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This paper studies a portfolio allocation problem, where the goal is to reach a prescribed wealth distribution at a final time. We study this problem with the tools of optimal mass transport. We provide a dual formulation which is solved with a gradient descent algorithm. This involves solving an associated Hamilton–Jacobi–Bellman and Fokker–Planck equations with a finite difference method. Numerical examples for various prescribed terminal distributions are given, showing that we can successfully reach attainable targets. We then consider adding consumption during the investment process, to take into account distributions that are either not attainable, or sub-optimal.

1. Introduction

Over the past decades, there has been a vast amount of research on portfolio allocation. Perhaps the most iconic result is the portfolio selection theory of Markowitz (1952), which states that investors should determine the allocation of wealth on the basis of the trade-off between *return* and *risk*. The classical objective function in a portfolio optimization problem is to maximize the expected return given a variance level. However, the first and second moments of the return of a portfolio are only a simplified description of portfolio wealth. Researchers have introduced objective functions that include more moments, such as skewness, to provide a more accurate statistical description of the distribution of the return (see, for example, Kraus and Litzenberger (1976) and Lee (1977)).

The whole distribution of the portfolio wealth would provide investors complete information. Instead of optimizing the first moments of the distribution, this paper introduces an objective function which includes a target distribution of the terminal wealth. We address the problem of controlling the portfolio allocation process to reach the prescribed terminal distribution. Of course, as we will see, not all terminal distributions are attainable.

On the one hand, this problem can be categorized as a stochastic control problem. The state variable (the wealth) is

driven by a process (the investment strategy) whose value is decided at any time $t \in [0, T]$. We treat the portfolio allocation process as a *control* for the investment process and aim to design the time path of the portfolio allocation process so that it steers the portfolio wealth from an initial state to a prescribed terminal distribution.

Designing a continuous semimartingale having prescribed distributions at given times can be addressed with optimal mass transport (OMT) theory. The optimal transport problem is an old problem first described in the work of Monge (1781), and later revisited by Kantorovich (1942) leading to the so-called Monge–Kantorovich formulation. A comprehensive review of the extensions and applications of the Monge–Kantorovitch problem can be found in the book by Rachev and Rüschendorf (1998) and the books of Villani (2003) and Villani (2008). The original formulation of the problem looks for a map $f : X \rightarrow Y$ that maps a distribution μ to another distribution ν. Later, Benamou and Brenier (2000) reinterpreted the problem in a fluid mechanics framework, where one is not looking only for an optimal transport map, but instead for the whole trajectory of the mass distribution over time. Brenier's contribution opened the way to the problem of continuous optimal transport.

Stochastic extensions of the discrete and continuous time OMT problem have since flourished, see e.g. Mikami and Thieullen (2006), Tan *et al.* (2013), Mikami (2015) and Henry-Labordère *et al.* (2016). Beyond its mathematical

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interest, the optimal mass transport problem has applications in many fields, in economy, meteorology, astrophysics (Brenier *et al.* 2003, Loeper 2006), image processing (Ferradans *et al.* 2014). In the field of finance, optimal transport has been applied in robust hedging (Dolinsky and Soner 2014, Henry-Labordère 2017) and volatility calibration (Guo *et al.* 2019). Optimal transport is also closely connected to the Schrödinger problem, see Léonard (2013) for a comprehensive survey.

The novelty of this paper is to provide a new perspective on portfolio optimization inspired by OMT. The prescribed distribution problem can be seen as the dual of the utility maximization problems. However, for many investors, utility functions are not necessarily the most natural way to specify their preferences. Directly choosing the wealth distribution might be more intuitive for investors and much easier for them to describe. Our approach also allows more flexible than simply specifying a particular prescribed distribution, as the $C(\rho_1, \bar{\rho}_1)$ term may be used to represent other types of constraints or penalty functions on the terminal wealth, as long as it remains a convex function. For example, the investor can set bounds on the density function of the terminal wealth. An investor must decide how to allocate her portfolio between a risky and a risk-free asset. The price of the risky asset is modelled by a semimartingale, with prescribed drift and diffusion coefficients. By controlling the portfolio allocation, the investor wants the distribution of the wealth to match, or be close to, a given target distribution. Depending on the risky asset diffusion coefficients, not all target distributions are attainable (think for example of too high an expected return versus variance), or optimal (one could reach a 'better' distribution than the target). We consider two different approaches: either relaxing the terminal constraint by penalization, or adding a consumption process.

In other related papers, Zhang *et al.* (2019) proposed an investment strategy which maximizes the expected portfolio value within a bounded targeted range. Chalabi and Wuertz (2012) sought the allocation that produces a portfolio that is the closest to a reference model while remaining as compatible as possible with the empirical data. They used a ϕ-divergence measure from information theory as the tool and did not consider any cost function in the formulation. The research by Zhou (2010), in the field of behavioural finance, discusses the continuous-time portfolio choice problem where the S-shape utility function leads to distributional constraints on the final asset. Chen *et al.* (2019) formulated a class of stochastic two-person zero-sum differential games, where a specification on a target terminal state distribution is imposed on the players. However, their paper only applies to linear quadratic games with Gaussian target distributions. Our problem has a general class of SDEs and cost functions. Chen *et al.* (2018) worked on guiding particles to a specific distribution. But the prescribed terminal distribution in their problem is restricted to Gaussian. By contrast, our method applies to a general choice of target distributions. Although some mathematical tools are clearly related to the previous papers by Guo and Loeper (2018) and Guo *et al.* (2019), this paper focuses on a new application—portfolio optimization (as opposed to model calibration for the aforementioned papers). In this problem the drift and volatility are unknown, but linked by a constraint (to ensure that the strategy is self-financed). This issue is absent from the calibration problem where only the volatility is unknown and the drift is given by the risk-neutral measure. We show here that this constraint can be embedded into the cost as a convex constraint and new features and numerical examples are added, where the investor can either inject or withdraw cash from the portfolio in order to reach a terminal target.

The rest of the paper is organized as follows. In Section 2, we formulate the problem. Then we introduce the dual formulation in Section 3. In Section 4, we provide a gradient descent algorithm to solve the dual problem, and the numerical results are presented in Section 5. We give examples for general target distributions with various penalty functionals in Section 5.1. The addition of cash savings/cash inputs are studied in Sections 5.2 and 5.3 respectively.

2. Problem formulation

Let $\mathcal{D}$ be a Polish space equipped with its Borel σ-algebra. We denote $C(\mathcal{D}; \mathbb{R})$ the space of continuous functions on $\mathcal{D}$ with values in $\mathbb{R}$, $C_b(\mathcal{D}; \mathbb{R})$ the space of bounded continuous functions and $C_0(\mathcal{D}; \mathbb{R})$ the space of continuous functions, vanishing at infinity. Let $\mathcal{P}(\mathcal{D})$ be the space of Borel probability measures on $\mathcal{D}$ with a finite second moment. Denote by $\mathcal{M}(\mathcal{D}; \mathbb{R})$ the space of finite signed measures on $\mathcal{D}$ with values in $\mathbb{R}$, $\mathcal{M}_+(\mathcal{D}; \mathbb{R}) \subset \mathcal{M}(\mathcal{D}; \mathbb{R})$ be the subset of non-negative measures. When $\mathcal{D}$ is compact, the topological dual of C_b is given by $C_b^*(\mathcal{D}; \mathbb{R}) = \mathcal{M}(\mathcal{D}; \mathbb{R})$. But when $\mathcal{D}$ is non-compact, $C_b^*(\mathcal{D}; \mathbb{R})$ is larger than $\mathcal{M}(\mathcal{D}; \mathbb{R})$. For convenience, we often use the notation $\mathcal{E} := [0, 1] \times \mathbb{R}$. We say that a function $\phi : \mathcal{E} \to \mathbb{R}$ belongs to $C_b^{1,2}(\mathcal{E})$ if $\phi \in C_b(\mathcal{E})$ and $(\partial_t \phi, \partial_x \phi, \partial_{xx} \phi) \in C_0(\mathcal{E}; \mathbb{R}, \mathbb{R}, \mathbb{R})$. Let $\mathbb{R}^+$ denote non-negative real numbers, and $\mathbb{S}^d$ denote the set of $d \times d$ symmetric positive semidefinite matrices.

Without loss of generality, we set the time horizon T to be 1. Let $\Omega := (\omega \in C([0, 1]; \mathbb{R}^d))$, we denote by $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,1]}$ the filtration generated by the canonical process. The process W is a d-dimensional standard Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

We consider a portfolio with d risky assets and one risk-free asset, the risk-free interest r being set to 0 for simplicity. The drift μ takes values in $\mathbb{R}^d$, the invertible matrix σ takes values in $\mathbb{R}^{d \times d}$ and the covariance matrix $\Sigma := \sigma \sigma^\mathsf{T}$ takes value in $\mathbb{S}^d$. We assume these are known processes. The price process of the risky assets is denoted by $S_t \in \mathbb{R}^d$ ($0 \le t \le 1$), and the ith element of S_t follows the semimartingale

$$\frac{\mathrm{d}S_t^i}{S_t^i} = \mu_t^i \,\mathrm{d}t + \sum_{j=1}^d \sigma_t^{ij} \,\mathrm{d}W_t^j, \quad 1 \le i \le d. \tag{1}$$

For $t \in [0, 1]$, the portfolio allocation strategy $\alpha_t \in \mathbb{R}^d$ represents the proportion of the total wealth invested into the d risky assets, and $1 - \sum_{i=1}^d \alpha_t^i$ is the proportion invested in the risk-free asset. We define the concept of *admissible control* as follows.

DEFINITION 2.1 *An admissible control process α for the investor on* [0, 1] *is a progressively measurable process with*

respect to $\mathbb{F}$, taking values in a compact convex set $K \subset \mathbb{R}^d$. The set of all admissible α is compact and convex, denoted by $\mathcal{K}$.

We denote by $X_t \in \mathbb{R}$ the portfolio wealth at time t. Starting from an initial wealth x_0, the wealth of the self-financing portfolio evolves as follows,

$$dX_t = X_t \alpha_t^\mathsf{T} \mu_t \, dt + X_t \alpha_t^\mathsf{T} \sigma_t \, dW_t, \tag{2}$$

$$X_0 = x_0. \tag{3}$$

2.1. Portfolio optimization with a prescribed terminal distribution

We denote by $\rho_t := \mathbb{P} \circ X_t^{-1} \in \mathcal{P}(\mathbb{R})$ the distribution of X_t. In this problem, we know the initial distribution of the portfolio wealth $\rho_0 \in \mathcal{P}(\mathbb{R})$ (ρ_0 can be a Dirac measure), we are given a prescribed terminal distribution $\bar{\rho}_1 \in \mathcal{P}(\mathbb{R})$ and a convex cost function $f(\alpha_t) : K \to \mathbb{R}$.

With ρ_0 and a process α, the realized terminal distribution of the portfolio wealth is $\rho_1 := \mathbb{P} \circ X_1^{-1}$ (ρ_1 is not necessarily the same as $\bar{\rho}_1$). We want ρ_1 to be close to our target $\bar{\rho}_1$, hence we introduce a functional $C(\rho_1, \bar{\rho}_1)$ to penalize the deviation of ρ_1 from $\bar{\rho}_1$. At the same time, we want to minimize the expectation of the transportation cost from ρ_0 to ρ_1. Combining the expected transportation cost and the penalty functional, our objective function is

$$\inf_{\alpha, \rho} \left\{ \mathbb{E}^\mathbb{P} \left[\int_0^T f(\alpha_t) \, dt \right] + C(\rho_1, \bar{\rho}_1) \right\}, \tag{4}$$

with the initial distribution

$$\rho(0, x) = \rho_0(x) \quad \forall x \in \mathbb{R}. \tag{5}$$

Note that the term $\mathbb{E}^\mathbb{P}[\int_0^T f(\alpha_t) \, dt]$ can add some coercivity to this problem. Without this term, the optimal α will almost surely be non-unique. The strict convexity of f will also make the resulting process α smoother.

Following closely Guo *et al.* (2019, Lemma 3.1), we state the result below.

LEMMA 2.1 *For $t \in [0, 1]$, an optimal strategy α_t can be chosen to be a Markovian process $\alpha(t, X_t)$.*

REMARK 2.1 This is actually a consequence of the following facts:

(a) the cost function only depends on α_t and the constraint only depends on X_1 via ρ_1 (and not on any path dependent variable);
(b) Jensen's inequality and the convexity of f.

The feasible (α, ρ) in (4) should satisfy the Fokker–Planck equation

$$\partial_t \rho(t, x) + \partial_x (\alpha(t, x)^\mathsf{T} \mu(t, x) x \rho(t, x))$$
$$- \frac{1}{2} \partial_{xx} (\alpha(t, x)^\mathsf{T} \Sigma(t, x) \alpha(t, x) x^2 \rho(t, x)) = 0 \quad \forall (t, x) \in \mathcal{E}. \tag{6}$$

However, the feasible set for (α, ρ) defined by equality (6) is not convex, which means we may not be able to find the optimal solution. To address this issue, we introduce the following definition.

NOTATION 2.1 *We define maps $\tilde{B}, \tilde{A} : \mathcal{E} \to \mathbb{R}$ as $\tilde{B}(t, x) := \alpha(t, x)^\mathsf{T} \mu(t, x) x$ and $\tilde{A}(t, x) := \alpha(t, x)^\mathsf{T} \Sigma(t, x) \alpha(t, x) x^2$. Then define $B(t, x) := \tilde{B}\rho$, $B \in \mathcal{M}(\mathcal{E}; \mathbb{R})$ and $A(t, x) := \tilde{A}\rho$, $A \in \mathcal{M}^+(\mathcal{E}; \mathbb{R})$. Densities B and A are absolutely continuous with respect to ρ.*

Note that ρ, B, A are densities. As a shorthand, we will write $d\rho = \rho \, dt \, dx, dA = A \, dt \, dx, dB = B \, dt \, dx$ in the subsequent sections. We show that B and A are connected in the following way:

PROPOSITION 2.1 *When $d > 1$ (resp. $d = 1$), the necessary and sufficient condition for the existence of an $\alpha(t, x) \in \mathbb{R}^d$ such that $A = \alpha(t, x)^\mathsf{T} \Sigma(t, x) \alpha(t, x) x^2 \rho$, $B = \alpha(t, x)^\mathsf{T} \mu(t, x) x \rho$ is $A \geq \frac{B^2}{\|v(t,x)\|^2 \rho}$ (resp. $A = \frac{B^2}{\|v(t,x)\|^2 \rho}$), where $v(t, x) := \Sigma(t, x)^{-\frac{1}{2}} \mu(t, x)$.*

Proof See Section A.1. ∎

Using notations $\rho, B, \tilde{B}$ and $A, \tilde{A}$, the Fokker–Planck equation (6) becomes linear and the SDE of the portfolio wealth reads

$$dX_t = \tilde{B}(t, X_t) \, dt + \tilde{A}^{\frac{1}{2}}(t, X_t) \, dW_t, \tag{7}$$

$$X_0 = x_0. \tag{8}$$

To attain a certain prescribed terminal distribution, as we see in Proposition 2.1, it is possible that the optimal drift $\tilde{B}(t, x)$ is not saturated, i.e. $\tilde{B}(t, x)^2 < \|v(t, x)\|^2 \tilde{A}(t, x)$. It means that the drift function in SDE (7) can be greater and eventually we should be able to reach a better terminal wealth with a higher expectation. Then, instead of using this unsaturated drift $\tilde{B}(t, x)$ to reach the prescribed terminal distribution, we can use the drift $\tilde{B}(t, x) = \|v(t, x)\| \sqrt{A}$ to attain a more ambitious distribution, and the extra part in the drift can be interpreted as cash saving. In this case, even when we have multiple assets ($d > 1$) in the portfolio, optimal portfolios should lie on the curve $\tilde{B}(t, x) = \|v(t, x)\| \sqrt{A}$, as in the $d = 1$ case. Any portfolio lying below the curve represents a less than ideal investment because for the same level of risk (variance), we could achieve a greater return. This is consistent with the efficient frontier in modern portfolio theory (Markowitz 1952).

REMARK 2.2 Due to the flexibility offered by the function C, our framework is in fact compatible with the classical Markowitz problem, where one aims to find the portfolio that minimizes the variance for a fixed level of expected returns. Note that, for a given level of $\mathbb{E}X_1$, minimizing the variance is equivalent to minimizing the second moment $\mathbb{E}X_1^2$. This can be achieved by setting f to be arbitrarily small and choosing C to be the following convex function in ρ_1,

$$C(\rho_1) = \begin{cases} \int_\mathbb{R} x^2 \, d\rho_1(x) & \text{if } \int_\mathbb{R} x \, d\rho_1(x) = R, \\ \infty & \text{otherwise,} \end{cases}$$

where $R \in \mathbb{R}$ is the desired expected return.

Our approach can also be adapted to other Markowitz-like criteria, even in some cases where the set of admissible target distributions is not convex. For example, suppose one wants to find the portfolio with the highest expected return subject to fixing the shape of the terminal wealth distribution. In other words, we want to maximize the expected return while fixing higher-order central moments. This can be achieved by iteratively applying our technique to a range of prescribed distributions, all with the same required shape but different averages, and locating the attainable target with the greatest expected return. Note that the attainability of a target distribution corresponds to the finiteness of the dual problem.

Now we define the concept of *cash saving* at time t as $c(t,x) := \|\nu(t,x)\|\sqrt{\tilde{A}(t,x)} - \tilde{B}(t,x)$. When the prescribed terminal distribution is not ambitious enough, to ensure we have as much cash saving as we can, we define the feasible set as $\Pi(t,x) := \{(\tilde{B}, \tilde{A}) : \tilde{A} \geq \frac{(\tilde{B}^+)^2}{\|\nu(t,x)\|^2}\}$ ($\tilde{B}^+(t,x) := \max(0, \tilde{B}(t,x))$) and we can see the set $\Pi(t,x)$ is convex.

In the following parts, we focus on a particular subset of f, which can be represented by $f(\alpha(t,x)^\mathsf{T} \Sigma(t,x)\alpha(t,x)x^2)$. To penalize functions out of the set $\Pi(t,x)$, we define a cost function $F : \mathcal{E} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+ \cup \{+\infty\}$ such that

$$F(p,q) = f(q) + \delta(p,q), \tag{9}$$

where $f : \mathcal{E} \times \mathbb{R} \to \mathbb{R}^+$ is a convex function and $\delta(p,q)$ is a delta function defined as

$$\delta(p,q) = \begin{cases} 0 & \text{if } (p,q) \in \Pi(t,x), \\ +\infty & \text{otherwise.} \end{cases} \tag{10}$$

For simplicity, we write $F(p,q) := F(t,x,p,q)$ if there is no ambiguity.

Now we are ready to introduce formally the problem:

PROBLEM 2.1 *Starting from an initial distribution ρ_0, with a prescribed terminal distribution $\bar{\rho}_1$ and a cost function (9), we want to solve the infimum of the functional*

$$V(\rho_0, \bar{\rho}_1) = \inf_{\rho,B,A} \int_{\mathcal{E}} F\left(\frac{B}{\rho}, \frac{A}{\rho}\right) \mathrm{d}\rho + C(\rho_1, \bar{\rho}_1) \tag{11}$$

over all $(\rho, B, A) \in (\mathcal{M}(\mathcal{E}))^3$ satisfying the constraints

$$\partial_t \rho(t,x) + \partial_x B(t,x) - \frac{1}{2}\partial_{xx} A(t,x) = 0 \quad \forall (t,x) \in \mathcal{E}, \tag{12}$$

$$\rho(0,x) = \rho_0(x) \quad \forall x \in \mathbb{R}. \tag{13}$$

Recall that the penalty function $C(\rho_1, \bar{\rho}_1)$ above penalizes deviation of ρ_1 from the desired final density $\bar{\rho}_1$.

2.2. Assumptions

ASSUMPTION 2.1 *The penalty functional $C(\cdot, \bar{\rho}_1) : \mathcal{P}(\mathbb{R}) \to \mathbb{R}^+$ is convex and lower semi-continuous with respect to the weak$-*$ convergence of ρ. We have $C(\rho_1, \bar{\rho}_1) = 0$ if and only if $\rho_1(x) = \bar{\rho}_1(x)$ almost everywhere.*

ASSUMPTION 2.2

(i) *The cost function $F(p,q)$ is non-negative, lower semi-continuous and strictly convex in (p,q).*

(ii) *The cost function $F(p,q)$ is coercive in the sense that there exist constants $m > 1$ and $C > 0$ such that*

$$|p|^m + |q|^m \leq C(1 + F(p,q)) \quad \forall (p,q) \in \Pi.$$

3. Duality

In this section, we introduce the dual problem to Problem 2.1, this allows us to give optimality condition for the primal problem. First of all, we find out the convex conjugate of the cost functional, which will be used in the later proof.

3.1. Convex conjugate

Define a function $G : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ as

$$G(u,b,a) = \sup_{(\rho,\tilde{B},\tilde{A}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}} \left\{ u\rho + b\tilde{B}\rho + a\tilde{A}\rho - F(\tilde{B},\tilde{A})\rho \right\}$$

$$= \sup_{\rho \in \mathbb{R}} \left\{ \rho \left[u + \sup_{\tilde{A} \geq \frac{(\tilde{B}^+)^2}{\|\nu(t,x)\|^2}} \left(b\tilde{B} + a\tilde{A} - F(\tilde{B},\tilde{A}) \right) \right] \right\}$$

$$= \sup_{\rho \in \mathbb{R}} \left\{ \rho \left[u + F^*(b,a) \right] \right\},$$

where F^* is the convex conjugate of F. Since $\rho(t,x)$ is non-negative, it is obvious that

$$G(u,b,a) = \begin{cases} 0 & \text{if } u + F^*(b,a) \leq 0 \ \forall (t,x) \in \mathcal{E}, \\ +\infty & \text{otherwise.} \end{cases}$$

If we restrict the domain of its convex conjugate to $(\mathcal{M}(\mathcal{E}))^3$, then

$$G^*(\rho,B,A) = \sup_{(u,b,a) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}} \left\{ u\rho + bB + aA : u + F^*(b,a) \leq 0 \right\}. \tag{14}$$

Because the function to be optimized is linear and $\rho(t,x) \geq 0$, we can see the optimal $u^* = -F^*(b,a)$ in (14). With F being convex and lower-semicontinuous, we have

$$G^*(\rho,B,A) = \sup_{(b,a) \in \mathbb{R} \times \mathbb{R}} \left\{ -F^*(b,a) + b\tilde{B} + a\tilde{A} \right\} \rho$$

$$= F\left(\frac{B}{\rho}, \frac{A}{\rho}\right)\rho.$$

We interchange the supremum and integral following Guo *et al.* (2019), Lemma A.1), and get

$$\int_{\mathcal{E}} F\left(\frac{B}{\rho}, \frac{A}{\rho}\right) \mathrm{d}\rho$$

$$= \sup_{(u,b,a) \in (C_b(\mathcal{E}))^3} \left\{ \int_{\mathcal{E}} u \, d\rho + b \, dB + a \, dA : u \right.$$

$$\left. + F^*(b,a) \leq 0 \right\}. \qquad (15)$$

Note that (u,b,a) are real numbers only in 3.1, and they are functions after the interchange, this will apply to the following sections.

3.2. Dual problem

Now we can state our main result. A key element in the dual problem is the Hamilton–Jacobi–Bellman (HJB) equation:

$$\partial_t \phi + \sup_{\tilde{A} \geq \frac{(\tilde{B}+)^2}{\|v(t,x)\|^2}} \left\{ \partial_x \phi \tilde{B} + \frac{1}{2} \partial_{xx} \phi \tilde{A} - F(\tilde{B}, \tilde{A}) \right\} = 0. \qquad (16)$$

If $\phi(t,x) \in C_b^{1,2}(\mathcal{E})$ is a solution of the HJB equation (16), Itô's formula yields,

$$\int_{\mathbb{R}} \phi_1 \, d\rho_1 - \phi_0 \, d\rho_0$$

$$= \int_{\mathcal{E}} \left(\partial_t \phi + \partial_x \phi \tilde{B} + \frac{1}{2} \partial_{xx} \phi \tilde{A} \right) d\rho$$

$$= \int_{\mathcal{E}} \left(-F^*(\partial_x \phi, \frac{1}{2} \partial_{xx} \phi) + \partial_x \phi \tilde{B} + \frac{1}{2} \partial_{xx} \phi \tilde{A} \right) d\rho$$

$$\leq \int_{\mathcal{E}} F(\tilde{B}, \tilde{A}) \, d\rho.$$

Adding the penalty functional to both sides yields

$$\int_{\mathbb{R}} \phi_1 \, d\rho_1 - \int_{\mathbb{R}} \phi_0 \, d\rho_0 + C(\rho_1, \bar{\rho}_1)$$

$$\leq \int_{\mathcal{E}} F(\tilde{B}, \tilde{A}) \, d\rho + C(\rho_1, \bar{\rho}_1). \qquad (17)$$

Taking the infimum of the left hand side of (17) over ρ_1 and taking the infimum of the right hand side of (17) over (ρ, B, A), we get

$$- C^*(-\phi_1) - \int_{\mathbb{R}} \phi_0 \, d\rho_0$$

$$\leq \inf_{(\rho, B, A) \in (\mathcal{M}(\mathcal{E}))^3} \int_{\mathcal{E}} F\left(\frac{B}{\rho}, \frac{A}{\rho}\right) d\rho + C(\rho_1, \bar{\rho}_1)$$

$$\leq V(\rho_0, \bar{\rho}_1).$$

The following result shows that optimizing the left hand side yields an equality.

THEOREM 3.1 (Duality) *When $C(\cdot, \bar{\rho}_1)$ is continuous, there holds*

$$V(\rho_0, \bar{\rho}_1) = \sup_{\phi} \left\{ -C^*(-\phi_1) - \int_{\mathbb{R}} \phi_0 \, d\rho_0 \right\}, \qquad (18)$$

where the supremum is taken over all $\phi(t,x) \in C_b^{1,2}(\mathcal{E})$ satisfying

$$\partial_t \phi(t,x) + \sup_{\tilde{A} \geq \frac{(\tilde{B}+)^2}{\|v(t,x)\|^2}} \left\{ \partial_x \phi \tilde{B} + \frac{1}{2} \partial_{xx} \phi \tilde{A} - F(\tilde{B}, \tilde{A}) \right\}$$

$$\leq 0, \quad \forall (t,x) \in \mathcal{E}. \qquad (19)$$

Proof This proof is an application of the Fenchel–Rockafellar duality theorem, e.g. Brezis (2010, Theorem 1.12). From the constraint (12), we have that for all $\phi \in C_b^{1,2}(\mathcal{E})$,

$$\int_{\mathbb{R}} \int_0^1 \phi \partial_t \rho + \phi \partial_x B - \frac{1}{2} \phi \partial_{xx} A \, dt \, dx = 0. \qquad (20)$$

Integrating by parts we obtain

$$\int_{\mathbb{R}} \phi_1 \, d\rho_1 - \phi_0 \, d\rho_0 - \int_{\mathcal{E}} \partial_t \phi \, d\rho + \partial_x \phi \, dB + \frac{1}{2} \partial_{xx} \phi \, dA = 0. \qquad (21)$$

Because of equation (15), we can reformulate the primal problem (11) as a saddle point problem:

$$V(\rho_0, \bar{\rho}_1)$$

$$= \inf_{\rho, B, A} \sup_{u + F^*(b,a) \leq 0} \int_{\mathcal{E}} u \, d\rho + b \, dB + a \, dA + C(\rho_1, \bar{\rho}_1). \qquad (22)$$

Adding the Lagrangian penalty (21) to the functional (22), then Problem 2.1 can be written as

$$V(\rho_0, \bar{\rho}_1) = \inf_{\rho, B, A} \sup_{u + F^*(b,a) \leq 0, \phi} \int_{\mathcal{E}} u \, d\rho$$

$$+ b \, dB + a \, dA + C(\rho_1, \bar{\rho}_1)$$

$$+ \int_{\mathbb{R}} \phi_1 \, d\rho_1 - \phi_0 \, d\rho_0 - \int_{\mathcal{E}} \partial_t \phi \, d\rho$$

$$+ \partial_x \phi \, dB + \frac{1}{2} \partial_{xx} \phi \, dA.$$

We write $C^*(r) : C_b(\mathbb{R}; \mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$ for the convex conjugate of functional $C(\rho_1, \bar{\rho}_1)$:

$$C^*(r) = \sup_{\rho_1 \geq 0} \left\{ \int_{\mathbb{R}} r \, d\rho_1 - C(\rho_1, \bar{\rho}_1) \right\}.$$

Here we define the functional $\alpha : (C_b(\mathcal{E}))^4 \to \mathbb{R} \cup \{+\infty\}$ by

$$\alpha(u, b, a, r) = \begin{cases} C^*(r) & \text{if } u + F^*(b,a) \leq 0, \\ +\infty & \text{otherwise.} \end{cases} \qquad (23)$$

Its convex conjugate $\alpha^* : (C_b^*(\mathcal{E}))^4 \to \mathbb{R} \cup \{+\infty\}$ is defined as

$$\alpha^*(\rho, B, A, \rho_1) = \sup_{u + F^*(b,a) \leq 0, r} \int_{\mathcal{E}} u \, d\rho + b \, dB + a \, dA$$

$$+ \left[\int_{\mathbb{R}} r \, d\rho_1 - C^*(r) \right]. \qquad (24)$$

If we restrict the domain to $(\mathcal{M}(\mathcal{E}))^4$, with (15) and Assumption 2.1, we have

$$\alpha^*(\rho, B, A, \rho_1)$$

$$= \begin{cases} \displaystyle\int_{\mathcal{E}} F\left(\frac{B}{\rho}, \frac{A}{\rho}\right) d\rho + C(\rho_1, \bar\rho_1) & \text{if } \rho \in \mathcal{M}_+ \text{ and } B \\ & \qquad = \tilde{B}\rho, A = \tilde{A}\rho, \\ +\infty & \text{otherwise.} \end{cases}$$

Indeed, if ρ is not positive in (24), we would let $b = a = 0$ and $u = -\lambda \mathbb{1}_O$ for some O such that $\rho(O) < 0$ and let $\lambda \to +\infty$. If B or A are not absolutely continuous with respect to ρ, we can find some O such that $\rho(O) = 0$ but $B(O) \neq 0$ or $A(O) \neq 0$. Then we let $u = -F^*(b, a)$ and $b = a = \lambda \mathbb{1}_O$, and $\alpha^*(\rho, B, A) \geq \lambda B(O) + \lambda A(O) \to +\infty$ by letting $\lambda \to \pm\infty$ depending on the sign of $B(O)$ and $A(O)$.

Next, we say that the set $(u, b, a, r) \in (C_b(\mathcal{E}))^4$ is *represented* by $\phi \in C_b^{1,2}(\mathcal{E})$ if

$$u = -\partial_t \phi, \quad b = -\partial_x \phi, \quad a = -\frac{1}{2}\partial_{xx}\phi, \quad r = \phi_1.$$

Then define $\beta : (C_b(\mathcal{E}))^4 \to \mathbb{R} \cup \{+\infty\}$ as follows,

$$\beta(u, b, a, r)$$

$$= \begin{cases} \displaystyle\int_{\mathbb{R}} \phi_0 \, d\rho_0 & \text{if } (u, b, a, r) \text{ is represented by} \\ & \qquad \phi \in C_b^{1,2}(\mathcal{E}), \\ +\infty & \text{otherwise.} \end{cases} \qquad (25)$$

Notice that β is well-defined, indeed, it does not depend on the choice of ϕ. If both ϕ, ψ represent u, b, a, r, then $\phi_1 = \psi_1 \forall x \in \mathbb{R}$, $\partial_t \phi(t, x) = \partial_t \psi(t, x)$, $\partial_x \phi(t, x) = \partial_x \psi(t, x)$, $\partial_{xx}\phi(t, x) = \partial_{xx}\psi(t, x) \ \forall (t, x) \in \mathcal{K}$. It follows that $\phi_0(x) = \psi_0(x) \forall x \in \mathbb{R}$. The set of represented functions (u, b, a, r) is a linear subspace. Hence β is convex and its convex conjugate $\beta^* : (C_b^*(\mathcal{E}))^4 \to \mathbb{R} \cup \{+\infty\}$ is

$$\beta^*(\rho, B, A, \rho_1)$$

$$= \sup_{u,b,a,r} \int_{\mathcal{E}} u \, d\rho + b \, dB + a \, dA + \int_{\mathbb{R}} r \, d\rho_1 - \phi_0 \, d\rho_0,$$

over all $(u, b, a, r) \in (C_b(\mathcal{E}))^4$ represented by $\phi \in C_b^{1,2}(\mathcal{E})$.

Or equivalently,

$$\beta^*(\rho, B, A, \rho_1) = \sup_\phi \int_{\mathcal{E}} -\partial_t \phi \, d\rho - \partial_x \phi \, dB - \frac{1}{2}\partial_{xx}\phi \, dA$$

$$+ \int_{\mathbb{R}} \phi_1 \, d\rho_1 - \phi_0 \, d\rho_0.$$

We find that $\beta^*(\rho, B, A, \rho_1) = 0$ if (ρ, B, A, ρ_1) satisfies (21), and $\beta^*(\rho, B, A, \rho_1) = +\infty$ otherwise.

Now we can express our objective functional $V(\rho_0, \bar\rho_1)$ as

$$V(\rho_0, \bar\rho_1)$$

$$= \inf_{(\rho,B,A)\in(\mathcal{M}(\mathcal{E}))^3} \left\{ \alpha^*(\rho, B, A, \rho_1) + \beta^*(\rho, B, A, \rho_1) \right\}$$

$$= \inf_{(\rho,B,A,\rho_1)\in(\mathcal{M}(\mathcal{E}))^4} \left\{ \alpha^*(\rho, B, A, \rho_1) + \beta^*(\rho, B, A, \rho_1) \right\}$$

$$= \inf_{(\rho,B,A,\rho_1)\in(C_b^*(\mathcal{E}))^4} \left\{ \alpha^*(\rho, B, A, \rho_1) + \beta^*(\rho, B, A, \rho_1) \right\}.$$

The second equality is because $\beta^*(\rho, B, A, \rho_1) = +\infty$ if ρ_1 does not equal to $\rho(t, x)$ at time $t = 1$. We prove the third equality in Section A.2.

We can let $\phi(t, x) = t$, then $u = -1$, $b = 0$, $a = 0$, $r = 1$. We can see $\alpha(-1, 0, 0, 1) = 1$ and it is continuous in (u, b, a, r) at this point, and $\beta(-1, 0, 0, 1) = 0$ being finite at this point. Finally, the conditions of Fenchel duality theorem in Brezis (2010, Theorem 1.12) are fulfilled, and it implies

$$V(\rho_0, \bar\rho_1)$$

$$= \inf_{(\rho,B,A,\rho_1)\in(C_b^*(\mathcal{E}))^4} \left\{ \alpha^*(\rho, B, A, \rho_1) + \beta^*(\rho, B, A, \rho_1) \right\}$$

$$= \sup_{(u,b,a,r)\in(C_b(\mathcal{E}))^4} \left\{ -\alpha(-u, -b, -a, -r) - \beta(u, b, a, r) \right\},$$

over the set (u, b, a, r) being represented by $\phi \in C_b^{1,2}(\mathcal{E})$, and satisfying $-u + F^*(-b, -a) \leq 0$.

Therefore we express $V(\rho_0, \bar\rho_1)$ in terms of ϕ:

$$V(\rho_0, \bar\rho_1) = \sup_{(u,b,a,r)\in(C_b(\mathcal{E}))^4} \left\{ -C^*(-r) - \int_{\mathbb{R}} \phi_0 \, d\rho_0 \right\}$$

$$= \sup_{\phi\in C_b^{1,2}(\mathcal{E})} \left\{ -C^*(-\phi_1) - \int_{\mathbb{R}} \phi_0 \, d\rho_0 \right\},$$

under the constraint $\partial_t \phi + F^*(\partial_x \phi, \frac{1}{2}\partial_{xx}\phi) \leq 0$. As a consequence of Fenchel duality theorem, the infimum in the primal problem is attained if finite. This completes the proof. ∎

Actually, using the same proof as in Guo *et al.* (2019), we can write the dual formulation in the following way:

COROLLARY 3.1 *When $C(\cdot, \bar\rho_1)$ is continuous, there holds*

$$V(\rho_0, \bar\rho_1) = \sup_{\phi_1} \left\{ -C^*(-\phi_1) - \int_{\mathbb{R}} \phi_0 \, d\rho_0 \right\}, \qquad (26)$$

where the supremum is running over all functions $\phi_1 \in C_b^2(\mathbb{R})$, and ϕ_0 is a viscosity solution of the Hamilton–Jacobi–Bellman equation

$$\begin{cases} -\phi_t - \sup_{\tilde{A} \geq \frac{(\tilde{B}^+)^2}{\|\nu(t,x)\|^2}} \left[\phi_x \tilde{B} + \frac{1}{2}\phi_{xx}\tilde{A} - F(\tilde{B}, \tilde{A}) \right] \\ \qquad = 0, & \text{in } [0, 1) \times \mathbb{R}, \\ \phi(1, x) = \phi_1(x), & \text{on } [1] \times \mathbb{R}. \end{cases} \qquad (27)$$

Because the minimal objective function (11) is a trade-off between the cost function and the penalty functional, the optimal ϕ_1 in the dual problem (26) will not in general ensure that ρ_1 reaches $\bar\rho_1$, unless the penalty functional goes to infinity for $\rho_1 \neq \bar\rho_1$. When $\bar\rho_1$ is attainable, it can be realized by choosing

the penalty functional as an indicator function

$$C(\rho_1, \bar{\rho}_1) = \begin{cases} 0 & \text{if } \rho_1 = \bar{\rho}_1, \\ +\infty & \text{if } \rho_1 \neq \bar{\rho}_1. \end{cases} \qquad (28)$$

Using the penalty functional (28) is equivalent to adding the terminal constraint $\rho_1 = \bar{\rho}_1$, $\forall x \in \mathbb{R}$. This also recovers our problem to the classical optimal transport problem.

COROLLARY 3.2 *When $C(\rho_1, \bar{\rho}_1)$ is defined as* (28), *there holds*

$$V(\rho_0, \bar{\rho}_1) = \sup_{\phi_1} \left\{ \int_{\mathbb{R}} \phi_1 \, d\bar{\rho}_1 - \phi_0 \, d\rho_0 \right\}, \qquad (29)$$

where the supremum is running over all $\phi_1 \in C_b^2(\mathbb{R})$ and ϕ_0 is a viscosity solution of the Hamilton–Jacobi–Bellman equation (27).

Proof This proof is very similar to the one of Theorem 3.1, hence the repetitive steps are omitted here. Being different from the proof of Theorem 3.1, in this case, we define the functional $\alpha : (C_b(\mathcal{E}))^4 \to \mathbb{R} \cup \{+\infty\}$ by

$$\alpha(u, b, a, r) = \begin{cases} \displaystyle\int_{\mathbb{R}} r \, d\bar{\rho}_1 & \text{if } u + F^*(b, a) \leq 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Then its convex conjugate of $\alpha^* : (C_b^*(\mathcal{E}))^4 \to \mathbb{R} \cup \{+\infty\}$ is

$$\alpha^*(\rho, B, A, \rho_1) = \sup_{u + F^*(b,a) \leq 0, r} \int_{\mathcal{E}} u \, d\rho + b \, dB$$
$$+ a \, dA + \int_{\mathbb{R}} r \, d\rho_1 - r \, d\bar{\rho}_1.$$

We restrict the domain to $(\mathcal{M}(\mathcal{E}))^4$, we have

$$\alpha^*(\rho, B, A, \rho_1)$$
$$= \int_{\mathcal{E}} F\left(\frac{B}{\rho}, \frac{A}{\rho}\right) d\rho + \sup_r \left\{ \int_{\mathbb{R}} r(d\rho_1 - d\bar{\rho}_1) \right\}$$
$$= \int_{\mathcal{E}} F\left(\frac{B}{\rho}, \frac{A}{\rho}\right) d\rho + C(\rho_1, \bar{\rho}_1).$$

Note that $\sup_r \int_{\mathbb{R}} r(d\rho_1 - d\bar{\rho}_1)$ is equal to 0 if $\rho_1 = \bar{\rho}_1 \forall x \in \mathbb{R}$ and is equal to $+\infty$ otherwise, which is equivalent to $C(\rho_1, \bar{\rho}_1)$ in (28). Define $\beta : (C_b(\mathcal{E}))^4 \to \mathbb{R} \cup \{+\infty\}$ by (25), let $\phi(t, x) = t$, the conditions of Fenchel duality theorem in Brezis (2010, Theorem 1.12) are fulfilled. Therefore, we get

$$V(\rho_0, \bar{\rho}_1)$$
$$= \sup_{(u,b,a,r) \in (C_b(\mathcal{E}))^4} \{-\alpha(-u, -b, -a, -r) - \beta(u, b, a, r)\}$$
$$= \sup_{(u,b,a,r) \in (C_b(\mathcal{E}))^4} \left\{ \int_{\mathbb{R}} r \, d\bar{\rho}_1 - \int_{\mathbb{R}} \phi_0 \, d\rho_0 \right\},$$

over the set (u, b, a, r) being represented by $\phi \in C_b^{1,2}(\mathcal{E})$, and satisfying $-u + F^*(-b, -a) \leq 0$. For the same reasons as in Corollary 3.1, we can express $V(\rho_0, \bar{\rho}_1)$ in terms of ϕ:

$$V(\rho_0, \bar{\rho}_1) = \sup_{\phi_1} \left\{ \int_{\mathbb{R}} \phi_1 \, d\bar{\rho}_1 - \int_{\mathbb{R}} \phi_0 \, d\rho_0 \right\},$$

where $\phi_0(x)$ is a viscosity solution of the Hamilton–Jacobi–Bellman equation (27). ∎

4. Numerical methods for the dual problem

There has been a vast amount of numerical algorithms for the optimal mass transport problem. Gradient descent based methods are widely used to solve the reformulated dual problem of the Monge–Kantorovich problem, for example, by Chartrand *et al.* (2009) and Tan *et al.* (2013). Cuturi (2013) looked at transport problems from a maximum entropy perspective and computed the OT distance through Sinkhorn's matrix scaling algorithm. This algorithm is also used for the entropic regularization of optimal transport by Benamou *et al.* (2019).

In this paper, we also use a gradient descent based method to solve the dual problem in Section 3. We know $\phi(t, x)$ is the solution of the HJB equation (27). For a given terminal function ϕ_1, we can calculate ϕ_0 by solving the HJB equation backward.

4.1. Finite difference scheme

First of all, to get $\phi(0, x)$, we solve the following PDE

$$\partial_t \phi + \sup_{\tilde{A} \geq \frac{(\tilde{B}^+)^2}{\|\nu(t,x)\|^2}} \left\{ \partial_x \phi \tilde{B} + \frac{1}{2} \partial_{xx} \phi \tilde{A} - F(\tilde{A}, \tilde{B}) \right\} = 0, \qquad (30)$$

with a given terminal boundary condition $\phi_1(x)$ backwardly, using an implicit finite difference scheme. We let

$$(\tilde{A}^*, \tilde{B}^*) = \underset{\tilde{A} \geq \frac{(\tilde{B}^+)^2}{\|\nu(t,x)\|^2}}{\operatorname{argmax}} \left\{ \partial_x \phi \tilde{B} + \frac{1}{2} \partial_{xx} \phi \tilde{A} - F(\tilde{A}, \tilde{B}) \right\}. \qquad (31)$$

In the numerical setting, we use N time steps and M space grid points. We use a constant time step Δt and a constant spatial step Δx. We discretize the PDE (30) using a forward approximation for $\partial_t \phi$, a central approximation for $\partial_x \phi$, and a standard approximation for $\partial_{xx} \phi$. With some manipulation, we get the discretized form of (30) as

$$\left(\frac{\Delta t \tilde{A}_i^{*^n}}{2(\Delta x)^2} - \frac{\Delta t \tilde{B}_i^{*^n}}{2\Delta x} \right) \phi_{i-1}^n + \left(-1 - \frac{\Delta t \tilde{A}_i^{*^n}}{(\Delta x)^2} \right) \phi_i^n$$
$$+ \left(\frac{\Delta t \tilde{A}_i^{*^n}}{2(\Delta x)^2} + \frac{\Delta t \tilde{B}_i^{*^n}}{2\Delta x} \right) \phi_{i+1}^n = -\phi_i^{n+1} + \Delta t F(\tilde{A}_i^{*^n}, \tilde{B}_i^{*^n}),$$
$$(32)$$

where the optimal controls $\tilde{A}^{*^n}$ and $\tilde{B}^{*^n}$ depend on ϕ^n. It is difficult to check the stability condition in our PDE because the optimal $\tilde{A}^{*^n}, \tilde{B}^{*^n}$ are unknown, but fortunately implicit finite

difference methods have a weaker requirement than explicit finite difference methods. At the n-th time step of the implicit finite difference method, although we do not have the true values for ϕ^n, we can make an initial guess of $(\tilde{A}^{*^n}, \tilde{B}^{*^n})_0$ using the known values ϕ^{n+1}, then use a fixed-point iteration scheme to generate a sequence $(\tilde{A}^{*^n}, \tilde{B}^{*^n})_{k,k=1,2,\dots}$ until $(\tilde{A}^{*^n}, \tilde{B}^{*^n})_k$ converges. The stopping criteria is when $\|\phi_k^n - \phi_{k-1}^n\|_2 \le 10^{-7}$. This method is also implemented in Guo *et al.* (2019).

With the optimal drift $\tilde{B}^*$ and diffusion $\tilde{A}^*$ known, we can now propagate forward with the Fokker–Planck equation (12) to find the empirical terminal density ρ_1. With an initial wealth x_0, the initial distribution ρ_0 is a Dirac Delta distribution $\delta(x - x_0)$. Since we used implicit finite difference to solve the HJB equation (30) backward, we use an explicit scheme for the forward Fokker–Planck equation (12).

4.2. *Optimization algorithm*

A key role in the gradient descent method is the optimality condition. By providing a gradient, the computation is faster and more accurate. For convenience, we define another function

$$\tilde{V}(\phi_1) := C^*(-\phi_1) + \int_{\mathbb{R}} \phi_0 \rho_0 \, dx, \tag{33}$$

and $V(\rho_0, \bar{\rho}_1) = -\inf_{\phi_1} \tilde{V}(\phi_1)$. Then we need to find an optimal ϕ_1 to minimize $\tilde{V}(\phi_1)$. The change of $\tilde{V}(\phi_1)$ caused by the change of ϕ_1 is

$$\delta \tilde{V}(\phi_1) = \delta C^*(-\phi_1) + \int_{\mathbb{R}} \rho_0 \frac{\partial \phi_0}{\partial \phi_1} \delta \phi_1 \, dx, \tag{34}$$

$$= \delta C^*(-\phi_1) + \int_{\mathbb{R}} \rho_0 \delta \phi_0 \, dx. \tag{35}$$

We know that $\phi(t, x)$ in (33) satisfies $F^*(\partial_x \phi, \frac{1}{2} \partial_{xx} \phi) = -\partial_t \phi$. If we add an arbitrarily small variation $\delta \phi$ to ϕ and denote $\partial_x \phi$ as p and $\frac{1}{2} \partial_{xx} \phi$ as q for short, then we get $\partial_p F^*(p, q) \partial_x \delta \phi + \frac{1}{2} \partial_q F^*(p, q) \partial_{xx} \delta \phi = -\partial_t \delta \phi$, which is equivalent to

$$\partial_t \delta \phi + \partial_x \delta \phi \tilde{B}^* + \frac{1}{2} \partial_{xx} \delta \phi \tilde{A}^* = 0, \tag{36}$$

where $\tilde{A}^*, \tilde{B}^*$ are the optimal controls, as defined in (31). Multiplying PDE (36) by density function $\rho(t, x)$ and with integration by parts, we have

$$\int_{\mathbb{R}} \rho_1 \delta \phi_1 - \rho_0 \delta \phi_0 \, dx - \int_{\mathbb{R}} \int_0^1 \delta \phi \partial_t \rho$$

$$+ \delta \phi \partial_x (\rho \tilde{B}^*) - \frac{1}{2} \delta \phi \partial_{xx} (\rho \tilde{A}^*) \, dx \, dt = 0.$$

Since the equation $\partial_t \rho + \partial_x (\rho \tilde{B}^*) - \frac{1}{2} \partial_{xx} (\rho \tilde{A}^*) = 0$ holds for admissible $(\rho, \tilde{A}^*, \tilde{B}^*)$, we get

$$\int_{\mathbb{R}} \rho_0 \delta \phi_0 \, dx = \int_{\mathbb{R}} \rho_1 \delta \phi_1 \, dx. \tag{37}$$

Substituting (37) into (34), we can see an optimal terminal function ϕ_1 should satisfy the optimality condition

$$\nabla_{\phi_1} \tilde{V} := \lim_{\delta \phi_1 \to 0} \frac{\delta \tilde{V}(\phi_1)}{\delta \phi_1} \tag{38}$$

$$= \lim_{\delta \phi_1 \to 0} \frac{\delta C^*(-\phi_1)}{\delta \phi_1} + \rho_1 = 0, \quad \forall x \in \mathbb{R}. \tag{39}$$

REMARK 4.1 When $C(\bar{\rho}_1, \rho_1)$ is defined as (28), the corresponding optimality condition is

$$\nabla_{\phi_1} \tilde{V} = -\bar{\rho}_1 + \rho_1 = 0, \quad \forall x \in \mathbb{R}. \tag{40}$$

Now we are ready to solve the dual problem numerically. In Algorithm 1, we state the gradient descent based algorithm to look for the optimal ϕ_1 in (26). It includes solving the HJB equation and the Fokker–Planck equation with a finite difference method combined with a fixed-point iteration, as described in Section 4.1. A similar numerical scheme can be found in Guo *et al.* (2019) for calibrating volatilities by optimal transport.

5. Numerical results

In this section, we will apply Algorithm 1 and demonstrate various numerical examples. Later, we will also consider the situations with cash saving and cash input during the investment process.

5.1. *Penalty functional with an intensity parameter*

Before we demonstrate the numerical results, we need to choose an appropriate penalty functional $C(\rho_1, \bar{\rho}_1)$. There are a range of methods to measure distribution discrepancy. A comprehensive survey on the distance or similarity measures between probability density functions (PDFs) is provided by Cha (2007). Note that our choice of penalty functional is not restricted to metrics, as long as $C(\rho_1, \bar{\rho}_1)$ satisfies Assumption 2.1 and describes similarity of the two PDFs.

The most intuitive choice is the L^2 norm of the difference. This quadratic function is convex and easy to implement. In the first example, we use the squared Euclidean distance as the penalty functional and $F(\tilde{A}, \tilde{B}) = (\tilde{A} - 0.2)^2 + (\tilde{B} - 0.2)^2$ as the cost function. We define the penalty functional as

$$C(\rho_1, \bar{\rho}_1) = \frac{\lambda}{2} \int_{\mathbb{R}} (\rho_1 - \bar{\rho}_1)^2 \, dx,$$

where the parameter λ can be regarded as the intensity of the penalty for the inconsistency. Then the dual problem (26) can be expressed explicitly as

$$V(\rho_0, \bar{\rho}_1) = \sup_{\phi_1} \left\{ \int_{\mathbb{R}} -\frac{1}{2\lambda} \phi_1^2 + \phi_1 \bar{\rho}_1 - \rho_0 \phi_0 \, dx \right\}. \tag{41}$$
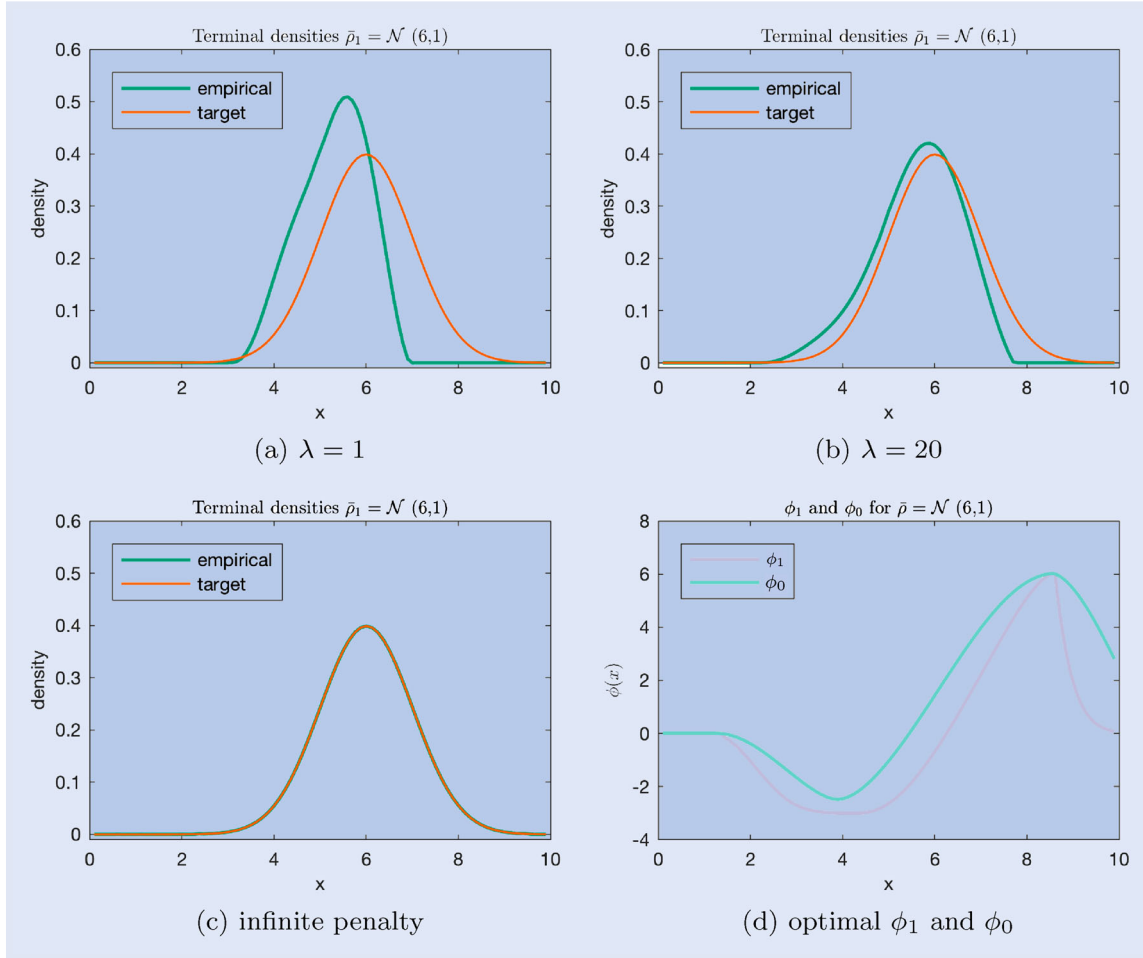
In this and the following numerical examples, we set the initial wealth $x_0 = 5, \mu = 0.1, \sigma = 0.1$. Figure 1(a) and (b) compares the empirical distribution of the terminal wealth

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Algorithm 1: A gradient descent based optimization scheme

Initial guess $\phi_N^1 := 0$

while $1 \le k \le$ *max iteration and* $\left\| \nabla_{\phi_N} \tilde{V}(\phi_N^k) \right\|_\infty > $ *tolerance* **do**

    Let $\phi_N = \phi_N^k$;

    for *time step* $n = N - 1 : 0:$ **do**

        Let $\phi_n = \phi_{n+1}$, solve the PDE (32) with $(\tilde{A}^{*^n}, \tilde{B}^{*^n})^0$ obtained from (31).

        Get the value vector ϕ_n^0;

        while $1 \le j \le$ *max iteration and* $\left\| \phi_n^j - \phi_n^{j-1} \right\|_2 > $ *tolerance* **do**

            Let $\phi_n = \phi_n^{j-1}$, solve the PDE (32) with $(\tilde{A}^{*^n}, \tilde{B}^{*^n})^j$ obtained from (31).

            Get the value vector ϕ_n^j;

            $j = j + 1$;

        end

        Let $\phi_n = \phi_n^j$, store the optimal controls $(\tilde{A}^{*^n}, \tilde{B}^{*^n}) = (\tilde{A}^{*^n}, \tilde{B}^{*^n})^j$;

    end

    Compute the empirical distribution ρ_1^k from ρ_0 with Fokker–Planck equation;

    Compute the gradient vector $\nabla_{\phi_N} \tilde{V}(\phi_N^k) = \left(\frac{\delta C^*(-\phi_N^k)}{\delta \phi_N^k} + \rho_1^k \right) \Delta x$;

    Update ϕ_N^{k+1} with Quasi-Newton Method using the gradient information $\nabla_{\phi_N} \tilde{V}(\phi_N^k)$;

    $k = k + 1$;

end

The optimal $\phi_N = \phi_N^k$.

Figure 1. Attainable example: $\bar{\rho}_1 = \mathcal{N}(6, 1)$. (a) $\lambda = 1$, (b) $\lambda = 20$, (c) infinite penalty and (d) optimal ϕ_1 and ϕ_0.
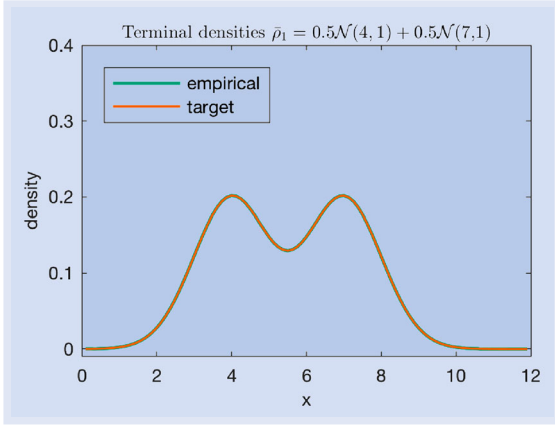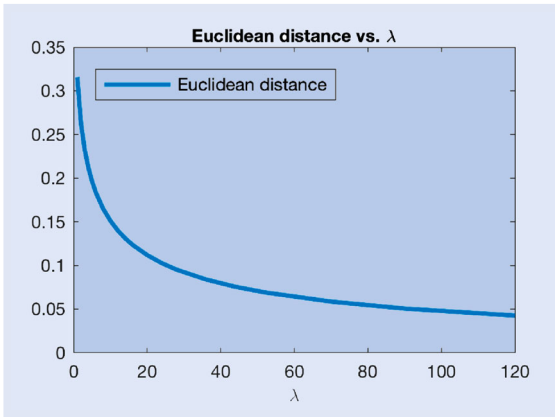
Figure 2. Mixture of normal distributions.

Figure 3. Distance metric vs. λ.

distribution ρ_1 from the target $\bar{\rho}_1$, the more similar the two distributions are, the smaller the relative entropy will be. This measurement is widely used in Machine Learning to compare two densities because it has the following advantages: (1) this function is non-negative; (2) for a fixed distribution $\bar{\rho}_1$, $C(\rho_1, \bar{\rho}_1)$ is convex in ρ_1; (3) $C(\rho_1, \bar{\rho}_1) = 0$ if and only if $\rho_1 = \bar{\rho}_1$ everywhere. There are also caveats to the implementation of this penalty function. We may face $0 \log 0$ or *division by zero* cases in practice; to address this, we can replace zero with an infinitesimal positive value.

In this case, the penalty functional is defined as

$$C(\rho_1, \bar{\rho}_1) = \int_{\mathbb{R}} \lambda \rho_1(x) \ln \left(\frac{\rho_1(x)}{\bar{\rho}_1(x)} \right) dx, \quad (42)$$

and the dual problem (26) can be expressed explicitly as

$$V(\rho_0, \bar{\rho}_1) = \sup_{\phi_1} \left\{ - \int_{\mathbb{R}} \lambda \exp \left(-\frac{\phi_1}{\lambda} - 1 \right) \bar{\rho}_1 - \phi_0 \rho_0 \, dx \right\}.$$

In figure 4, we compare the empirical terminal density ρ_1 and the target $\bar{\rho}_1$ when $C(\rho_1, \bar{\rho}_1)$ is defined by (42) and $F(\tilde{A}, \tilde{B}) = (\tilde{A} - 0.2)^2 + (\tilde{B} - 0.2)^2$. The initial wealth $x_0 = 5$ and we set $\lambda = 0.1$ in figure 4(a) and $\lambda = 10$ in figure 4(b).

5.2. Distribution of the wealth with cash saving

In this section, we consider the cash saving during the investment process. From previous parts, we have the constraint $(\tilde{B}^+)^2 \leq \|\nu(t,x)\|^2 \tilde{A}$. However, when the prescribed target $\bar{\rho}_1$ is not ambitious enough, we will find the optimal drift $\tilde{B}^*$ is not saturated, i.e. $(\tilde{B}^{*+})^2 < \|\nu(t,x)\|^2 \tilde{A}^*$ in (31). Our goal in this section is to show that we can reach a better terminal distribution, in the sense that the terminal wealth has a higher expected value, when we take cash saving into account.

Keep using the concept *cash saving* defined in Section 2.1, we denote $(C_t)_{t \in [0,1]}$ the accumulated cash saving up to time t. Then the evolution of C_t is

$$dC_t = \left(\|\nu(t,x)\| \sqrt{\tilde{A}^*} - \tilde{B}^* \right) dt,$$
$$C_0 = 0.$$

If we add up the cash saving C_t and the portfolio wealth X_t, we can get a new process *wealth with cash saving*. Define $X_t^c := X_t + C_t$, it is obvious to see that X_t^c follows the dynamics
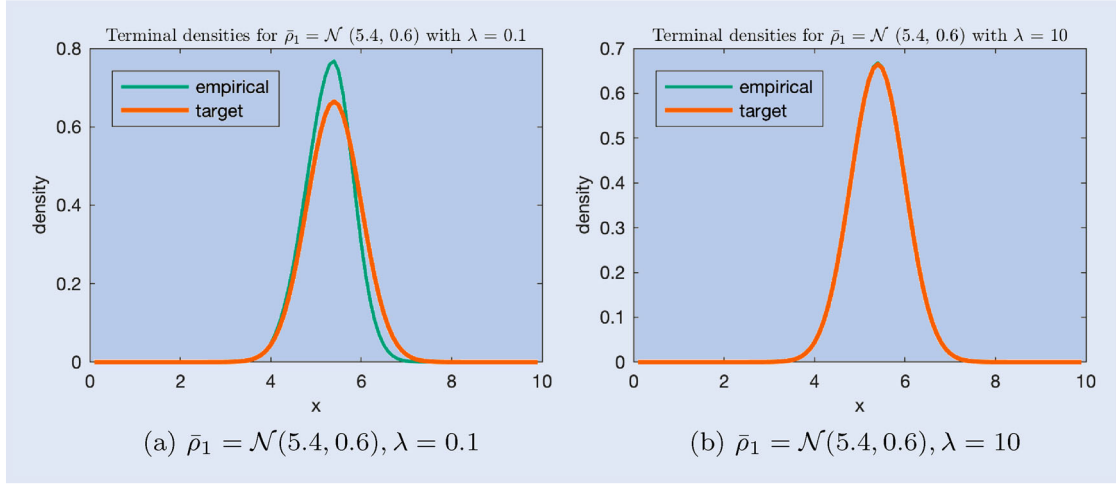
$$dX_t^c = \|\nu(t,x)\| \sqrt{\tilde{A}^*} \, dt + \tilde{A}^{*1/2} \, dW_t,$$
$$X_0^c = x_0.$$

Denote by $p(t,x) \in \mathcal{P}(\mathbb{R})$ the distribution of X_t^c at time t, then $p(t,x)$ satisfies the following Fokker–Planck equation

$$\partial_t p + \partial_x \left(\|\nu(t,x)\| \sqrt{\tilde{A}^*} p \right) - \frac{1}{2} \partial_{xx} \left(\tilde{A}^* p \right) = 0,$$
$$p_0(x) = \delta(x - x_0).$$

Therefore, after solving for the optimal $\tilde{A}^*$, $\tilde{B}^*$ according to (31) over time, we can find the densities of X_t as well

(ρ_1) and the prescribed terminal distribution ($\bar{\rho}_1$) for different intensities λ, where $\bar{\rho}_1 = \mathcal{N}(6, 1)$†. We can see that ρ_1 gets closer to $\bar{\rho}_1$ as we increase the intensity of the penalty. In figure 1(c) and (d), we use the penalty functional (28), which is equivalent to letting $\lambda = +\infty$. As shown in figure 1(c), this penalty functional makes ρ_1 attain the target $\bar{\rho}_1$, and the plot (figure 1(d)) illustrates the optimal function ϕ_1 and the corresponding ϕ_0 we got from Algorithm 1.

Compared to other research where the prescribed distributions are restricted to Gaussian, our method applies to a wide choice of $\bar{\rho}_1$, such as heavy-tailed and asymmetric distributions. In figure 2, we illustrate that we can attain the target when $\bar{\rho}_1$ is a mixture of two Normal distributions, in particular

$$\bar{\rho}_1(x) = 0.5\mathcal{N}(4, 1) + 0.5\mathcal{N}(7, 1).$$

In figure 3, we plot how the Euclidean distance $(\int_{\mathbb{R}} (\rho_1 - \bar{\rho}_1)^2 \, dx)^{\frac{1}{2}}$ changes with respect to λ. As we increase the intensity parameter λ, the Euclidean distance between ρ_1 and $\bar{\rho}_1$ decreases. As λ goes to infinity, the distance asymptotically goes to zero.

The Kullback–Leibler (K–L) divergence introduced in Kullback and Leibler (1951), is also known as relative entropy or information deviation. It measures the divergence of the

---

† We denote $\mathcal{N}(\mu, \sigma)$ a Normal distribution with mean μ and standard deviation σ.

Figure 4. K–L divergence as the penalty functional. (a) $\bar{\rho}_1 = \mathcal{N}(5.4, 0.6), \lambda = 0.1$ and (b) $\bar{\rho}_1 = \mathcal{N}(5.4, 0.6), \lambda = 10$.
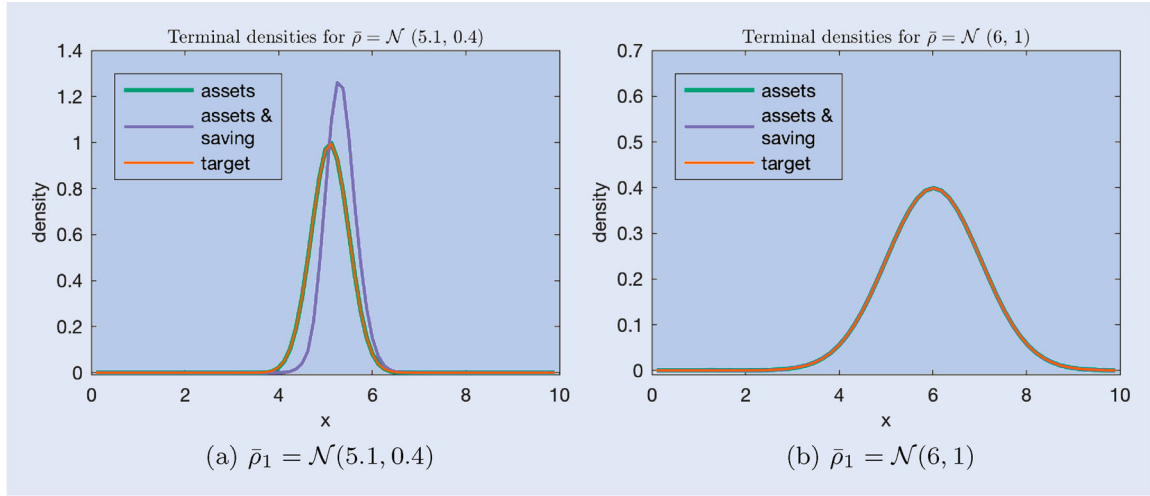
Figure 5. Compare terminal distributions with or without cash saving. (a) $\bar{\rho}_1 = \mathcal{N}(5.1, 0.4)$ and (b) $\bar{\rho}_1 = \mathcal{N}(6, 1)$.

as X_t^c. We keep using the squared Euclidean distance as the penalty functional and $F(\tilde{A}, \tilde{B}) = (\tilde{A} - 0.2)^2 + (\tilde{B} - 0.2)^2$ as the cost function. Figure 5 compares the densities for X_1 (terminal wealth), X_1^c (terminal *wealth with cash saving*) and the prescribed target density. In figure 5(a), with a rather conservative target $\bar{\rho}_1 = \mathcal{N}(5.1, 0.4)$, although ρ_1 has attained the target, the distribution for the *wealth with cash saving* gathers at a higher value. When we set a higher target $\bar{\rho}_1 = \mathcal{N}(6, 1)$, as in figure 5(b), we see there is no cash saved in the process since the paths for ρ_1 and p_1 overlapped.

5.3. *Distribution of the wealth with cash input*

As stated in Proposition 2.1, we always have $(\tilde{B}^+)^2 \leq \|v(t, x)\|^2 \tilde{A}$ for a self-financing portfolio. However, in this section, we remove the constraint $(\tilde{B}^+)^2 \leq \|v(t, x)\|^2 \tilde{A}$, and we allow $\tilde{B} \in \mathbb{R}$ instead. Then the part $(\tilde{B} - \|v(t, x)\|\sqrt{\tilde{A}})^+$ can be interpreted as the extra cash we invest during the process. In this case, theoretically, we can attain any prescribed target distribution as we want (see Tan *et al.* 2013, Remark 2.3). For the $\bar{\rho}_1$ which is unattainable by the self-financing portfolio, we can now attain it with the help of cash input. However, to limit the use of cash, we design a cost function

as follows,

$$
F(\tilde{A}, \tilde{B}) = \begin{cases}
K(\tilde{B}^2 - \|v(t, x)\|^2 \tilde{A}) \\
\quad + w\tilde{A}^2, & \forall \, \tilde{B} > \|v(t, x)\| \sqrt{\tilde{A}}, \\
w\tilde{A}^2, & \forall \, 0 \leq \tilde{B} \leq \|v(t, x)\| \sqrt{\tilde{A}}, \\
l\tilde{B}^2 + w\tilde{A}^2, & \forall \, \tilde{B} < 0,
\end{cases}
$$

(43)

where K, w, l are positive constants. In the cost function (43), we use the term $K(\tilde{B}^2 - \|v(t, x)\|^2 \tilde{A})$ to penalize the part $(\tilde{B} - \|v(t, x)\|\sqrt{\tilde{A}})^+$. By varying K, we can control the strength of penalty and hence control the cash input flow. When K is small, we are allowed to put in cash without being penalized excessively. When K is large, we have to pay a high price for the cash input; consequently, the usage is limited. The terms $w\tilde{A}^2$ and $l\tilde{B}^2$ add coercivity to the function to ensure the existence of the solution, we set w, l to be small positive real values.

With the optimal drift $\tilde{B}^* \in \mathbb{R}$ and diffusion $\tilde{A}^* \in \mathbb{R}^+$, the dynamics of the wealth X_t is

$$
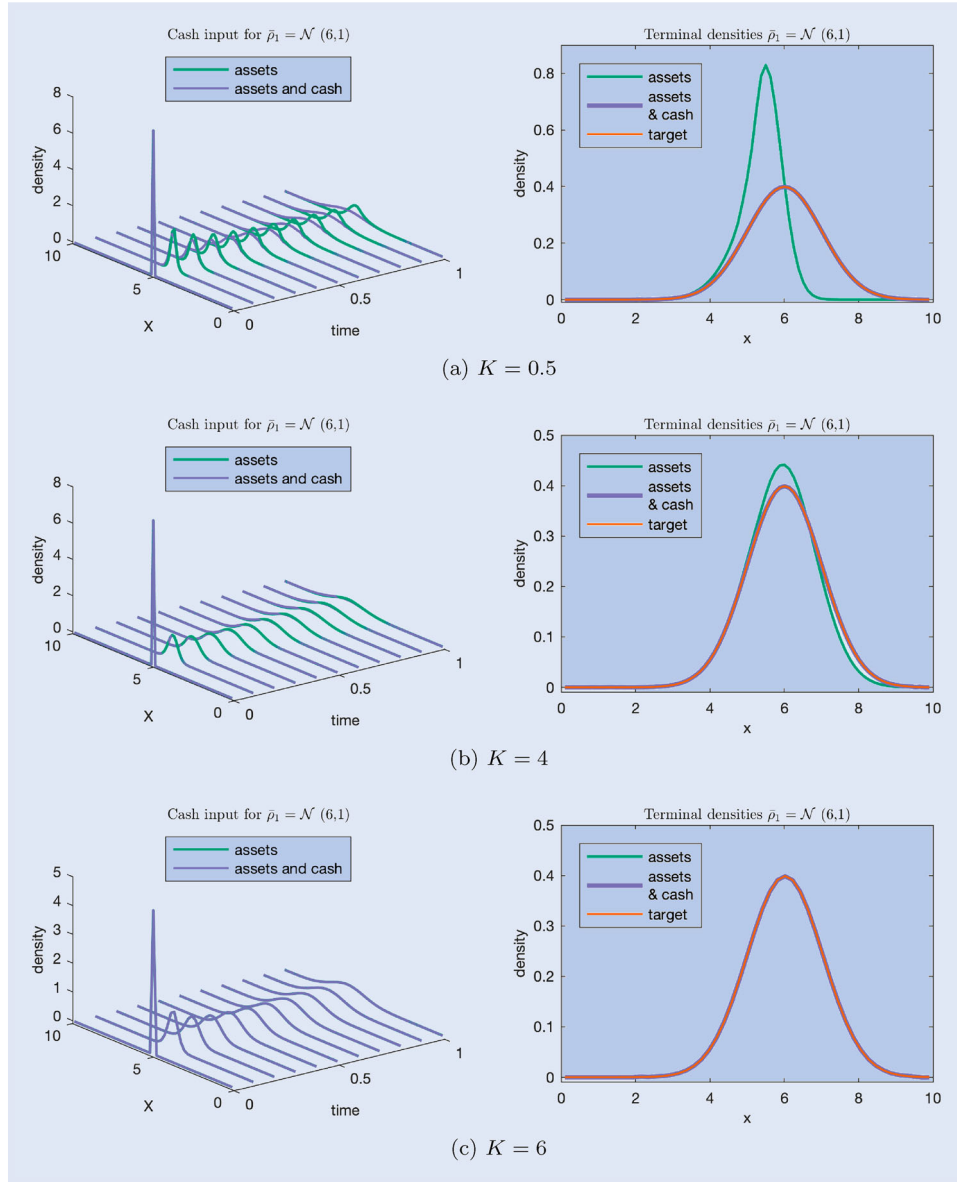dX_t = \tilde{B}^* \, dt + \sqrt{\tilde{A}^*} \, dW_t,
$$

$$
X_0 = x_0.
$$

Figure 6. Fixed K for an attainable target: $\bar{\rho}_1 = \mathcal{N}(6, 1)$. (a) $K = 0.5$, (b) $K = 4$ and (c) $K = 6$.

If there is no cash input, the maximum drift is $\|\nu(t,x)\|\sqrt{\tilde{A}}$. Denote $(I_t)_{t\in[0,1]}$ the accumulated cash input up to time t, and I_t follows the dynamics

$$dI_t = \left(\tilde{B}^* - \|\nu(t,x)\| \sqrt{\tilde{A}^*}\right)^+ dt,$$

$$I_0 = 0.$$

Define $X_t^I := X_t - I_t$ as the *path without the cash input*. Then the dynamics of X_t^I is

$$dX_t^I = \min\left(\tilde{B}^*, \|\nu(t,x)\| \sqrt{\tilde{A}^*}\right) dt + \sqrt{\tilde{A}^*} \, dW_t,$$

$$X_0^I = x_0.$$

Let the density of X_t^I be $q(t,x) \in \mathcal{P}(\mathbb{R})$, then $q(t,x)$ follows the following Fokker–Planck equation

$$\partial_t q + \partial_x \left[\min\left(\tilde{B}^*, \|\nu(t,x)\| \sqrt{\tilde{A}^*}\right) q\right] - \frac{1}{2}\partial_{xx}\left(\tilde{A}^* q\right) = 0,$$

$$q_0(x) = \delta(x - x_0).$$

Finally, we can see the effect of cash input by comparing $\rho_1(x)$ and $q_1(x)$.

5.3.1. Attainable target. In the first example, we aim at the terminal distribution $\bar{\rho}_1 = \mathcal{N}(6, 1)$, which is attainable by the self-financing portfolio. We use the squared Euclidean distance as the penalty functional and equation (43) as the cost function. Figure 6 demonstrates the time evolution of $q(t,x)$ (assets) and $\rho(t,x)$ (assets and cash input), and it compares $q_1(x)$, $\rho_1(x)$ and $\bar{\rho}_1(x)$ for various K values. At the beginning, we set the coefficient $K = 0.5$ in figure 6(a). There is a clear difference between the paths for *assets* and *assets and cash*, which means we have input a significant amount of cash over time. As we increase the value of K in figure 6(b), the difference between $q(t,x)$ and $\rho(t,x)$ becomes less obvious. When $K = 6$, the paths with or without cash input coincide in figure 6(c) because the high cost has prevented the cash input in
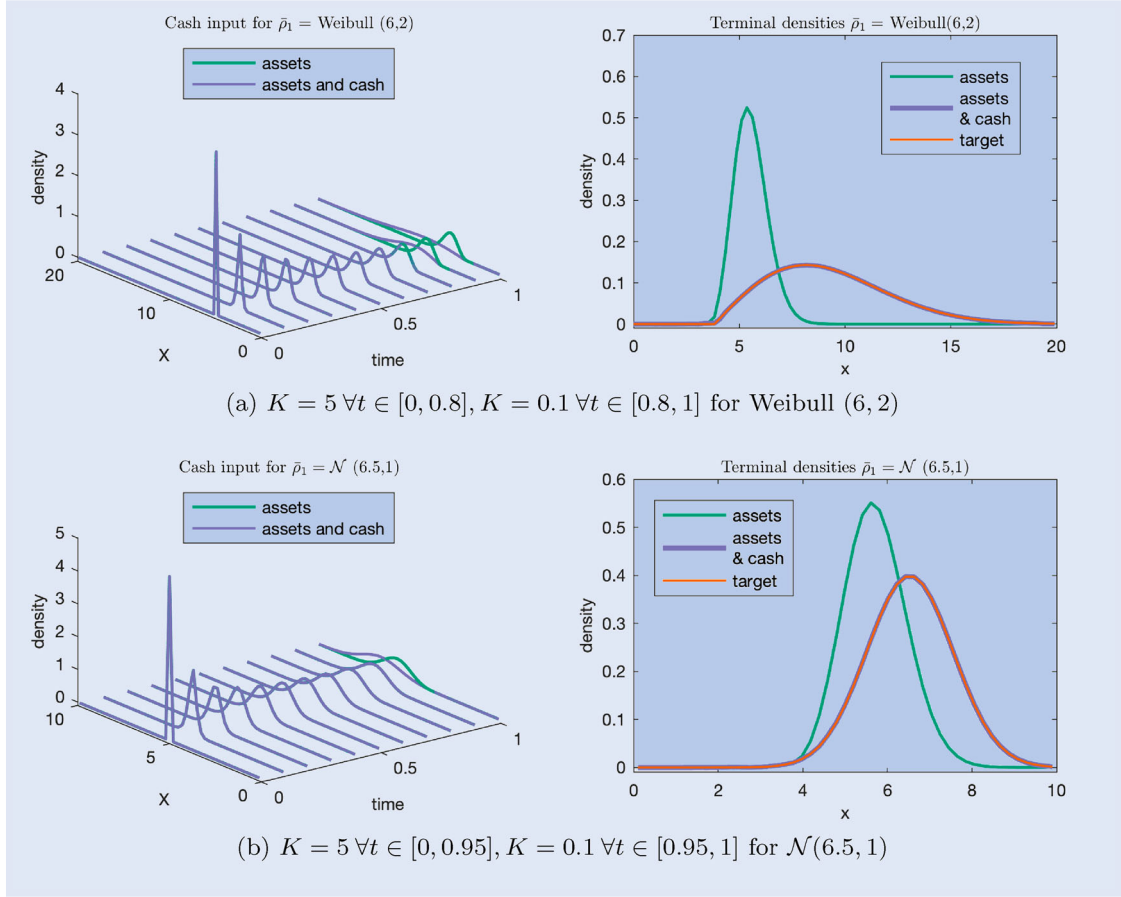
Figure 7. $K(t)$ for unattainable targets. (a) $K = 5 \, \forall t \in [0, 0.8, K = 0.1 \, \forall t \in [0.8, 1]$ for Weibull $(6, 2)$ and (b) $K = 5 \, \forall t \in [0, 0.95, K = 0.1 \, \forall t \in [0.95, 1]$ for $\mathcal{N}(6.5, 1)$.

this context. Since the target $\mathcal{N}(6, 1)$ is attainable, we can still reach it even without cash input, as shown in the second plot of figure 6(c).

5.3.2. Unattainable target. To see the effect of cash input, here we demonstrate an example with an unattainable target. For instance, we may target at a terminal distribution with no left tail but a heavy right tail, in other words, there is very little risk for the wealth to fall below some level. Therefore, we set $\bar{\rho}_1 = Weibull\,(6, 2)$ in figure 7(a), where $P(x < 4)$ is almost zero. In this example, the coefficient K in (43) is not a constant anymore. Instead, we let $K(t) : [0, 1] \to \mathbb{R}^+$ be a function of time so that we can control the cash input flow over time. We define $K(t) = 5$ for $t \in [0, 0.8]$ and $K(t) = 0.1$ for $t \in [0.8, 1]$. In the time-evolution plot (the left one of figure 7(a)), we can see that the paths for *assets* and *assets and cash* start to differentiate from $t = 0.8$. Similarly, we can see the same effect in figure 7(b), where we set $\bar{\rho}_1 = \mathcal{N}(6.5, 1)$ and we define $K(t) = 5$ for $t \in [0, 0.95]$ and $K(t) = 0.1$ for $t \in [0.95, 1]$. In these two examples, the targets $Weibull\,(6, 2)$ and $\mathcal{N}(6.5, 1)$ are unattainable under the constraint $(\tilde{B}^+)^2 \le \|\nu(t, x)\|^2 \tilde{A}$. However, we can make the empirical terminal density ρ_1 reach $\bar{\rho}_1$ by inputting cash wisely.

6. Conclusion

The ability to specify the whole portfolio final wealth distribution as a portfolio optimization target gives greater flexibility over classical objective functions such as expected utility, or moment-based objectives such as mean-variance and its extensions. In this article, we construct a portfolio and the dynamics of the portfolio wealth as a semimartingale. Starting from an initial wealth, by controlling the portfolio allocation process we are able to steer the portfolio wealth to a prescribed distribution at the terminal time. This problem is closely related to optimal mass transport (OMT). In the problem formulation, in addition to the conventional cost function in OMT, we also design a penalty functional to measure the divergence of the empirical terminal density from the prescribed one. We take into consideration possible consumption during the investment process, and show that we can actually reach a better terminal density when there is no consumption. When the target density is attainable, our problem can recover the classical OMT problem by choosing an indicator function as the penalty function. When the target terminal density is unattainable by the self-financing portfolio, we devise a strategy to reach it by allowing cash input during the investment process. We proved a duality result for the primal problem

and solved it with a gradient descent based algorithm. Our numerical results verify the accuracy and validity of this algorithm.

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No potential conflict of interest was reported by the author(s).

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Appendix

A.1.

Proof of Proposition 2.1 Firstly, we prove the necessity. We can use eigen-decomposition and write the covariance matrix $\Sigma(t, x) = Q\Lambda Q^{\mathsf{T}} = Q\Lambda^{\frac{1}{2}}\Lambda^{\frac{1}{2}}Q^{\mathsf{T}}$, where $Q \in \mathbb{R}^{d \times d}$ and the ith column of Q is the eigenvector q_i of $\Sigma(t, x)$, and $\Lambda \in \mathbb{R}^{d \times d}$ is the diagonal matrix whose diagonal elements are the corresponding eigenvalues, $\Lambda_{ii} = \lambda_i$.

For any given $\alpha(t, x) \in \mathbb{R}^d$, $A = \alpha(t, x)^{\mathsf{T}}Q\Lambda^{\frac{1}{2}}\Lambda^{\frac{1}{2}}Q^{\mathsf{T}}\alpha(t, x)x^2\rho$. We define $\beta := \alpha(t, x)^{\mathsf{T}}Q\Lambda^{\frac{1}{2}}$, then $A = \beta\beta^{\mathsf{T}}x^2\rho = \|\beta\|^2x^2\rho$, where $\|\cdot\|$ denotes the L^2 norm. Similarly,

$B = \alpha(t,x)^\mathsf{T} Q \Lambda^{\frac{1}{2}} (Q\Lambda^{\frac{1}{2}})^{-1} \mu(t,x) x\rho = \beta (Q\Lambda^{\frac{1}{2}})^{-1} \mu(t,x) x\rho$. Therefore we have the relationship between A and B as

$$B^2 = \left(\beta(Q\Lambda^{\frac{1}{2}})^{-1}\mu(t,x)\right)^2 x^2 \rho^2$$

$$\leq \|\beta\|^2 \left\|(Q\Lambda^{\frac{1}{2}})^{-1}\mu(t,x)\right\|^2 x^2\rho^2 = A\rho \left\|(Q\Lambda^{\frac{1}{2}})^{-1}\mu(t,x)\right\|^2,$$

$$\left\|(Q\Lambda^{\frac{1}{2}})^{-1}\mu(t,x)\right\|^2 A \geq \frac{B^2}{\rho}.$$

Define $\nu(t,x) := (Q\Lambda^{\frac{1}{2}})^{-1}\mu(t,x) = \Sigma(t,x)^{-\frac{1}{2}}\mu(t,x)$, we can write the above inequality as $A \geq \frac{B^2}{\|\nu(t,x)\|^2\rho}$.

For given (ρ, B, A) satisfying $A \geq \frac{B^2}{\|\nu(t,x)\|^2\rho}$, we want to show that there exists $\alpha(t,x) \in \mathbb{R}^d (d > 1)$, such that $A = \alpha(t,x)^\mathsf{T} \Sigma(t,x)\alpha(t,x)x^2\rho$, $B = \alpha(t,x)^\mathsf{T}\mu(t,x)x\rho$. First of all, since $\frac{A}{\rho} \geq 0$, there exists a vector $\beta \in \mathbb{R}^{1\times d}$ whose norm satisfies $\|\beta\|^2 = \frac{A}{x^2\rho}$. Then $\frac{B^2}{\|\nu(t,x)\|^2\rho} \leq A$ will be equivalent to $B^2 \leq \|\beta\|^2\|\nu(t,x)\|^2 x^2\rho^2$. With Cauchy–Schwarz inequality, $(\beta\nu(t,x))^2 x^2 \rho^2 \leq \|\beta\|^2\|\nu(t,x)\|^2 x^2\rho^2$ holds. Therefore there exists a vector $\beta \in \mathbb{R}^{1\times d}(d > 1)$ such that $B = \beta\nu(t,x)x\rho$ and $\|\beta\|^2 = \frac{A}{x^2\rho}$. With this β, there exists an $\alpha(t,x) = Q\Lambda^{-\frac{1}{2}}\beta^\mathsf{T}$.

The case for dimension $d = 1$ is trivial, hence omitted here. ∎

A.2.

PROPOSITION A.1 *We denote K_0 the set of $(u,b,a,r) \in (C_b(\mathcal{E}))^4$ that can be represented by $\phi \in C_b^{1,2}(\mathcal{E})$. Then we have*

$$\inf_{(\rho,B,A,\rho_1)\in(C_b^*(\mathcal{E}))^4} \left\{\alpha^*(\rho,B,A,\rho_1) + \beta^*(\rho,B,A,\rho_1)\right\}$$

$$= \inf_{(\rho,B,A,\rho_1)\in(\mathcal{M}(\mathcal{E}))^4} \left\{\alpha^*(\rho,B,A,\rho_1) + \beta^*(\rho,B,A,\rho_1)\right\}.$$

Proof Following closely the argument in Villani ([2003](#), Section 1.3), we define $C_0(\mathcal{E})$ the space of continuous functions on $\mathcal{E}$, going to 0 at infinity. For $(\rho,B,A,\rho_1) \in (C_b^*(\mathcal{E}))^4$, we decompose $(\rho,B,A,\rho_1) = (\hat{\rho},\hat{B},\hat{A},\hat{\rho}_1) + (\delta\rho,\delta B,\delta A,\delta\rho_1)$, where $(\hat{\rho},\hat{B},\hat{A},\hat{\rho}_1) \in (\mathcal{M}(\mathcal{E}))^4$. For any $(u,b,a,r) \in (C_0(\mathcal{E}))^4$, we have $\langle (u,b,a,r),(\delta\rho,\delta B,\delta A,\delta\rho_1)\rangle = 0$.

Because $\mathcal{M}(\mathcal{E};\mathbb{R})$ is a subset of $C_b^*(\mathcal{E};\mathbb{R})$, we naturally have

$$\inf_{(\rho,B,A,\rho_1)\in(C_b^*(\mathcal{E}))^4} \left\{\alpha^*(\rho,B,A,\rho_1) + \beta^*(\rho,B,A,\rho_1)\right\}$$

$$\leq \inf_{(\rho,B,A,\rho_1)\in(\mathcal{M}(\mathcal{E}))^4} \left\{\alpha^*(\rho,B,A,\rho_1) + \beta^*(\rho,B,A,\rho_1)\right\}. \quad (A1)$$

Now we look at the opposite direction of inequality (A1). For α^*, we have

$$\alpha^*(\rho,B,A,\rho_1)$$

$$= \sup_{(u,b,a,r)\in(C_b(\mathcal{E}))^4} \left\{\int_{\mathcal{E}} u\,\mathrm{d}\rho + b\,\mathrm{d}B + a\,\mathrm{d}A \right.$$

$$\left. + \left[\int_{\mathbb{R}} r\,\mathrm{d}\rho_1 - C^*(r)\right] : u + F^*(b,a) \leq 0\right\}$$

$$\geq \sup_{(u,b,a,r)\in(C_0(\mathcal{E}))^4} \left\{\int_{\mathcal{E}} u\,\mathrm{d}\rho + b\,\mathrm{d}B + a\,\mathrm{d}A \right.$$

$$\left. + \left[\int_{\mathbb{R}} r\,\mathrm{d}\rho_1 - C^*(r)\right] : u + F^*(b,a) \leq 0\right\}$$

$$= \sup_{(u,b,a,r)\in(C_0(\mathcal{E}))^4} \left\{\int_{\mathcal{E}} u\,\mathrm{d}\hat{\rho} + b\,\mathrm{d}\hat{B} + a\,\mathrm{d}\hat{A} \right.$$

$$\left. + \left[\int_{\mathbb{R}} r\,\mathrm{d}\hat{\rho}_1 - C^*(r)\right] : u + F^*(b,a) \leq 0\right\}$$

$$= \alpha^*(\hat{\rho},\hat{B},\hat{A},\hat{\rho}_1).$$

We know $\beta^*(\hat{\rho},\hat{B},\hat{A},\hat{\rho}_1) = 0$ if $(\hat{\rho},\hat{B},\hat{A},\hat{\rho}_1)$ satisfies (21), and $\beta^*(\hat{\rho},\hat{B},\hat{A},\hat{\rho}_1) = +\infty$ otherwise. When β^* is finite,

$$\int_{\mathcal{E}} u\,\mathrm{d}\hat{\rho} + b\,\mathrm{d}\hat{B} + a\,\mathrm{d}\hat{A} + \int_{\mathbb{R}} r\,\mathrm{d}\hat{\rho}_1 - \phi_0\,\mathrm{d}\rho_0$$

$$= 0 \quad \forall\,(u,b,a,r) \in K_0.$$

Then we have

$$\beta^*(\hat{\rho},\hat{B},\hat{A},\hat{\rho}_1)$$

$$= \sup_{(u,b,a,r)\in C_0\cap K_0} \int_{\mathcal{E}} u\,\mathrm{d}\hat{\rho} + b\,\mathrm{d}\hat{B} + a\,\mathrm{d}\hat{A} + \int_{\mathbb{R}} r\,\mathrm{d}\hat{\rho}_1 - \phi_0\,\mathrm{d}\rho_0$$

$$= \sup_{(u,b,a,r)\in C_0\cap K_0} \int_{\mathcal{E}} u\,\mathrm{d}\rho + b\,\mathrm{d}B + a\,\mathrm{d}A + \int_{\mathbb{R}} r\,\mathrm{d}\rho_1 - \phi_0\,\mathrm{d}\rho_0$$

$$\leq \sup_{(u,b,a,r)\in K_0} \int_{\mathcal{E}} u\,\mathrm{d}\rho + b\,\mathrm{d}B + a\,\mathrm{d}A + \int_{\mathbb{R}} r\,\mathrm{d}\bar{\rho}_1 - \phi_0\,\mathrm{d}\rho_0$$

$$= \beta^*(\rho,B,A,\rho_1).$$

This completes the proof. ∎