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Optimal characteristic portfolios

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Characteristic-sorted portfolios are the workhorses of modern empirical finance, deployed widely to evaluate anomalies and construct asset pricing models. We propose a new method for their estimation that is simple to compute, makes no ex-ante assumption on the nature of the relationship between the characteristic and returns, and does not require *ad hoc* selections of percentile breakpoints or portfolio weighting schemes. Characteristic portfolio weights are implied directly from data, through maximizing a Mean–Variance objective function with mean and variance estimated non-parametrically from the cross-section of assets. To illustrate the method, we evaluate the size, value and momentum anomalies and find overwhelming empirical evidence of the outperformance of our methodology compared to standard methods for constructing characteristic-sorted portfolios.

Keywords: Anomalies; Portfolio sorts; Size effect; Value effect; Momentum

JEL Classifications: G11, G12

1. Introduction

Portfolio sorting is an important tool of modern empirical finance. It has been used to test asset pricing theories, to construct profitable quantitative investment strategies and to identify empirical pricing anomalies based on stock characteristics. Researchers typically employ two methods to identify return predictors: (i) portfolio sorts based on single or multiple characteristics such as size or book-to-market and (ii) linear regression in the spirit of Fama and MacBeth (1973).

There is a vast and ongoing literature proposing novel quantitative factors that are constructed by uncovering empirical anomalies on stock characteristics. These anomalies are exploited for investment purposes or as alternative asset pricing models that build upon existing models and incorporate the additional factors. In this paper, rather than exploring different factor anomalies we study the formation of characteristic portfolios. The typical sorting procedure is to rank the cross-section of stocks according to a characteristic (or set of characteristics) and to construct a zero net investment portfolio, going long the top quantile portfolio and short the bottom quantile portfolio. This process is dependent on the specific choice of quantile breakpoints (e.g. deciles, terciles or quartiles) used to build the long and short strategies and is, therefore, subject to discretion in the choice of these tuning parameters. The weighting scheme also affects the formation of characteristic portfolios. Typical schemes are either equal weighting or value weighting. Both of these weighting schemes introduce ancillary factor exposures to a given characteristic portfolio. For example, value weighting a size portfolio is clearly counter-intuitive, reversing the return-driving signal within the cross-sectional spread of the portfolio. If the quantile portfolios contain few stocks/the number of quantiles is large, then the value weighting has less impact, as we are looking at the return differential between high and low quantiles. However, when the number of quantile portfolios becomes large, the portfolios also become less meaningful, economically, as they contain a small number of stocks and may not be investible for large investors. In our empirical analysis, we show that variation in decisions on breakpoints and weighting schemes is influential on the outcome of tests of anomalies and may be the difference between accepting and rejecting the hypothesis of a positive premium associated with a characteristic.

In this paper, we propose an alternative methodology to construct quantitative factors based on a mean–variance optimization procedure to obtain the weights allocated to long-minus-short portfolios. We are agnostic about the choice of empirical anomalies to construct the quantitative portfolios as our methodology can be applied to any characteristic portfolio, however, for illustrative purposes, we focus on Fama–French original factors: size, book-to-market and momentum. Our methodology extends the typical long-minus-short

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quantitative portfolios by allocating some weight to all of the assets in the cross-section and not only those in the top and bottom quantiles. Additionally, these weights are dynamic and can change across periods, allowing us to propose time-varying characteristic portfolios that adapt to changing market conditions. We consider a dynamic conditional mean–variance optimal strategy as a proxy for general 'characteristic-timed' strategy risk.

In an attempt to keep the characteristic portfolios economically meaningful ,† we adopt an approach where characteristic portfolios are formed over the constituent assets of a set filtered for micro-caps (without obfuscating the characteristic information with ancillary weighting schemes such as equal or value weighted).‡ This procedure does not require manual selection of the quantile breakpoints or of the portfolio weighting scheme, rather portfolio weights are implied from the data and are driven solely by the relationship between the target characteristic and asset returns.

Compared to the standard practice in ranked, sorted portfolios, our proposed methodology has three salient features:

(i) We do not overlay any fixed weighting schemes such as value-weighting or equal-weighting on the portfolio (these schemes are known to introduce other factor exposures to the anomaly returns), the weighting scheme is mean–variance optimal conditional on the attribute.
(ii) All assets in the cross-section are included in the characteristic portfolio (excluding micro-cap stocks), increasing the market capitalization available to absorb investment.§
(iii) Our portfolio choice is dynamically updated and estimated in the cross-section. It is robust to the problem of spurious time series correlations (see, e.g. Ferson *et al.* 2003, 2008) by construction. Furthermore, by constructing the portfolio weights from cross-sectional regressions we avoid the need to infer a stable relationship between returns and stock characteristics from time series data.

This final point may be seen as a potential weakness where there is a theoretical guidance for a stable relationship between the stock characteristic and its expected return, as our methodology is discarding this information. However, there is widespread evidence in the literature of time-varying risk premia (see, e.g. Ludvigson and Ng 2007, Gagliardini *et al.* 2016) and time-varying anomaly returns (see, e.g. Stambaugh *et al.* 2012, Jacobs 2015, Avramov *et al.* 2016, McGee and Olmo 2019). As our results are fully out of sample relative to portfolio formation and are not subjected to any fine-tuning of parameters we let the data speak to the strength of the approach.

The optimal mean–variance portfolio weights defining the characteristic-based portfolios are a function of the expected return and variance of the assets in the cross-section conditional on the stock characteristics. We propose nonparametric kernel methods for cross-sectional data to estimate these conditional moments of the distribution of returns. Empirically, these portfolios are shown to outperform state-of-the-art methods for exploiting stock anomalies for factor construction, using fundamental attributes. As we are testing a characteristic portfolio construction technique rather than replicating anomalies, we focus on traditional attributes commonly assumed to have a stable long-term relationship. Thus we focus on size, book-to-market and momentum anomalies.

Our proposed optimal non-parametric method works particularly well for the size anomaly, capturing a premium over the full sample with a test statistic of 4.33. None of the other standard rank, sorted portfolio methods tested resulted in a test statistic significant at the level modified for multiple hypothesis testing as per Harvey *et al.* (2016). The optimal non-parametric size portfolio constructed from maximizing the mean–variance objective function also has the best investment performance for a mean–variance investor and improves a standard four factor asset pricing model, dominating all other candidate size factors tested in pairwise tests of model squared Sharpe ratios, as per Barillas *et al.* (2019). The method also works well for the value attribute in terms of capturing the most significant return premium (test statistic 4.03) and investment performance. The method did not perform well, however, at capturing the momentum premium (test statistic 2.26, compared with a value of 5.03 for the method of Cattaneo *et al.* (2020)), suggesting that it performs better for fundamental rather than technical factors (that may be driven by time series rather than cross-sectional relationships).

The paper is structured as follows. Section 2 relates our approach for characteristic-based portfolio construction to recent literature on optimal portfolio sorting. Section 3 introduces the mean–variance procedure to construct optimal dollar-neutral portfolios based on stock characteristics. The section also discusses non-parametric models for capturing the relationship between stock returns and characteristics. Section 4 applies the portfolio methods proposed in Section 3 to three popular anomalies in the financial economics literature (size, value and momentum) and compares results against other widely used characteristic-portfolio construction methods. Section 5 presents the results of the empirical application to assess the performance of the different procedures for all common stocks from the CRSP database over the period 1963–2018 for size, value and momentum attributes. Section 6 concludes. The appendix contains the analysis of characteristic-based portfolios when the universe of assets includes micro-cap firms.

2. Literature review

Our paper is related to the fast-evolving literature developing characteristic-sorted portfolios for investment purposes and asset pricing. The empirical applications of these portfolios are too numerous to list but seminal contributions

† See Hou *et al.* (2020) for a discussion on the impact of microcap firms on characteristic-based portfolios.
‡ The method also facilitates the construction of economically meaningful portfolios with the potential for the addition of constraints to the optimisation, such as limits on the investment in any one firm.
§ Standard rank-sorted portfolios exclude the range of assets between the top and bottom quantiles. In the appendix, we present a robustness exercise that includes the effect of micro-cap stocks.

are Basu (1977), Banz (1981), Jegadeesh (1990), Fama and French (1992), and Jegadeesh and Titman (1993), among many others. Equally important is the contribution of this literature to empirical asset pricing models based on observable common factors, see Fama and French (1993) and Fama and French (2015) as seminal examples. The common factors are constructed as the returns of long-minus-short investment portfolios, where the sorting of assets in the portfolio depends on the specific choice of quantile breakpoints (e.g. deciles, terciles or quartiles).

Sorting into portfolios can be considered as a non-parametric alternative to imposing linearity on the relationship between returns and stock characteristics, (see, e.g. Fama and French 2008, Cochrane 2011). Freyberger *et al.* (2020) formalize this idea and show the equivalence between portfolio sorting based on stock characteristics and a linear regression model in which the regressors are indicator functions of the stock characteristics. Cattaneo *et al.* (2020) extend this idea and develop a general framework for portfolio sorting by casting the procedure as a non-parametric estimator. These authors allow for both estimated quantiles when forming the portfolios and additive linear-in-parameters conditioning variables entering the underlying model governing the relationship between returns and sorting characteristics. These authors develop optimal choices of the total number of portfolios to be used in empirical applications. Ledoit *et al.* (2019) propose efficient sorting portfolios when the cross-section of assets is very large and inverting the covariance matrix can be problematic. These authors obtain a characteristic-sorted portfolio with minimum variance among the set of dollar-neutral portfolios based on stock characteristics. Another recent contribution studying characteristic-based portfolios is Ammann *et al.* (2016) that shows that the introduction of a leverage constraint improves the practical implementation of characteristic-based portfolios by reducing transaction costs, negative portfolio weights, and a decrease in volatility and misspecification risk. A recent alternative to traditional rank sorting for constructing long-minus-short portfolios is proposed by Zhang *et al.* (2021). These authors propose a new listwise learn-to-rank loss function which aims to emphasize both the top and the bottom of a rank list.

The choice of optimal portfolio weights based on stock characteristics is not new. Hjalmarsson and Manchev (2012) study empirical mean–variance optimization under the assumption that the portfolio weights are direct functions of underlying stock characteristics such as value and momentum. Brandt *et al.* (2009) also assume parametric portfolio policies that exploit the characteristics of the cross-section of returns in an optimal asset allocation context. Aboussalah *et al.* (2021) apply mean–variance (and growth optimal investing) strategies in a cross-sectional setting.

Our method is different from these important contributions because we exploit the optimality of the portfolios in a cross-sectional setting to replace ad-hoc long-minus-short portfolio constructions used for empirical asset pricing. Our estimation approach is also very different from these authors as we rely on nonparametric kernel methods for estimating the mean and variance of the constituent portfolio returns conditional on the stock attributes.

The current paper is also related to a recent and very influential literature that adds statistical rigour to characteristic-based portfolio construction. Hou *et al.* (2020) identify 452 anomaly variables that constitute the basis in published studies for asset pricing models based on common factors or as investment factors exploiting such anomalies. These authors in a thorough replication study find that a majority of these anomalies do not replicate. Harvey *et al.* (2016) in a similar study also find that much of the predictive ability of characteristic-based portfolios is due to data mining procedures. To correct for this, these authors propose a multiple testing procedure that yields a larger critical value to validate the success of these portfolios, see also McLean and Pontiff (2016). In parallel, in the risk factor literature there have been a number of recent developments in testing asset pricing factor models. These models propose alternative metrics to test the gains of including additional factors in standard asset pricing factor models (see Barillas and Shanken 2018, Fama and French 2018, Barillas *et al.* 2019).

3. Optimal characteristic-based portfolios

3.1. Econometric model

Our strategy to construct an optimal characteristic-based portfolio shares some of the features of recent developments in the literature, see Hjalmarsson and Manchev (2012), Ledoit *et al.* (2019) or Cattaneo *et al.* (2020). The portfolio weights are the result of maximizing in each period a mean–variance objective function constructed from the cross-section of stock returns as

$$E\left[R^p_{t+1} \mid \mathbf{Z}_t\right] - \frac{\gamma}{2} V\left[R^p_{t+1} \mid \mathbf{Z_t}\right], \tag{1}$$

with $E[\cdot \mid \mathbf{Z_t}]$ and $V[\cdot \mid \mathbf{Z_t}]$ denoting the conditional mean and variance of the portfolio return given the information set $\mathbf{Z}_t$. This objective function can be interpreted as an individual's utility function conditional on the information set at time t. Similarly, the coefficient γ can be interpreted as the degree of individual's risk aversion. The vector $\mathbf{Z_t}$ contains a set of variables with power to predict variation in the expected return of the cross-section. In this paper, and following recent literature, see Cattaneo *et al.* (2020), we restrict the vector $\mathbf{Z_t}$ to consider only stock characteristics. Thus $\mathbf{Z}_t = [\mathbf{Z}_{1t}, \ldots, \mathbf{Z}_{pt}]'$, with p the number of stock characteristics, such that $\mathbf{Z}_{it} = [Z_{1,it}, \ldots, Z_{p,it}]'$, with $Z_{j,it}$ denoting characteristic j for asset i at time t; $\mathbf{z} = (z_1, \ldots, z_p)' \in \Omega \subset \mathbb{R}^p$ denotes the corresponding realizations of $\mathbf{Z}_t$ defined over a compact set Ω.

The portfolio return is defined as a weighted combination of all assets in the cross-section of stock returns. More formally, let $R^p_{t+1}(\mathbf{z}) = \int_\Omega w_t(\mathbf{z}) R_{t+1}(\mathbf{z}) d\mathbf{z}$, with $w_t(\mathbf{z})$ a weight function $\Omega \subset \mathbb{R}^p \to [a, b] \subset \mathbb{R}$ establishing the portfolio allocation as a function of $z \in \Omega$. The compact set $[a, b]$ implies that the portfolio allocation is bounded in this interval to avoid excessive leverage in the portfolio decisions. Note also that, by construction, $a < 0 < b$, to obtain a dollar-neutral portfolio, i.e. $\int_\Omega w_t(\mathbf{z}) d\mathbf{z} = 0$. In practice, we replace the continuum of assets by a cross-section of N_t assets available at time t such that the portfolio return is defined as

$$R^p_{t+1} = \sum_{i=1}^{N_t} w(\mathbf{Z}_{it}) R_{i,t+1}, \tag{2}$$

with $R_{i,t+1}$ the return on asset i at time $t+1$ and $\mathbf{Z}_{it}$ the corresponding vector of stock characteristics. Additionally, the portfolio weights are subject to the constraints

$$\sum_{i=1}^{N_t} w(\mathbf{Z}_{it}) = 0; \quad \sum_{w_i<0} |w(\mathbf{Z}_{it})| = \sum_{w_i>0} |w(\mathbf{Z}_{it})| = 1. \tag{3}$$

The first constraint imposes the dollar-neutrality condition of the portfolio. The second condition is to limit the amount of leverage, adds smoothness to the portfolio weights and is also a standard feature of traditional ranked sorted portfolios. The zero-sum constraint (3) implies that $w(\mathbf{Z}_{1t}) = -\sum_{i=2}^{N_t} w(\mathbf{Z}_{it})$.

Let $R^e_{i,t+1} = R_{i,t+1} - R_{1,t+1}$ denote the excess returns over a reference return $R_{1,t+1}$, for $i = 2, \ldots, N_t$, such that the portfolio return (2) can be expressed as $R^p_{t+1} = \sum_{i=2}^{N_t} w(\mathbf{Z}_{it}) R^e_{i,t+1}$. In a similar spirit to Cattaneo *et al.* (2020), we propose the following non-parametric predictive model for describing the relationship between portfolio returns and stock characteristics:

$$R^e_{i,t+1} = \mu_t(\mathbf{Z}_{it}) + \varepsilon_{i,t+1}, \quad \text{for } t = 1, \ldots, T, \tag{4}$$

where $\mu_t(\mathbf{Z}_{it}) = E[R^e_{i,t+1} \mid \mathbf{Z}_{it}]$ under the minimization of the mean square error. The function $\mu_t(\mathbf{Z}_{it})$ is continuously differentiable over Ω for each $t = 1, \ldots, T$. This function is the unknown object of interest that dictates how expected returns vary with the characteristic at each time t. The error term $\varepsilon_{i,t+1}$ satisfies that $E[\varepsilon_{i,t+1} \mid \mathbf{Z}_{it}] = 0$ and accommodates conditional heteroscedasticity such that $E[\varepsilon^2_{i,t+1} \mid \mathbf{Z}_{it}]$ can be a function of $\mathbf{Z}_{it}$. In this cross-sectional setting, we assume that all the cross dependence between the returns in the cross-section is captured by the conditioning information set $\mathbf{Z}_t$. This information set can be expanded to include observable common factors in the spirit of Fama and French (1993) and Fama and French (2015) and unobservable common factors obtained from principal components analysis, as in Bai (2009), Ando and Bai (2015) and Kelly *et al.* (2019). Hence, by assumption, the conditional covariance between the error terms is such that $E[\varepsilon_{i,t+1}\varepsilon_{j,t+1} \mid \mathbf{Z}_{it}, \mathbf{Z}_{jt}] = 0$ for all $i, j = 1, \ldots, N_t$, with $i \neq j$.†

Under these assumptions, the individual's optimization problem becomes

$$\max_{\{w(\mathbf{Z}_{2t}),\ldots,w(\mathbf{Z}_{N_t,t})\}} \left\{ E\left[\sum_{i=2}^{N_t} w(\mathbf{Z}_{it}) R^e_{i,t+1} \middle| \mathbf{Z}_t \right] - \frac{\gamma}{2} V\left[\sum_{i=2}^{N_t} w(\mathbf{Z}_{it}) R^e_{i,t+1} \middle| \mathbf{Z}_t \right] \right\}. \tag{5}$$

In matrix form, the solution to the optimization problem is $\mu_t = \gamma \Sigma_t w_t$, with μ_t a $(N_t - 1) \times 1$ vector that stacks the functions $\mu_t(Z_{it})$ for $i = 2, \ldots, N_t$. The matrix Σ_t is a $(N_t - 1) \times (N_t - 1)$ matrix with diagonal elements given by $V[R^e_{i,t+1} \mid \mathbf{Z}_{it}]$ and off-diagonal elements given by $Cov[R^e_{i,t+1}, R^e_{j,t+1} \mid \mathbf{Z}_{it}, \mathbf{Z}_{jt}]$, for $i = 2, \ldots, N_t$. The optimal weights that solve this system of equations are

$$w^\star_t = \frac{1}{\gamma} \Sigma_t^{-1} \mu_t, \tag{6}$$

with $w^\star_t = [w^\star(\mathbf{Z}_{2t}), \ldots, w^\star(\mathbf{Z}_{N_t,t})]'$ and $w^\star(\mathbf{Z}_{1t}) = -\sum_{i=2}^{N_t} w^\star(\mathbf{Z}_{it})$. This solution simplifies under the assumption that there is no cross-correlation between the stock returns in the cross-section conditional on the information set $\mathbf{Z}_t$. This assumption is standard in cross-sectional regression models in which the correct specification of the model implies, by construction, that the error terms are mutually uncorrelated‡ . We should note that we operate in a fully cross-sectional setting for each time $t = 1, \ldots, T$. In this case,

$$w^\star_t = \frac{1}{\gamma} D_t^{-1} \mu_t, \tag{7}$$

with D_t a diagonal matrix with elements given by $V[R^e_{i,t+1} \mid \mathbf{Z}_{it}]$, for $i = 2, \ldots, N_t$.

The optimal portfolio weight (7) is a function of the vector of stock characteristics $\mathbf{Z}_{it}$ given by

$$w^\star(\mathbf{Z}_{it}) = \frac{\mu_t(\mathbf{Z}_{it})}{\gamma V\left[R^e_{i,t+1} \mid \mathbf{Z}_t\right]}, \tag{8}$$

for $i = 2, \ldots, N_t$. For example, under conditional homoscedasticity of the cross-sectional returns, the weight function can be expressed as $w^\star(\mathbf{Z}_{it}) = \frac{\mu_t(\mathbf{Z}_{it})}{\gamma \sigma^2_\varepsilon}$, with σ^2_ε the variance of the error process, and the object of interest is to estimate the conditional mean process.

3.2. *Non-parametric estimation of mean and variance of returns*

We reproduce model (4) for convenience:

$$R^e_{i,t+1} = \mu_t(\mathbf{Z}_{it}) + \varepsilon_{i,t+1}, \quad \text{for } t = 1, \ldots, T,$$

where $\mathbf{Z}_{it} = [Z_{1,it}, \ldots, Z_{p,it}]'$ is a vector of stock characteristics.

The Nadaraya–Watson estimator (Nadaraya (1965)) of $E[R^e_{i,t+1} \mid \mathbf{Z}_t]$ is

$$\widehat{\mu}_t(\mathbf{Z}_{it}) = \frac{\frac{1}{N_t} \sum_{j=1}^{N_t} R^e_{j,t+1} K_{h_{1t}}\left(Z_{1,jt}, Z_{1,it}\right) \cdots K_{h_{pt}}\left(Z_{p,jt}, Z_{p,it}\right)}{\widehat{f}_{\mathbf{Z}_t}(\mathbf{Z}_{it})}, \tag{9}$$

† Although this assumption may seem very restrictive it is not unusual in empirical studies when the dimension of the cross-section of stocks is very large, see, e.g., Hjalmarsson and Manchev (2012). Furthermore, in practice, the information set proxied by $\mathbf{Z}_t$ may not be sufficient to capture all cross-sectional dependence, however, the use of cross-sectional data does not allow us to estimate the covariance terms using the time series dimension. Therefore, we acknowledge the importance of considering a suitable information set and the possibility of model misspecification affecting the optimal portfolio weights.

‡ Hjalmarsson and Manchev (2012) consider a similar assumption and ignore the conditional covariance matrix in an empirical analysis of international portfolio choice based on optimal characteristic-based portfolios.

with $\widehat{f}_{\mathbf{Z}_t}(\mathbf{Z}_{it}) = \frac{1}{N_t}\sum_{j=1}^{N_t} K_{h_{1t}}(Z_{1,jt}, Z_{1,it}) \cdots K_{h_{pt}}(Z_{p,jt}, Z_{p,it})$, where $K_h(Z,z) = \frac{1}{h}k(\frac{Z-z}{h})$ and $k(\cdot)$ is a univariate kernel function (e.g. uniform, triangle, Epanechnikov, Gaussian); h is a bandwidth parameter that determines the smoothing of the stock characteristic. In this case, each kernel function is characterized by different bandwidth coefficients $h_{1t}, \ldots, h_{pt}$ that depend on the marginal density function of the covariate and are time varying.†

Similarly, a suitable non-parametric estimator of the conditional variance of the cross-section of excess returns is

$$\widehat{\sigma}_t^2(\mathbf{Z}_{it}) = \frac{\frac{1}{N_t}\sum_{j=1}^{N_t} \widehat{\varepsilon}_{j,t+1}^2 \, K_{h_{1t}}\left(Z_{1,jt}, Z_{1,it}\right) \cdots K_{h_{pt}}\left(Z_{p,jt}, Z_{p,it}\right)}{\widehat{f}_{Z_t}(Z_{it})}, \tag{10}$$

with $\widehat{\varepsilon}_{i,t+1} = R_{i,t+1}^e - \widehat{\mu}_t(\mathbf{Z}_{it})$ the estimated residuals from model (4), for $i = 2, \ldots, N_t$. The bandwidth parameters for the estimation of the conditional mean and variance processes can be different across estimators.

More sophisticated models can be proposed to non-parametrically model the relationship between excess returns and vectors of stock characteristics. In a related context, Connor and Linton (2007) use semi-parametric methods to estimate such relationships. These authors propose multivariate kernel methods to form factor mimicking portfolios for the characteristics. Then they estimate factor returns and factor betas simultaneously using bilinear regression applied to the set of factor mimicking portfolio returns. However, as noted by Connor *et al.* (2012), a weakness of the Connor–Linton methodology is the reliance on multivariate kernel methods to create factor-mimicking portfolios. These multivariate kernel methods are affected by the curse of dimensionality (Stone 1980) and severely restrict the number of factors which can be consistently estimated.

An important recent contribution to obtain a satisfactory answer in a non-parametric framework is Freyberger *et al.* (2020). These authors propose non-parametric functions based on quadratic splines that fit the unknown function $\mu(\mathbf{Z}_{it})$ within small disjoint intervals. The method accommodates a number of stock characteristics that can be larger than the number of stocks in the cross-section. To do that, the authors impose an adaptive LASSO type regularization penalty function. A small dimensional alternative used in the non-parametric literature is non-parametric additive models. This possibility is already contemplated in Cattaneo *et al.* (2020) and Freyberger *et al.* (2020) using a non-optimal allocation of portfolio weights. The non-parametric additive model considers the effect of each characteristic on the cross-section of returns separately and does not allow for the possibility of interactions between stock attributes. The model is

$$R_{i,t+1}^e = \sum_{k=1}^{p} \mu_{kt}(Z_{k,it}) + \varepsilon_{i,t+1}, \quad \text{for } t = 1, \ldots, T. \tag{11}$$

This model can be estimated as in (4), with

$$\widehat{\mu}_{kt}(Z_{k,it}) = \frac{\frac{1}{N_t}\sum_{j=1}^{N_t} R_{j,t+1}^e \, K_{h_{kt}}\left(Z_{k,jt}, Z_{k,it}\right)}{\widehat{f}_{Z_{kt}}(Z_{k,it})}, \quad \text{for } k = 1, \ldots, p. \tag{12}$$

The case of a single characteristic is a particular example of non-parametric additive model for $p = 1$.

Our optimization strategy is agnostic about the estimation procedure, however, our empirical application will show that the additional flexibility offered by non-parametric methods can result in characteristic-based portfolios that perform better across a range of standard metrics. This finding is particularly important for mean–variance optimal portfolio decisions. There is an influential literature that highlights the difficulty of using suitable predictors of expected returns, see, for example, Best and Grauer (1991), Black and Litterman (1992), and more recently, Jagannathan and Ma (2003), among many others. These authors show that the choice of model and estimator for the expected returns may have sizeable effects on the optimal portfolio allocation of mean–variance investment strategies. In this respect, the choice of a non-parametric procedure to model the conditional excess returns on the portfolio assets may be a safe strategy as the methodology does not suffer from model risk, although estimation risk may be slightly higher due to the use of non-parametric convergence rates, as discussed in footnote 6.

Non-parametric methods suffer from the curse of dimensionality issues. In the empirical application, we overcome this by focusing on a single characteristic. Our application is therefore more grounded in the cross-sectional anomaly literature, motivated by recent work such as Hou *et al.* (2020). These authors show that cross-sectional regressions with many variables are excessively flexible. Leamer and Leonard (1983) show that inferences based on slopes from linear regressions are sensitive to the underlying specification. For example, two individually insignificant variables that are highly correlated can appear significant when used together. Hou *et al.* (2020) avoid this trap by using univariate regressions. Furthermore, given that one of our main contributions is to show the incremental value of using an optimal strategy to construct characteristic portfolios, it is sufficient to show the results obtained from one single characteristic. In the empirical application, we will separately explore size, value and momentum anomalies.

A second drawback common to non-parametric methods is the choice of the bandwidth parameters $(h_{1t}, \ldots, h_{pt})$. This choice is associated to the kernel functions $K_h(Z,z)$. Non-parametric kernel estimation has been established as being relatively insensitive to the choice of the kernel function.

† It is worth noting that this estimator exhibits some finite sample bias due to the choice of the bandwidth parameter. More specifically, $\widehat{\mu}_t(\mathbf{Z}_{it}) - \mu_t(\mathbf{Z}_{it}) = O_P(\sum_{i=1}^{q} h_s^2 + (nh_1 \cdots h_q)^{-1/2})$. Li and Racine (2007) show that if each bandwidth has the same order of magnitude, then the optimal choice of h_s that minimizes the mean square error (MSE) of the estimator $\widehat{\mu}_t(\mathbf{Z}_{it})$ is $h_s \sim n^{-1/(q+4)}$. The resulting MSE is, therefore, of order $O_P(n^{-4/(q+4)})$. In the empirical application, we will consider $q = 1$, in which case the MSE is $O_P(n^{-4/5})$, which quickly converges to zero given the large cross-sections ($n \equiv N_t$) considered in our paper for each period t.

The same cannot be said for bandwidth selection. There are different methods for optimally choosing the bandwidth parameter, namely, rule-of-thumb procedures, plug-in methods, least squares and maximum likelihood cross-validation methods. It is well known in the non-parametric econometrics literature that the optimal bandwidth parameter for Nadaraya–Watson type estimators for the conditional mean and variance estimators is $h = O(N_t^{-1/(4+p)})$, with p the number of regressors, see Li and Racine (2007) for an excellent monograph on the topic and Fan and Gijbels (1995) as a more specific reference for optimal choice of bandwidth parameter in regression models. A suitable choice for a regression model with one regressor is $h = cN_t^{-1/5}$, with c a positive constant. However, different procedures yield specific choices of the optimal bandwidth. A popular rule-of-thumb procedure is to choose the bandwidth parameter as $h = c\widehat{S}N_t^{-1/5}$, with $\widehat{S}$ the sample standard deviation of the stock attribute under consideration. Using this rule-of-thumb procedure, we propose the following bandwidth parameters: $h_{k,t} = \widehat{S}_{k,t}N_t^{-1/5}$, where $c = 1$ and $\widehat{S}^2_{k,t} = \frac{1}{N_t}\sum_{i=1}^{N_t} Z^2_{k,it}$, for $k = 1, \ldots, p$.†

4. Empirical application

In this section we apply the portfolio methods proposed in Section 3 to three popular anomalies in the financial economics literature (size, value and momentum) and compare results against other widely used characteristic portfolio construction methods.

4.1. Data

We use CRSP and Compustat data over the period July 1963 to November 2018. We include all common stocks (ordinary equity, CRSP sharecode 10 or 11) in the CRSP stock files. To avoid survivorship bias, all de-listed stocks are included and given the CRSP de-listed return. Monthly risk factor data and the risk-free rate are obtained from the Fama & French database on WRDS. We apply attribute specific filters, for the size analysis a stock must have a valid market capitalization at the end of the previous month. For the value attribute each included stock must have a valid book to market value calculated from Compustat data in December of year $y-1$. For the momentum attribute, for a stock to be included in a portfolio for month m (formed at the end of month $m-1$), must have a price for the end of month $t-13$ and a good return for $m-2$. More details on the process of obtaining attribute estimates are given in Section 4.3.

As per Hou *et al.* (2020), all attribute data are winsorized to the percentile range [1 99], and standardised by demeaning and dividing by the standard deviation. The final data used are in units of cross-sectional standard deviations of the attribute. For consistency, all portfolios are normalised such that their constituent long and short portfolios have unit investment weights: $+100\%/-100\%$, see portfolio constraints in (3). As described in Section 4.5, we perform our analysis on a data set excluding micro-cap stocks (with analysis on the unfiltered data set reported in the appendix).

4.2. Portfolio construction

It is important to benchmark our characteristic-based portfolios against existing methods. Hou *et al.* (2020) assess the profitability of a battery of investment factors constructed on more than four hundred anomalies. These authors entertain three different types of investment factors:

Rank sorted portfolios: Portfolios constructed through dividing the asset cross-section in deciles according to a given characteristic and going long the assets in the top decile and short the assets in the bottom decile. These portfolios are either market capitalization weighted or equally weighted and the net investment is zero.

OLS portfolio: These portfolios assume a linear relationship between the cross-section of stock returns and the stock characteristic. This strategy constructs two portfolios: an intercept and a slope portfolio. The return on these portfolios is $R^{ols}_{t+1} = B_t$, defined as $B_t = (X_t'X_t)^{-1}X_t'R_{t+1}$, where abusing of notation, we define $X_t = [1\ Z_t]$. The quantity 1 denotes a column vector of ones and Z_t a vector of same dimension stacking the stock characteristics for all the cross-section of stocks at time t. This portfolio construction satisfies the condition $W_t'X_t = I_2$, with $W_t = X_t(X_t'X_t)^{-1}$ and I_2 the 2×2 identity matrix, guaranteeing that the sum of the weights across stocks is equal to zero.

WLS portfolio: This portfolio is similar to the OLS portfolio, however, in this case assets are not equally weighted. Instead, the weights in this portfolio are determined according to market capitalization m_{it}. To do this, we define a diagonal matrix $M_t = diag[m_{1t}, \ldots, m_{N_t\,t}]$ and construct the portfolio as before: $R^{wls}_{t+1} = B_t$, with $B_t = (X_t'M_tX_t)^{-1}X_t'M_tR_{t+1}$, with $W_t = M_t'X_t(X_t'M_tX_t)^{-1}$ such that $W_t'X_t = I_2$.

As a recent contribution to the literature on characteristic-based portfolios, we also consider the non-parametric approach by Cattaneo *et al.* (2020). This is the fourth approach that we consider for comparison purposes:

Cattaneo et al. (2020): We also consider rank sorted portfolios constructed as per the method in Cattaneo *et al.* (2020). This method results in a much larger selected number of quantiles for rank sorting than standard methods (with fewer stocks within long and short portfolios). To replicate the investment strategy we implement the procedure in Appendix of their paper.‡ Figure 1 illustrates the optimal selected number of quantiles for each of size, value and momentum attributes over time using their approach.

In addition to these techniques, we add the characteristic portfolios using the methods proposed in Section 3. In total we compare seven different attribute portfolio constructions for each stock characteristic:

† To assess the robustness of our empirical estimates to the choice of $h_{k,t}$, we have also considered different values of c in the range $[0.5, 2]$. Unreported results show that the portfolio performance metrics obtained under different values of c are almost identical.

‡ We use the thresholds $\Phi^{-1}(0.05)$, $\Phi^{-1}(0.95)$ and perform a grid search up to a maximum quantile level of 400 as per their paper.

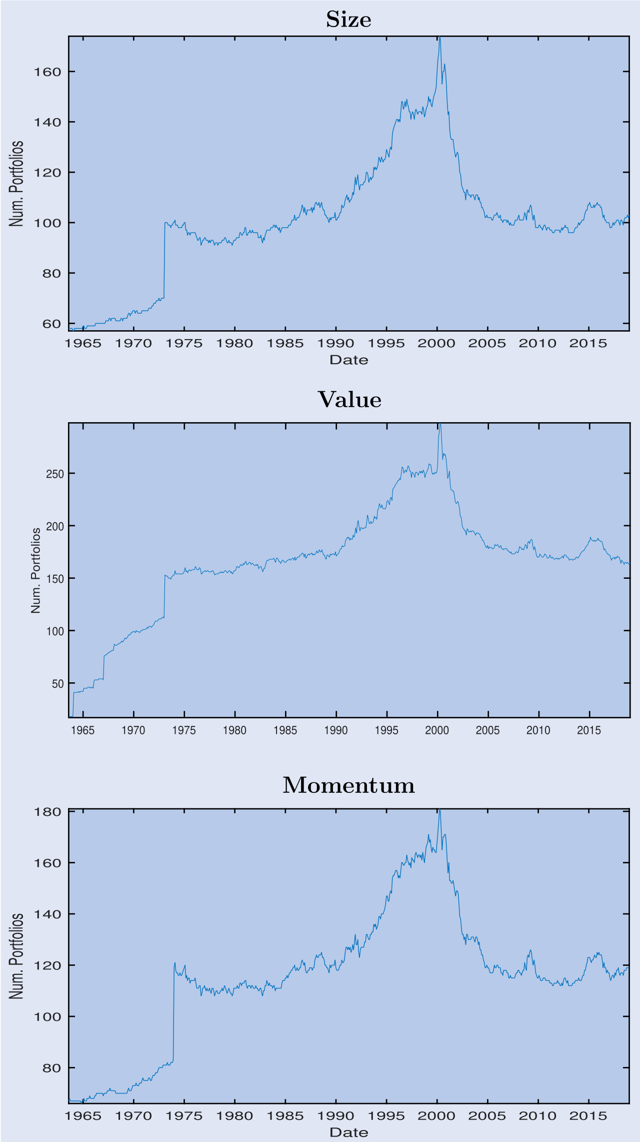

Figure 1. Optimal quantile numbers: The optimal number of quantile portfolios in characteristic-sorted portfolios found using the approach in Cattaneo *et al.* (2020) to minimize MSE of the anomaly premium.

MV_{NP}: The characteristic-based Mean–Variance optimised investment portfolio is given in expression (8). Mean and variance are estimated using non-parametric kernel methods, and the risk aversion coefficient γ is taken equal to 3.†

$FF_{SMB}/FF_{HML}/FF_{MOMO}$: Fama and French estimated size, value and momentum factors (where MOMO is the Carhart momentum factor).‡

OLS: The OLS-weight attribute portfolio defined above.

WLS: The WLS-weight attribute portfolio defined above.

RS_{CCFS}: The optimal in number of quantile, rank-sorted portfolios of Cattaneo *et al.* (2020).

RS_{EW}: The equally-weighted rank, sorted decile portfolio based on the attribute (e.g. for size, long stocks below the 10 percentile of market capitalization and short stocks above the 90 percentile).

RS_{CW}: The cap-weighted version of the ranked, sorted decile attribute portfolio RS_{EW}.

4.3. Example attributes

McLean and Pontiff (2016) and Hou *et al.* (2020) find that a majority of anomalies in the finance literature do not replicate. For the purposes of evaluating our approach we focus on three of the most popular attributes in financial economics that have been successfully replicated in the literature—the size, value and momentum anomalies.

4.3.1. Size. A size premium, where smaller firms earn higher returns than larger firms on average, has been widely documented in asset pricing research, dating back to Banz (1981). Cattaneo *et al.* (2020) examine the size anomaly using rank sorted portfolios with an optimal, time-varying number of ranking quantiles. They find that the size premium is highly significant and is robust to different sub-periods. However, they find that the size anomaly is not robust in sub-samples which exclude small firms.

We calculate Market Capitalization (hence forth ME) for each CRSP security as $|prc| * shrout$, where *prc* is the CRSP share price and *shrout* is the number of shares outstanding. As in McGee and Olmo (2019), we use the natural log of ME as our size attribute. As a benchmark portfolio, we consider the Fama and French *SMB* portfolio (see Fama and French 1993).

4.3.2. Value vs growth. The Value premium is a higher average return earned by value stocks over growth stocks. Hou *et al.* (2020) find a statistically significant premium associated with book to market, using annually sorted high-minus-low book-to-market decile portfolios. As a value attribute we estimate book to market as per Fama and French (1993). Book equity is calculated as $BE = SEQ + TXDITC - PS$, where *SEQ* is total parent stockholders' equity and *TXDITC* is deferred taxes and investment tax credit.§ For preferred stock, *PS*, we use the redemption value, *PSTKRV*, or the liquidation value, *PSTKL* or the par value, *PSTK*, in that order (see Fama and French 1993, p8). As we re-balance monthly we do not follow the Fama and French approach of using values sampled in June to allocate stocks to attribute portfolios for the following 12 months, we dynamically estimate using the book to market in December of year $y - 1$, and the ME at end of month, $m - 1$.

4.3.3. Momentum. The Momentum premium is where firms that have had better relative returns in the recent past have higher future relative returns, on average. Momentum anomalies fare well in terms of replicability in Hou *et al.* (2020), in particular price momentum. To construct a

† Davies (1981) suggests a relative risk aversion value in the range 3–4, we adopt the lower bound as equity investors are often diversified across portfolios including less risky assets.
‡ Data obtained from https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

§ All variable names correspond to the Compustat variable name definitions momentum attribute for each stock, we use the methodology in Carhart (1997). Specifically, we construct a 2–12 month momentum signal constructed as the 11-month return lagged by 1 month (to exclude reversal effects). As a benchmark we consider the Fama French MOMO portfolio, constructed using six sorted portfolios on size and momentum (see Fama and French 2012).†

4.4. Portfolio evaluation

The success of attribute portfolios is measured by the statistical significance of the estimated return premia, by assessing the corresponding *alpha* in Fama–French type asset pricing equations and by comparing a risk-adjusted investment metric (the information ratio). We should note the recent controversy on the appropriate choice of critical values for determining the significance of the estimates in these regressions, see McLean and Pontiff (2016), Harvey *et al.* (2016) and Hou *et al.* (2020), and we use a critical value adjusted for multiple hypothesis testing (t-stat > 2.78 at the 5% level).

Finally, we also compare the performance of each candidate portfolio in an asset pricing model. Recent research on evaluating factor models advocates a model selection approach through identifying the model whose factors yield the highest difference in squared Sharpe ratio (see, e.g. Barillas and Shanken 2018, Fama and French 2018, Barillas *et al.* 2019).

In this article, we are focusing on evaluating alternative factor portfolio construction methodologies. To assess the performance of the proposed optimal characteristic-based portfolios, we adopt an evaluation approach whereby we individually replace the risk factors in a four factor model (Fama and French 3-factor model with added momentum factor, $FF3M$), with each of the candidate attribute portfolios described in Section 4.2. We perform a non-nested model comparison as discussed in Barillas *et al.* (2019) over all models including all alternative attribute portfolio constructs for each attribute (size, value and momentum) to determine which factor construct yields the best asset pricing model.

4.5. Controlling for micro-cap stocks

It is standard procedure in creating rank sorted portfolios to value-weight or equal-weight stocks in the selected long-short quantile portfolios. Arguments for value-weighting include that it accurately reflects the wealth effect experienced by investors (Fama 1998). However, both of these standard approaches will introduce new factor exposures to the resulting portfolios. Value-weighting adds a negative exposure to the size attribute (within quantile portfolios). Equal-weighting is shown by Plyakha *et al.* (2015) to introduce a number of factor exposures along with a re-balancing return.

Our proposed optimal non-parametric characteristic portfolios assign to every stock a portfolio weight uniquely determined by its attribute score and avoids the introduction of additional factor exposures through the overlaying of value- or equal-weighting schemes. A legitimate concern is that

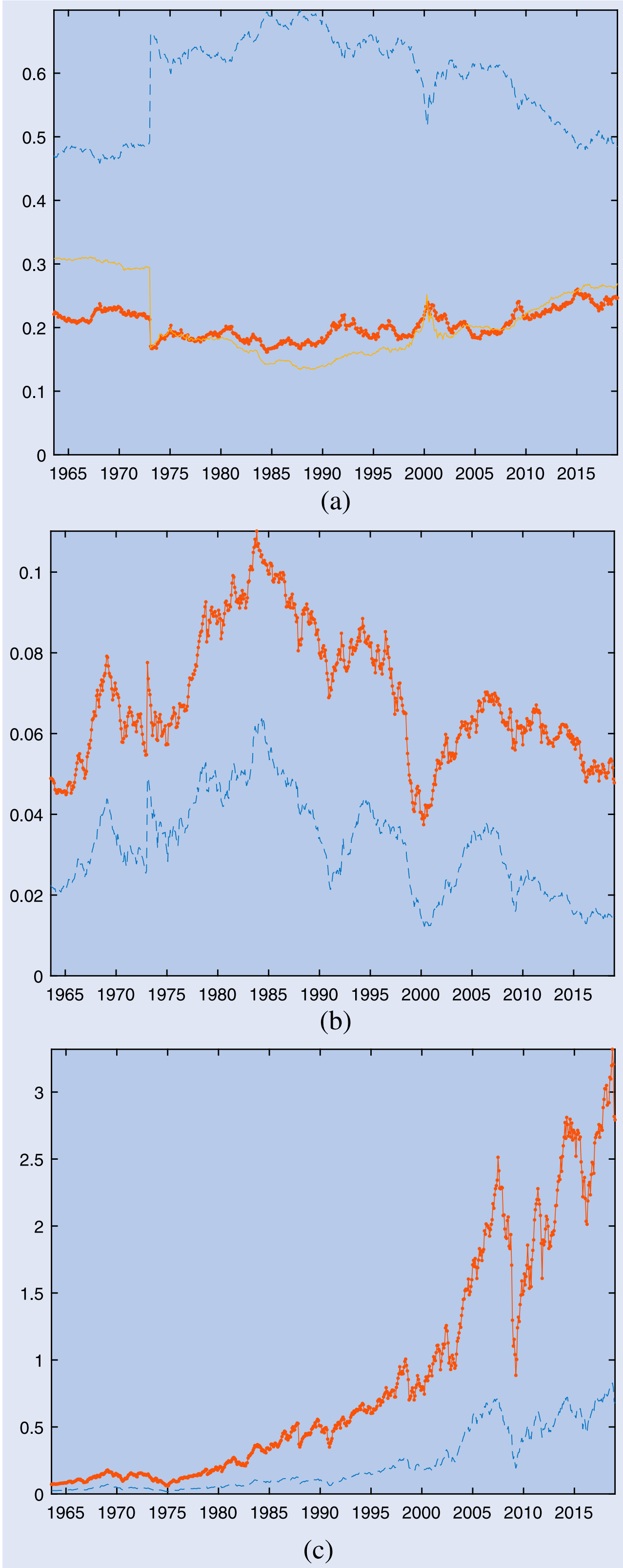

Figure 2. Micro-cap stock statistics (1963–2018): (A) the proportion of the total number of stocks made up by micro-caps (blue), small stocks (orange) and large stocks (yellow). (B) The proportion of the total market capitalization made up of micro-caps (blue) and small stocks (orange). (C) The 20 percentile (blue) and 50 percentile (orange) market capitalization breakpoints used in the size classifications (in billions).

† All Fama & French benchmark portfolios are downloaded from: https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

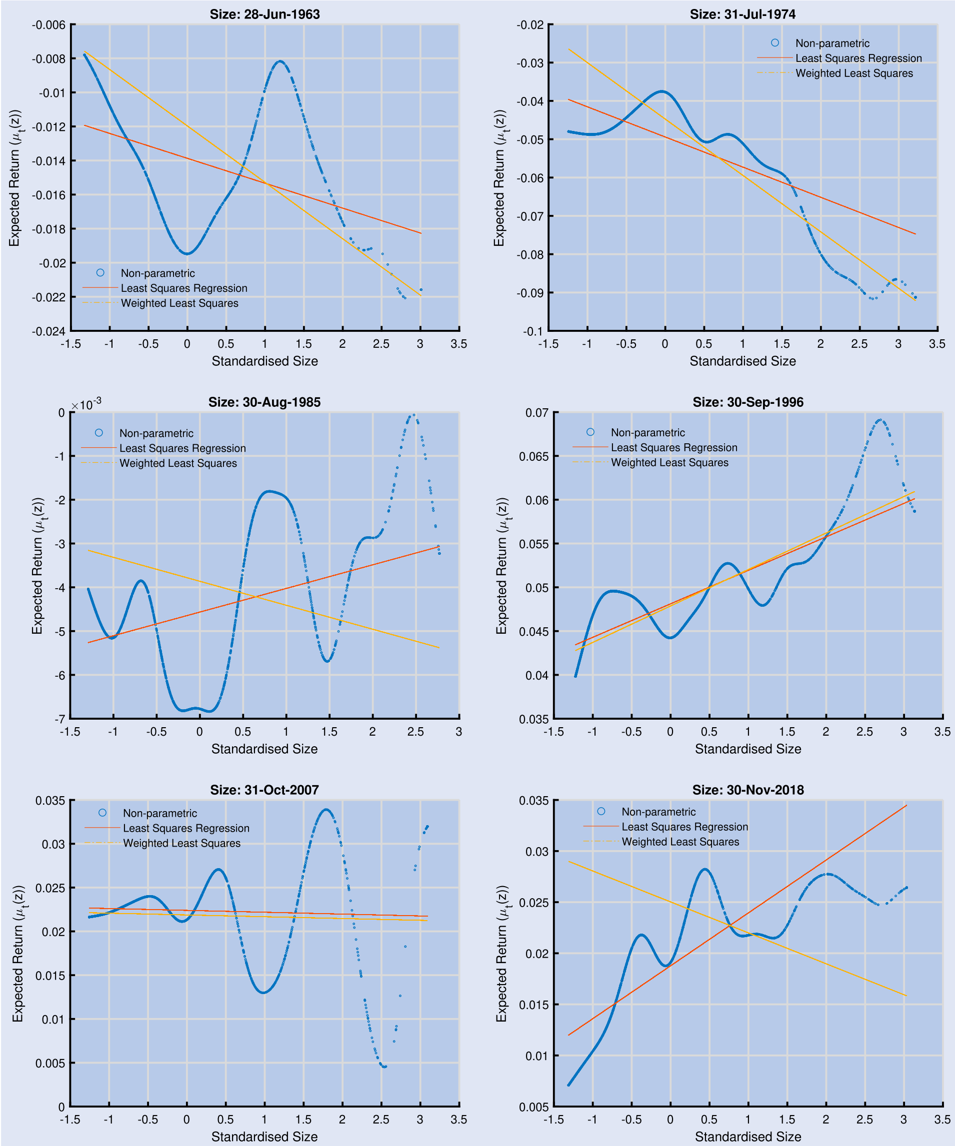

Figure 3. Size vs. expected return cross-sectional regressions: plots of the estimated relationship between the size attribute and realized returns using the non-parametric estimator for expression (8); least squares regression and weighted least squares (with stock market capitalizations used as the weights). The illustrated dates are sampled as 6 equally-spaced months over our full date range. The final panel, corresponding to November 2018, illustrates the sensitivity to estimation method as the OLS and WLS estimated slopes can be seen visually to be of opposite polarity.

this procedure could potentially result in large allocations to micro-cap stocks, small firms that are not economically significant. These firms could be difficult to invest in due to their low liquidity and potentially small market capitalization relative to the size of a fund's assets under management. Anomalies in micro-caps have been shown to be difficult to exploit in practice due to high transaction costs (Novy-Marx and Velikov 2016).

To address this, we adopt a filter as per Hou *et al.* (2020) that removes micro-caps from the sample. On each month, we exclude the stocks that have market capitalization smaller than the 20 percentile level of stocks listed on the NYSE.† On average, over our full sample, micro-caps make up 59% of the stocks‡ and 3.2% by market capitalization (see figure 2). For robustness, we also report summary results in the appendix on a data set with micro-caps included.

As we do not value weight the MV_{NP} portfolio, a potential criticism is that it may include unrealistic corner case allocations in small capitalization stocks. To address this we perform analysis on implied investment stakes in individual companies to check whether the scheme is making unrealistic allocations to individual securities. As a metric, we consider the maximum monthly allocation of each scheme, in terms of the percentage of market capitalization of each investment asset. To do this, we need to select a size of our fund or assets under management (AUM). We use a notional value of $500

† They also include a small stock classification of stocks larger than the 20th percentile and smaller than median and large stocks as those greater than the median.

‡ Consistent with the figures of 60% in Fama and French (2008) and 60.7% in Hou *et al.* (2020) (we note that the percentage is trending downward in the latter part of our sample).

Table 1. Size characteristic portfolio statistics: summary monthly return premia statistics and correlations for eight different size portfolio constructions (excess market return, FF_{MKT}, is also included for comparison).

Panel A. Means, Standard Deviations, and *t*-Statistics

Factor	Mean(%)	Std Dev.	*t*-Statistic	Info. Ratio
FF_{MKT}	0.51	4.39	3.02	0.40
FF_{SMB}	0.20	3.06	1.73	0.23
MV_{NP}	0.40	2.37	4.33	0.58
OLS	0.08	1.14	1.80	0.24
WLS	0.08	1.08	1.94	0.26
RS_{CCFS}	0.23	5.43	1.10	0.15
RS_{EW}	0.28	3.71	1.92	0.26
RS_{CW}	0.30	4.02	1.95	0.26

Panel B. Correlations

	FF_{SMB}	MV_{NP}	OLS	WLS	RS_{CCFS}	RS_{EW}	RS_{CW}
FF_{MKT}	0.29	0.10	0.33	0.36	0.34	0.34	0.36
FF_{SMB}		0.23	0.92	0.82	0.76	0.89	0.91
MV_{NP}			0.28	0.20	0.21	0.30	0.28
OLS				0.78	0.77	0.97	0.96
WLS					0.81	0.77	0.87
RS_{CCFS}						0.79	0.84
RS_{EW}							0.98

Note: FF_{SMB} is the Fama and French size factor; MV_{NP} is an optimal characteristic portfolio defined using the non-parametric estimator (8); OLS is the OLS-implied weight factor: $B_t = (X_t'X_t)^{-1}X_t'R_{t+1}$, with $X_t = [1\ Z_t]$; WLS is the least squares factor weighted by market capitalization and defined as $B_t = (X_t'M_tX_t)^{-1}X_t'M_tR_{t+1}$, with $W_t = M_t'X_t(X_t'M_tX_t)^{-1}$ and $W_t'X_t = I_2$; RS_{CCFS} is the rank sorted portfolio with a time varying number of quantile portfolios as per Cattaneo *et al.* (2020); RS_{EW} is the equally-weighted rank sorted size portfolio (short stocks above the 90 percentile market cap. value and long those below the 10 percentile) and RS_{CW} is the cap-weighted version of the same construct.

million in AUM, this value is adjusted backward in time for inflation from the end date of November 2018.†

5. Empirical results

In this section, we consider results for an attribute portfolio, based on a single asset attribute/return relationship, updated monthly. The 2-month lagged cross-sectional attribute vector and 1-month lagged return are used to estimate the return/attribute relationship: $R_{i,t-1} = \mu_{t-1}(Z_{i,t-2}) + \varepsilon_{t-1}$. Forecasts for the upcoming month, from t to $t+1$, and the corresponding Mean–Variance optimal asset weights, are then estimated by applying μ_{t-1} to Z_{t-1}, the 1-month lagged cross-sectional attribute vector.

5.1. *Size results*

We use the logarithm of 1-month lagged market equity as the attribute assumed to drive returns. Examples of the kernel estimated attribute/return relationship are compared with OLS and WLS estimates of the relationship in figure 3. The plots illustrate the dynamic nature of the monthly cross-sectional relationship between asset size and returns and highlight the sensitivity of the implied relationship to choices of estimation method and weighting schemes. In November 2018, for example, the OLS and WLS estimators yield slopes of opposite polarity for the relationship between size and expected return.

† As we are comparing across portfolios the relative implied percentages are important rather than the absolute values and the selected AUM is for illustration only.

Table 2. Size characteristic portfolio regressions: regression results for each of the seven size characteristic based portfolios, regressed against the Fama–French three-factor model with an added Carhart momentum factor ($FF3M$). P-values are in brackets.

Factor	α	β_{MKT}	β_{SMB}	β_{HML}	β_{MOMO}
Panel A. FF3M Regression Coefficients					
MV_{NP}	0.46***	− 0.01	0.17***	−0.07**	−0.10***
	(< 0.01)	(0.79)	(< 0.01)	(0.04)	(< 0.01)
OLS	0.01	0.02	0.34	0.04	− 0.03
	(0.63)	(< 0.01)	(< 0.01)	(< 0.01)	(< 0.01)
WLS	0.00	0.04***	0.28***	0.07***	−0.03***
	(0.89)	(< 0.01)	(< 0.01)	(< 0.01)	(< 0.01)
RS_{CCFS}	− 0.08	0.17***	1.33***	0.30***	−0.22***
	(0.55)	(< 0.01)	(< 0.01)	(< 0.01)	(< 0.01)
RS_{EW}	0.06	0.07***	1.07***	0.15***	−0.14***
	(0.32)	(< 0.01)	(< 0.01)	(< 0.01)	(< 0.01)
RS_{CW}	0.04	0.10***	1.18***	0.20***	−0.15***
	(0.46)	(< 0.01)	(< 0.01)	(< 0.01)	(< 0.01)

Table 3. Asset pricing model comparison of size characteristic portfolios: tests of equality of squared Sharpe ratios: *FF3M* is the four-factor model constructed by adding a Carhart momentum factor to the Fama–French three-factor model.

Model	FF3M+WLS	FF3M	FF3M+OLS	FF3M+RS_{CW}	FF3M+RS_{EW}	FF3M+MV_{NP}
Panel A. Differences in sample squared Sharpe ratios						
FF3M+RS_{CCFS}	0.002	0.002	0.002	0.003	0.003	**0.042****
FF3M+WLS		0.001	0.001	0.001	0.002	**0.041****
FF3M			0.000	0.001	0.001	**0.040****
FF3M+OLS				0.000	0.001	**0.040****
FF3M+RS_{CW}					0.001	**0.039****
FF3M+RS_{EW}						**0.039****
Panel B. P-values						
FF3M+RS_{CCFS}	0.534	0.462	0.476	0.418	0.406	0.014
FF3M+WLS		0.812	0.771	0.634	0.611	0.018
FF3M			0.825	0.671	0.571	0.018
FF3M+OLS				0.749	0.460	0.017
FF3M+RS_{CW}					0.610	0.018
FF3M+RS_{EW}						0.019

Note: The table shows pairwise tests of risk factor models constructed by replacing the Fama–French size factor with each of the other seven alternative size-based portfolios (row labels). A positive value indicates that the model in the column label outperforms the model in the corresponding row label in terms of the pairwise squared Sharpe ratio test of Barillas *et al.* (2019), P-values for the Sharpe difference are also displayed, $*$ indicates significance at the 10% level. The best performing portfolio in terms of the squared Sharpe metric is the MV_{NP} portfolio.

Summary statistics for the seven alternative size portfolio constructions are given in table 1. The highest returning size portfolio, with an average monthly return of 0.4%, is the MV_{NP} portfolio, defined in equation (8). The MV_{NP} portfolio has the only realized size return premium that is statistically significant, under a test statistic adjusted for multiple testing as per Harvey *et al.* (2016) (t-stats of 2.79 and 4.33 respectively). From an investment metric perspective, the MV_{NP} portfolio also has by far the highest information ratio across all portfolios tested (0.58). The MV_{NP} portfolio returns have a correlation value of 0.23 with the Fama and French (1993, 2015) size factor and have a low correlation with the market portfolio factor (0.10).

Table 2 shows the results of linear regressions of the size characteristic portfolios on a four-factor model consisting of the Fama–French three-factor model with an added Carhart momentum factor (henceforth referred to as *FF3M*). The MV_{NP} portfolio has a statistically significant *alpha* value of 0.46% monthly (P-value < 0.01), it also has a statistically insignificant beta to the market factor (P-value 0.79).

In table 3, the results of asset pricing tests on risk factor models including each size attribute are displayed. The benchmark model is the *FF3M* model and in each test the Fama–French size factor in the original model, FF_{SMB}, is replaced with an alternative construction, one of the other seven portfolio constructions presented in Section 4.2. The resulting alternative factor models are tested against each other in pairwise tests on the significance of improvement to the squared Sharpe ratio. The MV_{NP} portfolio is the only construct to significantly outperform the benchmark FF3M model at the 5% level (P-value of 0.018). Factor models including the variable also significantly outperform models including all the other size portfolio constructs.

Finally, we compare the economic significance of the strategy in terms of how much investment capital it can hold without over-investing in individual firms. Figure 4 shows that the MV_{NP} scheme would have lower maximum dollar fund allocations:

$$\max_i \left(\frac{\text{portfolio weighting in firm i} \times \text{fund size}}{\text{market capitalization firm i}} \right) \tag{13}$$

than a capitalization-weighted ranked, sorted decile portfolio scheme.†

5.2. *Value*

We use the ratio of the previous financial year's book equity (calculated as per Fama and French (1993)) to 1 month lagged market equity as the attribute assumed to drive returns. Examples of the kernel estimated attribute/return relationship, compared with OLS and WLS estimates of the relationship, are illustrated in figure 5. As with the size attribute, the plots illustrate the dynamic nature of the monthly cross-sectional relationship between value and returns. The WLS weighting scheme flips the polarity of the relationship in two out of six randomly selected months illustrated.

Summary statistics for the seven alternative value attribute portfolio constructions are given in table 4. The highest returning portfolio, with an average monthly return of 0.91%, is the RS_{CCFS}, however, the returns are highly volatile and the test statistic for the return premium does not meet the

† For comparison purposes, at time of writing, the largest mutual fund in the world is the Vanguard 500 Index Fund, with AUM of approximately \$300 billion. The fund's largest individual holding is a 5% allocation to Microsoft, which represents approximately a 1.5% share of the firm's market capitalization.

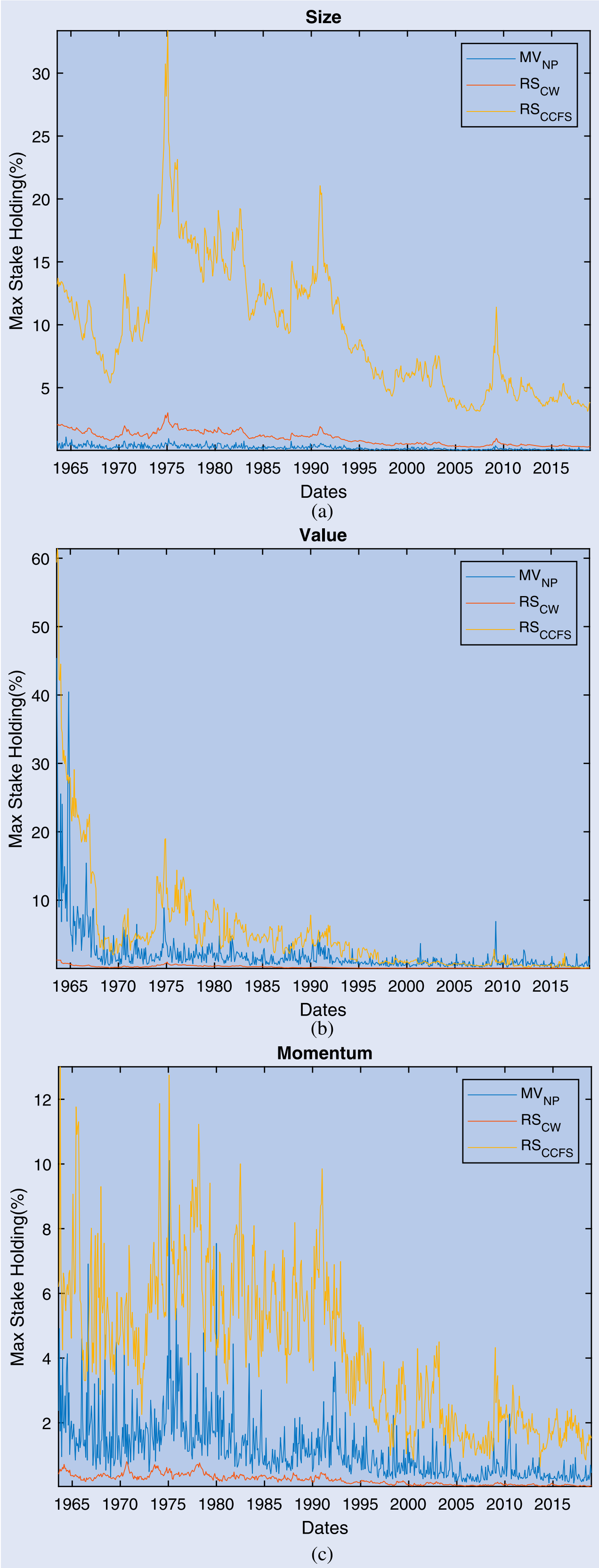

Figure 4. Maximum monthly stake holdings: the maximum percentage stake holding in a single company over time for each of the MV_{NP}, RS_{CW} and RS_{CCFS} portfolios and for size, value and momentum attributes. The values are estimated using a notional AUM figure of \$500 million adjusted for inflation back in time. The notional amount is for the purposes of relative comparison across the methods only.

threshold for significance at the 5% size, adjusted for multiple hypothesis testing as per Harvey *et al.* (2016) ($|t-stat| = 2.54$ vs. 2.78). Although it has a lower expected return, the premium for the MV_{NP} value portfolio is the most significant statistically (t-stat of 4.03). From an investment metric perspective, the MV_{NP} portfolio also has the highest information ratio (0.54). The portfolio returns have positive correlation (0.18) with the value factor of Fama and French (1993, 2015) and a negative correlation with the market factor (-0.28).

Table 5 shows the results of linear regressions of the value characteristic portfolios on a four-factor model consisting of the Fama–French three factor model with an added Carhart momentum factor. The MV_{NP} portfolio has a statistically significant *alpha* value of 0.36% monthly (P-value < 0.01). It also has an insignificant beta to the market factor (-0.02, P-value 0.53).

Table 6 displays the results of asset pricing tests on risk factor models including each alternative value attribute portfolio. The factor model including the MV_{NP} portfolio does not improve the benchmark $FF3M$ model despite the better investment metric performance of the portfolio standalone. This suggests that some of the additional Mean–Variance benefits of the standalone portfolio are captured across other factors in the benchmark model that may be correlated with MV_{NP}.† The best performing factor is the OLS factor. A four-factor model including this as the value factor outperforms all other models tested, however, the outperformance of the benchmark $FF3M$ model is not statistically significant at the 10% level (P-value 0.147).‡

5.3. Momentum

We use the Carhart momentum variable (the 1-month lag of the return over the previous 11 months) as the attribute assumed to drive returns. Examples of the kernel estimated attribute/return relationship, compared with OLS and WLS estimates of the relationship, are illustrated in figure 6. As with size and value attributes, there is variation in the estimated relationship between stock characteristics and returns depending on the method used to construct the portfolios, however, there is less divergence between value weighted estimates (WLS) and OLS estimates for the momentum attribute, with both having a common polarity across the illustrated sample.

Summary statistics for all tested factor constructions are given in table 7. The highest returning portfolio, with a huge average monthly return of 2.42%, is the RS_{CCFS} portfolio. The momentum attribute is quite robust to the portfolio construction method with 7 out of 10 portfolios having a risk premium with a t-stat adjusted for multiple hypothesis testing significant at the 5% level (2.78 or higher). From an investment metric perspective, the RS_{CCFS} portfolio has the highest information ratio (0.67), followed closely by an equally-weighted rank, sorted decile portfolio, RS_{EW}.

† For example, the Fama and French size factor, FF_{SMB}, is sorted by both size and value.

‡ It should be noted that the benchmark model includes factors that are sorted along multiple dimensions and we are constructing univariate sorted portfolio alternatives.

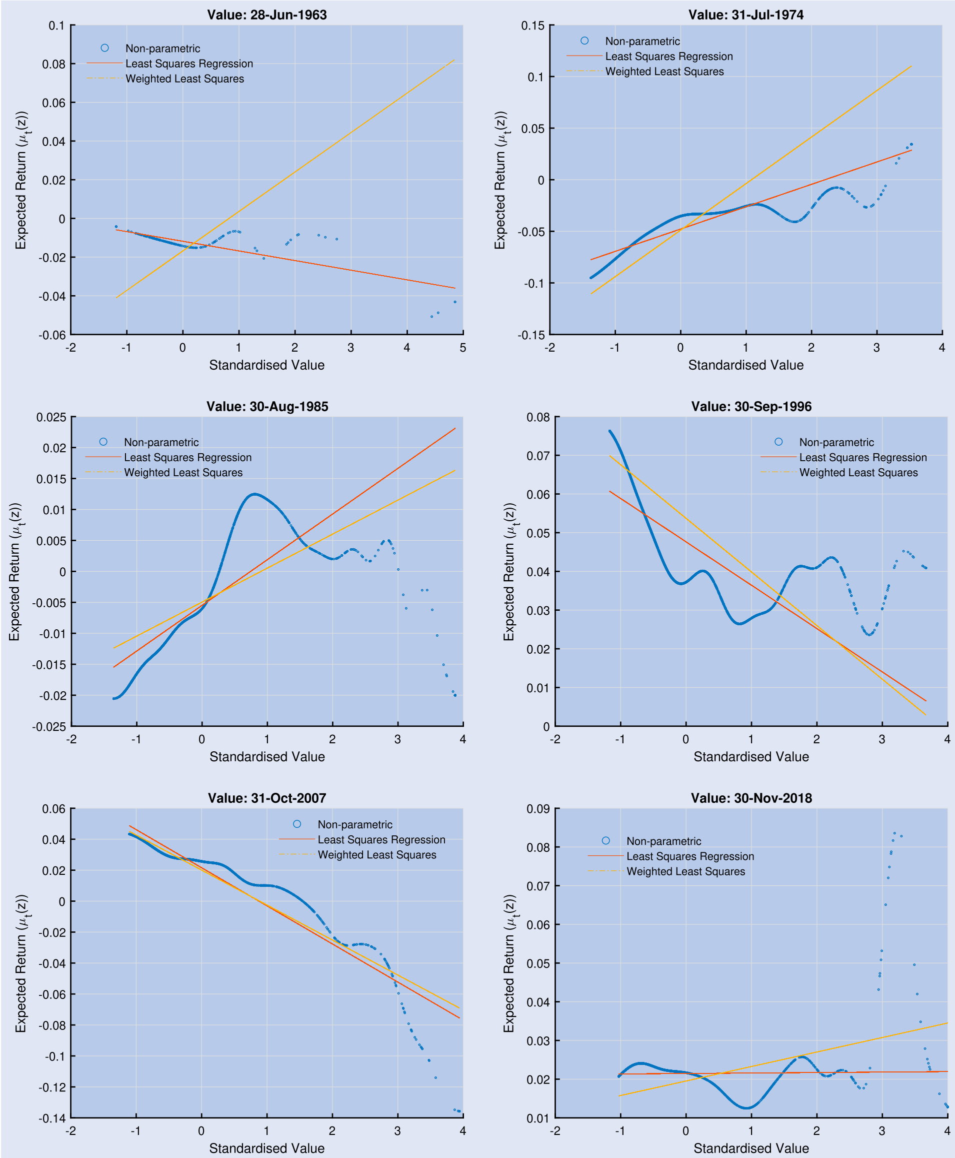

Figure 5. Value vs. expected return cross-sectional regressions: Plots of the estimated relationship between the book to market attribute and realized returns using: the non-parametric estimator; least squares regression and weighted least squares (with stock market capitalizations used as the weights). The illustrated dates are randomly sampled as 6 equally-spaced months from our full date range. The first panel, corresponding to June 1963, illustrates the sensitivity to estimation method as the OLS and WLS estimated slopes can be seen visually to be of opposite polarity.

Table 8 shows the results of linear regressions of the momentum-based portfolios on a four-factor model consisting of the Fama–French three-factor model with an added Carhart momentum factor. The RS_{CCFS} portfolio has a strongly statistically significant *alpha* value of 1.02% monthly (P-value < 0.01), it also has insignificant beta to the market (P-value 0.56).

Table 9 displays the results of asset pricing tests on risk factor models including each value attribute. The benchmark model is the $FF3M$ model and in each test the momentum factor in the original model is replaced with an alternative of the other seven portfolio constructions. The best performing factor model is the original $FF3M$ benchmark model, none of the other characteristic portfolio constructs improve performance in a linear risk factor model.†

Finally, in figure 4 the maximum stakeholding monthly in any single firm for an investor with AUM of $500 million is shown. From the early 90's onward this amount is in the range of 2–4%, while there is no accepted threshold in terms of economic significance for an anomaly portfolio, it is clear that there is a limit to the capacity of the investment portfolio that could easily be reached by a large investment fund.

† It should be noted that the benchmark in this case uses double sorts, controlling the momentum portfolio for market capitalisation.

Table 4. Value characteristic portfolio statistics: summary monthly return premia statistics and correlations for eight different value portfolio constructions (excess market return, FF_{MKT}, is also included for comparison).

Panel A. Means, standard deviations, and t-statistics

Factor	Mean	Std Dev.	t-Statistic	Info. Ratio
FF_{MKT}	0.51	4.39	3.02	0.40
FF_{HML}	0.32	2.80	2.98	0.40
MV_{NP}	0.53	3.39	4.03	0.54
OLS	0.14	1.48	2.38	0.32
WLS	0.10	1.67	1.56	0.21
RS_{CCFS}	0.91	9.29	2.54	0.34
RS_{EW}	0.32	5.87	1.39	0.19
RS_{CW}	0.18	5.36	0.88	0.12

Panel B. Correlations

	FF_{HML}	MV_{NP}	OLS	WLS	RS_{CCFS}	RS_{EW}	RS_{CW}
FF_{MKT}	− 0.26	− 0.10	− 0.21	− 0.09	− 0.08	− 0.23	− 0.06
FF_{HML}		0.18	0.81	0.78	0.55	0.79	0.76
MV_{NP}			− 0.05	− 0.10	− 0.08	− 0.05	− 0.08
OLS				0.86	0.70	0.97	0.85
WLS					0.69	0.85	0.94
RS_{CCFS}						0.71	0.71
RS_{EW}							0.86

Note: FF_{HML} is the Fama and French value factor; MV_{NP} is an optimal characteristic portfolio defined using the non-parametric estimator (8); OLS is the OLS-implied weight factor: $B_t = (X_t'X_t)^{-1}X_t'R_{t+1}$, with $X_t = [1\ Z_t]$; WLS is the least squares factor weighted by market capitalization and defined as $B_t = (X_t'M_tX_t)^{-1}X_t'M_tR_{t+1}$, with $W_t = M_t'X_t(X_t'M_tX_t)^{-1}$ and $W_t'X_t = I_2$; RS_{CCFS} is the rank sorted portfolio with a time varying number of quantile portfolios as per Cattaneo *et al.* (2020); RS_{EW} is the equally-weighted rank sorted value portfolio using decile portfolios and RS_{CW} is the cap-weighted version of the same construct.

Table 5. Value characteristic portfolio regressions: regression results for each of the seven size characteristic based portfolios, regressed against the Fama–French three-factor model with an added Carhart momentum factor ($FF3M$). P-values in brackets.

Factor	α (%)	β_{MKT}	β_{SMB}	β_{HML}	β_{MOMO}
Panel A. FF3M regression coefficients					
MV_{NP}	0.36**	-0.02	0.02	0.25***	0.14***
	(0.01)	(0.53)	(0.67)	(< 0.01)	(< 0.01)
OLS	0.13***	−0.02**	−0.05***	0.37***	−0.14***
	(< 0.01)	(0.01)	(< 0.01)	(< 0.01)	(< 0.01)
WLS	0.05	0.02*	0.02	0.43***	−0.16***
	(0.11)	(0.05)	(0.15)	(< 0.01)	(< 0.01)
RS_{CCFS}	0.95***	0.00	− 0.05	1.59***	−0.82***
	(0.00)	(0.96)	(0.59)	(< 0.01)	(< 0.01)
RS_{EW}	0.33***	−0.08***	−0.28***	1.41***	−0.57***
	(< 0.01)	(< 0.01)	(< 0.01)	(< 0.01)	(< 0.01)
RS_{CW}	0.04	0.07***	0.04	1.35***	−0.52***
	(0.69)	(0.00)	(0.33)	(< 0.01)	(< 0.01)

5.4. Discussion of results

We have tested the proposed methodology on three popular anomalies in the financial economics literature: size, value and momentum. For the size attribute, the optimal non-parametric portfolio results in a size return premium that has a larger test statistic than that of all other portfolio constructs tested. The portfolio also outperforms the competitors in investment metrics and when used as an asset pricing factor in a four-factor model, improves performance over models that include a large range of alternative size factor constructions, including the Fama and French size factor. The optimal non-parametric size portfolio also had lower implied maximum stakeholdings in individual companies (as a percentage of firm market capitalization) than a standard capitalization-weighted rank, sorted decile portfolio.

For the value attribute, the method also results in the highest test statistic for the associated return premium and the best mean–variance investment performance, however, the resulting portfolio did not improve the benchmark asset pricing model in asset pricing factor tests. The approach failed to capture a statistically significant momentum premium when adjusting for multiple hypothesis testing (t-stat $= 2.26 < 2.78$). The best performing construct for momentum was the procedure of Cattaneo *et al.* (2020), where a large number of quantile portfolios/smaller number of stocks in extreme quantiles, captured more extreme recent performers (t-stat = 5.03). Our findings suggest that the optimal non-parametric approach works better with portfolios constructed using firm fundamental attributes rather than technical factors, whose returns may be driven by time series rather than cross-sectional relationships. This finding also casts doubt on the optimality of standard ranked sorts for time series driven anomalies, as the main theoretical justification for their use is in capturing cross-sectional relationships.

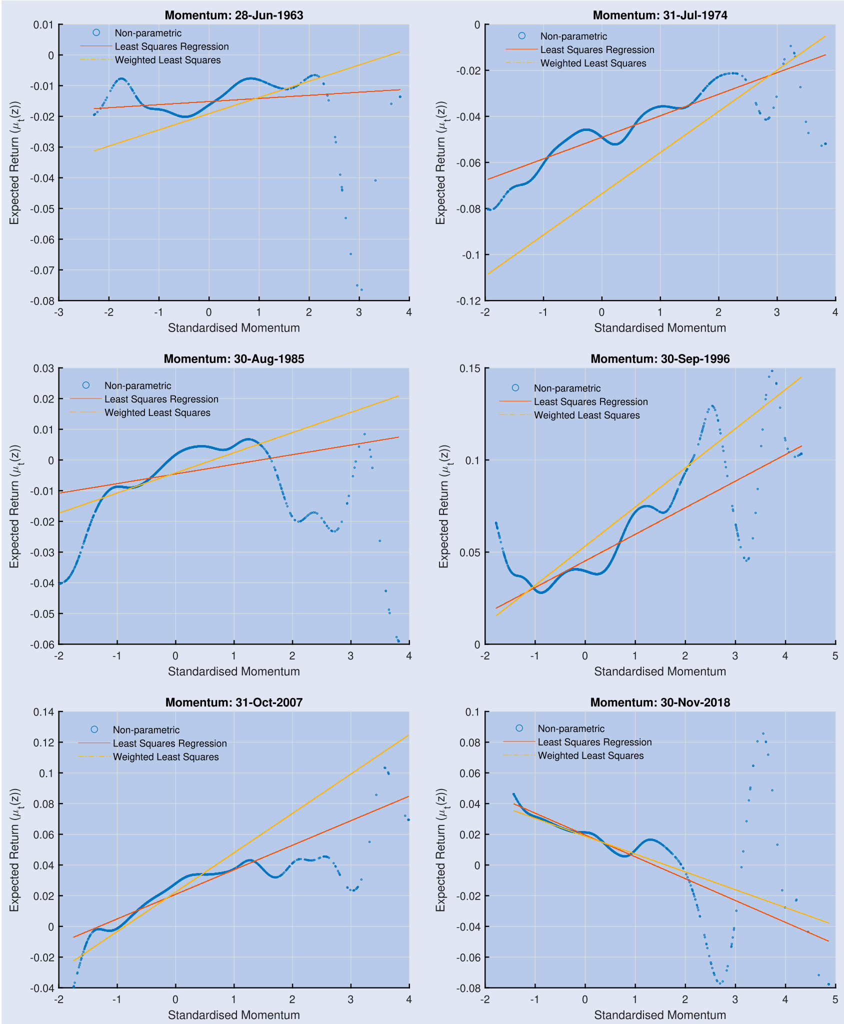

Figure 6. Momentum vs. expected return cross-sectional regressions: Plots of the estimated relationship between the momentum attribute and realized returns using the non-parametric estimator; least squares regression and weighted least squares (with stock market capitalizations used as the weights). The illustrated dates are randomly sampled as six equally-spaced months from our full date range.

Table 6. Asset pricing model comparison of value characteristic portfolios: Tests of equality of squared Sharpe ratios: *FF3M* is the four-factor model constructed by adding a Carhart momentum factor to the Fama–French three-factor model.

Model	FF3M+MV_{NP}	FF3M+RS_{CW}	FF3M+WLS	FF3M	FF3M+RS_{CCFS}	FF3M+RS_{EW}	FF3M+OLS
Panel A. Differences in sample squared Sharpe ratios							
FF3M+MV_{NP}		0.007	0.019	0.019	0.029	0.035*	0.057**
FF3M+RS_{CW}			0.012*	0.012	0.022	0.028**	0.049***
FF3M+WLS				0.000	0.010	0.016	0.038***
FF3M					0.010	0.016	0.038***
FF3M+RS_{CCFS}						0.006	0.028
FF3M+RS_{EW}							0.022***
Panel B. *P*-values							
FF3M+MV_{NP}		0.664	0.293	0.317	0.152	0.069	0.012
FF3M+RS_{CW}			0.064	0.319	0.145	0.011	0.001
FF3M+WLS				0.977	0.526	0.167	0.006
FF3M					0.565	0.158	0.003
FF3M+RS_{CCFS}						0.714	0.147
FF3M+RS_{EW}							0.004

Note: The table shows pairwise tests of risk factor models constructed by replacing the Fama–French value factor with each of the other seven alternative size-based portfolios (row labels). A positive value indicates that the model in the column label outperforms the model in the corresponding row label in terms of the pairwise squared Sharpe ratio test of Barillas *et al.* (2019), *P*-values for the Sharpe difference are also displayed, $*$ indicates significance at the 10% level. The best performing portfolio in terms of the squared Sharpe metric is the *OLS* portfolio.

Table 7. Momentum characteristic portfolio statistics: summary monthly return premia statistics and correlations for eight different momentum portfolio constructions (excess market return, FF_{MKT}, is also included for comparison).

Panel A. Means, standard deviations, and t-statistics

Factor	Mean	Std dev.	t-Statistic	Info. ratio
FF_{MKT}	0.51	4.39	3.02	0.40
FF_{MOMO}	0.66	4.17	4.10	0.55
MV_{NP}	0.37	4.20	2.26	0.30
OLS	0.30	1.89	4.04	0.54
WLS	0.30	2.28	3.41	0.46
RS_{CCFS}	2.42	12.43	5.03	0.67
RS_{EW}	1.23	6.56	4.86	0.65
RS_{CW}	1.22	7.13	4.40	0.59

Panel B. Correlations

	FF_{MOMO}	MV_{NP}	OLS	WLS	RS_{CCFS}	RS_{EW}	RS_{CW}
FF_{MKT}	– 0.13	– 0.14	0.00	0.01	– 0.07	– 0.09	– 0.09
FF_{MOMO}		0.27	0.91	0.91	0.72	0.93	0.91
MV_{NP}			0.21	0.23	0.24	0.26	0.26
OLS				0.92	0.75	0.97	0.90
WLS					0.72	0.89	0.93
RS_{CCFS}						0.78	0.74
RS_{EW}							0.92

Note: FF_{MOMO} is the Fama and French momentum factor; MV_{NP} is an optimal characteristic portfolio defined using the non-parametric estimator (8); OLS is the OLS-implied weight factor: $B_t = (X_t'X_t)^{-1}X_t'R_{t+1}$, with $X_t = [1\ Z_t]$; WLS is the least squares factor weighted by market capitalization and defined as $B_t = (X_t'M_tX_t)^{-1}X_t'M_tR_{t+1}$, with $W_t = M_t'X_t(X_t'M_tX_t)^{-1}$ and $W_t'X_t = I_2$; RS_{CCFS} is the rank sorted portfolio with a time varying number of quantile portfolios as per Cattaneo *et al.* (2020); RS_{EW} is the equally-weighted rank sorted size portfolio (short stocks above the 90 percentile market cap. value and long those below the 10 percentile) and RS_{CW} is the cap-weighted version of the same construct.

Table 8. Momentum risk factor regressions (July 1963–December 2018): regression of each of the seven momentum attribute based portfolios on the Fama–French three-factor model with the Carhart momentum factor.

Factor	α (%)	β_{MKT}	β_{SMB}	β_{HML}	β_{MOMO}
Panel A. FF3M regression coefficients					
MV_{NP}	0.18	−0.07*	– 0.05	0.15**	0.28***
	(0.28)	(0.09)	(0.34)	(0.01)	(< 0.01)
OLS	0.01	0.02***	0.09***	−0.06***	0.41***
	(0.64)	(< 0.01)	(< 0.01)	(< 0.01)	(< 0.01)
WLS	– 0.05	0.03***	0.11***	−0.05***	0.49***
	(0.13)	(< 0.01)	(< 0.01)	(< 0.01)	(< 0.01)
RS_{CCFS}	1.02***	0.05	0.01	– 0.11	2.13***
	(< 0.01)	(0.56)	(0.92)	(0.40)	(< 0.01)
RS_{EW}	0.27**	0.00	0.16***	−0.09**	1.46***
	(0.01)	(0.94)	(< 0.01)	(0.01)	(< 0.01)
RS_{CW}	0.19*	– 0.02	0.22***	−0.12***	1.55***
	(0.08)	(0.50)	(< 0.01)	(0.00)	(< 0.01)

6. Conclusion

This paper develops an optimal non-parametric procedure for constructing characteristic-based portfolios as a replacement for standard ranked sorting. In contrast to standard procedures such as long-minus-short portfolios sorted by deciles on the stock characteristics or more sophisticated versions given by OLS and WLS regression analysis, our characteristic-based portfolios are optimal in the sense that the allocation to the cross-section of assets is driven by a mean–variance objective function. One of the main features of this approach is that all the assets in the cross-section can potentially contribute to the attribute portfolio. In doing so, we avoid arbitrary choices of breakpoints (dividing the cross-section of returns into different deciles), and we avoid overlaying arbitrary weighting schemes that may obfuscate the attribute/return relationship. A difference to standard rank sorted portfolios is that the polarity of the return–characteristic relationship is not assumed to be static. A key contribution of the methodology is that it provides a non-parametric test bed for cross-sectional anomalies that is optimised to extract optimal performance for the candidate anomaly. The combination of a target of optimality with a non-parametric approach removes researcher discretion, the process is completely data-driven, robust to *p-hacking*† and can capture non-linear relationships.

† P-hacking is the practice of selecting parameters such as portfolio breakpoints or quantile definitions to maximize the apparent performance of a reported anomaly (see, e.g. Harvey 2017).

Table 9. Asset pricing model comparison of momentum characteristic portfolios: tests of equality of squared Sharpe ratios: *FF3M* is the four-factor model constructed by adding a Carhart momentum factor to the FamaFrench three-factor model.

Model	FF3M+RS_{EW}	FF3M+MV_{NP}	FF3M+WLS	FF3M+RS_{CCFS}	FF3M+RS_{CW}	FF3M+OLS	FF3M
Panel A. Differences in sample squared Sharpe ratios							
FF3M+RS_{EW}		0.003	0.003	0.019	0.031*	0.031*	0.035*
FF3M+MV_{NP}			0.001	0.016	0.028**	0.028**	0.032**
FF3M+WLS				0.016	0.028	0.028	0.032*
FF3M+RS_{CCFS}					0.012*	0.012*	0.016*
FF3M+RS_{CW}						0.000	0.004
FF3M+OLS							0.004
Panel B. p-values							
FF3M+RS_{EW}		0.803	0.447	0.177	0.070	0.073	0.058
FF3M+MV_{NP}			0.966	0.134	0.025	0.030	0.034
FF3M+WLS				0.297	0.121	0.125	0.099
FF3M+RS_{CCFS}					0.087	0.091	0.051
FF3M+RS_{CW}						0.940	0.611
FF3M+OLS							0.586

Note: The table shows pairwise tests of risk factor models constructed by replacing the momentum factor with each of the other seven alternative size-based portfolios (row labels). A positive value indicates that the model in the column label outperforms the model in the corresponding row label in terms of the pairwise squared Sharpe ratio test of Barillas *et al.* (2019), *P*-values for the Sharpe difference are also displayed, $*$ indicates significance at the 10% level. The best performing portfolio in terms of the squared Sharpe metric is the benchmark FF MOMO portfolio.

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Appendix. Including micro-caps

In this appendix, we report summary statistics for the analysis in the paper, repeated on an unfiltered data set that includes micro-caps (see tables A1–A3). As per the extant literature (see, e.g. Hou *et al.* 2020), we find that anomaly returns are much stronger when micro-caps are included in the analysis, but there are known issues around the costs of trade when including micro-caps (see Novy-Marx and Velikov 2016). In our own analysis of economic significance, we repeat the analysis in figure 4 of the main paper, with a data set including micro-caps. For the size attribute we find that the average monthly max stake holding in an individual firm, in a cap-weighted decile portfolio, is 11% for AUM of \$500 million. For our proposed optimal non-parametric portfolio, MV_{NP}, the equivalent figure is 56% and for the RS_{CCFS} portfolio of Cattaneo *et al.* (2020) the figure is 190%.

Table A1. Size characteristic portfolio statistics including microcaps: the equivalent results to Table 1, with no microcap filter applied to the data set.

Factor	Mean (%)	Std dev.	t-Statistic	Info. ratio
FF_{MKT}	0.51	4.39	3.02	0.40
FF_{SMB}	0.20	3.06	1.73	0.23
MV_{NP}	1.33	3.87	8.84	1.19
OLS	0.29	1.95	3.89	0.52
WLS	0.12	1.73	1.82	0.24
RS_{CCFS}	4.59	11.46	10.33	1.39
RS_{EW}	1.66	6.91	6.21	0.83
RS_{CW}	1.07	6.59	4.18	0.56

Table A2. Value characteristic portfolio statistics including microcaps: the equivalent results to table 4, with no microcap filter applied to the data set.

Factor	Mean (%)	Std dev.	t-Statistic	Info. ratio
FF_{MKT}	0.51	4.39	3.02	0.40
FF_{HML}	0.32	2.80	2.98	0.40
MV_{NP}	0.35	3.40	2.63	0.35
OLS	0.46	1.52	7.79	1.05
WLS	0.15	2.32	1.71	0.23
RS_{CCFS}	0.99	11.24	2.27	0.31
RS_{EW}	1.52	5.66	6.94	0.93
RS_{CW}	0.27	5.64	1.24	0.17

Table A3. Momentum characteristic portfolio statistics including microcaps: the equivalent results to table 7, with no microcap filter applied to the data set.

Factor	Mean (%)	Std dev.	t-Statistic	Info. ratio
FF_{MKT}	0.51	4.39	3.02	0.40
FF_{MOMO}	0.66	4.17	4.10	0.55
MV_{NP}	−0.14	4.73	−0.78	−0.11
OLS	0.25	1.98	3.22	0.43
WLS	0.37	2.43	3.96	0.53
RS_{CCFS}	2.67	12.11	5.70	0.76
RS_{EW}	0.77	7.58	2.62	0.35
RS_{CW}	1.71	8.34	5.28	0.71