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To cite this article: Christian Bongiorno & Damien Challet (2024) Covariance matrix filtering and portfolio optimisation: the average oracle vs non-linear shrinkage and all the variants of DCC-NLS, Quantitative Finance, 24:9, 1227-1234, DOI: [10.1080/14697688.2024.2372053](https://doi.org/10.1080/14697688.2024.2372053)

To link to this article: <https://doi.org/10.1080/14697688.2024.2372053>



Published online: 16 Jul 2024.



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Covariance matrix filtering and portfolio optimisation: the average oracle vs non-linear shrinkage and all the variants of DCC-NLS

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(Received 2 October 2023; accepted 19 June 2024)

The Average Oracle, a simple and very fast covariance filtering method, is shown to yield superior Sharpe ratios than the current state-of-the-art (and complex) methods, Dynamic Conditional Covariance coupled to Non-Linear Shrinkage (DCC-NLS). We pit all the known variants of DCC-NLS (quadratic shrinkage, gross-leverage or turnover limitations, and factor-augmented NLS) against the Average Oracle in large-scale randomized experiments. We find generically that while some variants of DCC-NLS sometimes yield the lowest average realized volatility, albeit with a small improvement, their excessive gross leverage and investment concentration, and their 10-time larger turnover contribute to smaller average portfolio returns, which mechanically result in smaller realized Sharpe ratios than the Average Oracle. We also provide simple analytical arguments about the origin of the advantage of the Average Oracle over NLS in a changing world.

Keywords: Covariance filtering; Portfolio optimization; Average Oracle; Non-linear shrinkage; Dynamical conditional covariance

1. Introduction

Portfolio optimization is a problem of fundamental importance. It rests on determining and exploiting the dependency structure between asset price returns, some encoded in the covariance matrix in a first approximation. The main objective of portfolio optimization is to minimize the volatility while enhancing returns, thereby improving the Sharpe Ratio. Given the inherent difficulty in predicting returns, the emphasis is often placed on effectively minimizing variance, highlighting the essential role of a reliable covariance matrix estimation. Focusing on reducing variance while monitoring other metrics such as realized profit, Sharpe Ratio, turnover, portfolio concentrations, and gross leverage is therefore crucial for mitigating the exposure to other sources of risk. In this context, a proper estimation of the covariance matrix turns out to be quite a challenging task. Recent progresses in covariance matrix filtering are nothing but remarkable (see Bun *et al.* 2017 for a review), culminating in the current state-of-the-art Dynamic Conditional Covariance coupled with Non-Linear Shrinkage (DCC-NLS) (Engle *et al.* 2019). While the

DCC part takes care of the dynamics, NLS proposes a target matrix that is much less noisy than a direct estimation. It rests on a series of results that show how to replace the eigenvalues of a covariance matrix with filtered ones to minimize the distance between the NLS-estimated matrix and the true one. Various extensions and refinements of these methods have been proposed to improve some aspects of the original DCC-NLS method (Ledoit and Wolf 2022a, 2022b).

Yet, DCC-NLS rests on strong assumptions. NLS, in particular, applies if the long-term true covariance matrix is constant, while the DCC part takes care of temporal variations. Although this seems the ideal combination, an alternative approach consists of accounting only for the average unconditional shift in the covariances among consecutive time periods, which is the basic idea behind the filtering method known as the Average Oracle (AO hereafter) (Bongiorno *et al.* 2023): it uses a fixed set of eigenvalues independent from the input and calibrated on an outdated dataset. Whereas this method seems to stand little chance against the sophistication of DCC-NLS, we show here that it consistently yields better portfolios than DCC-NLS, despite being orders of magnitude faster and simpler to implement. This raises substantial questions about the efficiency of complex methods in real scenarios and underscores the challenge of

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determining which information is relevant and preserved over time.

We compare various covariance filtering methods with randomized asset universes. A common approach consists in building portfolios with an investment universe made of the n stocks with the largest capitalization either at the time of writing or historically, and a single portfolio is computed at a given time, leading to a single set of performance metrics. We believe that this approach can be improved in several respects: high-capitalization stocks at a given date have, on average, outperformed during the past; second, having a single number for each metric does not allow us to measure confidence intervals without introducing assumptions such as normality or time invariance, both of them known to be false in real financial markets. To overcome these issues, we propose working with a universe composed of the top $N \gg 1$ capitalized stocks and then randomly sampling $n \ll N$ to build an investment universe. In addition, the universe of N stocks must be selected at the start of the investment period, using only the information available at the time. If the capitalization of some of the stocks changes dramatically, such stocks should be sold. Since $n \ll N$, computations can be carried out over many realizations, and the confidence intervals can be determined in a non-parametric way via bootstraps.

Our final contribution is to propose a simple mathematical covariance evolution framework able to explain why and when AO is superior to NLS. It turns out that what matters are not only the eigenvalues but also the fact that they are weighted by the average eigenvector overlap between two successive covariance matrices. When the typical rotation between two such matrices is larger than the random rotation due to estimation noise, and provided that the eigenvalues do not change too much, AO beats NLS.

2. Methods

2.1. Universe construction

To carry out our analysis, we outline here our data selection approach which is carefully crafted to mirror, as closely as possible, the actual way in which data selection is executed in real life, thus minimizing selection biases.

Consider day t , in-sample window $[t - \Delta t_{in}, t]$, and an out-of-sample window $]t, t + \Delta t_{out}]$. The selection of available assets will be the N most capitalized stocks of the US market on day t such that all the stocks are listed in $[t - \Delta t_{in}, t + \Delta t_{out}]$; using the fact that the chosen assets are listed in the future involves a limited amount of foresightedness. In addition, we ask that the selected assets have less than 20% zero or missing returns in $[t - \Delta t_{in}, t]$, and that any pair of assets should have a Pearson correlation coefficient smaller than 0.95 (De Nard *et al.* 2021).

2.2. Covariance filtering methods

We compare two main ways to estimate covariance matrices: DCC-NLS, a sophisticated dynamic econometric model, which consists in dynamic correlation targeting with the DCC

method (Engle 2002), where the target is the correlation matrix filtered with NLS (Ledoit and Wolf 2017), and the AO method (Bongiorno *et al.* 2023). NLS and AO first decompose an empirical correlation matrix E with the spectral theorem, i.e. write $E = \hat{C} = V\Lambda V^\dagger$, where V is the matrix of eigenvectors of E , Λ is a diagonal matrix containing the eigenvalues λ of E , and V^\dagger is V transposed, and then only modify the eigenvalues.

NLS provably optimally modifies the eigenvalues of E when the true correlation matrix is time-invariant, the matrix very large and provided that some moments of the returns r are finite. Optimality means that NLS brings $E_{NLS} = V\Lambda_{NLS}V^\dagger$ maximally close to C according to the Frobenius distance. The proof rests on the fact that λ_{NLS} tends to the Oracle eigenvalues $\text{diag}(V^\dagger CV)$, which can easily be shown to minimize the distance to C if one must keep V untouched.

When C is not constant instead, NLS ceases to be optimal. Indeed, its goal is to bring the empirical correlation matrix as close as possible to the current generating correlation matrix C , not to the future one. On the other hand, rewriting the Oracle eigenvalues as $\text{diag}(V^\dagger E_{out} V)$, where E_{out} is the empirical covariance matrix in the next period, makes it clear these eigenvalues contain future information and that having some way to approximate them will be beneficial. This is exactly what the Average Oracle does: it computes the average Oracle eigenvalues in a longer period of past data and therefore captures something of the average influence of the future on current eigenvalues (see Bongiorno *et al.* 2023). Once calibrated, the AO eigenvalues are kept constant. This means that when filtering E , one discards the empirical eigenvalues, keeps the eigenvectors V and plugs Λ_{AO} in. The only relevant parameters are the dimensions of the data matrix.

DCC-NLS, on the other hand, is computationally heavy and is considered the state-of-the-art method as regards covariance filtering. Several variants of this method have been introduced in order to improve its stability and performance (QIS and QuEST). This is why we tested the following methods:

- (1) **NotFilt1200** Sample covariance with $\Delta t_{in} = 1200$.
- (2) **NotFilt240** Sample covariance with $\Delta t_{in} = 240$.
- (3) **DCC-QIS** with quadratic shrinkage (Ledoit and Wolf 2022b).
- (4) **DCC-QuEST** DCC with the nonlinear shrinkage based on the QuEST function (Engle *et al.* 2019).
- (5) **1AFM-DCC-QuEST** DCC with the nonlinear shrinkage based on the QuEST function with a factor taken from the Fama and French (De Nard *et al.* 2021).
- (6) **1AFM-DCC-QIS** DCC with quadratic shrinkage with a factor taken from the Fama and French (De Nard *et al.* 2021).
- (7) **NLS1200** nonlinear shrinkage based on the QuEST function with $\Delta t_{in} = 1200$ (Ledoit and Wolf 2017).
- (8) **QIS1200** quadratic shrinkage with $\Delta t_{in} = 1200$ (Ledoit and Wolf 2022a).
- (9) **QuEST240** nonlinear shrinkage based on the QuEST function with $\Delta t_{in} = 240$ (Ledoit and Wolf 2017).
- (10) **QIS240** quadratic shrinkage with $\Delta t_{in} = 240$ (Ledoit and Wolf 2022a).

- (11) **AO1200** Average Oracle eigenvalues computed in the 1900–2000 period with $\Delta t_{in} = 1200$ (Bongiorno *et al.* 2023).
- (12) **AO240** Average Oracle eigenvalues calibrated in the 1900–2000 period with $\Delta t_{in} = 240$ (Bongiorno *et al.* 2023).

DCC, NLS, and QIS were run with the original MatLab implementation available (Ledoit 2022, Sheppard 2023).

We focus on global minimum variance portfolios (GMV) as they provide a natural test of the filtering abilities of a given method as regards covariance matrices: given covariance matrix Σ , the problem consists in solving

$$\min_w w^\dagger \Sigma w \quad (1)$$

$$\sum_{i=1}^N w_i = 1 \quad (2)$$

for long-short portfolios, with the additional constraint $w_i \geq 0$ for long-only portfolios.

3. Empirical analysis

3.1. Randomized experiments

According to Engle *et al.* (2019), DCC performs better for $\Delta t_{in} = 1200$ days; therefore, we use this calibration window. In this experiment, we fixed $n = 100$ assets randomly sampled from the top $N = 500$ stocks per capitalization at time t as described in Section 2.1. We tested both $\Delta t_{out} = 5$ and 20. We performed a near yearly stock rebalancing of 240 days, which consists of 48 weight rebalancing for the $\Delta t_{out} = 5$ and 12 rebalancing for $\Delta t_{out} = 20$. At each rebalancing, if some stocks of the portfolio exit the universe N of the top 500 capitalized stocks, such stocks will be sold and substituted with a random selection from the universe N defined on the current day. The initial day t is selected randomly in the range [2000-01-03, 2021-01-20], which includes the 2008-2009 and 2020 COVID crises.

To measure the respective performance of all the above methods, we set the transaction cost to 5 bps, which is considered a low transaction cost (Ledoit and Wolf 2022b); larger transactions would increase the advantage of AO over DCCs. Transaction costs are paid only at each rebalancing and also account for the stock price drift in the portfolio.

We performed 10, 000 simulations by randomly sampling t and the initial set of stocks n available at time t . Tables 1 and 2 report performance metrics of long-short portfolios, for weekly ($\Delta t = 5$ days) and monthly ($\Delta t = 20$ days) rebalancing. First, there is no significant difference between QIS and QuEST-based methods for any performance measure; for this reason, we refer to them as DCC-QIS/QuEST. DCC-QIS/QuEST and AO240 achieve the smallest realized volatility, which is not statistically different even after 10,000 simulations. We note that 240 timesteps correspond to the timescale fitted by the DCC process. In other words, the complex machinery of DCC-NLS models the covariance matrix at a similar timescale as AO240. What is still remarkable is that AO240, despite its simplicity, leads to the same result. We also note that AO1200 has only marginally higher volatility (approximately 7%).

There are large differences, however, in three key portfolio metrics. First, the Sharpe ratio of AO1200 is 14% ($\Delta t_{out} = 20$) and 37% ($\Delta t_{out} = 5$) larger than that of DCC-QIS/QuEST, which is a large and significant difference. Interestingly, QIS/QuEST1200 has a systematically higher SR than their respective DCC versions (yet smaller than those of AO). In passing, even an unfiltered covariance matrix with 1200 days of calibration yields a larger Sharpe ratio, which is consistent with the well-known fact that when the number of Δt_{in} is larger than about $10n$, covariance filtering is not needed.

Even more, the largest difference between AO and DCC-QIS/QuEST resides in the turnover. We report two kinds of turnover metrics. The first one measures the portfolio weights stability: it is defined as the average over all the rebalancing times $\tau = 1, 2$ (in units of rebalancing intervals) of $\sum_i |w_i(\tau) - w_i(\tau - 1)|$: if the GMV yields the same weights at rebalancing times τ and at the previous rebalancing time $\tau - 1$, the weight turnover is zero. The second measure also includes the natural change of asset prices which modifies the effective weights of each asset with respect to their initial

Table 1. $\Delta t_{out} = 5$, $n = 100$, average of 10, 000 simulations. GMV with long and short positions.

	SR	MEAN	VOL	Turnover	Turnover+drift	GrossLev	N_{eff}
NotFilt1200	1.101	0.098	0.113	0.174	0.250	2.765	6.368
NotFilt240	0.745	0.071	0.125	0.974	1.050	4.151	3.290
AO1200	1.229*	0.112*	0.114	0.068*	0.119*	1.702*	13.981*
AO240	1.167	0.095	0.105*	0.181	0.22	1.729	13.213
DCC-QIS	0.894	0.069	0.105*	1.308	1.335	2.543	5.856
DCC-QuEST	0.895	0.069	0.105*	1.307	1.333	2.542	5.852
QIS1200	1.123	0.098	0.112	0.150	0.221	2.536	7.931
QIS240	1.022	0.083	0.108	0.427	0.480	2.538	9.675
QuEST1200	1.122	0.098	0.112	0.156	0.226	2.531	7.992
QuEST240	1.027	0.084	0.108	0.402	0.456	2.512	10.011
DCC-QIS-1F	1.071	0.089	0.108	0.807	0.846	2.615	6.089
DCC-QuEST-1F	1.073	0.089	0.108	0.803	0.843	2.612	6.096

Note: All the values not statistically distinct from the best value according to a bootstrap confidence interval at 95% are marked in bold.

Table 2. $\Delta t_{out} = 20$, $n = 100$, randomized asset universe, average of 10, 000 simulations. GMV with long and short positions.

	SR	MEAN	VOL	Turnover	Turnover+drift	GrossLev	N_{eff}
NotFilt1200	1.111	0.101	0.116	0.414	0.639	2.763	6.376
NotFilt240	0.811	0.079	0.13	2.008	2.197	4.158	3.305
AO1200	1.230*	0.113*	0.117	0.173*	0.326*	1.705*	13.921*
AO240	1.156	0.095	0.109*	0.422	0.540	1.733	13.218
DCC-QIS	1.07	0.088	0.110*	2.082	2.117	2.544	5.962
DCC-QuEST	1.07	0.088	0.110*	2.080	2.115	2.543	5.958
QIS1200	1.132	0.101	0.115	0.358	0.569	2.535	7.938
QIS240	1.039	0.086	0.112	0.891	1.040	2.545	9.718
QuEST1200	1.132	0.101	0.115	0.358	0.568	2.531	8.000
QuEST240	1.043	0.086	0.112	0.833	0.985	2.519	10.058
DCC-QIS-1F	1.152	0.099	0.112	1.279	1.389	2.624	6.085
DCC-QuEST-1F	1.153	0.100	0.112	1.275	1.385	2.622	6.091

Note: All the values not statistically distinct from the best value according to a bootstrap confidence interval at 95% are marked in bold.

weight and is denoted by Turnover+drift in all the tables. It turns out that DCC-QIS/QuEST has a problem with the latter, as its Turnover+drift is 12 to 19 times larger than that of AO. This comes from two main differences. First, the gross leverage is approximately 45% larger for DCC-QIS/QuEST than AO1200. Second, the diversification of the portfolio from AO is almost twice that of DCC-QIS/QuEST.

The 1AFM slightly improves the performance of DCC-QIS/QuEST because it gives more stable portfolios, but at the cost of even larger gross leverage.

Finally, we report a portfolio diversification measure based on the effective number of assets for a portfolio with weights w_i on asset i (Bouchaud and Potters 2003), defined as

$$n_{eff} = \frac{1}{\sum_{i=1}^n w_i^2}. \quad (3)$$

In all the cases we tried, AO leads to a significantly better asset diversification than the other methods.

In summary, from the observations above, it seems that the DCC methods fail to give reliable (stable enough) estimates of the future covariance matrix and thus overbet on very concentrated portfolios that, unfortunately, change very quickly without capturing the relevant information, leading to poorer Sharpe ratios, higher gross-leverage, and low diversification.

3.2. Large portfolios

Engle *et al.* (2019) suggest that the DCC-NLS method performs better for large n . However, randomized experiments with $n = 1000$ would require a much larger stock universe (say, $N = 10,000$). But this is problematic for two additional reasons: first, DCC and NLS are very slow when n is large; second, assets with a small capitalization are not traded every day, which sometimes prevents a proper calibration of GARCH models.

Therefore, we test a more standard approach. We performed two simulations with $\Delta t_{out} = 5$ using only the top $n = 1000$ most capitalized stocks at each rebalancing time. Likewise, in the previous case, if a stock exits the top $n = 1000$ most capitalized universe, it is replaced by a new suitable asset. The simulations start on 2000-01-03 and end on 2021-12-30.

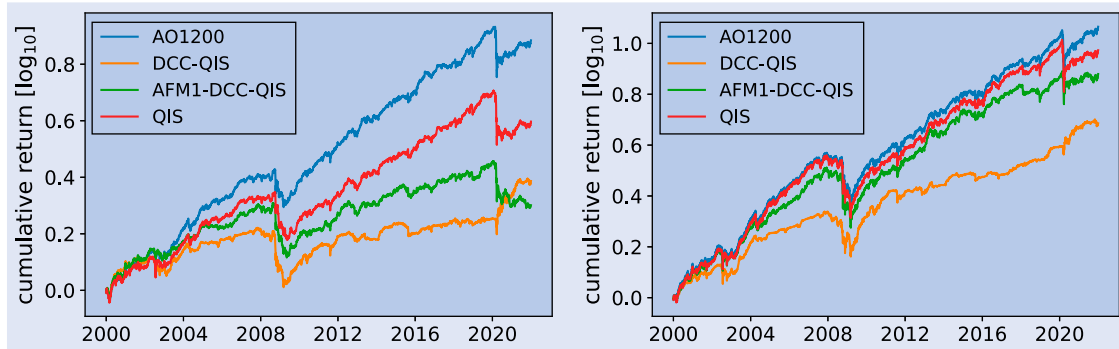
Figure 1 reports the results of these experiments. While the superiority of AO in this setting is visually obvious, particularly regarding long-short portfolios, tables 2 and 6 give a more detailed breakdown: for long-short portfolios, AO has a much better Sharpe ratio than QIS, DCC-QIS and AFM1-DCC-QIS (107%, 121.9%, and respectively). This comes both from larger average returns and smaller realized volatility. In addition, as in the random universe experiment, the turnover of AO is approximately smaller by a factor of at

Table 3. $\Delta t_{out} = 5$, $n = 100$, randomized asset universe, average of 10, 000 simulations. GMV with long-only positions.

	SR	MEAN	VOL	Turnover	Turnover+drift	N_{eff}
NotFilt1200	1.134	0.108	0.124	0.044	0.07	10.362
NotFilt240	1.072	0.092	0.116	0.149	0.169	9.294
AO1200	1.187*	0.114*	0.125	0.037*	0.064*	16.676*
AO240	1.122	0.098	0.116	0.106	0.127	15.528
DCC-QIS	1.041	0.084	0.110*	0.597	0.605	8.029
DCC-QuEST	1.040	0.084	0.110	0.597	0.606	7.995
QIS1200	1.133	0.107	0.124	0.043	0.069	11.694
QuEST1200	1.133	0.107	0.124	0.044	0.07	11.728
QIS240	1.079	0.093	0.116	0.126	0.146	14.719
QuEST240	1.076	0.093	0.116	0.127	0.148	14.588
DCC-QIS-1F	1.094	0.100	0.121	0.306	0.32	8.570
DCC-QuEST-1F	1.094	0.100	0.121	0.306	0.321	8.566
EQ	0.802	0.092	0.182	0.008	0.054	100

Table 4. $\Delta t_{out} = 20$, $n = 100$, randomized asset universe, average of 10, 000 simulations. GMV with long-only positions.

	SR	MEAN	VOL	Turnover	Turnover+drift	N_{eff}
NotFilt1200	1.124	0.108	0.127	0.115	0.197	10.346
NotFilt240	1.038	0.09	0.12	0.344	0.399	9.322
AO1200	1.177*	0.114*	0.127	0.097*	0.182*	16.611*
AO240	1.095	0.096	0.12	0.253	0.317	15.535
DCC-QIS	1.074	0.091	0.116*	0.926	0.931	8.130
DCC-QuEST	1.073	0.091	0.116*	0.927	0.932	8.096
QIS1200	1.122	0.107	0.126	0.111	0.194	11.679
QuEST1200	1.122	0.107	0.126	0.112	0.194	11.714
QIS240	1.049	0.090	0.120	0.289	0.350	14.773
QuEST240	1.046	0.090	0.120	0.288	0.349	14.646
DCC-QIS-1F	1.103	0.102	0.124	0.469	0.513	8.582
DCC-QuEST-1F	1.102	0.102	0.124	0.469	0.513	8.577
EQ	0.793	0.089	0.183	0.0288	0.156	100

Figure 1. Large portfolio case: $n = 1000$. Left plot: long-short portfolios, right plot: long-only portfolios. Portfolio rebalancing every week.Table 5. Large long-short portfolio case ($n = 1000$), single realization [2000–2021], the asterisk marks the best unconstrained method.

	SR	MEAN	VOL	Turnover	Turnover+drift	GrossLev	N_{eff}
NotFilt1200	-0.6	-0.097	0.162	7.235	7.282	17.562	2.013
AO1200	1.009*	0.093*	0.093	0.210	0.242	2.961*	46.147
QIS	0.689	0.063	0.092	0.729	0.761	4.859	29.179
DCC-QIS	0.484	0.041	0.084*	2.238	2.243	3.823	12.957
AFM1-DCC-QIS	0.37	0.032	0.088	2.558	2.572	6.375	11.452
AFM1-DCC-QIS-turn	0.812	0.072	0.089	0.433	0.279	6.408	11.704
DCC-QIS-turn	0.972	0.078	0.080	0.405	0.296	4.800	10.851
QIS-turn	0.804	0.074	0.092	0.354	0.294	4.809	29.575
DCC-QIS-gross	0.528	0.044	0.083	1.929	1.933	2.862	13.494
AFM1-DCC-QIS-gross	0.556	0.048	0.087	1.585	1.591	2.961	14.199
QIS-gross	0.752	0.07	0.092	0.416	0.438	2.961	37.43

least 10 than all the other methods, while having about half the gross leverage. Finally, the effective number of assets of AO is more than 20 times larger than those of the other methods.

The advantage of AO for long-only portfolios (table 6) is less impressive but still outperforms the other three methods in all metrics: its Sharpe ratio is significantly larger, once again because of both better average returns and realized volatility (except for DCC-QIS). AO has the same realized volatility as QIS and not filtered sample covariance (NotFilt1200). The turnover is also much smaller for AO, and the effective number of assets is much larger than all the other methods.

3.3. Reducing gross leverage and turnover for DCCs

One of the major problems we observed with DCCs is their very large turnover and gross leverage. In the presence of transaction costs, this problem mechanically reduces the average return. To give all possible chances to DCC-NLS, we now constrain the DCC methods to have a turnover or gross leverage bounded by those of AO1200 (we can only control the turnover in the case of long-only portfolios). We accounted for the fact that sometimes stocks exit the top 1000 and must be sold; accordingly, we only constrain the rebalancing that concerns stocks still in the top 1000 at the time of the rebalancing.

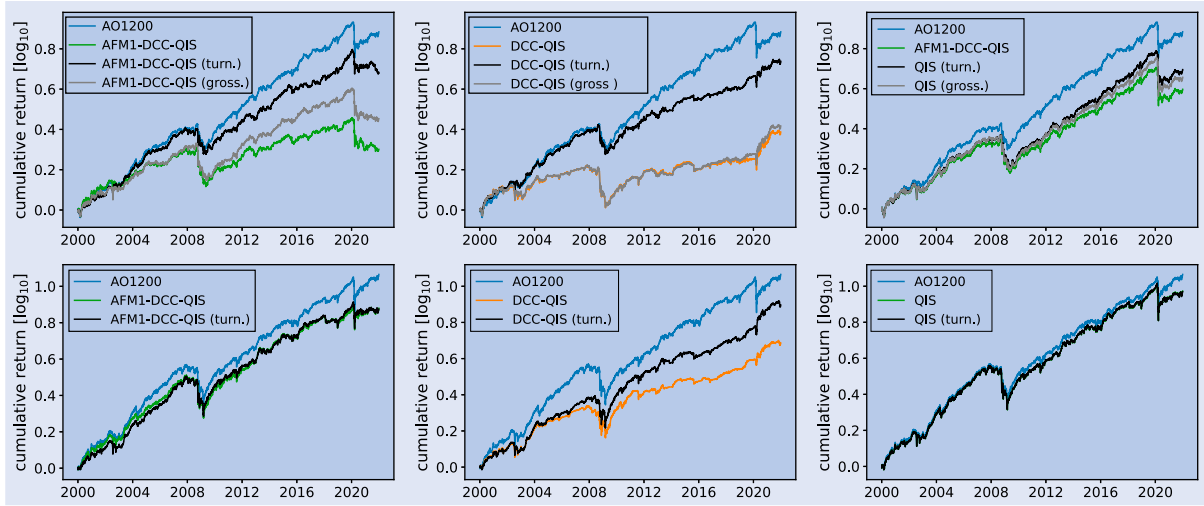


Figure 2. Large portfolio case ($n = 1000$). Upper plots: long-short portfolios, lower plots: long-only portfolios. Portfolio rebalancing every week.

Table 6. Large long-only portfolio case ($n = 1000$), single realization [2000–2021], the asterisk marks the best unconstrained method.

	SR	MEAN	VOL	Turnover	Turnover+drift	N_{eff}
NotFilt1200	0.772	0.095	0.124	0.117	0.122	18.142
AO1200	0.897*	0.112	0.125	0.083	0.089*	45.559
QIS	0.832	0.103	0.124	0.1	0.105	34.838
DCC-QIS	0.869	0.072	0.083	0.635	0.636	6.765
AFM1-DCC-QIS	0.801	0.093	0.116	0.473	0.474	12.402
AFM1-DCC-QIS-turn	0.777	0.092	0.119	0.1	0.089	14.816
DCC-QIS-turn	0.980	0.094	0.096	0.134	0.123	14.47
QIS-turn	0.826	0.102	0.124	0.087	0.088	34.651

While the unconstrained AO still beats all the DCC-NLS and NLS methods for long-short portfolios for all metrics except realized volatility, DCC-QIS, once constrained by AO1200, does bring the best Sharpe ratio, owing to its smaller realized volatility.

4. Average oracle vs NLS in a simplified dynamical world

The results above raise the question of why the Average Oracle performs better than non-linear shrinkage when the true covariance matrix is time-dependent. This section is devoted to capture the essence of the difference between what the Average Oracle computes and the assumptions of the non-linear shrinkage estimator.

Let us assume that one needs to estimate covariance matrix C between N time series in a time window of length T . The problem with many real-life systems is that the true covariance matrix C evolves as a function of time. Thus, over the calibration window of length T , the true covariance matrix may not be constant, which complexifies both its estimation and mathematical computations.

Instead of assuming that C evolves slowly as a function of time as DCC-NLS does, we assume that it changes at fixed times kT , $k \in 1, 2, \dots$, which also are opportunistically assumed to correspond to the end of successive estimation windows. Equivalently, time is split into slices indexed by k

and of length T : slice $k = [kT, (k+1)T]$. In other words, we assume that the true correlation matrix C is constant during a time slice and may only switch between S several possible values at the end of each time slice, possibly in a probabilistic way. This is a great simplification that makes it possible to derive simple bounds. The $S > 1$ different covariance matrices are denoted by C_s , $s = 1, \dots, S$. During time slice k , the true covariance matrix index is $s(k)$, the former being denoted by $C_{s(k)}$.

Assume that the current covariance matrix is $i = s(k)$. The spectral decomposition theorem states that $C_i = V_i \Lambda_i V_i^\dagger$, where the † symbol denotes the transpose operation, V_i is the eigenvector matrix of C_i , and Λ_i is the diagonal matrix of the eigenvalues of matrix i . We will also denote by λ_i the vector of eigenvalues of C_i .

At the end of the current period, the probability transition from the true covariance matrix with index $i = s(k)$ to the one with index $j = s(k+1)$ is denoted by $W_{i \rightarrow j}$ and the stationary frequency of state i $P_i = \frac{1}{N} \sum_{j=1}^S W_{j \rightarrow i}^\infty$. The eigenvectors of C_j are given by a rotation of those of C_i : the rotation matrix R_{ij} is defined as $R_{i \rightarrow j} = V_j V_i^\dagger$, so that $V_j = R_{i \rightarrow j} V_i$ and $V_i = R_{i \rightarrow j}^{-1} V_j$.

Estimating C_i yields \hat{V}_i and $\hat{\Lambda}_i$, noisy estimates of V_i and Λ_i . Computing the joint distribution of \hat{V}_i and $\hat{\Lambda}_i$ is in general a hard problem (that can be tackled by sophisticated mathematical methods (Bun *et al.* 2017)).

By definition, the AO eigenvalues will be the average of Oracle eigenvalues computed from the estimated eigenvalues

of $C_{s(k+1)}$ with the estimated eigenvectors of $C_{s(k)}$ over all possible pairs of correlation matrices, weighted by the transition probability, i.e.

$$E(\Lambda^{AO}) = \sum_{ij}^S E[\hat{P}_i \hat{W}_{i \rightarrow j} \text{diag}(\hat{V}_i^\dagger \hat{C}_j \hat{V}_i)]. \quad (4)$$

A welcome simplification comes from the relationship

$$\text{diag}(\hat{V}_i^\dagger \hat{C}_j \hat{V}_i) = (\hat{V}_i^\dagger \hat{V}_j)^{\circ 2} \hat{\lambda}_j = (\hat{R}_{i \rightarrow j})^{\circ 2} \hat{\lambda}_j$$

where $X \circ Y$ is the element-by-element Hadamar product and where λ_j is the diagonal part of Λ_j . Bongiorno *et al.* (2023) finds empirically that

$$E[(\hat{R}_{i \rightarrow j})^{\circ 2} \hat{\lambda}_j] \simeq E[(\hat{R}_{i \rightarrow j})^{\circ 2}] E[\hat{\lambda}_j]$$

to a very high accuracy, that is, that the eigenvector overlap and the eigenvalue estimates are independent. Finally, equation (4) becomes

$$E(\Lambda^{AO}) \simeq \sum_{ij}^S E[\hat{P}_i \hat{W}_{i \rightarrow j}] E[(\hat{R}_{i \rightarrow j})^{\circ 2}] E[\hat{\lambda}_j], \quad (5)$$

as the transition process between the true covariance matrices is independent from estimation noises.

AO eigenvalues must be computed over a long calibration interval which should include as many transitions as needed to have reliable estimates at least of the eigenvalues λ_j . When this calibration interval is infinitely long,

$$E(\Lambda^{AO}) \rightarrow \sum_{ij}^S P_i W_{i \rightarrow j} E[(R_{i \rightarrow j})^{\circ 2}] \lambda_j. \quad (6)$$

equations (5) and (6) make it clear that some cases simplify much the discussion: for example, when all the covariance matrix have the same eigenvalues, or if the overlap term does not depend on i and j .

The simplest case (and the worst one for NLS) is a world in which $C_{s(k+1)} \neq C_{s(k)}$. For $S = 2$, $s(k) = 1 + k \bmod 2$, the world is cyclic and the AO eigenvalues are given by

$$E(\Lambda^{AO}) = E \left[(\hat{R})^{\circ 2} \frac{\hat{\lambda}_1 + \hat{\lambda}_2}{2} \right] \rightarrow E[(R)^{\circ 2}] \cdot \frac{\lambda_1 + \lambda_2}{2}.$$

On the contrary, NLS only uses information from the last time slice: when the true covariance matrix is C_i during time slice k , its eigenvalues are given by

$$\hat{\Lambda}_i^{NLS} \rightarrow (\hat{V}_i^\dagger V_i)^{\circ 2} \hat{\lambda}_i \quad (7)$$

in the limit of large n and T at fixed n/T ratio. These eigenvalues minimize the Frobenius distance (average element-wise squares) $\|\hat{V}_i \Lambda_i^{NLS} \hat{V}_i^\dagger - C_i\|_2$. The magic of NLS is that it gives a closed formula for the eigenvalues that does not depend on the true matrix. It should be noted that $(\hat{V}_i^\dagger V_i)^{\circ 2}$ corresponds to the overlap matrix between the real eigenvectors and its estimates. Note as well that λ_i is a noisier estimate for NLS than

for AO in this context because it is performed over a single time slice.

The point is that the realized covariance matrix is different from C_i in a non-constant world and thus AO incorporates both the average eigenvalues and the average overlap of the eigenvectors between the calibration and test time windows. In the simple cyclic world defined above, half of the terms of the AO eigenvalues do account for the future transition: one has

$$\begin{aligned} & \|\hat{V}_1 \Lambda^{AO} \hat{V}_1^\dagger - C_2\|_2 \\ &= \left\| \hat{V}_1 E[(R)^{\circ 2}] \frac{\lambda_1}{2} \hat{V}_1^\dagger + V_1 E[(R)^{\circ 2}] \frac{\lambda_2}{2} \hat{V}_1^\dagger - C_2 \right\|_2 \end{aligned} \quad (8)$$

Now, $E[(R)^{\circ 2}] \lambda_2$ are the Oracle eigenvalues when the true covariance matrix is C_1 in the estimation time slice and C_2 in the test time slice, hence are Frobenius optimal. Rearranging the terms, one has

$$\begin{aligned} \|\hat{V}_1 \Lambda^{AO} \hat{V}_1^\dagger - C_2\|_2 &= \left\| \frac{1}{2} \left(\hat{V}_1 E[(R)^{\circ 2}] \lambda_1 \hat{V}_1^\dagger - C_2 \right) \right. \\ &\quad \left. + \frac{1}{2} \left(\hat{V}_1 E[(R)^{\circ 2}] \lambda_2 \hat{V}_1^\dagger - C_2 \right) \right\|_2 \end{aligned} \quad (9)$$

$$\leq \|\hat{V}_1 E[(R)^{\circ 2}] \lambda_1 \hat{V}_1^\dagger - C_2\|_2 \quad (10)$$

Finally

$$\|\hat{V}_1 E[(R)^{\circ 2}] \lambda_1 \hat{V}_1^\dagger - C_2\|_2 \leq \|\hat{V}_1 E[(\hat{V}_i^\dagger V_i)^{\circ 2}] \lambda_1 \hat{V}_1^\dagger - C_2\|_2$$

as $E[(\hat{V}_i^\dagger V_i)^{\circ 2}]$ is a random rotation $\widehat{\delta R}$ due to eigenvector estimation noise, and does not bring V_1 closer to V_2 while $E[(\hat{R})^{\circ 2}]$ is the composition of the true rotation from V_1 to V_2 followed by a random rotation $\widehat{\delta R'}$ of the same order of magnitude as $\widehat{\delta R}$. As a consequence

$$\|C_1^{AO} - C_2\|_2 \leq \|C_1^{NLS} - C_2\|_2 \quad (11)$$

$$\|C_2^{AO} - C_1\|_2 \leq \|C_2^{NLS} - C_1\|_2. \quad (12)$$

The above result only holds when the true covariance matrix systematically changes at the end of each time slice. It is easy to generalize it to a cyclic world with S states. When there is a finite probability that the true covariance matrix stays constant, there are cases when NLS performs better than AO, even in this simplifying context. Finally, the more generic case where the true covariance matrix may change slightly at each time step is currently out of reach of analytical methods.

5. Conclusion

Large-scale randomized experiments for mid-size portfolios (100 assets) allowed us to clarify the merits of a whole range of covariance filtering methods. Whereas the Average Oracle is a somewhat naive and very fast method not derived from an econometric framework, it still outperforms all the econometric methods tested here on four key portfolio metrics: Sharpe ratio, turnover, gross leverage, and diversification.

DCC-QIS/QuEST yields a small improvement in realized volatility in a few cases.

The contrast between the simplicity of the Average Oracle and the sophistication of DCC-QIS/QuEST begs to incorporate ingredients from econometric methods, i.e. to leverage the qualities of both approaches in order to yield even better covariance filtering tools. This is left for future work.

Acknowledgments

This publication used HPC resources from the ‘Mésocentre’ computing center of CentraleSupélec and École Normale Supérieure Paris-Saclay supported by CNRS and Région Île-de-France.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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