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When to efficiently rebalance a portfolio

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A constant weight asset allocation is a popular investment strategy and is optimal under a suitable continuous model. We study the tracking error for the target continuous rebalancing strategy by a feasible discrete-in-time rebalancing under a general multi-dimensional Brownian semimartingale model of asset prices. In a high-frequency asymptotic framework, we derive an asymptotically efficient sequence of simple predictable strategies.

Keywords: Discretization of stochastic integrals; Asymptotic analysis; Constant weight asset allocation; Impulse control; Pearson's inequality

1. Introduction

Consider a multi-dimensional risky asset $S = (S^1, \dots, S^d)^\top$ and a risk-free asset S^0 with

$$\frac{\mathrm{d}S_t^i}{S_t^i} = \mu_t^i\,\mathrm{d}t + \sum_{j=1}^m \sigma_t^{ij}\,\mathrm{d}W_t^j, \quad \frac{\mathrm{d}S_t^0}{S_t^0} = \mu_t^0\,\mathrm{d}t \tag{1}$$

where $(W^1, \dots, W^m)$ is an m-dimensional standard Brownian motion, and μ^i and σ^{ij} are locally bounded adapted processes with

$$\Sigma_t = [\Sigma_t^{ij}], \quad \Sigma_t^{ij} := \sum_{k=1}^m \sigma_t^{ik}\sigma_t^{jk}$$

being positive definite for all $t \geq 0$. For any d dimensional locally bounded adapted process $\pi = (\pi^1, \dots, \pi^d)^\top$ and a locally bounded adapted process c, the equation

$$\frac{\mathrm{d}V_t}{V_t} = \sum_{i=0}^d \pi_t^i \frac{\mathrm{d}S_t^i}{S_t^i} - c_t\,\mathrm{d}t \tag{2}$$

describes the dynamics of the wealth process V associated with a self-financing strategy under the consumption plan c and the admissibility constraint $V > 0$, where $\pi^0 = 1 - \sum_{i=1}^d \pi^i$. For each i, π^i represents the ratio of the wealth invested in S^i to the total wealth V.

A constant weight asset allocation refers to such an investment strategy that π is kept constant in time, and appears as, for example, the growth optimal portfolio strategy when

$$\theta := \Sigma^{-1}(\mu - r)$$

is constant, where $\mu = (\mu^1, \dots, \mu^d)^\top$ and $r = \mu^0(1, \dots, 1)^\top$. Indeed,

$$\log V_T = \log V_0 + \int_0^T \frac{\mathrm{d}V_t}{V_t} - \frac{1}{2}\int_0^T \pi_t^\top \Sigma_t \pi_t\,\mathrm{d}t$$

and so, under a suitable admissibility condition,

$$\mathsf{E}\left[\log \frac{V_T}{V_0}\right] = \int_0^T \mathsf{E}\left[-\frac{1}{2}(\pi_t - \theta_t)^\top \Sigma_t (\pi_t - \theta_t) + \frac{1}{2}\theta_t^\top \Sigma_t \theta_t + \mu_t^0 - c_t\right]\mathrm{d}t,$$

which is maximized by $\pi = \theta$. The simplest concrete model with constant θ is the Black-Scholes model, where μ, r and Σ are constant. Under the Black-Scholes model, the optimal strategy of the consumption and investment problem is known to be proportional to the constant vector θ under power utilities (Karatzas and Shreve 1998), or more generally, the Epstein-Zin stochastic differential utilities (Kraft *et al.* 2017), or even under relative performance criteria (Lacker and Zariphopoulou 2019). Also under model uncertainty, the superiority of an equal-weighted portfolio, also known as the 1/*N* portfolio, has been documented in the literature (e.g. DeMiguel *et al.* 2009). Beyond these theoretical frameworks, a constant weight asset allocation has been popular in the asset management industry, dating back to Talmud (1200 BC–500

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AD) (Gibson 2007). In this paper, we assume a continuous-time constant weight strategy (π, c) to be given for whatever reason and consider how to implement it under a general Brownian semimartingale model (1) and (2).

Denote by $H = (H^1, \ldots, H^d)^\top$, $H^i := V\pi^i/S^i$, the numbers of shares associated with the asset allocation strategy π. Notice that H is not of finite variation even though π is so. Indeed, we see in section 2 that the quadratic covariation of H is nondegenerate. Now the question is how to implement H in reality, where a continuous adjustment of portfolio is infeasible. Asset re-allocations have to be discrete in time and should be as less frequent as possible to avoid various kind of costs. Then the question is when and how to rebalance a portfolio efficiently.

Finding an efficient discrete-in-time rebalancing strategy amounts to finding an efficient approximation to a stochastic integral by one with a simple predictable integrand. In the case of $d = 1$, an asymptotically efficient sequence of simple predictable approximations was derived in (Fukasawa 2011a, 2011b, 2014). An extension to the multi-dimensional case in a hedging context was given by Gobet and Landon (2014), which however does not cover investment strategies such as constant weight asset allocations. In this paper, we give an extension to this missing direction. Further, in contrast to Gobet and Landon (2014), we do not restrict candidate strategies to discretization schemes but discuss asymptotic efficiency in a broader class of simple predictable strategies. From a mathematical point of view, this extension involves a novel inequality for centered moments of a general random vector that generalizes Pearson's inequality for one-dimensional kurtosis and skewness.

For the multi-dimensional Black-Scholes model, an asymptotic analysis of the optimal consumption investment problem under fixed transaction costs was given in Altarovici *et al.* (2015). Under the fixed transaction costs, the number of rebalancing penalizes the total wealth. The asymptotic solution of Altarovici *et al.* (2015) is a discretization of the Merton portfolio, a constant weight strategy that is optimal in the frictionless market, by a sequence of stopping times. Although our optimization problem is different from Altarovici *et al.* (2015), our solution has a similar structure to that of Altarovici *et al.* (2015), obtained by solving the same algebraic Riccati equation.

In section 2, we compute the quadratic covariation $\langle H, H\rangle$ of H when π is positive. We observe that the covariation matrix is nondegenerate under (2) with (1) if $\pi^i > 0$ for all $i = 1, \ldots, d$ and $\pi^0 \neq 0$. In section 3, we state our main result relying on the nondegeneracy condition on H under a more abstract framework of continuous semimartingales than (1) and (2). In section 4, we derive an asymptotically efficient strategy and discuss the efficiency loss of the equidistant discretization. In section 5, we observe from numerical experiments that the asymptotically efficient strategy is indeed effective in practical situations. In section 6, we give the proof of the main theorem stated in section 2. In section 7, we prove an inequality for centered moments of a general random vector that generalizes Pearson's inequality for one-dimensional kurtosis and skewness.

2. The structure of the continuous strategy

Here we compute the quadratic covariations of the process $H = (H^1, \ldots, H^d)^\top$, $H^i = V\pi^i/S^i$, which plays a key role in our analysis in the next section. Let $\{e_i\}_{i=1}^d$ denote the standard basis of $\mathbb{R}^d$, $I = (e_1, \ldots, e_d)$ denote the $d \times d$ identity matrix, $\mathbf{1} = \sum_{i=1}^d e_i = (1, \ldots, 1)^\top$, and $\operatorname{diag}(H)$ denote the $d \times d$ diagonal matrix with diagonal elements H, that is,

$$e_i^\top \operatorname{diag}(H) e_j = H^i e_i^\top e_j.$$

Lemma 1 *Assume π^i to be a positive constant for each $i = 1, \ldots, d$. Under* (1) *and* (2),

$$\mathrm{d}\langle H, H\rangle_t = J_t \, \mathrm{d}t, \tag{3}$$

where

$$J = \operatorname{diag}(H)(\pi\mathbf{1}^\top - I)^\top \Sigma (\pi \mathbf{1}^\top - I) \operatorname{diag}(H). \tag{4}$$

Further, $\det J_t \neq 0$ *for all* $t \geq 0$ *if and only if* $\pi^0 \neq 0$.

Proof Recall that $H^i = V\pi^i/S^i$, so that

$$\begin{aligned} \mathrm{d}\langle \log H^i, \log H^j\rangle_t &= \mathrm{d}\langle \log V - \log S^i, \log V - \log S^j\rangle_t \\ &= (\pi - e_i)^\top \Sigma_t (\pi - e_j) \, \mathrm{d}t, \end{aligned}$$

Therefore,

$$\mathrm{d}\langle H^i, H^j\rangle_t = (U_t^i)^\top U_t^j \, \mathrm{d}t, \quad U_t^i = H_t^i \Sigma_t^{1/2} (\pi - e_i),$$

which implies (3) noting that $J = (U)^\top U$, $U = (U^1, \ldots, U^d)$.

Since Σ is assumed to be positive definite, it is clear that $\det J_t \neq 0$ if and only if $\det(\pi\mathbf{1}^\top - I) \neq 0$. For any $a \in \mathbb{R}^d$, $a^\top(\pi\mathbf{1}^\top - I) = (a^\top \pi)\mathbf{1}^\top - a^\top$. Therefore the column vectors of $\pi\mathbf{1}^\top - I$ is linearly dependent if and only if $\mathbf{1}^\top \pi = 1$. Therefore $\det(\pi\mathbf{1}^\top - I) \neq 0$ if and only of $\pi^0 = 1 - \mathbf{1}^\top \pi \neq 0$. ■

3. The main result

Here we give a mathematical formulation of the problem and then state our main result. Let $(\Omega, \mathcal{F}, \mathsf{P}, \{\mathcal{F}_t\}_{t \in [0,1]})$ be a filtered probability space satisfying the usual assumptions. A simple predictable process is a stochastic process of the form

$$X = \sum_{i=0}^{\infty} \xi_i 1_{((\tau_i, \tau_{i+1}]]},$$

where $\{\tau_i\}_{i \geq 0}$ is a nondecreasing sequence of stopping times taking values in $[0, 1]$ and ξ_i is an $\mathcal{F}_{\tau_i}$ measurable d-dimensional random variable. For X of the above form and for a d-dimensional continuous semimartingale S, the stochastic integral $X \cdot S$ is defined by

$$(X \cdot S)_t = \sum_{i=0}^{\infty} \xi_i^\top (S_{\tau_{i+1} \wedge t} - S_{\tau_i \wedge t}).$$

For given d-dimensional continuous semimartingales H and S, we are interested in an efficient approximation to the stochastic integral $H \cdot S$ by a sequence $X^n \cdot S$, where X^n are simple predictable processes.

We say an adapted process X is locally bounded if there is a nondecreasing sequence of $[0, 1]$-valued stopping times $\{\tau_i\}$ such that the stopped process X^{τ_i} is bounded for each i and that for each $\omega \in \Omega$ there exists $N(\omega) \in \mathbb{N}$ such that $\tau_{N(\omega)}(\omega) = 1$. Denote by $\mathcal{S}_d^{\geq}$ and $\mathcal{S}_d^{>}$ respectively the sets of $d \times d$ nonnegative definite matrices and positive definite matrices.

Assumption 1 *There exist an $\mathcal{S}_d^{>}$-valued continuous adapted process J, an $\mathcal{S}_d^{\geq}$-valued continuous adapted process K, and a continuous nondecreasing adapted process A such that*

$$\mathrm{d}\langle H, H\rangle = J\,\mathrm{d}A, \quad \mathrm{d}\langle S, S\rangle = K\,\mathrm{d}A.$$

The finite variation part of H is absolutely continuous with respect to A and the associated Radon-Nikodym derivative is locally bounded.

Under (1) with (2), by lemma 1, for a constant weight asset allocation strategy π with $\pi^i > 0$ for $i = 1, \ldots, d$ and $\pi^0 \neq 0$, assumption 1 is satisfied with $A_t = t$ and $K = \mathrm{diag}(S)\Sigma\,\mathrm{diag}(S)$. It is however violated when, say, $\pi^0 = 0$, since the matrix J given by (4) becomes singular. Note that assumption 1 is stable against a continuous time-change. This means that if the assumption is true for (H, S), it remains true for the time-changed process (H_A, S_A) for any continuous non-decreasing process A. This time-change process A is not necessarily absolutely continuous and so, the time-changed model S_A can represent, for example, a multi-dimensional version of the hyper rough Heston model (Jusselin and Rosenbaum 2020).

For positive continuous adapted processes Q and N fixed and for a simple predictable process X, we introduce the cost functionals $Q[X]$ and $N[X]$ respectively of approximation error and of approximation effort as

$$Q[X] = \int_0^1 Q_t\,\mathrm{d}\langle H \cdot S - X \cdot S\rangle_t,$$
$$N[X] = \sum_{t \in (0,1)} N_t 1_{\{|\Delta X_t| \neq 0\}}.$$

In particular, if $N = 1$ then $N[X]$ counts the number of jumps of X, that is, the number of rebalancing in our financial context, and if Q is the density process of an equivalent martingale measure Q for S then $\mathsf{E}[Q[X]] = \mathsf{E}_{\mathsf{Q}}[(H \cdot S - X \cdot S)_1^2]$. Note that the expected approximation error $\mathsf{E}[Q[X]]$ can be arbitrarily made small by taking X sufficiently close to H, while it inevitably makes the expected approximation effort $\mathsf{E}[N[X]]$ large because H has a nondegenerate quadratic variation. We then seek an efficient frontier for the trade-off between $\mathsf{E}[Q[X]]$ and $\mathsf{E}[N[X]]$. We take an asymptotic approach to have an explicit solution.

Definition 1 *We say a sequence of simple predictable processes X^n is admissible if*

1. *X^n is locally bounded for each n,*
2. $\sup_{t\in[0,1]} |X_t^n - H_t| \to 0$ *in probability as $n \to \infty$,*
3. $\mathsf{E}[Q[X^n]] < \infty$ *and* $\frac{Q[X^n]}{\mathsf{E}[Q[X^n]]}$ *is uniformly integrable.*

Now we state our main result, of which the proof is deferred to section 6.

Theorem 1 *Let H and S be d-dimensional continuous semimartingales satisfying assumption 1, and let Q and N be positive continuous adapted processes. Then, for any admissible sequence X^n,*

$$\varliminf_{n\to\infty} \mathsf{E}[N[X^n]]\mathsf{E}[Q[X^n]] \geq \mathsf{E}\left[\int_0^1 N_t^{1/2} Q_t^{1/2} \mathrm{tr}(L_t J_t)\,\mathrm{d}A_t\right]^2, \tag{5}$$

where $L = \ell(J, K)$ and ℓ is the solution map given in lemma 2.

Lemma 2 *For any $J \in \mathcal{S}_d^{>}$ and $K \in \mathcal{S}_d^{\geq}$, there exists a unique $L = \ell(J, K) \in \mathcal{S}_d^{\geq}$ such that*

$$2\,\mathrm{tr}(LJ)L + 4LJL = K.$$

Further, the map ℓ is continuous on $\mathcal{S}_d^{>} \times \mathcal{S}_d^{\geq}$, and $L \in \mathcal{S}_d^{>}$ if $K \in \mathcal{S}_d^{>}$.

Lemma 2 is a straightforward extension of Lemma 3.1 of Gobet and Landon (2014) that dealt with the case that J is the identity. The proof of lemma 2 reduces to that case by considering $\tilde{L} := J^{1/2} L J^{1/2}$ and so omitted. This algebraic Riccati equation first appeared in Atkinson and Wilmott (1995) to describe an approximate solution to the variational inequality for an optimal consumption investment problem under the Black-Scholes model with fixed-type transaction costs. The existence of the solution with an efficient computational algorithm was given in Atkinson and Wilmott (1995). More specifically, it is given by $L = J^{-1/2}\tilde{L}J^{-1/2}$ with

$$\tilde{L} = P\,\mathrm{diag}(\lambda_1, \ldots, \lambda_d)P^\top,$$

where the matrix P diagonalizes K as

$$K = P\,\mathrm{diag}(k_1, \ldots, k_d)P^\top, \quad PP^\top = I,$$

$k_j, j = 1, \ldots, d$, are the eigenvalues of Σ, and

$$\lambda_j = \tfrac{1}{4}\left(-t + \sqrt{t^2 + 4k_j}\right) \tag{6}$$

for the unique solution t of the equation

$$(d + 4)t = \sum_{j=1}^d \sqrt{t^2 + 4k_j}.$$

The same algebraic Riccati equation naturally appeared in Altarovici *et al.* (2015).

Remark 1 Theorem 1 extends 1-dimensional results in Fukasawa (2011a, 2014). A related central limit theorem with a pathwise version of (5) for the 1-dimensional case is given in Fukasawa (2011b). Multi-dimensional extensions of the pathwise version are given in Gobet and Landon (2014), Gobet and Stazhynski (2018), where the integrand H is assumed to be of the form $H_t = v(t, S_t)$ and the sequence X^n is assumed to be of the form $X_t^n = H_{\tau_j^n}$ for $t \in [\tau_j^n, \tau_{j+1}^n)$ for a sequence of stopping times $\{\tau_j^n\}$. In Gobet and Stazhynski (2018), the covariation of $H_t = v(t, S_t)$ is allowed to be degenerate in contrast to assumption 1.

4. Efficient and inefficient strategies

4.1. An asymptotically efficient sequence

Here we show that the sequence

$$X^n = \sum_{i=0}^{\infty} \xi_i^n 1_{((\tau_i^n, \tau_{i+1}^n]]} \tag{7}$$

defined by

$$\xi_j^n = H_{\tau_j^n},$$

$$\tau_{j+1}^n = \inf\{t > \tau_j^n;\ (H_t - \xi_j^n)^\top L_{\tau_j^n}(H_t - \xi_j^n) = \epsilon_n Q_{\tau_j^n}^{-1/2} N_{\tau_j^n}^{1/2}\} \wedge 1 \tag{8}$$

and $\tau_0^n = 0$ with a deterministic positive sequence ϵ_n with $\epsilon_n \to 0$ as $n \to \infty$ is asymptotically efficient, where $L = \ell(J, K)$ and ℓ is the solution map given in lemma 2.

Theorem 2 *Let H and S be d-dimensional continuous semi-martingales satisfying assumption* 1 *with K being $\mathcal{S}_d^>$-valued, and let Q and N be positive continuous adapted processes. For the sequence X^n defined by* (7) *with* (8)*, we have*

$$\epsilon_n^{-1} Q[X^n] \to \int_0^1 N_t^{1/2} Q_t^{1/2} \operatorname{tr}(L_t J_t)\, \mathrm{d}A_t \tag{9}$$

and

$$\epsilon_n N[X^n] \to \int_0^1 N_t^{1/2} Q_t^{1/2} \operatorname{tr}(L_t J_t)\, \mathrm{d}A_t \tag{10}$$

in probability as $n \to \infty$.

Proof We are going to apply Itô's formula to the function $f(x) = (x^\top L x)^2$ for a $d \times d$ matrix L and $x \in \mathbb{R}^d$. Note that

$$\nabla f(x) = 4\sqrt{f(x)} L x, \quad \nabla^2 f(x) = 4\sqrt{f(x)} L + 8(Lx)(Lx)^\top.$$

Now, by Itô's formula,

$$\begin{aligned}
&((H_{\tau_{j+1}^n} - \xi_j^n)^\top L_{\tau_j^n}(H_{\tau_{j+1}^n} - \xi_j^n))^2 \\
&\quad = ((H_{\tau_j^n} - \xi_j^n)^\top L_{\tau_j^n}(H_{\tau_j^n} - \xi_j^n))^2 \\
&\qquad + 4\int_{\tau_j^n}^{\tau_{j+1}^n} (H_t - \xi_j^n)^\top L_{\tau_j^n}(H_t - \xi_j^n)(H_t - \xi_j^n)^\top L_{\tau_j^n}\, \mathrm{d}H_t \\
&\qquad + \int_{\tau_j^n}^{\tau_{j+1}^n} (H_t - \xi_j^n)^\top (2\operatorname{tr}(L_{\tau_j^n} J_t) L_{\tau_j^n} + 4 L_{\tau_j^n} J_t L_{\tau_j^n}) \\
&\qquad \times (H_t - \xi_j^n)\, \mathrm{d}A_t.
\end{aligned} \tag{11}$$

For (8),

$$\begin{aligned}
&((H_{\tau_{j+1}^n} - \xi_j^n)^\top L_{\tau_j^n}(H_{\tau_{j+1}^n} - \xi_j^n))^2 \\
&\quad = \epsilon_n Q_{\tau_j^n}^{-1/2} N_{\tau_j^n}^{1/2} (H_{\tau_{j+1}^n} - H_{\tau_j^n})^\top L_{\tau_j^n}(H_{\tau_{j+1}^n} - H_{\tau_j^n}), \\
&((H_{\tau_j^n} - \xi_j^n)^\top L_{\tau_j^n}(H_{\tau_j^n} - \xi_j^n))^2 = 0
\end{aligned}$$

and so,

$$\begin{aligned}
\epsilon_n^{-1} Q[X^n] &= \epsilon_n^{-1} \int_0^1 (H_t - X_t^n)^\top K_t (H_t - X_t^n) Q_t\, \mathrm{d}A_t \\
&= \sum_{j=0}^{\infty} Q_{\tau_j^n}^{1/2} N_{\tau_j^n}^{1/2} (H_{\tau_{j+1}^n} - H_{\tau_j^n})^\top L_{\tau_j^n}(H_{\tau_{j+1}^n} - H_{\tau_j^n}) \\
&\quad + E_1^n + E_2^n,
\end{aligned}$$

where

$$E_1^n = \epsilon_n^{-1} \sum_{j=0}^{\infty} \int_{\tau_j^n}^{\tau_{j+1}^n} (H_t - H_{\tau_j^n})^\top E_t^{n,j} (H_t - H_{\tau_j^n})\, \mathrm{d}A_t,$$

$$\begin{aligned}
E_2^n &= 4\epsilon_n^{-1} \sum_{j=0}^{\infty} \int_{\tau_j^n}^{\tau_{j+1}^n} (H_t - H_{\tau_j^n})^\top L_{\tau_j^n}(H_t - H_{\tau_j^n})(H_t - H_{\tau_j^n})^\top \\
&\qquad \times L_{\tau_j^n}\, \mathrm{d}H_t,
\end{aligned}$$

$$E_t^{n,j} = K_t^\top J_t K_t Q_t - (2\operatorname{tr}(L_{\tau_j^n} J_t) L_{\tau_j^n} + 4 L_{\tau_j^n} J_t L_{\tau_j^n}) Q_{\tau_j^n}.$$

Using Lemma 3.4 of Fukasawa (2014), we can show that $\sup_{j \geq 0}(\tau_{j+1}^n - \tau_j^n) \to 0$ as $n \to \infty$ in probability. Since J, K and Q are continuous and $L = \ell(J, K)$ with ℓ being continuous by lemma 2, we then have

$$\sup_{t \in [0,1], j \geq 0} |E_t^{n,j} 1_{\{\tau_j^n \leq t < \tau_{j+1}^n\}}| \to 0$$

in probability. Note also that

$$\sup_{t \in [0,1], j \geq 0} \epsilon_n^{-1} (H_t - H_{\tau_j^n})^\top (H_t - H_{\tau_j^n}) 1_{\{\tau_j^n \leq t < \tau_{j+1}^n\}} < \infty \tag{12}$$

under (8). These imply that $E_1^n \to 0$ in probability. We also have $E_2^n \to 0$ in probability because

$$\epsilon_n^{-2} \int_0^1 ((H_t - X_t^n)^\top (H_t - X_t^n))^3 \operatorname{tr}(J_t)\, \mathrm{d}A_t \to 0$$

in probability by (12) again, with the aid of the Lenglart inequality. Here, we also have used that the finite variation part of H is absolutely continuous with respect to A and the associated Radon-Nikodym derivative is locally bounded. We then conclude (9).

To see (10), observe that

$$\begin{aligned}
\epsilon_n N[X^n] &= \sum_{j=0}^{\infty} N_{\tau_{j+1}^n} Q_{\tau_j^n}^{1/2} N_{\tau_j^n}^{-1/2} (H_{\tau_{j+1}^n} - H_{\tau_j^n})^\top \\
&\quad \times L_{\tau_j^n}(H_{\tau_{j+1}^n} - H_{\tau_j^n}).
\end{aligned} \tag{13}$$

under (8). ■

Theorem 3 *Consider a constant weight asset allocation strategy under* (1) *and* (2)*. Assume that Q, N and the largest eigenvalue of Σ are bounded. Assume also that Q, N and the smallest eigenvalue of Σ are lower bounded away from 0. Then, the sequence X^n defined by* (7) *with* (8) *is admissible and attains the equality in* (5).

Proof Here we use C to denote a generic constant that does not depend on n but may vary line by line. First, we show that X^n is admissible. For each n, X^n is locally bounded because so is H. Since $K = \mathrm{diag}(S)\Sigma\,\mathrm{diag}(S)$, under (8), by lemma 3 below,

$$\sup_{t\in[0,1]} |H_t - X^n_t|^2 \le C\epsilon_n \max_{t\in[0,1],i=1,\dots,d} \frac{1}{|S^i_t|^2} \max_{t\in[0,1]} j_{\max}(t) \to 0,$$

where $j_{\max}(t)$ denotes the largest eigenvalue of J_t. By theorem 2, using Fatou's lemma, we have

$$\liminf_{n\to\infty} \epsilon_n^{-1}\mathsf{E}[Q[X^n]] \ge \mathsf{E}\left[\int_0^1 N_t^{1/2} Q_t^{1/2}\,\mathrm{tr}(L_t J_t)\,\mathrm{d}A_t\right] > 0.$$

Therefore, for the admissibility of X^n, it remains to show that $\epsilon_n^{-1}Q[X^n]$ is uniformly integrable. By lemma 3 again,

$$\begin{aligned}\epsilon_n^{-1}Q[X^n] &\le C \sup_{t\in[0,1]} |H_t - X^n_t|^2 \\ &\le C \max_{t\in[0,1],i=1,\dots,d} \frac{1}{|S^i_t|^2} \max_{t\in[0,1]} j_{\max}(t) \qquad (14)\end{aligned}$$

for all n. By (4),

$$j_{\max}(t) = \max_{x\neq 0} \frac{x^\top J_t x}{x^\top x} \le C \sup_{t\in[0,1]} |H_t|^2. \qquad (15)$$

Since $H^i = V\pi^i/S^i$ with (1) and (2), the right hand side of (14) is integrable.

Now, we are going to show that the equality is attained in (5). Since we have already seen that $\epsilon_n^{-1}Q[X^n]$ is uniformly integrable, we have

$$\lim_{n\to\infty} \epsilon_n^{-1}\mathsf{E}[Q[X^n]] = \mathsf{E}\left[\int_0^1 N_t^{1/2} Q_t^{1/2}\,\mathrm{tr}(L_t J_t)\,\mathrm{d}A_t\right]$$

by theorem 2. It remains then to show that

$$\lim_{n\to\infty} \epsilon_n\mathsf{E}[N[X^n]] = \mathsf{E}\left[\int_0^1 N_t^{1/2} Q_t^{1/2}\,\mathrm{tr}(L_t J_t)\,\mathrm{d}A_t\right]. \qquad (16)$$

By (13),

$$\begin{aligned}\epsilon_n N[X^n] &\le C \max_{t\in[0,1],i=1,\dots,d} \frac{1}{|S^i_t|^2} \max_{t\in[0,1]} j_{\max}(t) \\ &\quad\times \sum_{j=0}^{\infty} |H_{\tau^n_{j+1}} - H_{\tau^n_j}|^2.\end{aligned}$$

By (15), $\sup_{t\in[0,1]} j_{\max}(t)$ is L^p integrable for any $p > 1$. In order to show that $\epsilon_n N[X^n]$ is uniformly integrable, we are going to show

$$\mathsf{E}\left[\left(\sum_{j=0}^{\infty} |H_{\tau^n_{j+1}} - H_{\tau^n_j}|^2\right)^p\right] \le C$$

for some $p > 1$. There exists an equivalent measure P' to P such that H is a martingale under P' and $\mathrm{d}\mathsf{P}/\mathrm{d}\mathsf{P}'$ is L^q integrable under P' for any $q > 1$. We have

$$\begin{aligned}&\mathsf{E}\left[\left(\sum_{j=0}^{\infty} |H_{\tau^n_{j+1}} - H_{\tau^n_j}|^2\right)^{3/2}\right] \\ &\le \mathsf{E}'\left[\left(\frac{\mathrm{d}\mathsf{P}}{\mathrm{d}\mathsf{P}'}\right)^4\right]^{1/4} \mathsf{E}'\left[\left(\sum_{j=0}^{\infty} |H_{\tau^n_{j+1}} - H_{\tau^n_j}|^2\right)^2\right]^{3/4},\end{aligned}$$

where E' denotes the expectation under P'. By Lemma 1.7.3 of Nagai (1999), we have

$$\mathsf{E}'\left[\left(\sum_{j=0}^{\infty} |H_{\tau^n_{j+1}} - H_{\tau^n_j}|^2\right)^2\right] \le C\mathsf{E}'[|H_1 - H_0|^4] < \infty.$$

Thus we conclude that $\epsilon_n N[X^n]$ is uniformly integrable. Then (16) follows from theorem 2. ∎

LEMMA 3 *Let $\ell_{\min}$ and $k_{\min}$ denote the smallest eigenvalues of $L = \ell(J, K)$ and K respectively. Let $j_{\max}$ denote the largest eigenvalue of J. Then,*

$$\ell_{\min} \ge \frac{2}{1+\sqrt{17}} \frac{k_{\min}}{j_{\max}\mathrm{tr}(K^{1/2})}.$$

Proof By (6), we have

$$0 \le t = \sum_{j=1}^{d} \lambda_j = \sum_{j=1}^{d} \frac{k_j}{t + \sqrt{t^2 + 4k_j}} \le \frac{1}{2}\sum_{j=1}^{d} \sqrt{k_j} = \frac{\mathrm{tr}(K^{1/2})}{2}$$

and so that

$$\lambda_j = \frac{k_j}{t + \sqrt{t^2 + 4k_j}} \ge \frac{2}{1+\sqrt{17}} \frac{k_j}{\mathrm{tr}(K^{1/2})}.$$

Since

$$\ell_{\min} = \min_{x\neq 0} \frac{x^\top L x}{x^\top x} \ge \min_{y\neq 0} \frac{y^\top \tilde{L} y^\top}{y^\top y} \min_{x\neq 0} \frac{x^\top J^{-1} x}{x^\top x} = \frac{\lambda_{\min}}{j_{\max}},$$

we obtain the result. ∎

REMARK 2 Under (1) and (2),

$$V_t = V_0 \exp\left\{\sum_{i=0}^{d} \pi^i_u \frac{\mathrm{d}S^i_u}{S^i_u} - \int_0^t \left(c_u + \frac{1}{2}\pi_u^\top \Sigma_u \pi_u\right)\mathrm{d}u\right\}.$$

Notice that

$$H^i_{\tau^n_j} = \frac{\pi^i_{\tau^n_j}}{S^i_{\tau^n_j}} V_{\tau^n_j} = \hat{\xi}^{n,i}_j + \frac{\pi^i_{\tau^n_j}}{S^i_{\tau^n_j}}(V_{\tau^n_j} - V^n_{\tau^n_j}),$$

where $\hat{\xi}^{n,i}_j = \pi^i_{\tau^n_j} V^n_{\tau^n_j} / S^i_{\tau^n_j}$ is the number of share to invest $\pi^i_{\tau^n_j}$ portion of the total wealth

$$\begin{aligned}V^n_{\tau^n_j} &= V_0 + \int_0^{\tau^n_j} \frac{V^n_t - (X^n_t)^\top S_t}{S^0_t}\,\mathrm{d}S^0_t + \int_0^{\tau^n_j} (X^n_t)^\top\,\mathrm{d}S_t \\ &\quad - \int_0^{\tau^n_j} c_t\,\mathrm{d}t \qquad (17)\end{aligned}$$

in S^i at time τ^n_j.

4.2. The equidistant discretization

Here we compute the efficiency loss for the equidistant discretization strategy

$$\xi_j^n = H_{\tau_j^n}, \quad \tau_j^n = \frac{j}{n}$$

under the additional assumption that $A_t = t$.

THEOREM 4 *Let H and S be d-dimensional continuous semimartingales satisfying assumption 1 with $A_t = t$ for $t \in [0, 1]$ and J and K being h-Hölder continuous for some $h > 0$. Let Q and N be positive h-Hölder continuous adapted processes. Then,*

$$nQ[X^n] \to \int_0^1 Q_t(\mathrm{tr}(L_tJ_t)^2 + 2\mathrm{tr}(L_tJ_tL_tJ_t))\,\mathrm{d}t \tag{18}$$

and

$$n^{-1}N[X^n] \to \int_0^1 N_t\,\mathrm{d}t \tag{19}$$

in probability as $n \to \infty$.

Proof Under the additional assumption of $A_t = t$, we know that S and H are Brownian semimartingales and in particular their sample paths are $1/2 - \epsilon$ Hölder continuous almost surely for any $\epsilon > 0$. Therefore, using (11), we have

$$\begin{aligned} nQ[X^n] &= n\int_0^1 (H_t - X_t^n)^\top K_t(H_t - X_t^n)Q_t\,\mathrm{d}A_t \\ &= n\sum_{j=0}^{\infty} Q_{\tau_j^n}((H_{\tau_{j+1}^n} - H_{\tau_j^n})^\top L_{\tau_j^n}(H_{\tau_{j+1}^n} - H_{\tau_j^n}))^2 + E^n \end{aligned}$$

with E^n converging to 0 in probability. On the other hand, for $L, J \in \mathcal{S}_d^{\geq}$ and a Gaussian random vector $X = (X_1, \dots, X_d) \sim \mathcal{N}(0, J)$, we have

$$\begin{aligned} \mathsf{E}\left[\left(\sum_{i,j=1}^d X_iX_jL^{ij}\right)^2\right] &= \sum_{i,j,k,l=1}^d \mathsf{E}[X_iX_jX_kX_l]L^{ij}L^{kl} \\ &= \sum_{i,j,k,l=1}^d (\mathsf{E}[X_iX_j]\mathsf{E}[X_kX_l] + \mathsf{E}[X_iX_k]\mathsf{E}[X_jX_l] \\ &\quad + \mathsf{E}[X_iX_l]\mathsf{E}[X_kX_j])L^{ij}L^{kl} \\ &= \mathrm{tr}(LJ)^2 + 2\mathrm{tr}(LJLJ) \end{aligned}$$

by Isserlis' theorem. Then by a standard argument in the high-frequency data analysis (see e.g. Aït-Sahalia and Jacod 2014 or Jacod and Protter 2012), we obtain

$$\begin{aligned} &n\sum_{j=0}^{\infty} Q_{\tau_j^n}((H_{\tau_{j+1}^n} - H_{\tau_j^n})^\top L_{\tau_j^n}(H_{\tau_{j+1}^n} - H_{\tau_j^n}))^2 \\ &\quad\to \int_0^1 Q_t(\mathrm{tr}(L_tJ_t)^2 + 2\mathrm{tr}(L_tJ_tL_tJ_t))\,\mathrm{d}t \end{aligned}$$

in probability. Thus we conclude (18), while (19) is trivial. ■

The efficiency loss for the equidistant discretization can be decomposed into two parts. First,

$$\begin{aligned} &\mathsf{E}\left[\int_0^1 N_t^{1/2}Q_t^{1/2}\,\mathrm{tr}(L_tJ_t)\,\mathrm{d}t\right]^2 \\ &\quad\leq \mathsf{E}\left[\int_0^1 N_t\,\mathrm{d}t\right]\mathsf{E}\left[\int_0^1 Q_t\mathrm{tr}(L_tJ_t)^2\,\mathrm{d}t\right] \end{aligned}$$

by the Cauchy-Schwarz inequality. Second,

$$\begin{aligned} &\mathsf{E}\left[\int_0^1 Q_t\,\mathrm{tr}(L_tJ_t)^2\,\mathrm{d}t\right] \\ &\quad\leq \mathsf{E}\left[\int_0^1 Q_t\,\mathrm{tr}(L_tJ_t)^2\left(1 + \frac{2\mathrm{tr}(L_tJ_tL_tJ_t)}{\mathrm{tr}(L_tJ_t)^2}\right)\mathrm{d}t\right]. \end{aligned}$$

The loss from the first inequality is due to that the equidistant scheme does not take the time varying nature of $N^{1/2}Q^{1/2}\mathrm{tr}(LJ)$ into account. The loss from the second inequality is due to the use of deterministic time (or more generally, strongly predictable time; see Aït-Sahalia and Jacod 2014). Indeed, the factor $1 + 2/d$ for the case of LJ being the identity matrix coincides with the ratio of the asymptotic variance of the equidistant Euler-Maruyama scheme for discretizing stochastic differential equations to that of its hitting time counterpart given by Fukasawa and Obłój (2020).

5. Numerical experiments

In this section, we examine numerically the efficiency of the strategy (8) compared with the equidistant discretization under the Black-Scholes model.

5.1. 2-dimensional case

First, we consider the case $d = m = 2$. We have simulated 10 000 sample paths of $S = (S^1, S^2)^\top$ from (1) with the parameters

$$\mu^0 = 0, \quad S_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mu = \begin{pmatrix} 0.2 \\ 0.3 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0.1 & 0.1 \\ 0 & 0.2 \end{pmatrix}$$

with discretization size $\Delta t = 0.01$ on the time interval $[0, 1]$. We consider the growth optimal portfolio

$$\pi = \Sigma^{-1}(\mu - r) = \begin{pmatrix} 0.02 & 0.02 \\ 0.02 & 0.04 \end{pmatrix}^{-1}\begin{pmatrix} 0.2 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$

with $\pi^0 = -9$. We take $Q = N = 1$ for cost functions and $\epsilon_n = 0.1$. Our efficient strategy (8) is therefore

$$\begin{aligned} &\xi_j^n = H_{\tau_j^n}, \\ &\tau_{j+1}^n = \inf\{t > \tau_j^n;\ (H_t - \xi_j^n)^\top L_{\tau_j^n}(H_t - \xi_j^n) \geq 0.1\} \wedge 1 \end{aligned} \tag{20}$$

with $\tau_0^n = 0$, where t is limited on the grid $\{0, 0.01, 0.02, \dots, 0.99, 1\}$. Here, we numerically solve (2) with $c = 0$ and $V_0 =$

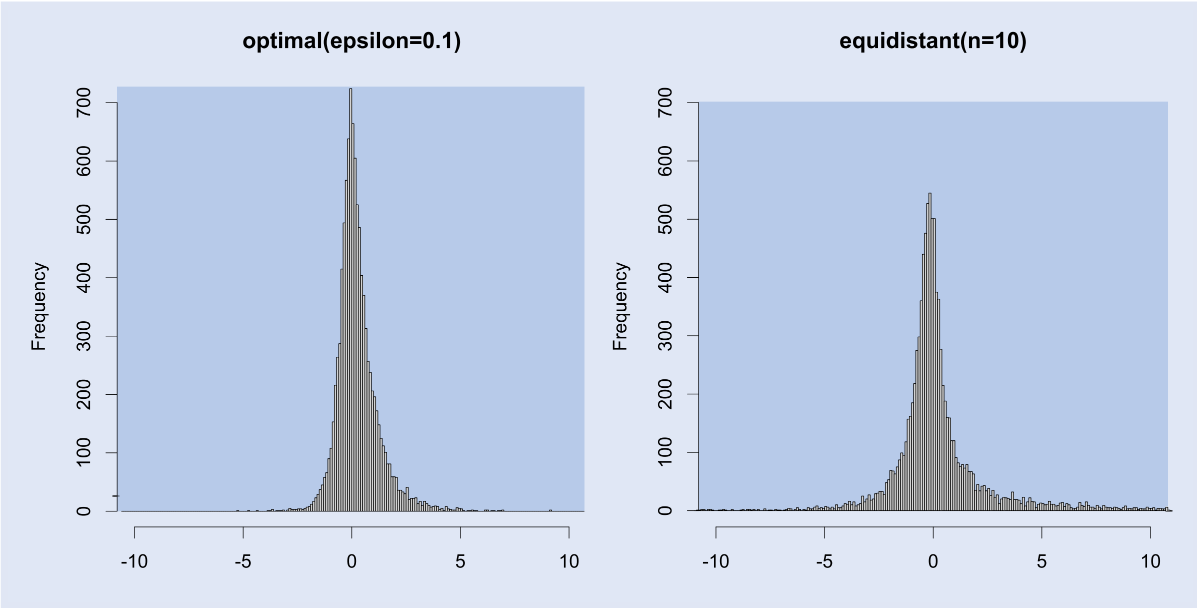

Figure 1. Histogram of $V - V^n$.

Table 1. The mean of the error $V - V^n$ and the mean squared error.

	Efficient	Equidistant
Mean	0.2543986	0.9792202
MSE	0.9824209	47.8588

1 pathwise (with the same discretization size $\Delta t = 0.01$ using the simulated sample paths of S) to compute the values of V and then H. For the computation of L, we follow the idea of Atkinson and Wilmott (1995), which we have already used in the proof of lemma 3. The size of ϵ_n controls how frequently we rebalance, and our choice of $\epsilon_n = 0.1$ has resulted in the average number of rebalancing 9.7612 for the simulated 10 000 sample paths. Therefore we also construct the equidistant discretization of H with the number of rebalancing $n = 10$ for the same sample paths for a fare comparison.

Figure 1 shows the histogram of the tracking error $V_1 - V_1^n$, where V_1^n is the terminal wealth associated with the efficient strategy (20) in the left figure, and it is with the equidistant discretization ($n = 10$) in the right figure. It is clearly seen that the tracking error is more concentrated around 0 and has a lighter tail for the efficient strategy (left). Table 1 shows the Monte-Carlo estimates of $\mathsf{E}[V_1 - V_1^n]$ and $\mathsf{E}[(V_1 - V_1^n)^2]$ from the 10 000 samples.

Figure 2 shows the histogram of the relative tracking error $(V_1 - V_1^n)/V_1$, where V_1^n is again, the terminal wealth associated with the efficient strategy (20) for the left figure, and with the equidistant discretization ($n = 10$) for the right figure. Although the relative error is not the objective function in our definition of efficiency, it is again observed that the relative

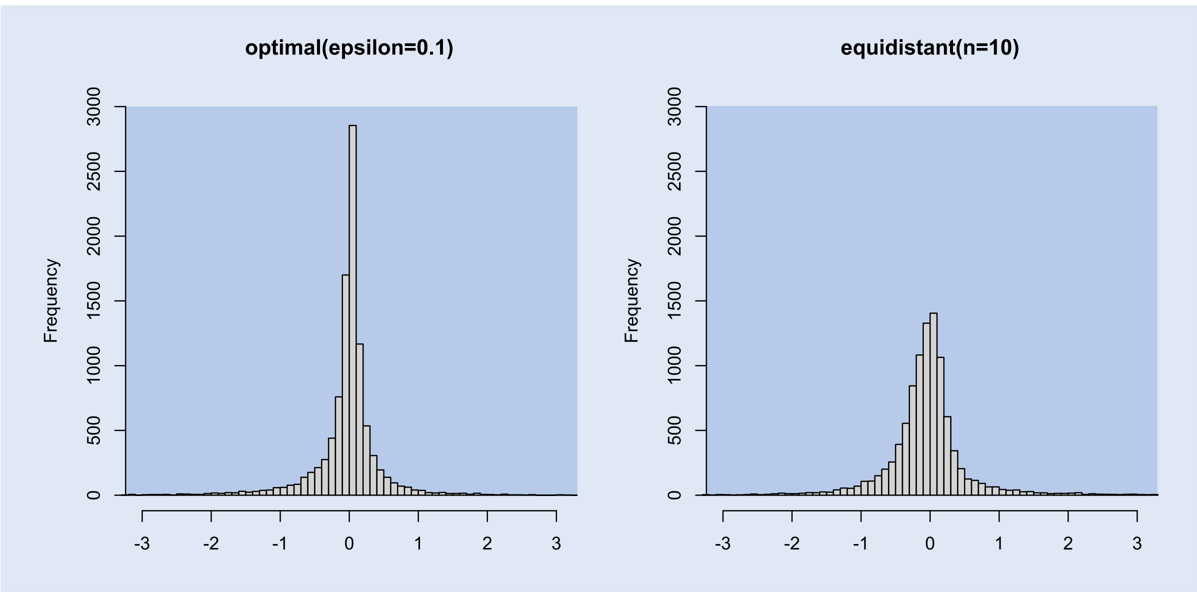

Figure 2. Histogram of $(V - V^n)/V$.

Table 2. The mean of the relative error $(V - V^n)/V$ and the mean squared relative error.

	Efficient	Equidistant
Mean	0.003018715	– 0.001761582
MSE	0.6918363	1.322438

Table 3. The mean of the error $V - V^n$ and the mean squared error.

	Efficient	Equidistant
Mean	0.001996048	0.01321023
MSE	0.004135032	0.02211172

tracking error is more concentrated around 0 and has a lighter tail for the efficient strategy (20). Table 2 shows the Monte-Carlo estimates of $\mathsf{E}[(V_1 - V_1^n)/V_1]$ and $\mathsf{E}[(V_1 - V_1^n)^2/V_1^2]$ from the 10 000 samples. The superiority of (20) is significant.

5.2. 50-*dimensional case*

Here we extend the analysis to the higher dimensional case of $d = m = 50$. We have simulated 10 000 sample paths of $S = (S^1, \dots, S^{50})^\top$ from (1) with the parameters

$$\mu^0 = 0, \quad S_0^i = \begin{cases} 1 & i \text{ is odd}, \\ 2 & i \text{ is even}, \end{cases}$$

$$\sigma^{ij} = \begin{cases} 0.1 & i < j, \\ 0.2 & i = j, \\ 0 & i > j, \end{cases} \quad \mu = \sigma\sigma^\top\theta, \quad \theta^i = 0.04$$

with discretization size $\Delta t = 0.01$ on the time interval $[0, 1]$. We consider the growth optimal portfolio

$$\pi^i = \begin{cases} 0.04 & i = 1, \dots, 50, \\ -19 & i = 0. \end{cases}$$

We take $Q = N = 1$ for cost functions and $\epsilon_n = 0.008$. This choice in (8) resulted in the average number of rebalancing 9.6751. We therefore compare it with the equidistant discretization with $n = 10$ again.

Figure 3 shows the histogram of the tracking error $V_1 - V_1^n$, where V_1^n is the terminal wealth associated with the efficient strategy in the left figure, and it is with the equidistant discretization in the right figure. It is again clearly seen that the tracking error is more concentrated around 0 and has a lighter tail for the efficient strategy (left). Table 3 shows the Monte-Carlo estimates of $\mathsf{E}[V_1 - V_1^n]$ and $\mathsf{E}[(V_1 - V_1^n)^2]$ from the 10 000 samples.

Figure 4 shows the histogram of the relative tracking error $(V_1 - V_1^n)/V_1$, where V_1^n is again, the terminal wealth associated with the efficient strategy for the left figure, and with the equidistant discretization for the right figure. Although the relative error is not the objective functional in our definition of the efficiency, it is again observed that the relative tracking error is more concentrated around 0 and has a lighter tail for the efficient strategy. Table 4 shows the Monte-Carlo estimates of $\mathsf{E}[(V_1 - V_1^n)/V_1]$ and $\mathsf{E}[(V_1 - V_1^n)^2/V_1^2]$ from the 10 000 samples.

5.3. *Summary and comments*

Both the cases of $d = 2$ and $d = 50$ under the Black-Scholes model, the asymptotically efficient strategy (8) has exhibited significant improvements in reducing the tracking error for the growth optimal portfolio over regular rebalancing. Regarding the time interval $[0, 1]$ as a one-year length, a rebalance occurs per 1.2 months on average under our choices of ϵ_n. These numerical experiments suggest that the asymptotic analysis of this paper provides practical approximations of optimal rebalancing times in realistic situations.

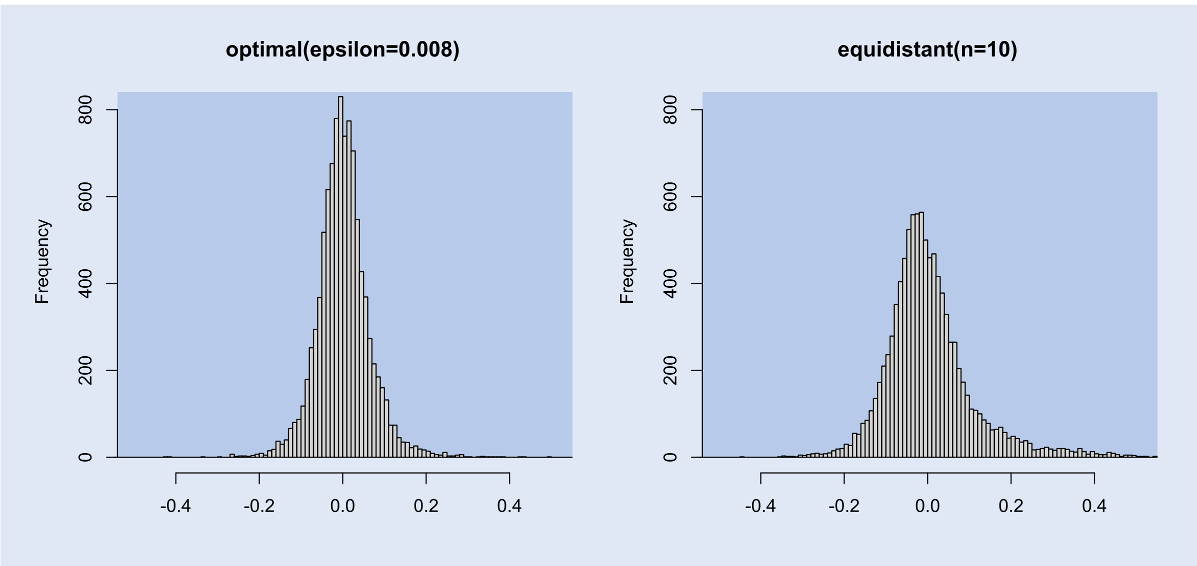

Figure 3. Histogram of $V - V^n$.

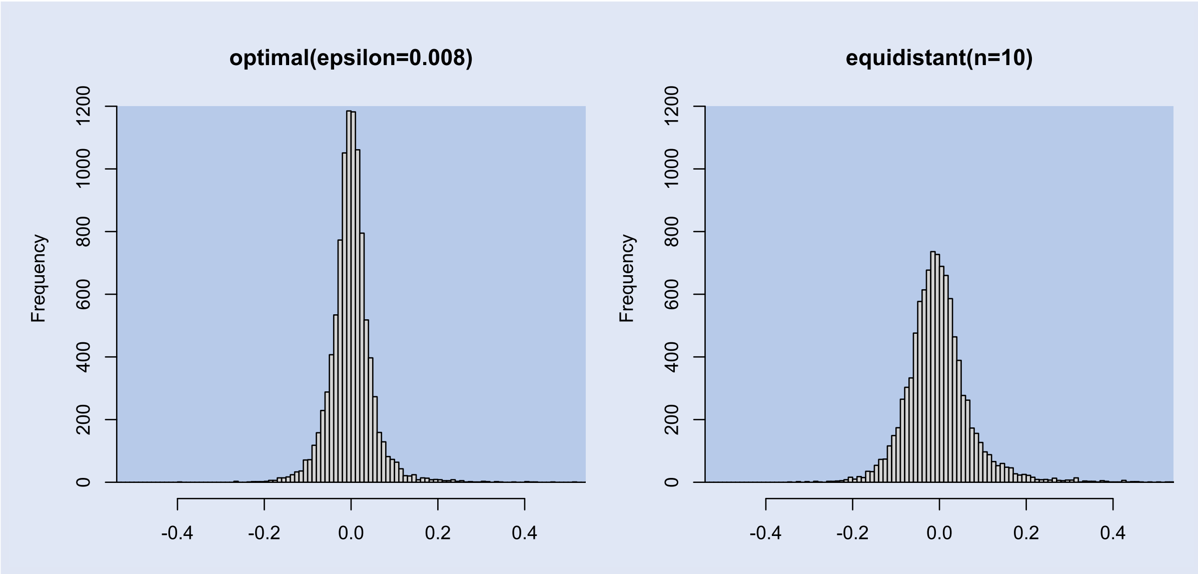

Figure 4. Histogram of $(V - V^n)/V$.

Table 4. The mean of the relative error $(V - V^n)/V$ and the mean squared relative error.

	Efficient	Equidistant
Mean	0.0004519002	− 0.001055898
MSE	0.00267405	0.006681065

The relation between the size of ϵ_n and the average number of rebalancing in (8) depends on d, π and Σ in particular. In practice we need to determine ϵ_n based on simulations to adjust the average number of rebalancing, or the average total cost of rebalancing, to fall within an acceptable range.

In practice, we also need to estimate Σ. The substitution of an estimator $\hat{\Sigma}$ to Σ would not cause a serious efficiency loss in usual situations because Σ can be estimated by using high-frequency data, such as, 5 minute returns of S, that are much more frequent than rebalancing times. It is well-known that the convergence rate of the realized covariance estimator to the quadratic covariation is the square root of the number of data; see e.g. Aït-Sahalia and Jacod (2014). For such a rebalancing frequency as once a month or less, the estimation error of Σ is negligible at least in our asymptotic framework. A care is however necessary when d is large, since the realized covariance might be too noisy in high-dimensions; see e.g. Buccheri and Mboussa Anga (2022) for a recent remedy for this problem. Note also that a model-adaptive optimal discretization is studied in Gobet and Stazhynski (2019), where the unknown parameters are simultaneously estimated in the efficient discretization of a stochastic integral.

6. Proof of theorem 1

It suffices to consider a case where $\mathsf{E}[N[X^n]]\mathsf{E}[Q[X^n]]$ converges. Then, since $Q[X^n]/\mathsf{E}[Q[X^n]]$ is uniformly integrable so is $\mathsf{E}[N[X^n]]Q[X^n]$. By localization, we can also assume without loss of generality that all the locally bounded processes are bounded, and that all the positive continuous processes, including the smallest eigenvalues of $\mathcal{S}^{\geq}_d$ valued continuous processes J and K, are bounded away from 0. Let

$$X^n = \sum_{j=0}^{\infty} \xi^n_j 1_{((\tau^n_j,\tau^n_{j+1}]]}$$

and

$$Y^n = \sum_{j=0}^{\infty} Y_{\tau^n_j} 1_{((\tau^n_j,\tau^n_{j+1}]]}$$

for $Y = J, K, L$ and Q. Since $\sup_{0\leq t\leq 1} |X^n_t - H_t| \to 0$ in probability, we have that $\sup_{j\geq 0} |\tau^n_{j+1} - \tau^n_j| \to 0$ in probability and as a result, $\sup_{0\leq t\leq 1} |Y^n_t - Y_t| \to 0$ in probability for $Y = J, K, L$ and Q. We refer to Fukasawa (2014) for more technical details on these observations in the one-dimensional case; the proofs are trivially extended to the multi-dimensional case.

By (11), we have

$$\begin{aligned}
&((H_{\tau^n_{j+1}} - \xi^n_j)^\top L_{\tau^n_j}(H_{\tau^n_{j+1}} - \xi^n_j))^2 \\
&= ((H_{\tau^n_j} - \xi^n_j)^\top L_{\tau^n_j}(H_{\tau^n_j} - \xi^n_j))^2 \\
&\quad + 4\int_{\tau^n_j}^{\tau^n_{j+1}} (H_t - X^n_t)^\top L^n_t (H_t - X^n_t)(H_t - X^n_t)^\top L^n_t \, dH_t \\
&\quad + \int_{\tau^n_j}^{\tau^n_{j+1}} (H_t - X^n_t)^\top (2\mathrm{tr}(L^n_t J_t) L^n_t + 4 L^n_t J_t L^n_t) \\
&\quad \times (H_t - X^n_t)\, dA_t
\end{aligned}$$

and so,

$$\begin{aligned}
Q[X^n] &= \int_0^1 (H_t - X^n_t)^\top K_t (H_t - X^n_t) Q_t \, dA_t \\
&= \sum_{j=0}^{\infty} G^n_j Q_{\tau^n_j}(((\Delta^n_j + \delta^n_j)^\top(\Delta^n_j + \delta^n_j))^2 - ((\delta^n_j)^\top \delta^n_j)^2)
\end{aligned}$$

$$+E_1^n + E_2^n,$$

where

$$\Delta_j^n = L_{\tau_j^n}^{1/2}(H_{\tau_{j+1}^n} - H_{\tau_j^n}),$$
$$\delta_j^n = L_{\tau_j^n}^{1/2}(H_{\tau_j^n} - \xi_j^n),$$
$$E_1^n = \int_0^1 (H_t - X_t^n)^\top (K_t Q_t - (2\mathrm{tr}(L_t^n J_t)L_t^n + 4L_t^n J_t L_t^n) \times Q_t^n G_t^n)(H_t - X_t^n)\,\mathrm{d}A_t,$$
$$E_2^n = 4\sum_{j=0}^\infty G_j^n \int_{\tau_j^n}^{\tau_{j+1}^n} (H_t - X_t^n)^\top L_t^n (H_t - X_t^n)(H_t - X_t^n)^\top \times L_t^n\,\mathrm{d}H_t,$$
$$G^n = \sum_{j=0}^\infty G_j^n 1_{((\tau_j^n, \tau_{j+1}^n]]},$$
$$G_j^n = \exp\left\{-\int_{\tau_j^n}^{\tau_{j+1}^n} G_t^\top J_t^{-1}\,\mathrm{d}M_t - \frac{1}{2}\int_{\tau_j^n}^{\tau_{j+1}^n} G_t^\top J_t^{-1} G_t\,\mathrm{d}A_t\right\}$$

and M and G are respectively the local martingale part of H and the Radon-Nikodym derivative of the finite variation part of H with respect to A.

Since $L = \ell(J, K)$ and ℓ is continuous by lemma 1, we have

$$\sup_{0\le t\le 1} |K_t Q_t - (2\mathrm{tr}(L_t^n J_t)L_t^n + 4L_t^n J_t L_t^n)Q_t^n|$$
$$= \sup_{0\le t\le 1} |(2\mathrm{tr}(L_t J_t)L_t + 4L_t J_t L_t)Q_t - (2\mathrm{tr}(L_t^n J_t)L_t^n + 4L_t^n J_t L_t^n)Q_t^n| \to 0$$

in probability. Together with $\sup_{0\le t\le 1}|G_t^n - 1| \to 0$ and the uniform integrability of $\mathsf{E}[N[X^n]]Q[X^n]$, we deduce $\mathsf{E}[N[X^n]]\mathsf{E}[|E_1^n|] \to 0$.

Define probability measures Q_j^n by

$$\frac{\mathrm{d}\mathsf{Q}_j^n}{\mathrm{d}\mathsf{P}} = G_j^n.$$

By the Girsanov-Maruyama theorem, $H_{\cdot\wedge\tau_{j+1}^n} - H_{\cdot\wedge\tau_j^n}$ is a martingale under Q_j^n for each $j \ge 0$. This implies $\mathsf{E}[E_2^n] = 0$ and

$$\mathsf{E}\left[\sum_{j=0}^\infty G_j^n Q_{\tau_j^n}(((\Delta_j^n + \delta_j^n)^\top(\Delta_j^n + \delta_j^n))^2 - ((\delta_j^n)^\top \delta_j^n)^2)\right]$$
$$= \mathsf{E}\left[\sum_{j=0}^\infty Q_{\tau_j^n}\mathsf{E}_{\mathsf{Q}_j^n}[(((\Delta_j^n + \delta_j^n)^\top(\Delta_j^n + \delta_j^n))^2 - ((\delta_j^n)^\top \delta_j^n)^2) \mid \mathcal{F}_{\tau_j^n}]\right].$$

Here we have used the fact that all the partial sums of the infinite series are uniformly bounded as shown by rewriting them as integrals using Itô's formula. Further by lemma 4 in section 7, this expectation is lower bounded by

$$\mathsf{E}\left[\sum_{j=0}^\infty Q_{\tau_j^n}\mathsf{E}_{\mathsf{Q}_j^n}[(\Delta_j^n)^\top \Delta_j^n \mid \mathcal{F}_{\tau_j^n}]^2\right].$$

Thus,

$$\lim_{n\to\infty}\mathsf{E}[N[X^n]]\mathsf{E}[Q[X^n]]$$
$$\ge \overline{\lim_{n\to\infty}}\,\mathsf{E}[N[X^n]]\mathsf{E}\left[\sum_{j=0}^\infty Q_{\tau_j^n}\mathsf{E}_{\mathsf{Q}_j^n}[(\Delta_j^n)^\top \Delta_j^n \mid \mathcal{F}_{\tau_j^n}]^2\right]$$
$$\ge \overline{\lim_{n\to\infty}}\,\mathsf{E}\left[\sum_{j=0}^\infty N_{\tau_j^n}^{1/2} Q_{\tau_j^n}^{1/2}\mathsf{E}_{\mathsf{Q}_j^n}[(\Delta_j^n)^\top \Delta_j^n \mid \mathcal{F}_{\tau_j^n}]\right]^2$$
$$= \overline{\lim_{n\to\infty}}\,\mathsf{E}\left[\sum_{j=0}^\infty N_{\tau_j^n}^{1/2} Q_{\tau_j^n}^{1/2} G_j^n \int_{\tau_j^n}^{\tau_{j+1}^n} \mathrm{tr}(L_{\tau_j^n} J_t)\,\mathrm{d}A_t\right]^2$$
$$= \mathsf{E}\left[\int_0^1 N_t^{1/2} Q_t^{1/2}\mathrm{tr}(L_t J_t)\,\mathrm{d}A_t\right]^2$$

with the aid of the Cauchy-Schwarz inequality.

7. Kurtosis-Skewness inequality

Here we prove an inequality for the centered fourth and third moments of a general random vector. This is a version of multi-variate Pearson's inequality; see Móri et al. (1993), Ogasawara (2017) for related preceding results.

Lemma 4 *Let Δ be a d-dimensional L^4 random variable with $\mathsf{E}[\Delta] = 0$ and $\delta \in \mathbb{R}^d$. Then,*

$$\mathsf{E}[((\Delta + \delta)^\top(\Delta + \delta))^2] - (\delta^\top \delta)^2 \ge \mathsf{E}[\Delta^\top \Delta]^2.$$

Proof We have

$$\mathsf{E}[((\Delta + \delta)^\top(\Delta + \delta))^2] - (\delta^\top \delta)^2$$
$$= \mathsf{E}[(\Delta^\top \Delta + 2\delta^\top \Delta + \delta^\top \delta)^2] - (\delta^\top \delta)^2$$
$$= \mathsf{E}[(\Delta^\top \Delta)^2] + 4\delta^\top \mathsf{E}[\Delta\Delta^\top]\delta + 4\mathsf{E}[\delta^\top \Delta(\Delta^\top \Delta)] + 2\delta^\top \delta \mathsf{E}[\Delta^\top \Delta].$$

Taking the gradient with respect to δ,

$$2(4\mathsf{E}[\Delta\Delta^\top]) + 2\mathsf{E}[\Delta^\top \Delta])\delta + 4\mathsf{E}[\Delta(\Delta^\top \Delta)]$$

and so, the minimum is attained at

$$\delta = -(2\mathsf{E}[\Delta\Delta^\top] + \mathsf{E}[\Delta^\top \Delta])^{-1}\mathsf{E}[\Delta(\Delta^\top \Delta)].$$

Substitute this to get

$$\mathsf{E}[((\Delta + \delta)^\top(\Delta + \delta))^2] - (\delta^\top \delta)^2$$

$$\geq \mathsf{E}[(\Delta^\top\Delta)^2] - \mathsf{E}[(\Delta^\top\Delta)\Delta^\top](\mathsf{E}[\Delta\Delta^\top] + \tfrac{1}{2}\mathsf{E}[\Delta^\top\Delta]I)^{-1}\mathsf{E}[\Delta(\Delta^\top\Delta)].$$

The result then follows from the lemma 5. ■

Lemma 5 *Let Δ be a d-dimensional L^4 random variable with* $\mathsf{E}[\Delta] = 0$ *and* $\delta \in \mathbb{R}^d$. *Then,*

$$\mathsf{E}[(\Delta^\top\Delta)^2] - \mathsf{E}[(\Delta^\top\Delta)\Delta^\top](\mathsf{E}[\Delta\Delta^\top] + D)^{-1}\mathsf{E}[\Delta(\Delta^\top\Delta)] \geq \mathsf{E}[\Delta^\top\Delta]^2$$

for any $D \in \mathcal{S}_d^>$.

Proof For any $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^d$,

$$\mathsf{E}[(\alpha(\Delta^\top\Delta - \mathsf{E}[\Delta^\top\Delta]) + \beta^\top\Delta)^2] \geq 0$$

The left-hand side is a quadratic form with respect to the symmetric matrix

$$\begin{pmatrix} \mathsf{E}[(\Delta^\top\Delta - \mathsf{E}[\Delta^\top\Delta])^2] & \mathsf{E}[\Delta^\top(\Delta^\top\Delta)] \\ \mathsf{E}[\Delta(\Delta^\top\Delta)] & \mathsf{E}[\Delta\Delta^\top] \end{pmatrix}$$

and the above nonnegativity implies that the matrix is nonnegative definite. Therefore the matrix

$$\begin{pmatrix} \mathsf{E}[(\Delta^\top\Delta - \mathsf{E}[\Delta^\top\Delta])^2] & \mathsf{E}[\Delta^\top(\Delta^\top\Delta)] \\ \mathsf{E}[\Delta(\Delta^\top\Delta)] & \mathsf{E}[\Delta\Delta^\top] + D \end{pmatrix}$$

is also nonnegative definite and so, has a nonnegative determinant. By the determinant formula for block matrices, the determinant is computed as

$$|\mathsf{E}[\Delta\Delta^\top] + D| \times (\mathsf{E}[(\Delta^\top\Delta - \mathsf{E}[\Delta^\top\Delta])^2] - \mathsf{E}[(\Delta^\top\Delta)\Delta^\top](\mathsf{E}[\Delta\Delta^\top] + D)^{-1}\mathsf{E}[\Delta(\Delta^\top\Delta)]),$$

which implies the claim. ■

Remark 3 As easily seen from the proof, the equality is attained in lemma 5 when $\Delta^\top\Delta = \mathsf{E}[\Delta^\top\Delta]$, or equivalently, Δ is supported on a sphere. We apply the inequality in section 6 for $\Delta = L_{\tau_j^n}^{1/2}(X_{\tau_{j+1}^n} - X_{\tau_j^n})$, so we have $\Delta^\top\Delta = \mathsf{E}[\Delta^\top\Delta]$ when $X_{\tau_{j+1}^n} - X_{\tau_j^n}$ is supported on an ellipsoid characterized by $L_{\tau_j^n}$. This explains the construction of our efficient strategy (8) in section 4.

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