) = \dfrac{1}{2\gamma}\cdot\dfrac{1-\varphi(t)}{(MN+M+N)^2-\varphi(t)} \\ b(t) = \dfrac{\sigma_1^2+\sigma_3^2}{2\gamma(MN+M+N)^2}\left[T-t-\dfrac{\eta}{2\gamma}\cdot(MN+M+N+1)^2\ln\dfrac{(MN+M+N)^2-\varphi(t)}{(MN+M+N)^2-1}\right] \end{cases}.$$

*Proof* See appendix. ∎

This result covers two classic scenarios, and we state them in the next two corollaries.

Corollary 4.2 (Nash equilibrum) *In the case of $N$ equal players, the closed-loop strategy $\xi^i$ for the ith player is given by*

$$\xi_t^{i,*} = \frac{1}{\eta}\cdot\frac{(N-1)\Lambda_t}{N^2-e^{-\frac{N-1}{N+1}\frac{2\gamma}{\eta}(T-t)}}, \quad i=1,2,\ldots,N. \tag{23}$$

*Proof* Take $M=0$. The solution can also be obtained using a similar proof as before. ∎

This corollary also holds for the case of a single informed trader. By taking the limit as $N\to 1$, we obtain (15).

Corollary 4.3 (one-leader Stackelberg-Nash equilibrium) *In the case of a single leader and N followers, the closed-loop strategy $v$ for the leader and $\xi^i$ for the ith follower are given by*

$$\begin{aligned} v_t^* &= (N+1)\xi_t^{1,*}, \\ \xi_t^{i,*} &= \frac{1}{\eta}\cdot\frac{2N\Lambda_t}{(2N+1)^2-e^{-\frac{N}{N+1}\frac{2\gamma}{\eta}(T-t)}}, \quad i=1,2,\ldots,N. \end{aligned} \tag{24}$$

*Proof* Take $M=1$. ∎

It's clear in figure 2 to see that the number of informed traders in the market significantly affects the trading rates of traders. We separately study the impact of the number of leaders $M$ and the number of followers $N$ on the equilibrium strategies in (22). We also show the trajectories of $\mathbb{E}[\Lambda_t]$ in figure 3 to study price efficiency with different $M$ and $N$. In summary, competition with each other has made it no longer advisable to trade uniformly. Traders are likely to trade more rapidly in the early stage in order to fill orders at a good price. With more leaders or followers engaging in competition, the overall trading rate of informed traders goes up, making $\Lambda_t$ move towards zero more rapidly.

Leaders can occupy a larger portion of the market compared to followers because they have priority in decision making. In figure 2, we find an interesting behavior of leaders. Given the number of leaders, with an increasing number of followers, the leader trades at a faster rate initially. It is clear from this trading pattern that leaders do not give up their status in the market to additional followers. Instead, they further occupy a larger portion of the market, and the remaining followers

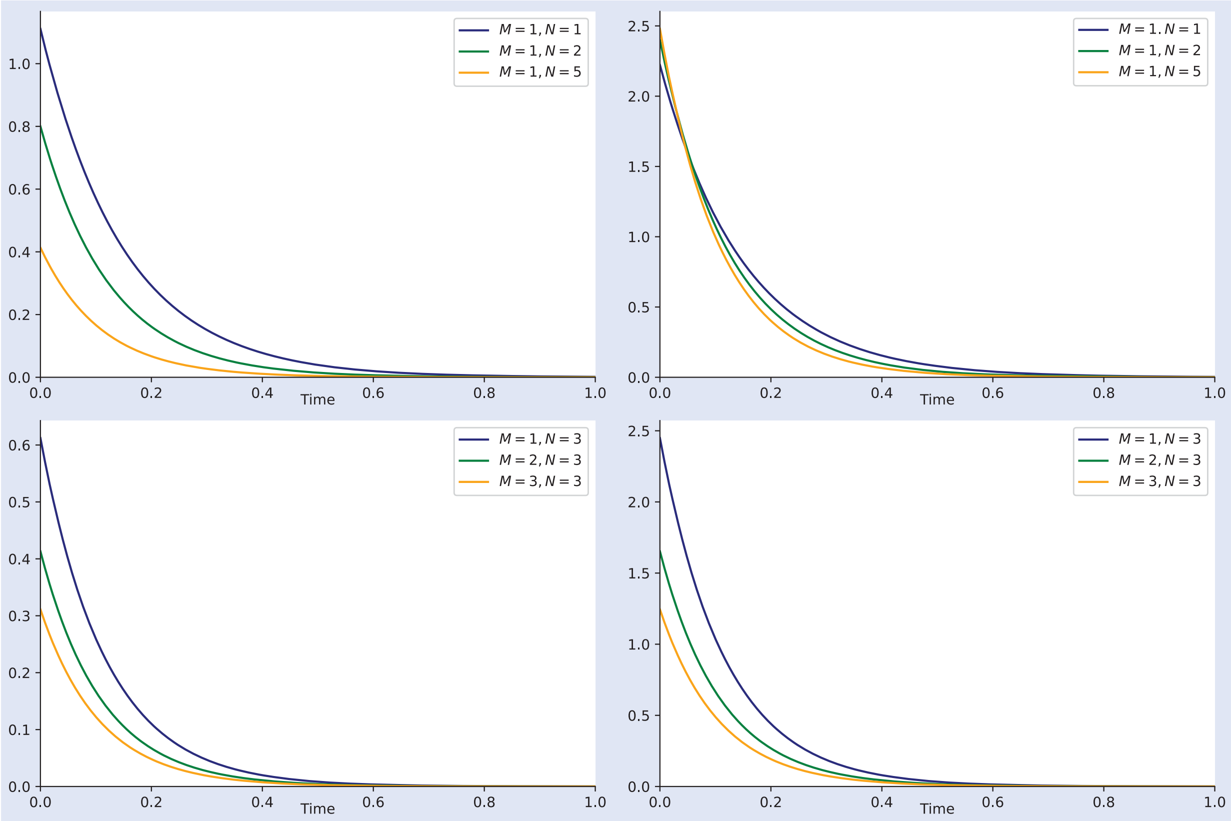


Figure 2. Sample trajectories of $\mathbb{E}[\xi^*]$(left panel) and $\mathbb{E}[\nu^*]$(right panel) in (22). The parameters are the same as in figure 1.

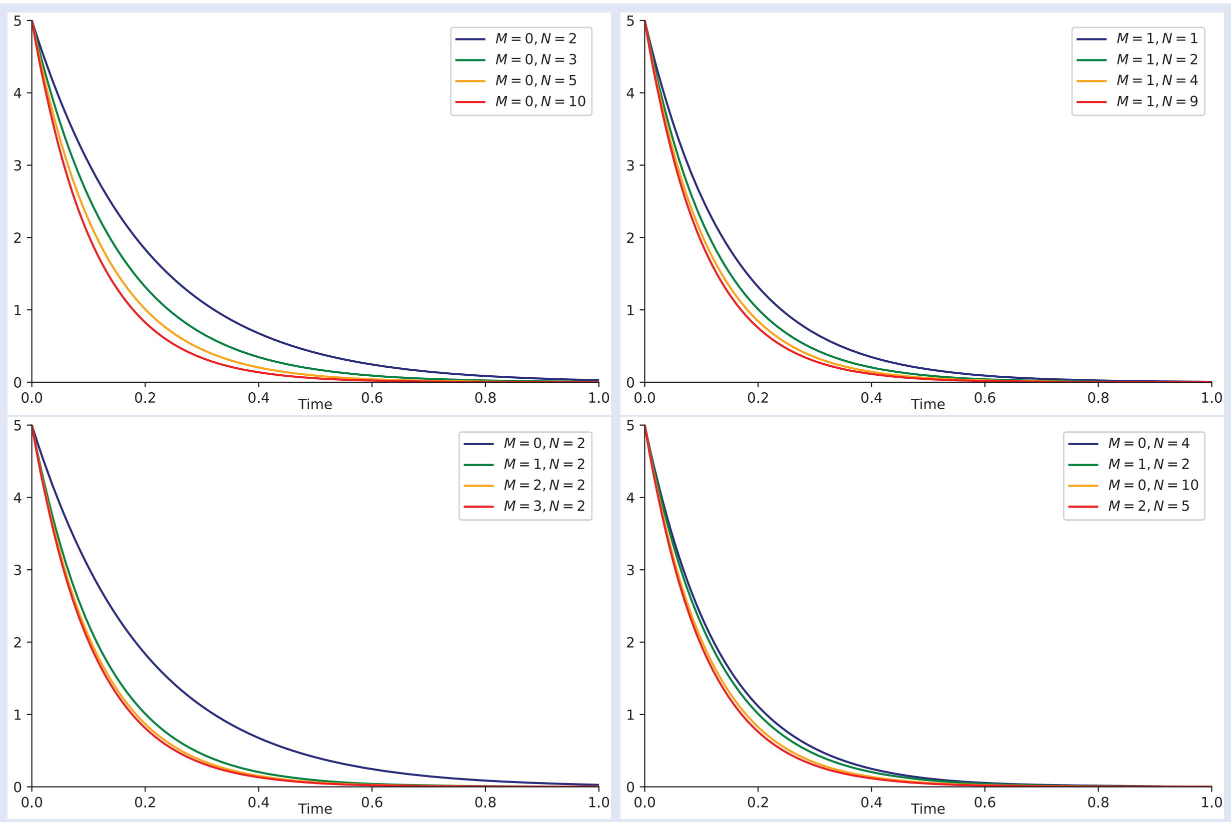


Figure 3. Sample trajectories of $\mathbb{E}[\Lambda_t]$ affected by the trading strategy (22). The parameters are the same as in figure 1.

are forced to make compromises. In addition, leaders make concessions to additional leaders, and followers make concessions to both leaders and followers. Although leaders occupy a larger portion of the market and therefore earn more than followers (exactly $N+1$ times more), all traders will end up earning less with more intense competition (a larger $M$ or $N$). Note that the expected revenue $V(0, \Lambda_0)$ for the leader and the expected revenue $U(0, \Lambda_0)$ for the follower both tend to be zero as $M+N \to \infty$. It means that the fierce competition causes all informed traders to fill orders at a very expensive transaction price, leading to little expected revenue for each trader.

Although competition reduces the revenue of informed traders, it makes the information more rapidly incorporated into the price, leading to a more efficient price. Next, we will mathematically quantify price efficiency under different competition structures. With (8) and (22), we have

$$\begin{aligned} \mathrm{d}\Lambda_t &= -\gamma(Mv_t^* + N\xi_t^*)\,\mathrm{d}t + \sigma_1\,\mathrm{d}B_t - \sigma_3\,\mathrm{d}W_t \\ &= -\frac{(MN+M+N-1)(MN+M+N)}{(MN+M+N)^2 - \varphi(t)} \cdot \frac{\gamma}{\eta}\Lambda_t\,\mathrm{d}t \\ &\quad + \sigma_1\,\mathrm{d}B_t - \sigma_3\,\mathrm{d}W_t. \end{aligned}$$

with the fact

$$\frac{(MN+M+N-1)(MN+M+N)}{(MN+M+N)^2 - \varphi(t)} \in \left(1 - \frac{1}{MN+M+N}, 1 - \frac{1}{MN+M+N+1}\right],$$

we define

$$\theta_{M,N} = \left(1 - \frac{1}{MN+M+N}\right) \cdot \frac{\gamma}{\eta}. \tag{25}$$

As a result, when $N, M$ is large, $\Lambda_t$ approximately follows

$$\mathrm{d}\Lambda_t = -\theta_{M,N}\Lambda_t\,\mathrm{d}t + \sigma_1\,\mathrm{d}B_t - \sigma_3\,\mathrm{d}W_t, \tag{26}$$

and we also have

$$\mathrm{d}S_t = \theta_{M,N}(P_t - S_t)\,\mathrm{d}t + \sigma_3\,\mathrm{d}W_t.$$

$\Lambda_t$ mean reverts to 0, which is interpreted in the model setup in Donnelly and Lorig (2020) as a trading opportunity that does not necessarily persist, and our model suggests that the opportunity is fading probably because there are other informed traders competing for this trading opportunity. The mean-reverting property is different from that obtained by Holden and Subrahmanyam (1992) and Back *et al.* (2000), who showed that in the face of identical information, equilibrium does not exist and all information is revealed instantaneously. As the price impact model fully accounts for transaction costs, we derive explicit equilibrium strategies and we find that $S_t$ mean-reverts to $P_t$ instead of instantly changing to $P_t$. When $\eta \to 0$ (no transaction cost), trading strategies in (22) no longer exists and $\Lambda_t$ immediately changes to zero (see figure 4), which recovers the result in Holden and Subrahmanyam (1992) and Back *et al.* (2000). Given a positive $\eta$, the mean-reverting property of the price has actually been discovered and used for a long time (see Carcano *et al.* 2005, Zhang and Zhang 2008, Bergault *et al.* 2022 and references therein), especially in some commodity markets and commodity futures markets. Since the value of the commodity itself varies in a regular manner, traders are generally aware of its approximate fundamental value, which is very similar to our assumption about multiple informed traders. In addition, by (26) net order flow $q_t$ $(= Mv_t^* + N\xi_t^*)$ approximately satisfies

$$\mathrm{d}q_t = \frac{\theta_{M,N}}{\gamma}\,\mathrm{d}\Lambda_t = -\theta_{M,N}q_t\,\mathrm{d}t + \frac{\theta_{M,N}}{\gamma}(\sigma_1\,\mathrm{d}B_t - \sigma_3\,\mathrm{d}W_t).$$

$q_t$ is mean-reverting to zero, which matches the assumptions about net order flow in Bechler and Ludkovski (2015) and Ma *et al.* (2020). Furthermore, the diffusion term of net order flow is correlated with the diffusion term of the price. And the more volatile the information is (a larger $\sigma_1$), the less the correlation is.

Since the price is mean-reverting to its expected value, then the faster it reverts, the more efficient the price is. Therefore, we can use $\theta_{M,N}$ as a quantified index to measure price efficiency. To begin with, The parameters $\gamma$ and $\eta$ describe market liquidity. A larger $\frac{\gamma}{\eta}$ indicates a larger $\theta_{M,N}$, and according to figure 4 we can conclude that small transaction costs leads to good price efficiency. What's more, given $M+N=\pi$ $(\pi \geq 2)$, $\theta_{0,\pi} = (1-\frac{1}{\pi}) \cdot \frac{\gamma}{\eta}$ and $\theta_{1,\pi-1} = (1-\frac{1}{2\pi-1}) \cdot \frac{\gamma}{\eta}$, which are price efficiency in the case of no leader and one leader. It's obvious that $\theta_{0,\pi} < \theta_{1,\pi-1}$. Therefore, we can see that the presence of a leader promotes price efficiency. In fact, according to (25), we can easily figure out that theoretically the best competition structure is $\frac{\pi}{2}$ leaders and $\frac{\pi}{2}$ followers. Given the total number of informed traders, an excessive number of leaders will compete with each other, making their individual status lower and hence reducing price efficiency. And it is also clear from the third panel in figure 3 that the presence of a leader contributes to great price efficiency and that subsequent additional leaders do not have a dramatic effect on efficiency. Therefore, a small number of leaders is sufficient and they can significantly increase price efficiency. In addition, as shown in the last panel in figure 3, $\theta_{0,4} < \theta_{1,2}$ and $\theta_{0,10} < \theta_{2,5}$. The competition structure plays a larger role in price efficiency than the total number of informed traders. Finally, regardless of the competition structure, when the total number of informed traders tends to infinity $(M+N \to \infty)$, $\theta_{M,N} \to \frac{\gamma}{\eta}$. We assert that this is the supreme price efficiency under the linear price impact model resulting from competition among risk-neutral informed traders.

## 5. Trading patterns arising from inventory management

In this section, we specifically consider the case of risk-averse informed traders. For the sake of subsequent analysis, we only consider Nash competition. We use a quadratic terminal penalty $-\frac{\beta}{2}Q_T^2$ to represent risk aversion. On the one hand, it can represent the aversion to information inaccuracy $\varepsilon$, as

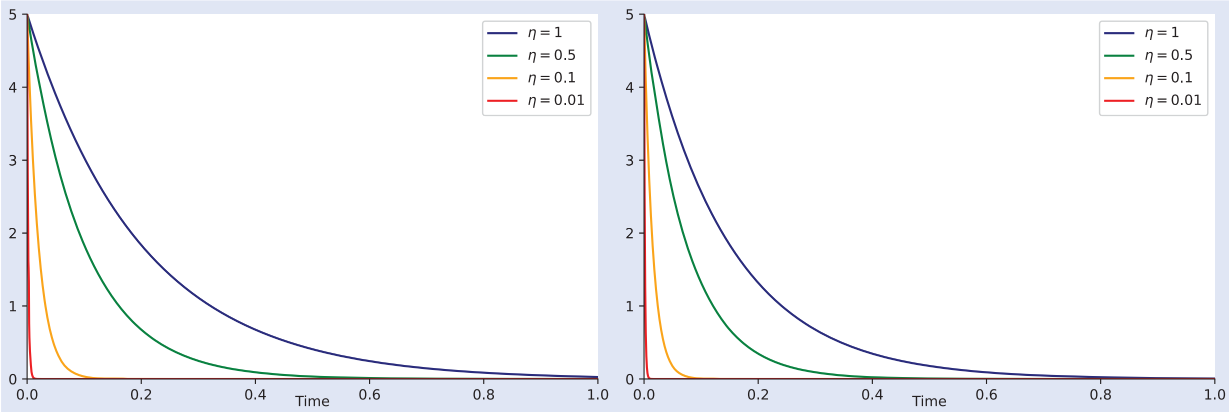


Figure 4. Sample trajectories of $\mathbb{E}[\Lambda_t]$ affected by the trading strategy (22) with different $\eta$. Other parameters are the same as in figure 1. $M, N = 0, 2$ on the left and $M, N = 1, 1$ on the right.

described in section 3. On the other hand, we would like to emphasize that, as used in Herrmann *et al.* (2020), it can also help inventory management under multiple rounds of competition. In fact, we can consider intraday trading for HFTs. Daily trading is a game, so it is difficult for HFTs to devote much time to unwinding, leading to the fact that the terminal position of this round will be the initial position of the next round. And in the terminal wealth in (9), the term $X_0^i + P_0 Q_0^i$ indicates that the initial position has a direct impact on the wealth. Therefore, for HFTs, inventory management is extremely necessary. Due to the complexity of the model, we give closed-form open-loop deterministic strategies in order to better illustrate trading patterns of informed traders within a single round as well as across multiple rounds. The extensions like the running penalty and adaptive strategies are left for future research.

$N$ informed traders compete for liquidity in the market. We focus on the open-loop strategy which is a deterministic function of time $t$. According to Carlin *et al.* (2007), the problem faced by the $i$th trader is

$$\sup_{v^i \in \mathcal{B}} \mathbb{E}\left[-\frac{\beta_i}{2}(Q_T^i)^2 + \int_0^T \left[-\eta v_t^i \sum_{j=1}^N v_t^j + \Lambda_t v_t^i\right] \mathrm{d}t\right], \quad i = 1, 2, \ldots, N, \tag{27}$$

where $\mathcal{B} = \{v \text{ is continuous on } [0,T] : \int_0^T |v_t|^2 \mathrm{d}t < +\infty\}$ and the positive constant $\beta_i$ measures the degree of aversion of the $i$th trader. The definition of the Nash equilibrium can be seen as the definition of the multiple-leader Stackelberg-Nash equilibrium in the absence of the leader, so we don't dwell on it here. The solution to (27) is shown in the following proposition.

Proposition 5.1 *The problem* (27) *admits the unique open-loop Nash equilibrium. The open-loop strategy $v^i$ for the ith trader is given by*

$$v_t^i = a \cdot e^{-\frac{N-1}{N+1}\frac{\gamma}{\eta}t} + b_i \cdot e^{\frac{\gamma}{\eta}t}, \tag{28}$$

*where*

$$a = \frac{N-1}{N+1} \cdot \frac{\gamma}{\eta} \cdot \left(\sum_{i=1}^N \frac{\Lambda_0 - \beta_i Q_0^i}{(\gamma+\beta_i)e^{\frac{\gamma}{\eta}T} - \beta_i}\right) \cdot \left(\sum_{i=1}^N \frac{\gamma N + \beta_i - (\gamma+\beta_i)e^{-\frac{N-1}{N+1}\frac{\gamma}{\eta}T}}{(\gamma+\beta_i)e^{\frac{\gamma}{\eta}T} - \beta_i}\right)^{-1},$$

$$b_i = -\frac{\gamma}{\eta} \cdot \frac{\frac{N+1}{N-1}\frac{\eta}{\gamma} a\left[\gamma N + \beta_i - (\gamma+\beta_i)e^{-\frac{N-1}{N+1}\frac{\gamma}{\eta}T}\right] - \Lambda_0 + \beta_i Q_0^i}{(\gamma+\beta_i)e^{\frac{\gamma}{\eta}T} - \beta_i}.$$

*Proof* See appendix. ■

This general result is too complex to be analyzed effectively. Therefore, we impose some constraints and give the following corollaries to facilitate the analysis of trading patterns arising from inventory management during informed trading.

Corollary 5.2 *Consider $N = 2$, $\beta_1 = 0$, $\beta_2 = +\infty$. The open-loop strategy for the ith trader is given by*

$$v_t^i = a \cdot e^{-\frac{1}{3}\frac{\gamma}{\eta}t} + b_i \cdot e^{\frac{\gamma}{\eta}t},$$

*where*

$$a = \frac{1}{3\eta} \cdot \frac{(\Lambda_0 - \gamma Q_0^2)e^{\frac{\gamma}{\eta}T} - \Lambda_0}{3e^{\frac{\gamma}{\eta}T} + e^{-\frac{\gamma}{3\eta}T} - 2e^{\frac{2\gamma}{3\eta}T} - 2},$$

$$b_1 = \frac{\Lambda_0 - 3\eta a(2 - e^{-\frac{\gamma}{3\eta}T})}{\eta e^{\frac{\gamma}{\eta}T}},$$

$$b_2 = -\frac{\gamma Q_0^2 + 3\eta a(1 - e^{-\frac{\gamma}{3\eta}T})}{\eta(e^{\frac{\gamma}{\eta}T} - 1)}.$$

*Proof* Take the limit as $\beta_2 \to +\infty$. ■

In this extreme scenario, the first trader (he) is completely risk neutral and does not care about his own position, while the second trader (she) enforces her own position to be zero at time $T$. The significant difference between the two traders

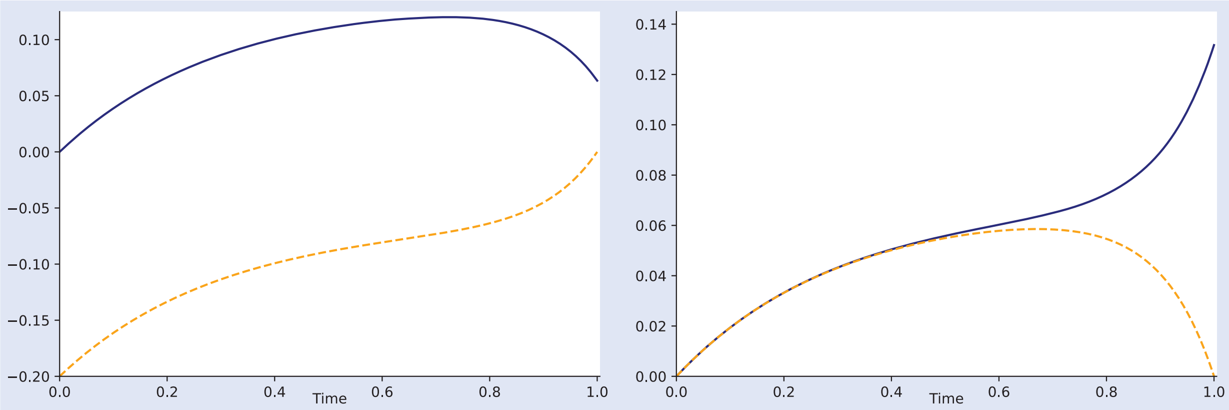


Figure 5. Sample trajectories of positions. Solid and dashed lines correspond to the first and second trader respectively. $\Lambda_0 = 2$. $Q_0^2 = -0.2$ on the left and $Q_0^2 = 0$ on the right. Other parameters are the same as in figure 1.

allows us to clearly see how predatory trading is carried out. Figure 5 illustrates two different cases. On the left panel, the second trader has a negative initial position and therefore has buying pressure. The first trader will first make an over-buy to compete for liquidity with the second trader, and then sell some orders at the end to provide liquidity to the second trader. Finally the first trader will hold a positive position because $\Lambda_0 > 0$. On the right panel, the position of the second trader begins at 0 and ends in 0. However, she will buy at the beginning to compete for liquidity and sell at the end, instead of doing nothing. And the first trader will accelerate buying at the last moment taking advantage of the liquidity the second trader provides. As a result, the first trader is a predator in the first case and the second trader is a predator in the second case. We can see that predatory trading can also occur in informed trading. Actually, the second term in the equilibrium strategy (28) is the mathematical expression of such predatory behavior. Whether traders are preyed upon or prey upon others depends on the initial positions they hold. Note that this result also applies to scenarios with different $\beta_i$s or with identical $\beta_i$s but different $Q_0^i$s. In general, a lighter position allows the trader to be more flexible during informed trading. To maintain a lighter position across multiple rounds of competition, risk management (a positive $\beta$) is a necessity.

In addition, we find a more interesting trading pattern. Consider that there exists only the second trader in the market. Then she will upwind her position with the TWAP strategy (take the aversion $\beta \to +\infty$ in the trading strategy (14)). If her initial position is zero, then her optimal strategy is to do nothing. In other words, the information is of no value to her due to her position target $Q_T^2 = 0$. On the right panel, we know that if the first trader knows the information, the second trader can profit by predatory trading. Therefore, when the second trader is the only one in the market who knows the information, she will actively leak it to one or more traders for the purpose of gaining or expanding profits. This behavior is called rational information leakage (see Indjejikian *et al.* 2014 and references therein), which can promote the efficiency of turning private information into profit. Strategies on how to leak information are beyond the scope of this paper. However, we believe that this pattern holds true for traders needing inventory management (the aversion $\beta > 0$ instead of being $+\infty$). Moreover, traders don't have to leak the information at the initial time. In fact, she can first open a position at a low transaction cost via an order execution strategy, and then leak the information. While other traders compete with the information, she closes this position.

Corollary 5.3 *Consider* $\beta_i = \beta \quad (i = 1, 2, \ldots, N)$. *The open-loop strategy for the ith trader is given by*

$$v_t^i = a \cdot e^{-\frac{N-1}{N+1}\frac{\gamma}{\eta}t} + b_i \cdot e^{\frac{\gamma}{\eta}t},$$

*where*

$$a = \frac{N-1}{N+1} \cdot \frac{\gamma}{\eta} \cdot \left(\Lambda_0 - \beta \frac{\sum_{j=1}^N Q_0^j}{N}\right) \cdot \left(\gamma N + \beta - (\gamma + \beta) e^{-\frac{N-1}{N+1}\frac{\gamma}{\eta}T}\right)^{-1},$$

$$b_i = \frac{\gamma}{\eta} \cdot \beta \left(\frac{\sum_{j=1}^N Q_0^j}{N} - Q_0^i\right) \cdot \left((\gamma + \beta) e^{\frac{\gamma}{\eta}T} - \beta\right)^{-1}.$$

*Furthermore, if* $\beta = 0$, *then we have*

$$a = \frac{N-1}{N+1} \cdot \frac{\Lambda_0}{\eta} \cdot \left(N - e^{-\frac{N-1}{N+1}\frac{\gamma}{\eta}T}\right)^{-1}, \quad b_i = 0.$$

*If* $\beta = +\infty$, *then we have*

$$a = -\frac{N-1}{N+1} \cdot \frac{\gamma}{\eta} \cdot \frac{\sum_{j=1}^N Q_0^j}{N} \cdot \left(1 - e^{-\frac{N-1}{N+1}\frac{\gamma}{\eta}T}\right)^{-1},$$

$$b_i = \frac{\gamma}{\eta} \cdot \left(\frac{\sum_{j=1}^N Q_0^j}{N} - Q_0^i\right) \left(e^{\frac{\gamma}{\eta}T} - 1\right)^{-1},$$

*which is exactly the general result in Carlin et al.* (2007).

*Proof* Direct substitution. ■

In this scenario, all informed traders have the same level of risk aversion. When they are risk neutral ($\beta = 0$), they pay no attention to positions, but only to the information, and no predatory trading occurs in this case. When they care only

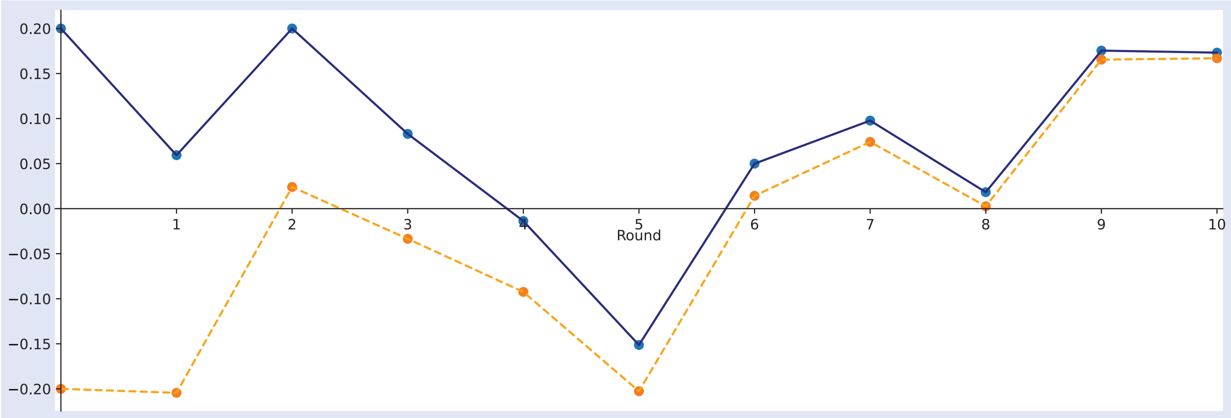


Figure 6. Sample trajectories of positions under multiple rounds of the two-player game.

about positions at all ($\beta = +\infty$), information is of no value to them and the problem turns into an order execution problem. We are particularly concerned with the situation when they need to manage inventories during informed trading. In this case, they are concerned about both positions and the information. Thus, the game between them in multiple rounds becomes complex but interesting.

We assume that in multiple rounds of competition, informed traders don't have time to fully commit to closing their positions. Therefore, the terminal position of this round is the initial position of the next round. Given the initial position, the terminal position is given by

$$Q_T^i = Q_0^i + \int_0^T v_t^i \,\mathrm{d}t.$$

With straightforward computations, we have

$$Q_T^i - Q_T^j = \frac{\gamma e^{\frac{\gamma}{\eta}T}}{\gamma e^{\frac{\gamma}{\eta}T} + \beta(e^{\frac{\gamma}{\eta}T} - 1)}(Q_0^i - Q_0^j).$$

$\beta > 0$ yields

$$\frac{\gamma e^{\frac{\gamma}{\eta}T}}{\gamma e^{\frac{\gamma}{\eta}T} + \beta(e^{\frac{\gamma}{\eta}T} - 1)} \in (0, 1).$$

As a result, predatory trading reduces the difference between the positions of any two traders, and after many rounds all traders' positions tend to be identical. And the larger the aversion $\beta$ is, the faster the convergence speed is. In addition, denoting $\sum_{j=1}^N Q_t^j$ by $Q_t$ we also have

$$Q_T - Q_0 = -k_1 \cdot Q_0 + k_2 \cdot \Lambda_0$$

where

$$k_1 = \frac{\beta(1 - e^{-\frac{N-1}{N+1}\frac{\gamma}{\eta}T})}{\gamma(N - e^{-\frac{N-1}{N+1}\frac{\gamma}{\eta}T}) + \beta(1 - e^{-\frac{N-1}{N+1}\frac{\gamma}{\eta}T})},$$

$$k_2 = \frac{N(1 - e^{-\frac{N-1}{N+1}\frac{\gamma}{\eta}T})}{\gamma(N - e^{-\frac{N-1}{N+1}\frac{\gamma}{\eta}T}) + \beta(1 - e^{-\frac{N-1}{N+1}\frac{\gamma}{\eta}T})}.$$

In every round, we assume $\Lambda_0$ is normally distributed with a constant variance. $\beta > 0$ yields $k_1 \in (0, 1)$, and $k_1$ increases with $\beta$. As a result, the sum of positions of informed traders is mean-reverting and such mean reversion is more pronounced with growing risk aversion. Based on the convergence between traders, we know that each trader's position is approximately mean-reverting. Finally, figure 6 visualizes this interaction between positions across multiple rounds of competition in the case of two traders.

## 6. Conclusion

In this paper, we formulate informed trading in the linear price impact model framework. We derive the closed-form feedback strategy in the case of a single trader risk averse to dynamic information. The risk-averse trader reveals information more slowly and thus contributes less to price efficiency. In the case of multiple risk-neutral traders, closed-loop strategies we derive form a multiple-leader Stackelberg-Nash equilibrium. Competition contributes much to price efficiency and the presence of a leader further helps promote price efficiency. Due to transaction costs, the information is no longer revealed instantaneously, and we derive the empirically observed mean-reverting behavior of price dynamics and order flows. In the case of risk-averse traders, we derive the unique open-loop deterministic equilibrium strategies. Inventory management triggers predatory trading and active information leakage during informed trading. What's more, the positions of traders with the same level of risk aversion tend to be identical and mean-reverting to zero under multiple rounds of competition. Finally, we would like to fully extend the case of a single risk-averse trader to explore adaptive equilibrium strategies for multiple risk-averse traders in our future works.

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## Appendix. Proofs

### *A.1. Proof of proposition 3.1*

With the ansatz $V(t,\Lambda,Q) = a(t)\Lambda^2 + b(t)Q^2 + c(t)\Lambda Q + d(t)$, $v^*$ satisfies

$$v^* = \frac{\Lambda + V_Q - \gamma V_\Lambda}{2\eta} = \frac{f\Lambda + gQ}{2\eta}$$

where $f(t) = 1 - 2\gamma a(t) + c(t)$ and $g(t) = 2b(t) - \gamma c(t)$. Substitute the ansatz into (12), and solving (12) is now equivalent to solving the following ODE system:

$$\begin{cases} a' + \dfrac{f^2}{4\eta} = 0 \\ b' + \dfrac{g^2}{4\eta} - \lambda\sigma_1^2 = 0 \\ c' + \dfrac{fg}{2\eta} \qquad\qquad = 0 \\ d' + (\sigma_1^2 + \sigma_3^2)a \quad = 0 \end{cases}$$

with the terminal condition

$$b(T) = -\frac{\beta}{2}\sigma_2^2, \quad a(T) = c(T) = d(T) = 0.$$

Note that no equation can be solved independently. Next we will show the ODE system has a unique solution, and it's most critical to first solve $f$ and $g$. With some straightforward computations, we have

$$\begin{cases} f' = \dfrac{f(\gamma f - g)}{2\eta} \\ g' = \dfrac{g(\gamma f - g)}{2\eta} + 2\lambda\sigma_1^2 \end{cases}$$

We define $h(t) = \gamma f(t) - g(t)$, and we then have

$$h' = \frac{h^2}{2\eta} - 2\lambda\sigma_1^2$$

and $h(T) = \delta$, where $\delta = \gamma + \beta\sigma_2^2$. We can easily obtain

$$h(t) = 2\eta\alpha \cdot \frac{\delta\cosh\big(\alpha(T-t)\big) + 2\eta\alpha\sinh\big(\alpha(T-t)\big)}{\delta\sinh\big(\alpha(T-t)\big) + 2\eta\alpha\cosh\big(\alpha(T-t)\big)}$$

where $\alpha = \sqrt{\frac{\lambda\sigma_1^2}{\eta}}$. With $f' = \frac{fh}{2\eta}$ and $f(T) = 1$, we get

$$f(t) = \frac{2\eta\alpha}{\delta\sinh\big(\alpha(T-t)\big) + 2\eta\alpha\cosh\big(\alpha(T-t)\big)}.$$

With $g(t) = \gamma f(t) - h(t)$, we obtain (13). As $f(t)$ and $g(t)$ are continuous on $[0,T]$, they are bounded on $[0,T]$. By simple integration we can find the unique expressions of $a(t)$, $b(t)$, $c(t)$ and $d(t)$ from the ODE system, and they are all differentiable and bounded on $[0,T]$. As a result, our candidate $V(t,\Lambda,Q)$ is differentiable in $t$ and smooth in the state, and then conditions for verification of the HJB equation are met. Finally, we can conclude that $v^*$ in (13) is an optimal Markovian control.

### *A.2. Proof of proposition 4.1*

The step of solving the Stackelberg-Nash equilibrium is generally to first consider the optimal strategies of the followers given the leaders' strategies, and then to solve the optimal strategies of the leaders. We first focus on (21). Given the leaders' strategies and other followers' strategies, the term in $\sup_{\xi^i\in\mathbb{R}}\{\cdot\}$ is a quadratic function of $\xi^i$. Strict concavity leads to the unique maximum point, and by differentiating we have $N$ equations

$$-\eta\left(\sum_{j=1}^{M} v^j + \sum_{j=1}^{N}\xi^j\right) - (\gamma U_\Lambda^i + \eta\xi^i) + \Lambda = 0, \quad i = 1,2,\ldots,N.$$

Collect $N$ such equations together, and we can uniquely solve $\xi^i$ ($i = 1,2,\ldots,N$). Since these followers are all risk-neutral, their strategies only depend on $t$ and $\Lambda$, which are shared among the followers. Under this symmetric game, we can easily figure out all $\xi^i$s should be the same, denoted by $\xi$. Consequently, $U^i$s should also be the same, denoted by $U$. Therefore, the solution has a simple expression

$$\xi^* = \frac{\Lambda - \gamma U_\Lambda - \eta\sum_{j=1}^M v^j}{(N+1)\eta}.$$

We substitute it into (20) and obtain

$$V_t^i + \frac{1}{2}(\sigma_1^2+\sigma_3^2)V_{\Lambda\Lambda}^i + \sup_{v^i\in\mathbb{R}}\Bigg\{-(\gamma V_\Lambda^i + \eta v^i) \times \left(\frac{1}{N+1}\sum_{j=1}^M v^j + \frac{N(\Lambda-\gamma U_\Lambda)}{(N+1)\eta}\right) + \Lambda v^i\Bigg\} = 0.$$

Strict concavity leads to the unique maximum point, and by differentiating we have $M$ equations

$$-\left(\frac{\eta}{N+1}\sum_{j=1}^M v^j + \frac{N(\Lambda-\gamma U_\Lambda)}{(N+1)}\right) - \frac{1}{N+1}(\gamma V_\Lambda^i + \eta v^i) + \Lambda = 0, \quad i = 1,2,\ldots,M.$$

Similar to the earlier analysis, we use $v$ to denote all $v^i$s and use $V$ to denote all $V^i$s. Therefore, The solution has a simple expression

$$v^* = \frac{\Lambda + N\gamma U_\Lambda - \gamma V_\Lambda}{(M+1)\eta}.$$

With the ansatz $V = (N+1)U$, we have

$$v^* = (N+1)\xi^* = \frac{\Lambda - \gamma U_\Lambda}{(M+1)\eta},$$

and all HJB equations in (20) and (21) become an identical equation

$$U_t + \frac{1}{2}(\sigma_1^2+\sigma_3^2)U_{\Lambda\Lambda} - \frac{MN+M+N-1}{(M+1)(N+1)}\cdot\frac{\gamma}{\eta}(\Lambda-\gamma U_\Lambda)U_\Lambda + \frac{(\Lambda-\gamma U_\Lambda)^2}{(M+1)^2(N+1)^2\eta} = 0.$$

With the ansatz $U = a(t)\Lambda^2 + b(t)$, we have the following ODE system:

$$\begin{cases} a' - \dfrac{MN+M+N-1}{(M+1)(N+1)} \cdot \dfrac{\gamma}{\eta}(1-2\gamma a)2a \\ \qquad + \dfrac{(1-2\gamma a)^2}{(M+1)^2(N+1)^2\eta} = 0 \\ b' + (\sigma_1^2 + \sigma_3^2)a = 0 \end{cases}$$

with the terminal condition

$$a(T) = b(T) = 0.$$

Define $f(t) = 1 - 2\gamma a(t)$, and we have $f(T) = 1$ and

$$\xi^* = \frac{f\Lambda}{(M+1)(N+1)\eta}.$$

In addition, $f(t)$ satisfies a Bernoulli differential equation

$$\frac{\eta}{2\gamma} f' = \frac{(MN+M+N)^2}{(M+1)^2(N+1)^2} f^2 - \frac{MN+M+N-1}{(M+1)(N+1)} f,$$

which yields

$$f(t) = \frac{(MN+M+N)^2 - 1}{(MN+M+N)^2 - e^{-\frac{MN+M+N-1}{MN+M+N+1}\frac{2\gamma}{\eta}(T-t)}}.$$

As a result, we obtain (22). We also have

$$\begin{cases} a(t) = \dfrac{1-f(t)}{2\gamma} \\ b(t) = (\sigma_1^2 + \sigma_3^2)\displaystyle\int_t^T a(u)\,\mathrm{d}u \end{cases}$$

and

$$\frac{V(t,\Lambda)}{N+1} = U(t,\Lambda) = a(t)\Lambda^2 + b(t).$$

It can be verified that our value functions and the equilibrium strategies in (22) satisfy the system formed by (20) and (21). And according to the earlier deduction, it can also be verified that neither a leader nor a follower has an incentive to deviate from the equilibrium because the current trading rate already maximizes the corresponding goal. As a result, a closed-loop multiple-leader Stackelberg-Nash equilibrium is formed by the equilibrium strategies in (22).

### *A.3. Proof of proposition 5.1*

The proof is very similar to the one in Carlin *et al.* (2007). We denote $\sum_{j=1}^N v_t^j$ by $Y_t$ and denote $\sum_{j=1}^N Q_t^j$ by $Q_t$. With

$$-\frac{\beta_i}{2}(Q_T^i)^2 = -\beta_i \int_0^t Q_t^i v_t^i \,\mathrm{d}t - \frac{\beta_i}{2}(Q_0^i)^2,$$

the problem (27) is equivalent to

$$\sup_{v^i \in \mathcal{B}} \int_0^T \left\{-\eta v_t^i Y_t + [\Lambda_0 - \gamma(Q_t - Q_0)]v_t^i - \beta_i Q_t^i v_t^i\right\} \mathrm{d}t,$$
$$i = 1, 2, \ldots, N.$$

We define

$$F(Q_t^i, v_t^i, t) = -\eta v_t^i Y_t + [\Lambda_0 - \gamma(Q_t - Q_0)]v_t^i - \beta_i Q_t^i v_t^i.$$

With Euler equation $\frac{\partial F}{\partial Q_t^i} - \frac{\mathrm{d}}{\mathrm{d}t} \cdot \frac{\partial F}{\partial v_t^i} = 0$, we have

$$\gamma\,\mathrm{d}Q_t + \eta\,\mathrm{d}Y_t + \eta\,\mathrm{d}v_t^i - \gamma v_t^i\,\mathrm{d}t = 0, \quad i = 1, 2, \ldots, N,$$

which is exactly the equations obtained in Carlin *et al.* (2007). The general solution is hence given by

$$v_t^i = a \cdot e^{-\frac{N-1}{N+1}\frac{\gamma}{\eta}t} + b_i \cdot e^{\frac{\gamma}{\eta}t}$$

where constants $a, b_i$ satisfy

$$\sum_{i=1}^N b_i = 0.$$

With the restriction $\frac{\partial F}{\partial v^i}|_{t=T} = 0$, we can express $b_i$s by $a$, and eliminate $b_i$s by adding all of them together to finally get the expression of $a$. The expressions of $a$ and $b_i$s are shown in (28). It's easy to verify that given other traders' trading rates, every trader's objective is strictly concave when all $\beta_i$s are positive. As a result, there exists at most one open-loop Nash equilibrium (see Schied and Zhang 2017, Voß 2022), and thus the Nash equilibrium formed by the equilibrium strategies in (28) is unique.