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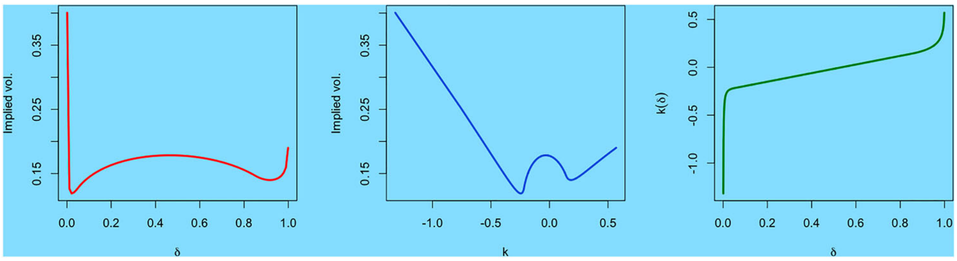

Smiles in delta

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Fukasawa introduced in Fukasawa [The normalizing transformation of the implied volatility smile. *Math. Finance*, 2012, **22**(4), 753–762] two necessary conditions for no butterfly arbitrage on a given implied volatility smile which require that the functions d_1 and d_2 of the Black–Scholes formula have to be decreasing. In this article, we characterize the set of smiles satisfying these conditions, using the parametrization of the smile in delta. We obtain a parametrization of the set of such smiles via one real number and three positive functions, which can be used by practitioners to calibrate a weak arbitrage-free smile. We also show that such smiles and their symmetric smiles can be transformed into smiles in the strike space by a bijection. Our result motivates the study of the challenging question of characterizing the subset of butterfly arbitrage-free smiles using the parametrization in delta.

Keywords: Implied volatility; Volatility smile; Delta; Butterfly arbitrage

JEL Classification: G13, C60, C63

1. Introduction

FX OTC options are quoted in delta through the At-The-Money volatility, and Risk Reversal and Strangle prices for different delta points. On Equity and Commodity markets, many trading firms analyze the risk of options portfolios at a given maturity on a grid of deltas instead of a grid of strikes. This fundamentally relates to the high-view of the risk as being driven at first order by a delta risk and a Vega risk, which underpins in particular CME's SPAN methodology, which, although dated back to 1988, is still widely used in CCPs.

From this perspective, a first immediate question is whether the transformation of a smile $\hat{\sigma}$ in the strike space to a smile σ in the delta space is well-defined. We show that it is the case under the assumption that the map $k \to \delta(k) := N(d_1(k, \hat{\sigma}(k)))$ is decreasing, where $d_1 = -\frac{k}{\hat{\sigma}(k)} + \frac{\hat{\sigma}(k)}{2}$ denotes the classical quantity in the Black–Scholes formula, N the Gaussian cumulative density function, and k the log-forward moneyness. Thanks to a result of Fukasawa (2012), we know that this property is fulfilled by smiles with no butterfly arbitrage. Given the fact that a smile has no butterfly arbitrage if and only if the symmetrical smile (in log forward-moneyness) has no butterfly arbitrage, such smiles will also have the property that the map $k \to \bar{\delta}(k) := N(d_2(k, \hat{\sigma}(k)))$ is decreasing, since $d_1(k, \hat{\sigma}(-k))$ equals $-d_2(-k, \hat{\sigma}(-k))$.

The possibility of transforming a smile in the delta space into a smile in the strike space is also studied in this work, and a characterization of such smiles is obtained. This result sustains the widespread practice of the industry to calibrate smiles in the delta space to recover smiles in the strike space used to determine margins of options in strike. The present paper does not look at the question of reconstructing a smile in strike from the ATM, butterfly and risk reversal market quotes in delta, which is sustained in Reiswich and Uwe (2012). The two main differences are that we work with full smiles in delta instead of the standard FX market quotes, and also that we account for no butterfly arbitrage.

The second immediate question is how the no butterfly arbitrage condition translates in the delta space. Surprisingly, this question is essentially an open one. Some results on the absence of arbitrage for implied volatility surfaces in the delta space can be found in Lucic (2021). We don't address the no butterfly arbitrage in delta in this work, but we consider instead the weaker condition obtained by Fukasawa that the two mappings $k \to d_{1,2}(k, \hat{\sigma}(k))$ are decreasing, and obtain in Theorem 5.2 an explicit parametrization of the smiles in delta fulfilling those conditions. This family can be useful in

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practice, since such smiles are expected to be not too far from fully (strongly) no arbitrage ones. Practitioners wishing to calibrate arbitrage-free smiles should then take into consideration the parametrization in Theorem 5.2 since every arbitrage-free smile in delta can be represented through it.

From a theoretical view point, another take on our work is to consider the open question of parametrizing all the smiles with no butterfly arbitrage. Such a parametrization would allow practitioners to calibrate implied volatility smiles with the certainty of fulfilling no butterfly arbitrage and without range restrictions coming from the choice of a particular model (like Gatheral's SVI, see for example Gatheral 2011). This question is probably a difficult one, and our work can be seen as a solution to the same question for a notion of *weak arbitrage*, through a move to the delta space. This could suggest that there is some hope to solve also the initial question in the delta space.

We start in Section 2 with a description of the delta notation and a detailed discussion of the no butterfly arbitrage conditions in the delta space.

In Section 3.1 we characterize the set of smiles in delta that can be converted to smiles in log-forward moneyness, i.e. the set of smiles that allow to unambiguously define a delta function $k \to \delta(k)$ from the relation $\delta(k) = N(d_1(k, \sigma(\delta(k)))$. Looking at a similar question but from the strike perspective, Section 3.2 achieves the characterization of the set of smiles in delta which correspond to an existing smile in strike, i.e. smiles that are defined from a strike function $\delta \to k(\delta)$ satisfying $\delta = N(d_1(k(\delta), \hat{\sigma}(k(\delta)))$. The two sets are shown to coincide, so that a smile in delta belonging to them can be transformed into a smile in strike and re-transformed in the original smile in delta. We synthetize in Section 3.4.2 a practical methodology to calibrate smiles satisfying this property.

Section 4 summarizes some properties of the smiles in delta which have a corresponding smile in strike. In particular, we write in the delta notation the notions of maximum and minimum points in the smiles, the second order approximation of the smiles around 0 and the Lee asymptotic conditions in Lee (2004).

Section 5 deals with the set of smiles in delta which satisfy the weak no arbitrage conditions of monotonicity of the functions d_1 and d_2. These smiles contain the subset of the butterfly arbitrage-free smiles and are for this reason of interest for practitioners calibrating smiles in the delta notation. In particular, the result in Theorem 5.2 shows that such set can be parametrized by a real number and three positive functions:

$$\sigma(\delta)\sqrt{T} = \begin{cases} N^{-1}(\delta) + \sqrt{N^{-1}(\delta)^2 + 2\left(\int_\delta^{\frac{1}{2}} \lambda(x)\,\mathrm{d}x + \int_{\frac{1}{2}}^{\tilde{\delta}} \mu(x)\,\mathrm{d}x\right)} & \text{if } \delta \le \frac{1}{2}, \\ N^{-1}(\delta) + \sqrt{2\int_\delta^{\tilde{\delta}} \mu(x)\,\mathrm{d}x} & \text{if } \frac{1}{2} < \delta \le \tilde{\delta}, \\ N^{-1}(\delta) - \sqrt{2\int_{N^{-1}(\tilde{\delta})}^{N^{-1}(\delta)} x\beta(x)\,\mathrm{d}x} & \text{if } \delta > \tilde{\delta}. \end{cases}$$

Table 1. Symbols used in the article and their meaning.

Symbol	Meaning
σ	Smile in delta
$\hat{\sigma}$	Smile in log-forward moneyness, here abbreviated as smile in strike
Σ	Set of smiles in delta
$\mathbb{R}^+$	Real numbers strictly larger than 0
$\mathbb{R}^+_*$	Non-negative real numbers
$\mathcal{C}(\mathcal{A}, \mathcal{B})$	Set of continuous functions from the open interval $\mathcal{A}$ to the open interval $\mathcal{B}$
D	Subscript for sets related to the transformation of a smile in delta into a smile in strike
K	Subscript for sets and functions related to the transformation of a smile in strike into a smile in delta
a.e.	almost everywhere

The functions λ, μ and β are required to satisfy some weak conditions (positiveness and diverging integrals) that can be easily achieved. We also compare our result with the parametrization in the strike space obtained by Lucic in Theorem 2.2 of Lucic (2021). In Section 5.2 we describe a calibration routine which can be adopted by practitioners to fit market smiles in delta with weak arbitrage-free functions.

The section ends with practical examples of smiles in the weak no arbitrage set. In particular, both skew-shaped smiles and W-shaped smiles can be easily obtained with appropriate choices of the functions λ, μ and β (see Section 5.3).

2. Notations and preliminaries

In table 1 we summarize the notations that will be used in the article.

For a fixed maturity T, we denote by $C_{\mathrm{BS}}(k, \hat{\sigma}(k))$ the Black–Scholes pricing formula for a call option with maturity T, strike $F_0(T)e^k$, forward value $F_0(T)$, discounting factor $D_0(T)$, and implied volatility $\hat{\sigma}(k)$:

$$C_{\mathrm{BS}}(k, \hat{\sigma}(k)) = D_0(T)F_0(T)\left(N(d_1(k, \hat{\sigma}(k)) - e^k N(d_2(k, \hat{\sigma}(k))\right),$$

$$d_{1,2}(k, \hat{\sigma}(k)) = -\frac{k}{\hat{\sigma}(k)\sqrt{T}} \pm \frac{\hat{\sigma}(k)\sqrt{T}}{2}.$$

For easier notation, we will sometimes denote with $d_1(k)$ and $d_2(k)$ the functions $d_1(k, \hat{\sigma}(k))$ and $d_2(k, \hat{\sigma}(k))$ respectively.

Let a number $C(k)$ lie strictly between $D_0(T)(F_0(T) - K)^+$ and $D_0(T)F_0(T)$. Then the *implied volatility* $\hat{\sigma}(k)$ is well defined in $]0, \infty[$ by the equation $C_{\mathrm{BS}}(k, \hat{\sigma}(k)) = C(k)$. In turn, this defines the quantity *delta* by

$$\delta(k) := N(d_1(k, \hat{\sigma}(k))). \tag{1}$$

Remark 2.1 Observe that the pair $(\delta(k), \hat{\sigma}(k))$ allows to recover the log-forward moneyness k, indeed

$$k = \left(-N^{-1}(\delta(k)) + \frac{\hat{\sigma}(k)\sqrt{T}}{2}\right)\hat{\sigma}(k)\sqrt{T}.$$

2.1. *No butterfly arbitrage and convexity of Call prices*

Under the hypothesis of a perfect market for the underlying asset and for the Call options a Call price function with respect to the strike is free of butterfly arbitrage if and only if (cf. e.g. Tehranchi 2020) it is

(i) convex,
(ii) non-increasing,
(iii) contained in the interval $]D_0(T)(F_0(T)-K)^+, D_0(T)F_0(T)[$.

In the case of Call prices specified with the Black–Scholes formula through an implied volatility, the third property is automatically satisfied. Indeed, the Black–Scholes formula is increasing with respect to the implied volatility and it tends to the two bounds when the implied volatility goes to 0 and ∞ respectively. Given that the third property is granted, then the first property implies the second since an increasing and convex function cannot be bounded.

Therefore, in our context, there is no butterfly arbitrage if and only if Call prices are convex.

As shown in Theorem 2.9 condition (IV3) of Roper (2010), in the case of twice differentiable implied volatility functions, the requirement of convexity corresponds to the requirement that the function

$$\hat{\sigma}''(k) + d_1'(k,\hat{\sigma}(k))d_2'(k,\hat{\sigma}(k))\hat{\sigma}(k) \tag{2}$$

is non-negative. This requirement is sometimes called the Durrleman condition (Durrleman 2010).

2.2. *Behavior of d_1 and d_2 under convexity assumptions*

As shown in section 2.3 of Martini and Mingone (2022), the condition of vanishing Call prices for increasing strikes is not necessary for the absence of arbitrage. Such condition is one-to-one with the behavior of the function d_1 at ∞. In particular, it holds true if and only if the property

P1. $\lim_{k\to\infty} d_1(k,\hat{\sigma}(k)) = -\infty$

is satisfied. This follows from the fact that the arithmetic mean exceeds the geometric mean, so that the function $d_2(k,\hat{\sigma}(k))$ is smaller than $-\sqrt{2k}$ and goes to $-\infty$ at ∞. For negative z, the Mills ratio $\frac{N(z)}{n(z)}$, where n is the Gaussian pdf, satisfies $\frac{N(z)}{n(z)} < \frac{1}{|z|}$. Observe that $n(d_2(k,\hat{\sigma}(k)) < n(-\sqrt{2k}) = \frac{e^{-k}}{\sqrt{2\pi}}$, so that the quantity $e^k N(d_2(k,\hat{\sigma}(k))$ is smaller than the Mills ratio applied to $z = d_2(k,\hat{\sigma}(k))$ (divided by the constant $\sqrt{2\pi}$), which in turn is smaller than $(|d_2(k,\hat{\sigma}(k))|\sqrt{2\pi})^{-1}$. Then, the value of a Call price at ∞ is

$$C(\infty) = D_0(T)F_0(T)N(d_1(\infty)). \tag{3}$$

From now on and only in this section, we work under the assumption that Call prices are convex, i.e. that the function $K \to C(K) = C_{\mathrm{BS}}(k,\hat{\sigma}(k))|_{k=\log\frac{K}{F_0(T)}}$ is convex. Under the hypothesis of convex prices and **P1**, for every $K \geq 0$, Call prices have the representation

$$C(K) = \int_{\mathbb{R}_*^+} (x-K)^+ \mu(\mathrm{d}x)$$

where μ is a probability measure supported by $\mathbb{R}_*^+$. Then, using the identity $(x-K)^+ = \int_K^\infty \mathbb{1}_{y<x}\,\mathrm{d}y$ and Fubini's theorem, the following formula holds:

$$C(K) = \int_K^\infty (1 - F_\mu(x))\,\mathrm{d}x,$$

where $F_\mu(K) = \mu([0,K])$ is the CDF of the measure μ. Taking the right derivative of the above formula, it holds $C'_+(K) = -1 + F_\mu(K)$, so that $\mu(\{0\}) = F_\mu(0) = 1 + C'_+(0)$. Observe that the bounds of the Call prices function imply that its slope for K going to 0 lies between -1 and 0. Then, the limit corresponds to -1 if and only if there is no mass of the underlying in 0. This in turn is equivalent to the fact that the left limit of the function d_2 is ∞:

P2. $\lim_{k\to-\infty} d_2(k,\hat{\sigma}(k)) = \infty$,

which follows from the fact that $d_2(-\infty) = -N^{-1}(\mu(\{0\}))$ as shown in proposition 2.4 of Fukasawa (2010).

Remark 2.2 **P1** and **P2** are not necessary conditions for butterfly arbitrage-free Call prices. **P1** implies that Call prices vanish for increasing strikes. Under the assumption of convex prices, **P2** implies that Call prices have a slope of -1 for null strikes and that there is no mass of the underlying in 0.

P1 and **P2** imply some easy consequences that we state in the following proposition. The Lee conditions can be found in Lee (2004), Lemmas 3.1 and 3.3.

Proposition 2.1 ***P1*** *holds if and only if $d_1(k)$ is surjective, and* ***P2*** *holds if and only if $d_2(k)$ is surjective. Furthermore,*

- ***P1*** *implies the Lee right wing condition, i.e. $\hat{\sigma}(k)\sqrt{T} < \sqrt{2k}$ for k large enough;*
- ***P2*** *implies the Lee left wing condition, i.e. $\hat{\sigma}(k)\sqrt{T} < \sqrt{-2k}$ for k small enough.*

Proof Since the arithmetic mean exceeds the geometric mean, the function $k \to d_1(k,\hat{\sigma}(k))$ is greater than $\sqrt{-2k}$ for every $k \leq 0$ (see Lemma 3.5 of Fukasawa 2012). As a consequence, $d_1(k,\hat{\sigma}(k))$ always goes to ∞ as k goes to $-\infty$. Similarly with the same proof, the function $d_2(k,\hat{\sigma}(k))$ always goes to $-\infty$ at ∞ since it is smaller than $-\sqrt{2k}$. Then, under **P1** and **P2**, the functions $d_1(k)$ and $d_2(k)$ are surjective. The if implication is trivial.

Since the function $k \to d_1(k,\hat{\sigma}(k))$ goes to $-\infty$ on the right under **P1**, it must be negative for k large enough, which implies the right wing Lee condition. Similarly, since the function $k \to d_2(k,\hat{\sigma}(k))$ explodes on the left, it must be positive for k small enough and the left wing Lee condition holds. ■

Observe that Lee shows that the left wing condition holds for every arbitrage-free smile if and only if $P(S_T = 0) < \frac{1}{2}$, and in particular that no mass in 0 implies the left wing condition, as in the above proposition. Furthermore, the proof for the only if implication can also be found in Lemmas 3.2 and 3.5 of Fukasawa (2012).

We will often use necessary conditions for the absence of butterfly arbitrage found by Fukasawa in Theorem 2.8 of Fukasawa (2012). We recall them in the following generalized remark from Lemma 3.1 of Martini and Mingone (2022).

REMARK 2.3 Weak no butterfly arbitrage conditions/Fukasawa necessary no butterfly arbitrage conditions If $k \to C(k)$ is convex and $k \to \hat{\sigma}(k)$ is differentiable, then

F1 $k \to d_1(k, \hat{\sigma}(k))$ is strictly decreasing,
F2 $k \to d_2(k, \hat{\sigma}(k))$ is strictly decreasing.

2.3. *No butterfly arbitrage in the delta notation*

As discussed above, a difficult challenge is to parametrize the set of implied volatility functions corresponding to functions $k \to C(k)$ with no butterfly arbitrage. We will denote with Σ_{A} the set of such implied volatility functions (in the delta notation $\delta \to \sigma(\delta)$) satisfying also **P1** and **P2**. In the following, we will show that these delta smiles can be transformed into strike smiles and, given the above study, they will guarantee the monotonicity and surjectivity of functions $d_1(k, \hat{\sigma}(k))$ and $d_2(k, \hat{\sigma}(k))$. Even though we will not reach the aim of parametrizing the set Σ_{A}, we will be able to achieve the parametrization of the larger set of smiles satisfying the two monotonicity conditions on the functions $d_1(k, \hat{\sigma}(k))$ and $d_2(k, \hat{\sigma}(k))$.

2.3.1. The range of the functions δ and $\bar{\delta}$.

In this section we consider properties **P1** and **P2** in the delta notation, since they influence the range of the function $\delta(k)$ in equation (1) and of its symmetric delta $\bar{\delta}(k)$ defined as

$$\bar{\delta}(k) := N(d_1(k, \hat{\bar{\sigma}}(k))) \tag{4}$$

where

$$\hat{\bar{\sigma}}(k) = \hat{\sigma}(-k).$$

PROPOSITION 2.2 *The function* $\delta(k)$ *defined in equation* (1) *has range* $]\frac{C(\infty)}{D_0(T)F_0(T)}, 1[$*, and this coincides with* $]0, 1[$ *if and only if* ***P1*** *holds.*

The function $\bar{\delta}(k)$ *defined in equation* (4) *has range* $]1 + C'(0), 1[$*, and this coincides with* $]0, 1[$ *if and only if* ***P2*** *holds.*

Proof It is easy to see that the function $\delta(k)$ goes to 0 as k goes to ∞ if and only if **P1** holds. The condition $\delta(-\infty) = 1$ corresponds to the fact that $d_1(k, \hat{\sigma}(k))$ goes to ∞ as k decreases, but this is always true. More precisely, from equation (3) it follows

$$C(\infty) = D_0(T)F_0(T)\delta(\infty)$$

so that the range of $\delta(k)$ is $]\frac{C(\infty)}{D_0(T)F_0(T)}, 1[$.

Secondly, the assumption **P2** is the equivalent of the assumption **P1** for the symmetric smile $\hat{\bar{\sigma}}(k) = \hat{\sigma}(-k)$. Indeed, it holds $d_1(k, \hat{\bar{\sigma}}(k)) = -d_2(-k, \hat{\sigma}(-k))$, so that condition **P2** is equivalent to requiring that the d_1 function for the symmetric smile, i.e. $d_1(k, \hat{\bar{\sigma}}(k))$, satisfies **P1**. It is immediate that

$$\bar{\delta}(\infty) = 1 + C'(0)$$

since $d_2(-\infty) = -N^{-1}(1 + C'(0))$. On the other hand, the limit of $\bar{\delta}(k)$ for k decreasing is still 1. Then, the range of the symmetric delta $\bar{\delta}(k)$ is $]1 + C'(0), 1[$. ■

3. From a smile in delta to a smile in strike and vice versa

3.1. *From a smile in delta to a smile in strike*

In the industry, it is not an unusual practice to calibrate volatility smiles in the delta parametrization, instead of the usual strike one, especially when dealing with Forex products. When options are quoted on a grid of maturities and deltas, such choice is natural and easy to be exploited. When, on the other hand, options are quoted on a grid of maturities and strikes, the permutation between delta and strike smiles is not straightforward.

Indeed, it is firstly necessary to transform data in strikes into data in deltas. For a fixed maturity, the procedure reads:

(i) compute volatilities $\hat{\sigma}(k)$ for the quoted strikes $F_0(T)e^k$ using the inversion of the Black–Scholes pricing formula;
(ii) compute corresponding deltas $\delta(k)$ with equation (1) and uniquely associate them to the smile values;
(iii) interpolate pairs $(\delta(k), \hat{\sigma}(k)) = (\delta, \sigma(\delta))$ with a chosen method in order to recover the continuous smile $\delta \to \sigma(\delta)$.

At this point, different operations can be done using the smiles in delta, such as collecting historical values, stressing data, doing statistics, and so on. When it is necessary to come back to the strike notation, for example to compute a stressed option price with known strike, the smile in delta must have the ability to be converted into a smile in strike.

In this section we aim to find conditions under which any positive and continuous function $\delta \to \sigma(\delta)$ defined on $]0, 1[$ allows to recover a function $k \to \hat{\sigma}(k)$.

DEFINITION 3.1 *We call* Σ_D *the set of positive and continuous delta smiles* $\delta \to \sigma(\delta)$ *defined on* $]0, 1[$ *for which there exists a surjective mapping* $k \to \delta(k)$ *defined on* $\mathbb{R}$ *satisfying* $\delta(k) = N(d_1(k, \sigma(\delta(k))))$.

In particular,

$$\Sigma_D := \Big\{\delta \to \sigma(\delta) \in \mathcal{C}(]0, 1[, \mathbb{R}^+) \,\Big|\, \forall k \in \mathbb{R}\, \exists! \delta(k) \,| \\ \delta(k) = N(d_1(k, \sigma(\delta(k)))), \{\delta(k) \,|\, k \in \mathbb{R}\} =]0, 1[\Big\}.$$

For a smile in Σ_D, the corresponding smile in strike is defined as $\hat{\sigma}(k) := \sigma(\delta(k))$. Note that there is indeed a question, because in the above procedure it could happen that two different pairs $(\delta, \sigma(\delta))$ produce the same strike, meaning that there is no way to define the value $\hat{\sigma}(k)$. The second condition defining Σ_D is added in order to avoid degenerate cases.

Let us define the function

$$l(\delta) := \left(N^{-1}(\delta) - \frac{\sigma(\delta)\sqrt{T}}{2} \right) \sigma(\delta)\sqrt{T}. \qquad (5)$$

In the following Lemma, we characterize the set Σ_D through the function $l(\delta)$.

LEMMA 3.1 *A continuous and positive delta smile $\delta \to \sigma(\delta)$ belong to Σ_D if and only if the function $\delta \to l(\delta)$ is strictly increasing and surjective onto $\mathbb{R}$.*

In other words,

$$\Sigma_D = \left\{ \delta \to \sigma(\delta) \in \mathcal{C}(]0,1[, \mathbb{R}^+) \mid l \text{ strictly increasing and surjective onto } \mathbb{R} \right\}.$$

Proof Reformulating the conditions of Σ_D, we ask that given a continuous positive function $\delta \to \sigma(\delta)$ defined on $]0,1[$,

- for every $k \in \mathbb{R}$ there exists a unique $\delta(k)$ such that $-k = l(\delta(k))$,
- the mapping $k \to \delta(k)$ is surjective in $]0,1[$.

This implies that the function $k \to l(\delta(k))$ must be well-defined, monotonic and surjective. For a delta smile in Σ_D, the function $k \to \delta(k)$ is monotonic and surjective, so that we are requiring $\delta \to l(\delta)$ to be monotonic and surjective from the interval $]0,1[$ to $]-\infty, +\infty[$. Since $l(0)$ is negative, l must be strictly increasing.

These conditions are necessary but also sufficient. Indeed, the uniqueness of $\delta(k)$ follows from the fact that if for a fixed k there are two different $\delta(k)$, say δ_1 and δ_2, then $l(\delta_1) \neq l(\delta_2)$ since l is monotonic. However, $l(\delta(k)) = -k$, so the two values of l should be the same. The existence of $\delta(k)$ is guaranteed by the surjectivity of l.

For the second condition, firstly observe that from the relation $-k = l(\delta(k))$, it follows $l(\delta(\infty)) = -\infty$ and since l is injective, it must hold $\delta(\infty) = 0$. Similarly we can show $\delta(-\infty) = 1$. These observations guarantee the full range of $\delta(k)$. ■

3.2. From a smile in strike to a smile in delta

Starting from a smile in strike $k \to \hat{\sigma}(k)$ and making k move in $\mathbb{R}$, one obtains a collection of pairs $(\delta(k), \hat{\sigma}(k))$. There is no guarantee that such a collection allows to define a function $\delta \to \sigma(\delta)$. Indeed, in order to define a function in δ, for every $\delta \in]0,1[$, there must exist a unique k such that $\delta = N(d_1(k, \hat{\sigma}(k)))$. In such a way one can define unambiguously a function σ by the equality $\sigma(\delta) = \hat{\sigma}(k)$ for all ks. We define Σ_K the set of delta smiles obtained from a smile in strike when this condition holds.

DEFINITION 3.2 *We call Σ_K the set of positive and continuous delta smiles $\delta \to \sigma(\delta)$ defined on $]0,1[$ for which there exist a positive and continuous strike smile $k \to \hat{\sigma}(k)$ and a surjective mapping $\delta \to k(\delta)$ such that $\delta = N(d_1(k(\delta), \hat{\sigma}(k(\delta))))$ and $\hat{\sigma}(k(\delta)) = \sigma(\delta)$.*

In particular,

$$\begin{aligned}
\Sigma_K := \Big\{ \delta \to \sigma(\delta) \in \mathcal{C}(]0,1[, \mathbb{R}^+) \;\Big|\; & \exists k \to \hat{\sigma}(k) \in \mathcal{C}(\mathbb{R}, \mathbb{R}^+) \\
& s.t.\ \forall \delta \in]0,1[\ \exists! k(\delta)\ s.t. \\
& \delta = N(d_1(k(\delta), \hat{\sigma}(k(\delta)))), \\
& \hat{\sigma}(k(\delta)) = \sigma(\delta), \\
& \{k(\delta) \mid \delta \in]0,1[\} = \mathbb{R} \Big\}.
\end{aligned}$$

This definition can be simplified thanks to the following Lemma:

LEMMA 3.2 *A continuous and positive delta smile $\delta \to \sigma(\delta)$ belongs to Σ_K if and only if there exists a continuous and positive strike smile $k \to \hat{\sigma}(k)$ with $\hat{\sigma}(k) = \sigma(N(d_1(k, \hat{\sigma}(k))))$ and such that the function $k \to d_1(k, \hat{\sigma}(k))$ is strictly decreasing and surjective.*

In other words,

$$\begin{aligned}
\Sigma_K = \Big\{ \delta \to \sigma(\delta) \in \mathcal{C}(]0,1[, \mathbb{R}^+) \;\Big|\; & \exists k \\
& \to \hat{\sigma}(k) \in \mathcal{C}(\mathbb{R}, \mathbb{R}^+)\ s.t. \\
& k \to d_1(k, \hat{\sigma}(k)) \text{ is strictly decreasing and surjective}, \\
& \hat{\sigma}(k) = \sigma(N(d_1(k, \hat{\sigma}(k)))) \Big\}.
\end{aligned}$$

Proof The existence and uniqueness of a $k(\delta)$ in the first condition defining Σ_K can be translated in requiring that the map $k \to N(d_1(k, \hat{\sigma}(k)))$ is strictly monotonic. The second condition defining Σ_K requires the surjectivity of $\delta \to k(\delta)$. Equivalently, the two conditions hold if and only if the map $k \to d_1(k, \hat{\sigma}(k))$ is strictly monotonic and surjective in $]-\infty, \infty[$. The decreasing behavior is due to the formula defining the map $k \to d_1(k, \hat{\sigma}(k))$. ■

REMARK 3.1 In Lemma 3.2 we have shown that all smiles in delta living in Σ_K are such that their smile in strike satisfies **P1** and **F1**, i.e. $d_1(k, \hat{\sigma}(k))$ is decreasing and surjective.

3.3. Reversibility of smiles in delta to smiles in strike

We now look at the relation between smiles in delta that can be transformed into smiles in strike and smiles in strike that can be transformed in smiles in delta. It turns out that the image of the latter set in the space of smiles in delta actually coincides with the former set. In other words, a smile in delta obtained through a smile in strike can be re-transformed into the original smile in strike. Also, any smile in delta that can be transformed in a smile in strike can be recovered from its transformation into a smile in strike.

PROPOSITION 3.3 *It holds $\Sigma_D = \Sigma_K$ and $\delta \to k(\delta)$ is the inverse of $k \to \delta(k)$.*

Proof We firstly prove $\Sigma_K \subset \Sigma_D$. For a function $\delta \to \sigma(\delta)$ in Σ_K and a fixed $k \in \mathbb{R}$, there exists a unique $\delta_K(k)$ such that $\delta_K(k) = N(d_1(k, \hat{\sigma}(k)))$. Given the definition of $\sigma(\delta)$, it holds $\hat{\sigma}(k) = \sigma(\delta_K(k))$, so that $\delta_K(k) = N(d_1(k, \sigma(\delta_K(k))))$. Suppose there is a second $\tilde{\delta}$ such that $\tilde{\delta} = N(d_1(k, \sigma(\tilde{\delta})))$. For such $\tilde{\delta}$ there is a unique $k(\tilde{\delta})$ such that $\tilde{\delta} = N(d_1(k(\tilde{\delta}), \hat{\sigma}(k(\tilde{\delta}))))$, furthermore $\hat{\sigma}(k(\tilde{\delta})) =$

$\sigma(\tilde{\delta})$. Then, $N(d_1(k, \sigma(\tilde{\delta}))) = \tilde{\delta} = N(d_1(k(\tilde{\delta}), \sigma(\tilde{\delta})))$, from which it immediately follows $k = k(\tilde{\delta})$. Then for k, it both holds $\delta_K(k) = N(d_1(k, \hat{\sigma}(k))$ and $\tilde{\delta} = N(d_1(k, \hat{\sigma}(k)))$, so that $\delta_K(k) = \tilde{\delta}$. The function $\delta_K(k)$, in particular, corresponds to the function $\delta(k)$ defining the set Σ_D.

On the other hand, we now look at the relation $\Sigma_D \subset \Sigma_K$. Let $\delta \to \sigma(\delta)$ in Σ_D, then we can define a function $k \to \hat{\sigma}(k)$ such that $\hat{\sigma}(k) := \sigma(\delta(k))$, where $\delta(k)$ is the only δ satisfying $\delta = N(d_1(k, \sigma(\delta))$. If two values k_1 and k_2 have the same delta $\delta(k_1) = \delta(k_2)$, then they also have the same volatility since $\hat{\sigma}(k_1) = \sigma(\delta(k_1)) = \sigma(\delta(k_2)) = \hat{\sigma}(k_2)$. From Remark 2.1, the pair $(\delta(k_1), \hat{\sigma}(k_1)) = (\delta(k_2), \hat{\sigma}(k_2))$ is associated to a unique log-forward moneyness, so that $k_1 = k_2$. Then $k \to \delta(k)$ is injective. The function $d_1(k) = d_1(k, \hat{\sigma}(k))$ is surjective since for any $\delta \in]0, 1[$ there exists a k such that $\delta = \delta(k)$ so that $\delta = N(d_1(k, \sigma(\delta)) = N(d_1(k, \hat{\sigma}(k))$. To prove the monotonicity, firstly observe that for any k there is a unique $\delta(k)$ such that $\delta(k) = N(d_1(k, \sigma(\delta(k))))$, which is equivalent to write

$$l(\delta(k)) = -k. \tag{6}$$

Secondly, from the definition of $\hat{\sigma}(k)$, it holds

$$\delta(k) = N(d_1(k, \hat{\sigma}(k))). \tag{7}$$

The functions $l(\delta)$, $\delta(k)$ and $d_1(k)$ are monotonic into an open interval, so they are a.e. differentiable. Taking derivatives with respect to k (in the complementary of the zero measure set where these derivatives are not defined) in both equations (6) and (7), we find $l'(\delta(k))\frac{\mathrm{d}\delta}{\mathrm{d}k}(k) = -1$ and $\frac{\mathrm{d}\delta}{\mathrm{d}k}(k) = n(d_1(k))d_1'(k)$, so in particular

$$d_1'(k)l'(\delta(k)) = -\frac{1}{n(d_1(k))}.$$

Since $\delta \to \sigma(\delta)$ lives in Σ_D, l is increasing so d_1 is decreasing and $\sigma(\delta)$ lives also in Σ_K. The inverse of the function $k \to \delta(k)$, in particular, corresponds to the function $\delta \to k(\delta)$ defined in Σ_K. ■

REMARK 3.2 From now on, we will generally denote the identical sets Σ_D and Σ_K as Σ.

REMARK 3.3 As a side-product of the proof of Proposition 3.3, the functions $k \to \delta(k)$ and $\delta \to k(\delta)$ are both a.e. differentiable. Furthermore, in the proof we have shown that a smile in strike transformed into a smile in delta, can be re-transformed into a smile in strike and such smile necessarily coincides with the initial one.

3.4. Calibration of a delta smile in Σ

Suppose we want to calibrate a smile in delta to market quotes given in either the strike or the delta variable, in such a way that the calibrated smile can be converted to a smile in strike and vice versa, i.e. that the smile lives in Σ. We could go through the following steps:

(i) (a) consider the market discrete pillars in delta notation $\{(\delta_i, \sigma_i)\}_i$;
  (b) if the market discrete pillars are in strike notation $\{(k_i, \sigma_i)\}_i$, convert them to the pillars in delta notation $\{(\delta_i, \sigma_i)\}_i$ by defining $\delta_i = N(d_1(k_i, \sigma_i))$;

(ii) compute the pillars $\{(\delta_i, l_i)\}_i$ with $l_i = (N^{-1}(\delta_i) - \frac{\sigma_i\sqrt{T}}{2})\sigma_i\sqrt{T}$;

(iii) interpolate/extrapolate an increasing and surjective function $\delta \to l(\delta)$, given pillars in point (ii).

The last natural point would be to recover a function $\delta \to \sigma(\delta)$ by the calibrated function $l(\delta)$. However, this reduces to solve the equation

$$\frac{\sigma^2 T}{2} - N^{-1}(\delta)\sqrt{T}\sigma + l(\delta) = 0,$$

which could have none, one or two solutions. In the following section we study the problem of existence and uniqueness of the solution σ.

3.4.1. Conditions on *l* for the existence of σ.

PROPOSITION 3.4 *Let $l :]0, 1[\to \mathbb{R}$ an increasing and surjective function. The equation*

$$\frac{\sigma^2 T}{2} - N^{-1}(\delta)\sqrt{T}\sigma + l(\delta) = 0 \tag{8}$$

has at least one solution σ for every $\delta \in]0, 1[$ if and only if $l(\frac{1}{2}) < 0$ and $l(\delta) \leq \frac{N^{-1}(\delta)^2}{2}$ for all $\delta > \frac{1}{2}$. The continuous solution $\delta \to \sigma(\delta)$ is

$$\left(N^{-1}(\delta) + \sqrt{N^{-1}(\delta)^2 - 2l(\delta)}\right)/\sqrt{T} \tag{9}$$

for $\delta \leq \frac{1}{2}$ and it could switch between equation (9) *and*

$$\left(N^{-1}(\delta) - \sqrt{N^{-1}(\delta)^2 - 2l(\delta)}\right)/\sqrt{T} \tag{10}$$

at any $\tilde{\delta} > \frac{1}{2}$ such that $l(\tilde{\delta}) = \frac{N^{-1}(\tilde{\delta})^2}{2}$.

Proof For a fixed $\delta \in]0, 1[$, the two admissible solutions to equation (8) are

$$\sigma_{\pm}(\delta) = \frac{N^{-1}(\delta)}{\sqrt{T}} \pm \frac{\sqrt{N^{-1}(\delta)^2 - 2l(\delta)}}{\sqrt{T}}.$$

For the existence, the delta of the equation must be non-negative and at least one of the two solutions must be positive, so

$$N^{-1}(\delta)^2 - 2l(\delta) \geq 0,$$
$$N^{-1}(\delta) \pm \sqrt{N^{-1}(\delta)^2 - 2l(\delta)} > 0,$$

where the sign depends on the chosen solution.

Firstly, when $\delta \leq \frac{1}{2}$, the quantity $N^{-1}(\delta)$ is non-positive and the σ_- solution is negative, so it can be discarded. Instead, the σ_+ solution is well-defined and positive if and only if $l(\delta) < 0$. Since l is increasing, the latter condition is equivalent to $l(\frac{1}{2}) < 0$.

When $\delta > \frac{1}{2}$, then $N^{-1}(\delta)$ is positive and both the solutions could be valid. Under the requirement $l(\delta) \leq \frac{N^{-1}(\delta)^2}{2}$, the σ_+ solution is always positive while the σ_- solution becomes positive when l becomes positive, and it will stay positive since l is increasing.

The possibility to pass from the σ_+ to the σ_- solution at a point $\tilde{\delta}$ must be guaranteed by continuity of the volatility on $\tilde{\delta}$. Then, it should hold $\sigma_+(\tilde{\delta}) = \sigma_-(\tilde{\delta})$, or $2l(\tilde{\delta}) = N^{-1}(\tilde{\delta})^2$ and $\sigma(\tilde{\delta})\sqrt{T} = N^{-1}(\tilde{\delta})$. Rewriting equation (8) as

$$l(\delta) = \left(N^{-1}(\delta) - \frac{\sigma(\delta)\sqrt{T}}{2}\right)\sigma(\delta)\sqrt{T}$$

and evaluating in $\tilde{\delta}$, one finds $l(\tilde{\delta}) = \frac{\sigma(\tilde{\delta})^2 T}{2}$. If there exists such a point $\tilde{\delta}$, then either the solution keeps being σ_+ after $\tilde{\delta}$, or it switches to σ_-. If there is no such point $\tilde{\delta}$, the solution remains σ_+. ■

As an immediate consequence of Proposition 3.4, under the requirements $l\left(\frac{1}{2}\right) < 0$ and $l(\delta) \leq \frac{N^{-1}(\delta)^2}{2}$, if there are no $\tilde{\delta}$ such that $l(\tilde{\delta}) = \frac{N^{-1}(\tilde{\delta})^2}{2}$, the σ solution is unique and it coincides with equation (9). If there are one or more points $\tilde{\delta}$, the uniqueness is not guaranteed since the σ solution could either switch between equations (9) and (10) or not do the switch. For the uniqueness of the solution, more requirements on the solution itself are needed.

In the following lemma we show that smiles in delta living in Σ actually satisfy the requirements $l\left(\frac{1}{2}\right) < 0$ and $l(\delta) \leq \frac{N^{-1}(\delta)^2}{2}$, so that calibrating a function $l(\delta)$ which satisfies such conditions and is increasing and surjective guarantees the existence of at least one solution to equation (8), and this solution lives in Σ. We will show in Lemma 5.1 that if we require the solution σ to satisfy **P1**, **P2**, **F1** and **F2**, then the uniqueness is satisfied.

LEMMA 3.5 *For every $\delta \to \sigma(\delta) \in \Sigma$, the function l satisfies $l(\frac{1}{2}) < 0$ and $l(\delta) \leq \frac{N^{-1}(\delta)^2}{2}$ for all $\delta > \frac{1}{2}$.*

Proof From the definition of Σ_K, it follows that its smiles are such that for any $\delta \in]0,1[$ there is a unique $k(\delta)$ satisfying $\delta = N(d_1(k(\delta), \hat{\sigma}(k(\delta))))$. In particular, by the definition of the smile $\delta \to \sigma(\delta)$, it holds $\delta = N(d_1(k(\delta), \sigma(\delta)))$, or $\frac{\sigma(\delta)^2 T}{2} - N^{-1}(\delta)\sqrt{T}\sigma(\delta) - k(\delta) = 0$. Then, $\sigma = \sigma(\delta)$ is a solution of equation (8) and the conclusion follows from Proposition 3.4.

The statement can also be proven by hand. Indeed, $l(\frac{1}{2})$ is equal to $-\frac{\sigma(\frac{1}{2})^2 T}{2}$, which is always negative, and the second condition reads

$$\left(N^{-1}(\delta) - \frac{1}{2}\sigma(\delta)\sqrt{T}\right)\sigma(\delta)\sqrt{T} \leq \frac{N^{-1}(\delta)^2}{2},$$

which simplifying becomes $(N^{-1}(\delta) - \sigma(\delta)\sqrt{T})^2 \geq 0$, which is always verified. ■

3.4.2. Application: calibration of delta smiles in Σ. We now reconsider the calibration of a smile in delta started in the introduction to this section. The aim here is to calibrate a delta smile which can be transformed into a smile in strike and vice versa, and in particular the smile in strike satisfies **F1** and **P1** (i.e. the function $d_1(k)$ is decreasing and surjective). We have seen that belonging to Σ guarantees the existence of a solution $\sigma(\delta)$ to equation (8) but not its uniqueness. In the following calibration procedure, we target one of the possible solutions. The reason behind the choice of this particular solution is that it approaches the weak arbitrage-free solution as we will see in Lemma 5.1.

The calibration methodology follows the steps:

(i) (a) consider the market discrete pillars in delta notation $\{(\delta_i, \sigma_i)\}_i$;
  (b) if the market discrete pillars are in strike notation $\{(k_i, \sigma_i)\}_i$, convert them to the pillars in delta notation $\{(\delta_i, \sigma_i)\}_i$ by defining $\delta_i = N(d_1(k_i, \sigma_i))$;

(ii) compute the pillars $\{(\delta_i, l_i)\}_i$ with $l_i = (N^{-1}(\delta_i) - \frac{\sigma_i\sqrt{T}}{2})\sigma_i\sqrt{T}$;

(iii) given the pillars in point (ii), interpolate/extrapolate a function $\delta \to l(\delta)$ such that
  (a) $l(0) = -\infty, l(1) = +\infty$,
  (b) l strictly increasing,
  (c) $l(\frac{1}{2}) < 0$,
  (d) $l(\delta) \leq \frac{N^{-1}(\delta)^2}{2}\ \forall \delta > \frac{1}{2}$,
  (e) $\exists! \tilde{\delta} |\, l(\tilde{\delta}) = \frac{N^{-1}(\tilde{\delta})^2}{2}$.

This would guarantee that the smile $\delta \to \sigma(\delta)$ defined as

$$\sigma(\delta)\sqrt{T} = \begin{cases} N^{-1}(\delta) + \sqrt{N^{-1}(\delta)^2 - 2l(\delta)} & \text{if } \delta \leq \tilde{\delta}, \\ N^{-1}(\delta) - \sqrt{N^{-1}(\delta)^2 - 2l(\delta)} & \text{if } \delta > \tilde{\delta}. \end{cases}$$

lives in Σ, i.e. that the smile in delta can be transformed into a smile in strike and reverted to the original smile in delta.

REMARK 3.4 The calibration here described requires the knowledge of market points $\{(\delta_i, \sigma_i)\}_i$ (or equivalently $\{(k_i, \sigma_i)\}_i$). However, in the FX market smiles are quoted in terms of ATM volatility, risk reversals and strangles associated to a specific delta.

The market point $(\delta_{\text{ATM}}, \sigma_{\text{ATM}})$ is easily retrieved by $\delta_{\text{ATM}} = N(\frac{\sigma_{\text{ATM}}\sqrt{T}}{2})$. Under the approximation that the δ risk reversal volatility is the difference between the volatility at δ and the volatility at $1-\delta$, while the strangle volatility is the average between these two volatilities, the market points $(\delta, \sigma(\delta))$ and $(1-\delta, \sigma(1-\delta))$ are easily calculated.

If the approximation is not used, there is ambiguity in the definition of market points associated to δ and $1-\delta$. Indeed, it is known (cf. e.g. Clark 2011) that there are 4 different strikes associated to risk reversals and strangles (and so 4 different deltas), so that the situation is much more intricate.

To our knowledge, market providers such as Bloomberg make available $\{(\delta_i, \sigma_i)\}_i$ market quotes (which the provider deducts under its internal models from standard FX quotes), so that no ambiguity arises.

3.4.3. Application: inversion of delta smiles in Σ into strike smiles. We have seen in Proposition 3.3 that smiles in delta living in Σ can be converted into smiles in strike satisfying **P1** and **F1**, using the relation $\hat{\sigma}(k) = \sigma(\delta(k))$. The function $\delta(k)$ is the inverse of $k(\delta) = -l(\delta)$. Then, if we are able to calibrate the inverse of $-l(\delta)$ for a given smile σ in delta living in Σ, we can directly calibrate a smile $\hat{\sigma}$ in strike satisfying **P1** and **F1**. To do so, we can use a monotonic preserving interpolation and a well-chosen fine enough grid for the δ_i points.

The inversion of the smile σ in delta living in Σ can then be performed following the steps:

(i) consider the fine enough discrete pillars in delta notation $\{(\delta_i, \sigma_i)\}_i$;

(ii) compute the pillars $\{(-l_i, \delta_i)\}_i$ with $l_i = (N^{-1}(\delta_i) - \frac{\sigma_i\sqrt{T}}{2})\sigma_i\sqrt{T}$;

(iii) given the pillars in point (ii), interpolate/extrapolate a function $-l \to \delta(-l)$ such that
  (a) $\delta(-\infty) = 1, \delta(\infty) = 0$,
  (b) δ strictly decreasing.

If the delta grid is fine enough and in particular it includes the points $(-l(\frac{1}{2}), \frac{1}{2})$ and $(-\frac{N^{-1}(\tilde{\delta})^2}{2}, \tilde{\delta})$, the interpolated function $\delta(-l)$ is the inverse of $l(\delta)$. Then, the smile defined as

$$\hat{\sigma}(k) = \sigma(\delta(k))$$

is the corresponding smile in strike of the original smile in delta.

Following the steps above we can transform any smile in delta living in Σ into its smile in strike satisfying **P1** and **F1** in an efficient way and without involving any optimization routine.

4. Qualitative properties of the smile: from k to δ

In this section we look at the qualitative properties of the smile in delta resulting from the transformation of a smile in strike with decreasing $k \to d_1(k, \hat{\sigma}(k))$ function.

We start with an easy consequence to Proposition 3.3.

Proposition 4.1 *For every smile $\sigma(\delta) \in \Sigma$ it holds*

- $\sigma'(\delta) \geq 0$ *if and only if* $\hat{\sigma}'(k(\delta)) \leq 0$ *and points of minimum (respectively maximum)* $\bar{\delta}$ *for* $\sigma(\delta)$ *are points of minimum (resp.maximum)* $k(\bar{\delta})$ *for* $\hat{\sigma}(k)$;
- $\hat{\sigma}'(k) \geq 0$ *if and only if* $\sigma'(\delta(k)) \leq 0$ *and points of minimum (respectively maximum)* $\bar{k}$ *for* $\hat{\sigma}(k)$ *are points of minimum (resp.maximum)* $\delta(\bar{k})$ *for* $\sigma(\delta)$.

Proof Firstly observe that $\hat{\sigma}(k) = \sigma(\delta(k))$ is differentiable being composition of differentiable functions.

Since $\hat{\sigma}(k(\delta)) = \sigma(\delta)$, taking derivatives with respect to δ implies

$$\begin{aligned}\sigma'(\delta) &= \hat{\sigma}'(k(\delta))k'(\delta)\\ &= -\hat{\sigma}'(k(\delta))l'(\delta)\end{aligned}$$

where we have used $l(\delta) = -k(\delta)$ as in equation (6). Smiles in Σ have increasing $l(\delta)$, so the sign of $\sigma'(\delta)$ is opposite to the sign of $\hat{\sigma}'(k(\delta))$. Furthermore, if $\bar{\delta}$ is a point of minimum for $\sigma(\delta)$, then for every δ in a neighborhood of $\bar{\delta}$, it holds $\sigma(\bar{\delta}) < \sigma(\delta)$. Using the relation $\hat{\sigma}(k(\delta)) = \sigma(\delta)$, it follows $\hat{\sigma}(k(\bar{\delta})) < \hat{\sigma}(k(\delta))$. Since the function $d_1(k, \hat{\sigma}(k))$ is monotonic and surjective, it is continuous, so also $k(\delta) = d_1^{-1}(N^{-1}(\delta))$ is continuous. Then for every k in a neighborhood of $k(\bar{\delta})$ it holds $\hat{\sigma}(k(\bar{\delta})) < \hat{\sigma}(k)$, so $k(\bar{\delta})$ is a point of minimum for $\hat{\sigma}(k)$. Similarly for points of maximum.

The proof is similar for the second point, using the relation $\hat{\sigma}(k) = \sigma(\delta(k))$ and the fact that $\delta'(k) = \frac{1}{k'(\delta(k))} = -\frac{1}{l'(\delta(k))}$. ∎

We have already seen in Proposition 2.1 that under **P1** and **P2**, the left and right wing Lee conditions hold, i.e. that the limits of $\frac{\hat{\sigma}(k)^2T}{k}$ at $\pm\infty$ are bounded by 2. We now look at what these limits correspond to in the δ notation.

Proposition 4.2 Lee moment formula in delta *Let $\sigma(\delta) \in \Sigma$ and $\hat{\sigma}(k)$ the corresponding smile in strike. The left wing Lee condition $\frac{\hat{\sigma}(k)^2T}{k} > -2$ for sufficiently large $-k$ holds if and only if $\frac{\sigma(\delta)\sqrt{T}}{N^{-1}(\delta)} < 1$ for δ near 1. The right wing Lee condition $\frac{\hat{\sigma}(k)^2T}{k} < 2$ for sufficiently large k holds.*

Furthermore,

$$\limsup_{k\to-\infty} \frac{\hat{\sigma}(k)^2T}{k} = a \iff \liminf_{\delta\to 1} \frac{\sigma(\delta)\sqrt{T}}{N^{-1}(\delta)} = -\frac{2a}{2-a},$$

and

$$\limsup_{k\to\infty} \frac{\hat{\sigma}(k)^2T}{k} = b \iff \liminf_{\delta\to 0} \frac{\sigma(\delta)\sqrt{T}}{N^{-1}(\delta)} = -\frac{2b}{2-b}.$$

Proof Since $\sigma(\delta) \in \Sigma$, there exists a smile in strike $\hat{\sigma}(k)$ with strictly decreasing surjective function $k \to d_1(k, \hat{\sigma}(k))$. Thanks to the surjectivity of d_1, the right wing Lee condition is satisfied as shown in Proposition 2.1.

For every k it holds $\frac{\hat{\sigma}(k)^2T}{k} = \frac{\sigma(\delta(k))^2T}{k(\delta(k))}$, where $\delta(k) = N(d_1(k, \hat{\sigma}(k))$ and $k(\delta)$ is its inverse. Since $\delta(k)$ is surjective decreasing, it goes to 1 when k goes to $-\infty$ and to 0 when k goes to ∞. Also, since it is monotonic continuous, the left wing Lee condition holds if and only if $\frac{\sigma(\delta)^2T}{k(\delta)} > -2$ for every δ near 1, and it holds

$$\limsup_{k\to-\infty} \frac{\hat{\sigma}(k)^2T}{k} = \limsup_{\delta\to 1} \frac{\sigma(\delta)^2T}{k(\delta)}$$

and similarly for $k \to \infty$. From equation (6), we can substitute $k(\delta)$ with $-l(\delta)$, which is defined in equation (5), so that we are now studying the quantity $-\frac{\sigma(\delta)\sqrt{T}}{N^{-1}(\delta) - \frac{\sigma(\delta)\sqrt{T}}{2}}$. Since $l(\delta)$ is increasing surjective, the denominator is negative for small δ and positive for large δ. Then, it is easy to see that the left wing Lee condition holds if and only if $\frac{\sigma(\delta)\sqrt{T}}{N^{-1}(\delta)} < 1$.

The argument of the limit becomes $(\frac{1}{2} - \frac{N^{-1}(\delta)}{\sigma(\delta)\sqrt{T}})^{-1}$, so that the two limits superior are equal to

$$\left(\frac{1}{2} - \limsup_{\delta} \frac{N^{-1}(\delta)}{\sigma(\delta)\sqrt{T}}\right)^{-1}.$$

This quantity is equal to c (either a for $\delta \to 1$ or b for $\delta \to 0$) if and only if $\limsup_{\delta} \frac{N^{-1}(\delta)}{\sigma(\delta)\sqrt{T}} = \frac{c-2}{2c}$ and the conclusion follows. The reasoning still holds for $c = 0$. ∎

From the above proposition it follows that Lee conditions applied to the total variance limits (i.e. $a \in [-2, 0]$ and $b \in [0, 2]$), translate in the delta notation into the requirement that the limit at 0 of $\frac{\sigma(\delta)\sqrt{T}}{N^{-1}(\delta)}$ is negative while the limit at 1 is positive and smaller than 1. The first condition, i.e. the right wing Lee condition, is automatically granted by the sign of the function $N^{-1}(\delta)$. This is what we expected using Proposition 2.1 since a smile in delta which can be transformed in a smile in strike has surjective function $k \to d_1(k, \hat{\sigma}(k))$. The second condition, i.e. the left wing Lee condition, is not automatically granted.

REMARK 4.1 In section 4.1 of Schlüter and Fischer (2009), it is shown that the high Gaussian quantile can be asymptotically written as $N^{-1}(\delta) = \sqrt{-2\log(1-\delta)} + o(N^{-1}(\delta))$. With a similar reasoning, it is easy to prove that for low Gaussian quantiles it holds $N^{-1}(\delta) = -\sqrt{-2\log(\delta)} + o(N^{-1}(\delta))$. Then, the limit inferior in Proposition 4.2 can be substituted with

$$\liminf_{\delta\to 1} \frac{\sigma(\delta)\sqrt{T}}{\sqrt{-2\log(1-\delta)}} \quad \text{and} \quad \liminf_{\delta\to 0} \frac{\sigma(\delta)\sqrt{T}}{-\sqrt{-2\log(\delta)}}.$$

We now look at the expansion of a smile in delta around the ATM point for a given expansion of the corresponding smile in strike.

PROPOSITION 4.3 *Let $\sigma(\delta) \in \Sigma$ and $\hat{\sigma}(k)$ the corresponding smile in strike, and suppose $\hat{\sigma}(k)$ is twice differentiable in a neighborhood of $k = 0$. If*

$$\hat{\sigma}(k)\sqrt{T} = a_0 + a_1 k + a_2 k^2 + o(k^2)$$

then

$$\begin{aligned}\sigma(\delta)\sqrt{T} = a_0 &- \frac{2a_0a_1}{n\left(\frac{a_0}{2}\right)(2-a_0a_1)}(\delta - \delta_{\mathrm{ATM}}) \\ &+ \frac{a_0(-a_0^3a_1^3 + 4a_0^2a_1^2 - 4a_0a_1 + 16a_0a_2 + 16a_1^2)}{2n\left(\frac{a_0}{2}\right)^2(2-a_0a_1)^3} \\ &\times (\delta - \delta_{\mathrm{ATM}})^2 \\ &+ o((\delta - \delta_{\mathrm{ATM}})^2)\end{aligned}$$

where $\delta_{\mathrm{ATM}} = N(\frac{a_0}{2})$ is the delta ATM point.

Proof The delta ATM point is $\delta_{\mathrm{ATM}} = \delta(0) = N(d_1(0, \hat{\sigma}(0))) = N(\frac{a_0}{2})$. Observe that from equation (6), it holds $l(\delta) = -k(\delta)$. We will use the following relations:

$$\sigma(\delta) = \hat{\sigma}(k(\delta)) \tag{11}$$

$$\sigma'(\delta) = -\hat{\sigma}'(k(\delta))l'(\delta) \tag{12}$$

$$\sigma''(\delta) = \hat{\sigma}''(k(\delta))l'(\delta)^2 - \hat{\sigma}'(k(\delta))l''(\delta). \tag{13}$$

The first and second derivatives of $l(\delta)$ can be computed from the definition in equation (5) as

$$l'(\delta) = \left(\frac{1}{n(N^{-1}(\delta))} - \frac{\sigma'(\delta)\sqrt{T}}{2}\right)\sigma(\delta)\sqrt{T} + l(\delta)\frac{\sigma'(\delta)}{\sigma(\delta)}$$

$$\begin{aligned}l''(\delta) = &\left(\frac{N^{-1}(\delta)}{n(N^{-1}(\delta))^2} - \frac{\sigma''(\delta)\sqrt{T}}{2}\right)\sigma(\delta)\sqrt{T} \\ &+ \left(2l'(\delta) - l(\delta)\frac{\sigma'(\delta)}{\sigma(\delta)}\right)\frac{\sigma'(\delta)}{\sigma(\delta)} + l(\delta)\frac{\mathrm{d}}{\mathrm{d}\delta}\frac{\sigma'(\delta)}{\sigma(\delta)}.\end{aligned}$$

From equation (11), it follows that the constant coefficient of the expansion of $\sigma(\delta)\sqrt{T}$ is a_0.

Observe that $l(\delta(0)) = -k(\delta(0)) = 0$, so that $l'(\delta(0))$ has only one term. For the first order coefficient, from equation (12) we obtain $\sigma'(\delta(0))\sqrt{T} = -a_1 l'(\delta(0))$. Substituting in the expression for the derivative of $l(\delta)$ and solving for $l'(\delta(0))$, it follows

$$l'(\delta(0)) = \frac{2a_0}{n\left(\frac{a_0}{2}\right)(2-a_0a_1)},$$

so that we find the first order coefficient of the expansion of $\sigma(\delta)\sqrt{T}$.

Finally, the second order coefficient can be found from the expression for the second derivative of $l(\delta)$ evaluated in $\delta(0)$. As in the previous steps, substituting $\sigma''(\delta(0))\sqrt{T}$ with $2a_2 l'(\delta(0))^2 - a_1 l''(\delta(0))$ from equation (13) and solving for $l''(\delta(0))$, we find

$$l''(\delta(0)) = \frac{a_0\left(a_0(2-a_0a_1)^2 - 8(a_0^2a_2 + 2a_1)\right)}{n\left(\frac{a_0}{2}\right)^2(2-a_0a_1)^3}$$

and from this the expression for the second order coefficient of the expansion of $\sigma(\delta)\sqrt{T}$. ■

5. Weak arbitrage-free smiles in delta

The parametrization of the set Σ_{A} of butterfly arbitrage-free smiles is hard. Indeed, twice differentiable functions $\sigma(\delta)$ belong to Σ_{A} if and only if they satisfy the delta version of the requirement of positivity of equation (2). Observe that such condition can be written only for the subset of Σ_{A} of twice differentiable functions since it involves second derivatives of the smile.

Consider the smile inversion $k \to \hat{\sigma}(-k) =: \hat{\hat{\sigma}}(k)$. Arbitrage-free smiles are such that their inverse smile is still arbitrage-free. Even though we do not parametrize the set Σ_{A}, we show that the subset of Σ which is closed under symmetry can be parametrized, and this could make a step forward to the search of arbitrage-free smiles in delta. We define such set Σ_{WA} since it is the set of smiles satisfying the Weak Arbitrage-free conditions **F1** and **F2** (plus the surjectivity of such functions, i.e. properties **P1** and **P2**). As we showed before, the requirement defining Σ is that there exists a smile $k \to \hat{\sigma}(k)$ such that the function $d_1(k) = d_1(k, \hat{\sigma}(k))$ is decreasing and surjective. For the inverse smile, we are asking that the function $\bar{d}_1(k) = d_1(k, \hat{\bar{\sigma}}(k))$ is decreasing and surjective. It is easy to see that $\bar{d}_1(k) = -d_2(-k)$.

Taking the function $\sigma(\delta) = \hat{\sigma}(k(\delta))$ where $k(\delta) = d_1^{-1}(N^{-1}(\delta))$, the requirement that $d_2(k)$ is decreasing and surjective corresponds to the requirement that $\delta \to d_2(k(\delta))$ is increasing and surjective. It holds $d_2(k) = d_1(k) - \hat{\sigma}(k)\sqrt{T}$, so in the delta notation the requirement is that the function

$$m(\delta) := N^{-1}(\delta) - \sigma(\delta)\sqrt{T}$$

is increasing and surjective.

We can then define the subset of Σ closed for smile inversion as in the following:

Definition 5.1 *We call Σ_{WA} the set of Weak Arbitrage-free delta smiles, i.e. the set of continuous and positive delta smiles $\delta \to \sigma(\delta)$ for which there exist a positive and continuous strike smile $k \to \hat{\sigma}(k)$ with $\hat{\sigma}(k(\delta)) = \sigma(\delta)$ where $k(\delta) = d_1^{-1}(N^{-1}(\delta))$, which satisfy **P1**, **P2**, **F1**, **F2**, i.e. the functions $k \to d_{1,2}(k, \hat{\sigma}(k))$ are strictly decreasing surjective. In particular,*

$$\begin{aligned}\Sigma_{\mathrm{WA}} := \big\{\delta \to \sigma(\delta) \in \mathcal{C}(]0,1[,\mathbb{R}^+) \;\big|\; \exists k \\ \to \hat{\sigma}(k) \in \mathcal{C}(\mathbb{R},\mathbb{R}^+) \text{ s.t.} \\ k \to d_1(k, \hat{\sigma}(k)) \text{ is strictly decreasing surjective}, \\ k \to d_2(k, \hat{\sigma}(k)) \text{ is strictly decreasing surjective}, \\ \hat{\sigma}(k(\delta)) = \sigma(\delta), k(\delta) = d_1^{-1}(N^{-1}(\delta))\big\}.\end{aligned}$$

It is easy to see that if $\sigma(\delta)$ belongs to Σ_{WA}, then it is possible to define the smile $\bar{\sigma}(\delta) := \hat{\bar{\sigma}}(\bar{k}(\delta))$ where $\bar{k}(\delta) = \bar{d}_1^{-1}(N^{-1}(\delta)) = -d_2^{-1}(-N^{-1}(\delta))$, so that $\bar{\sigma}(\delta) = \hat{\sigma}(d_2^{-1}(-N^{-1}(\delta)))$ and $\bar{\sigma}(\delta)$ belongs to Σ_{WA}.

Remark 5.1 For smiles living in Σ_{WA}, the functions $d_1(k, \hat{\sigma}(k))$ and $d_2(k, \hat{\sigma}(k))$ are strictly decreasing and surjective, so they are a.e. differentiable. As a consequence, the smile $\hat{\sigma}(k) = d_1(k, \hat{\sigma}(k)) - d_2(k, \hat{\sigma}(k))$ is a.e. differentiable and so is $\sigma(\delta) = \hat{\sigma}(d_1^{-1}(N^{-1}(\delta)))$.

5.1. Parametrization of Σ_{WA}

We showed in Lemma 3.5 that the properties **P1** and **F1** for a smile guarantee the existence of a σ solution to equation (8). We now see that smiles satisfying also **P2** and **F2**, i.e. living in Σ_{WA}, guarantee the existence and uniqueness of a σ solution.

Lemma 5.1 *For every $\delta \to \sigma(\delta) \in \Sigma_{\mathrm{WA}}$, the function l satisfies $l(\frac{1}{2}) < 0$, $l(\delta) \le \frac{N^{-1}(\delta)^2}{2}$ for all $\delta > \frac{1}{2}$ and there exists a unique $\tilde{\delta}$ such that $l(\tilde{\delta}) = \frac{N^{-1}(\tilde{\delta})^2}{2}$. Furthermore*

$$\sigma(\delta)\sqrt{T} = \begin{cases} N^{-1}(\delta) + \sqrt{N^{-1}(\delta)^2 - 2l(\delta)} & \text{if } \delta \le \tilde{\delta}, \\ N^{-1}(\delta) - \sqrt{N^{-1}(\delta)^2 - 2l(\delta)} & \text{if } \delta > \tilde{\delta}. \end{cases} \quad (14)$$

Proof Since Σ_{WA} is a subset of Σ, the conditions $l(\frac{1}{2}) < 0$ and $l(\delta) \le \frac{N^{-1}(\delta)^2}{2}$ for all $\delta > \frac{1}{2}$ are satisfied by Lemma 3.5.

Since $d_2(k)$ is a decreasing and surjective function, the left wing Lee condition $\hat{\sigma}(k)\sqrt{T} < \sqrt{-2k}$ for sufficiently large $-k$ holds. Since $l(\delta(k)) = -k$, the condition becomes $\sigma(\delta(k))\sqrt{T} < \sqrt{2l(\delta(k))}$ for k negative enough. Given the monotonicity of the function $\delta(k)$, this is equivalent to requiring $\sigma(\delta)\sqrt{T} < \sqrt{2l(\delta)}$ for sufficiently large δ, in particular $\delta > \frac{1}{2}$. The σ solution of equation (8) can be either of the form σ_+ equation (9) or σ_- equation (10). Computing the square of $\sigma_\pm$, the Lee condition is

$$N^{-1}(\delta)\left(N^{-1}(\delta) \pm \sqrt{N^{-1}(\delta)^2 - 2l(\delta)}\right) < 2l(\delta).$$

Since $\delta > \frac{1}{2}$, the quantity $N^{-1}(\delta)$ is positive and dividing we get

$$\pm\sqrt{N^{-1}(\delta)^2 - 2l(\delta)} < \frac{2l(\delta)}{N^{-1}(\delta)} - N^{-1}(\delta).$$

The right hand side is positive if and only if $2l(\delta) > N^{-1}(\delta)^2$, which cannot hold. Then the σ_+ solution does not satisfy the left wing Lee condition for small k. On the other hand, the σ_- solution always satisfies it since the above inequality holds true if and only if $2l(\delta) < N^{-1}(\delta)^2$.

Then, the σ solution is equal to σ_+ for δ smaller than a certain $\tilde{\delta}$ such that $l(\tilde{\delta}) = \frac{N^{-1}(\tilde{\delta})^2}{2}$ and then it switches to σ_-. The uniqueness of such point $\tilde{\delta}$ follows from the monotonicity of $d_2(k)$. Indeed, if two points $\tilde{\delta}$ and $\hat{\delta}$ satisfy $l(\delta) = \frac{N^{-1}(\delta)^2}{2}$, then they also satisfy $\sigma(\delta)\sqrt{T} = N^{-1}(\delta)$. In the log-forward moneyness notation, there exist $\tilde{k} = k(\tilde{\delta})$ and $\hat{k} = k(\hat{\delta})$ which satisfy $\hat{\sigma}(k)\sqrt{T} = d_1(k)$, or $d_2(k) = 0$. Since d_2 is one-to-one, $\tilde{k} = \hat{k}$ and $\tilde{\delta} = \hat{\delta}$. ■

Thanks to the above result, it is possible to parametrize the smile σ living in Σ_{WA} using a parametrization of the function l, which has to be increasing and surjective and has to satisfy the conditions of existence and uniqueness of Lemma 5.1.

We finally state how to parametrize the set Σ_{WA}, which is our main result.

Theorem 5.2 Parametrization of weak arbitrage-free smiles *A smile $\sigma(\delta)$ belongs to Σ_{WA} if and only if it can be parametrized as*

$$\sigma(\delta)\sqrt{T} = \begin{cases} N^{-1}(\delta) + \sqrt{N^{-1}(\delta)^2 + 2\left(\int_\delta^{\frac{1}{2}} \lambda(x)\,\mathrm{d}x + \int_{\frac{1}{2}}^{\tilde{\delta}} \mu(x)\,\mathrm{d}x\right)} & \text{if } \delta \le \frac{1}{2}, \\ N^{-1}(\delta) + \sqrt{2\int_\delta^{\tilde{\delta}} \mu(x)\,\mathrm{d}x} & \text{if } \frac{1}{2} < \delta \le \tilde{\delta}, \\ N^{-1}(\delta) - \sqrt{2\int_{N^{-1}(\tilde{\delta})}^{N^{-1}(\delta)} x\beta(x)\,\mathrm{d}x} & \text{if } \delta > \tilde{\delta}, \end{cases} \quad (15)$$

where

- $\tilde{\delta} \in]\frac{1}{2}, 1[$;
- *λ is an a.e. continuous positive function defined on $]0, \frac{1}{2}]$ such that $\int_0^{\frac{1}{2}} \lambda(x)\,\mathrm{d}x = \infty$;*
- *μ is an a.e. continuous positive function defined on $[\frac{1}{2}, \tilde{\delta}[$;*

- β *is an a.e. continuous function defined on* $]N^{-1}(\tilde{\delta}), \infty[$ *such that* $\beta(x) \in]0,1[$ *a.e. and* $\int_{N^{-1}(\tilde{\delta})}^{\infty} x\beta(x)\,\mathrm{d}x = \int_{N^{-1}(\tilde{\delta})}^{\infty} x(1-\beta(x))\,\mathrm{d}x = \infty$.

Proof Let $\sigma(\delta) \in \Sigma_{\mathrm{WA}}$, then $\sigma(\delta)$ is a.e. differentiable for Remark 5.1, and has the form in equation (14) for Lemma 5.1. The function $l(\delta)$ is increasing and surjective and it satisfies $l(\frac{1}{2}) < 0$, $l(\delta) \leq \frac{N^{-1}(\delta)^2}{2}$ for all $\delta > \frac{1}{2}$, and there exists a unique $\tilde{\delta}$ such that $l(\tilde{\delta}) = \frac{N^{-1}(\tilde{\delta})^2}{2}$. Since $d_2(k)$ is decreasing and surjective, the function $m(\delta) = N^{-1}(\delta) - \sigma(\delta)\sqrt{T}$ is increasing and surjective. Substituting with the expression for $\sigma(\delta)$, $m(\delta)$ can be re-written as $\mp\sqrt{N^{-1}(\delta)^2 - 2l(\delta)}$, where the sign is negative for $\delta \leq \tilde{\delta}$ and positive otherwise. Its derivative, when it is defined, is $\mp\frac{\frac{N^{-1}(\delta)}{n(N^{-1}(\delta))} - l'(\delta)}{\sqrt{N^{-1}(\delta)^2 - 2l(\delta)}}$, and it is positive. Equivalently, the derivative of l satisfies

$$
\begin{aligned}
l'(\delta) &> \frac{N^{-1}(\delta)}{n(N^{-1}(\delta))} \text{ a.e.} \quad \text{if } \delta < \tilde{\delta},\\
l'(\delta) &< \frac{N^{-1}(\delta)}{n(N^{-1}(\delta))} \text{ a.e.} \quad \text{if } \delta > \tilde{\delta}
\end{aligned} \tag{16}
$$

The first inequality is weaker than $l'(\delta) > 0$ a.e. if $\delta < \frac{1}{2}$. Consider then $\delta \in [\frac{1}{2}, \tilde{\delta}[$. The first inequality implies that there is a positive and a.e. continuous function $\mu(\delta)$ on $[\frac{1}{2}, \tilde{\delta}[$, such that $l'(\delta) = \frac{N^{-1}(\delta)}{n(N^{-1}(\delta))} + \mu(\delta)$. Taking the integral from δ to $\tilde{\delta}$ results into $l(\delta) = \frac{N^{-1}(\delta)^2}{2} - \int_{\delta}^{\tilde{\delta}} \mu(x)\,\mathrm{d}x$. If $\delta \leq \frac{1}{2}$, the fact that $l'(\delta)$ is positive can be written as $l'(\delta) = \lambda(\delta)$ where $\lambda(\delta)$ is an a.e. continuous positive function defined on $]0, \frac{1}{2}]$. Taking the integral between δ and $\frac{1}{2}$ implies $l(\delta) = l(\frac{1}{2}) - \int_{\delta}^{\frac{1}{2}} \lambda(x)\,\mathrm{d}x$. Substituting with the value of $l(\frac{1}{2})$ in the expression with μ, it holds $l(\delta) = -\int_{\delta}^{\frac{1}{2}} \lambda(x)\,\mathrm{d}x - \int_{\frac{1}{2}}^{\tilde{\delta}} \mu(x)\,\mathrm{d}x$. Since $l(\delta)$ is surjective and $l(\frac{1}{2})$ is finite, then $\int_0^{\frac{1}{2}} \lambda(x)\,\mathrm{d}x = \infty$. Similarly, for $\delta > \tilde{\delta}$, the property of the derivative of $l(\delta)$ implies $l(\delta) = \frac{N^{-1}(\delta)^2}{2} - \int_{\tilde{\delta}}^{\delta} \eta(x)\,\mathrm{d}x$ for an a.e. continuous positive function $\eta(\delta)$ defined on $]\tilde{\delta}, 1[$. Also, since $l'(\delta) > 0$, $\eta(\delta)$ is smaller than $\frac{N^{-1}(\delta)}{n(N^{-1}(\delta))}$, so $\eta(\delta) = \alpha(\delta)\frac{N^{-1}(\delta)}{n(N^{-1}(\delta))}$ where $\alpha(\delta)$ is a function strictly bounded between 0 and 1 and a.e. continuous on $]\tilde{\delta}, 1[$. Since $\int_{\tilde{\delta}}^{\delta} \alpha(x)\frac{N^{-1}(x)}{n(N^{-1}(x))}\,\mathrm{d}x = \int_{N^{-1}(\tilde{\delta})}^{N^{-1}(\delta)} x\alpha(N(x))\,\mathrm{d}x$ and $\frac{N^{-1}(\delta)^2}{2} = \int_{N^{-1}(\tilde{\delta})}^{N^{-1}(\delta)} x\,\mathrm{d}x + \frac{N^{-1}(\tilde{\delta})^2}{2}$, it holds $l(\delta) = \frac{N^{-1}(\tilde{\delta})^2}{2} + \int_{N^{-1}(\tilde{\delta})}^{N^{-1}(\delta)} x(1 - \alpha(N(x))\,\mathrm{d}x$ for $\delta > \tilde{\delta}$. The function $\beta(x) = \alpha(N(x))$ is strictly bounded between 0 and 1 and a.e. continuous on $]N^{-1}(\tilde{\delta}), \infty[$. Furthermore, $l(1) = \infty$, then $\int_{N^{-1}(\tilde{\delta})}^{\infty} x(1-\beta(x))\,\mathrm{d}x = \infty$. In order to have $m(1) = \infty$, it must hold $\int_{N^{-1}(\tilde{\delta})}^{\infty} x\beta(x)\,\mathrm{d}x = \infty$.

On the other hand, if a function $\sigma(\delta)$ has the form in equation (15), the function $l(\delta)$ is

$$
l(\delta) = \begin{cases} -\displaystyle\int_{\delta}^{\frac{1}{2}} \lambda(x)\,\mathrm{d}x - \int_{\frac{1}{2}}^{\tilde{\delta}} \mu(x)\,\mathrm{d}x & \text{if } \delta \leq \frac{1}{2},\\ \dfrac{N^{-1}(\delta)^2}{2} - \displaystyle\int_{\delta}^{\tilde{\delta}} \mu(x)\,\mathrm{d}x & \text{if } \frac{1}{2} < \delta \leq \tilde{\delta},\\ \dfrac{N^{-1}(\tilde{\delta})^2}{2} + \displaystyle\int_{N^{-1}(\tilde{\delta})}^{N^{-1}(\delta)} x(1-\beta(x))\,\mathrm{d}x & \text{if } \delta > \tilde{\delta}. \end{cases}
$$

Given the hypothesis on the parameters, it is easy to show that $l(\delta)$ is a.e. differentiable and $l'(\delta) > 0$ for every δ where the derivative is defined. Also, $l(0) = -\int_0^{\frac{1}{2}} \lambda(x)\,\mathrm{d}x = -\infty$ and $l(1) = \infty$. So far, we have proven $\sigma(\delta) \in \Sigma$. The only requirement left is $m(\delta)$ increasing and surjective. The monotonicity holds since inequalities in equation (16) are verified. For the surjectivity, $m(0) = -\sqrt{N^{-1}(0)^2 + 2(\int_0^{\frac{1}{2}} \lambda(x)\,\mathrm{d}x + \int_{\frac{1}{2}}^{\tilde{\delta}} \mu(x)\,\mathrm{d}x)}$ which is $-\infty$, while $m(1)$ is equal to $\sqrt{2\int_{N^{-1}(\tilde{\delta})}^{\infty} x\beta(x)\,\mathrm{d}x}$ which diverges. ∎

A first consequence of Theorem 5.2 is that the given reparametrization of the smiles in delta guarantees a simpler calibration than the one in Section 3.4.2 as we will see in Section 5.2. Indeed, conditions on λ, μ and β are more easily verified in practice.

Interestingly, in the attempt of characterizing the Fukasawa conditions **F1** and **F2** considering the parametrization in log-forward moneyness, Lucic has found similar results in Theorem 2.2 of Lucic (2021). In particular, Lucic's theorem states that an a.e. differentiable smile $\hat{\sigma}(k)$ has both $d_1(k, \hat{\sigma}(k))$ and $d_2(k, \hat{\sigma}(k))$ strictly decreasing if and only if it can be parametrized as

$$
\hat{\sigma}(k)\sqrt{T} = \begin{cases} -\sqrt{2k - \phi(k)} + \sqrt{-\phi(k)} & \text{if } k \leq k_*,\\ \sqrt{2k - \phi(k)} + \sqrt{-\phi(k)} & \text{if } k_* < k \leq k^*,\\ \sqrt{2k + \phi(k)} - \sqrt{\phi(k)} & \text{if } k > k^* \end{cases} \tag{17}
$$

where $k_* < 0 < k^*$ and $\phi(k)$ is a continuous increasing function such that $\phi(k_*) = 2k_*$, $\phi(k^*) = 0$, and $\phi'(k) > 2$ for $k < k_*$, $\phi'(k) < 2$ for $k_* < k \leq k^*$.

Note that results in Theorem 5.2 have been achieved independently and looking solely at the delta parametrization. It can be shown that Theorem 5.2 and Lucic's theorem are equivalent when some requirements on the limits at infinity of the functions ϕ and $2k - \phi(k)$ are added. Indeed, the additional conditions are needed under **P1** and **P2** and guarantee the surjectivity of functions $d_1(k, \hat{\sigma}(k))$ and $d_2(k, \hat{\sigma}(k))$. In the following proposition we explain how to pass from the parametrization in Theorem 5.2 to Lucic's parametrization equation (17). The proof can be found in Appendix 1

PROPOSITION 5.3 *A smile* $\sigma(\delta) \in \Sigma_{\mathrm{WA}}$ *with parametrization as in Theorem* 5.2 *has corresponding smile in strike* $\hat{\sigma}(k)$ *with parametrization equation* (17) *where* $\phi(k) := -N^{-1}(l^{-1}(-k))|N^{-1}(l^{-1}(-k))|$ *and*

$$
l(\delta) := \begin{cases} -\displaystyle\int_{\delta}^{\frac{1}{2}} \lambda(x)\,\mathrm{d}x - \int_{\frac{1}{2}}^{\tilde{\delta}} \mu(x)\,\mathrm{d}x & \textit{if } \delta \leq \frac{1}{2},\\ \dfrac{N^{-1}(\delta)^2}{2} - \displaystyle\int_{\delta}^{\tilde{\delta}} \mu(x)\,\mathrm{d}x & \textit{if } \frac{1}{2} < \delta \leq \tilde{\delta},\\ \dfrac{N^{-1}(\tilde{\delta})^2}{2} + \displaystyle\int_{N^{-1}(\tilde{\delta})}^{N^{-1}(\delta)} x(1-\beta(x))\,\mathrm{d}x & \textit{if } \delta > \tilde{\delta}. \end{cases} \tag{18}
$$

Vice versa, consider a smile $\hat{\sigma}(k)$ *with parametrization equation* (17) *and surjective function* $\phi(k)$ *such that* $2k -$

$\phi(k)$ goes to ∞ for k going to $-\infty$. Then, it has corresponding smile in delta $\sigma(\delta)$ with parametrization as in Theorem 5.2 where

$$\lambda(\delta) := -\frac{2\chi(\delta)}{\phi'\left(\phi^{-1}(\chi(\delta)^2)\right) n(\chi(\delta))}$$
$$\mu(\delta) := \frac{\chi(\delta)}{n(\chi(\delta))}\left(\frac{2}{\phi'\left(\phi^{-1}(-\chi(\delta)^2)\right)} - 1\right) \qquad (19)$$
$$\beta(x) := 1 - \frac{2}{\phi'\left(\phi^{-1}(-x^2)\right)}$$

and $\chi(\delta) = N^{-1}(\delta)$.

5.1.1. Requirements on the parameters. The relation of the functions λ, μ and β with l and m is detailed in Appendix 2.

In Theorem 5.2, the positivity of parameter λ and the requirement $\beta < 1$ are directly linked to the fact that the function $l(\delta)$ must be increasing. The positivity of μ and β is instead connected with the monotonicity of the function $m(\delta)$.

The requirement $\int_0^{\frac{1}{2}} \lambda(x)\,\mathrm{d}x = \infty$ comes from the fact that $l(0) = -\infty$ and it also implies $m(0) = -\infty$.

The requirement $\int_{N^{-1}(\tilde{\delta})}^{\infty} x(1-\beta(x))\,\mathrm{d}x$ arises to satisfy $l(1) = \infty$. Observe that if $\lim_{x\to\infty}\beta(x) < 1$, the requirement is automatically satisfied, but this is a sufficient and not necessary property of β. Indeed, the choice of $\beta(x) = 1 - \frac{c}{x}$ for a positive constant $c \le N^{-1}(\tilde{\delta})$ still satisfies conditions of Theorem 5.2 even though β has right limit equal to 1.

The last requirement $\int_{N^{-1}(\tilde{\delta})}^{\infty} x\beta(x)\,\mathrm{d}x = \infty$ originates from $m(1) = \infty$. Similarly as before, taking $\lim_{x\to\infty}\beta(x) > 0$ automatically satisfies this requirement, even though it is not a necessary condition on β. As a counterexample, we could indeed take $\beta(x) = \frac{c}{x}$ with c positive and smaller than $N^{-1}(\tilde{\delta})$.

REMARK 5.2 If we are requiring the delta smile to not have discontinuity points, we should add additional conditions on parameters. In particular, the function $l(\delta)$ should be differentiable also in $\frac{1}{2}$ and $\tilde{\delta}$. Which means that it should hold $\lambda(\frac{1}{2}) = \mu(\frac{1}{2})$ and $\mu(\tilde{\delta}) = -\frac{N^{-1}(\tilde{\delta})}{n(N^{-1}(\tilde{\delta}))}\beta(N^{-1}(\tilde{\delta})) = 0$ since all the functions are positive.

5.2. Application: calibration of weak arbitrage-free smiles in delta

In this section we develop a similar calibration procedure as the one in Section 3.4.2 but for weak arbitrage-free smiles, i.e. smiles living in Σ_{WA}.

There are two methodologies that can be designed. The first one is tricky to be implemented. Indeed, it reconsiders the calibration in Section 3.4.2 and, in order to have that the smile in delta lives in Σ_{WA}, it requires to add in step 3. the conditions:

(f) $l'(\delta) > \frac{N^{-1}(\delta)}{n(N^{-1}(\delta))}$ a.e. for $\delta \in]\frac{1}{2}, \tilde{\delta}[$,
(g) $l'(\delta) < \frac{N^{-1}(\delta)}{n(N^{-1}(\delta))}$ a.e. for $\delta > \tilde{\delta}$,

so that the function $m(\delta)$ is increasing and surjective.

Interpolating a function l which satisfies these requirements is not immediate. For this reason, a second more cunning calibration methodology can be implemented, using Theorem 5.2. The target of such calibration routine are the functions λ, μ and β. These functions must satisfy the requirements in Theorem 5.2 in order to guarantee that the smile in equation (15) can be transformed into a smile in strike satisfying the conditions of bijectivity of the functions $d_1(k, \hat{\sigma}(k))$ and $d_2(k, \hat{\sigma}(k))$.

The steps to be performed become:

- (i) (a) consider the market discrete pillars in delta notation $\{(\delta_i, \sigma_i)\}_i$;
  - (b) if the market discrete pillars are in strike notation $\{(k_i, \sigma_i)\}_i$, convert them to the pillars in delta notation $\{(\delta_i, \sigma_i)\}_i$ by defining $\delta_i = N(d_1(k_i, \sigma_i))$;
- (ii) given the pillars in point (i), interpolate/extrapolate a function $\delta \to \sigma(\delta)$ defined as in Theorem 5.2 such that
  - (a) $\tilde{\delta} \in]\frac{1}{2}, 1[$;
  - (b) λ is a positive function defined on $]0, \frac{1}{2}]$ such that $\int_0^{\frac{1}{2}} \lambda(x)\,\mathrm{d}x = \infty$;
  - (c) μ is a positive function defined on $[\frac{1}{2}, \tilde{\delta}[$;
  - (d) β is a function defined on $]N^{-1}(\tilde{\delta}), \infty[$ such that $\beta(x) \in]0, 1[$, and $\int_{N^{-1}(\tilde{\delta})}^{\infty} x\beta(x)\,\mathrm{d}x = \int_{N^{-1}(\tilde{\delta})}^{\infty} x(1-\beta(x))\,\mathrm{d}x = \infty$.

The requirements on the functions λ, μ and β can be easily achieved parametrizing these functions. As an example, for chosen positive degrees $n_\lambda, m_\lambda > 1$, n_μ, m_μ and n_β, we could define

$$\lambda(\delta) = a_{n_\lambda}\delta^{n_\lambda} + \cdots + a_1 x + a_0 + \frac{a_{-1}}{\delta} + \cdots + \frac{a_{-m_\lambda}}{\delta^{m_\lambda}}$$
$$\mu(\delta) = c_{n_\mu}\delta^{n_\mu} + \cdots + c_1\delta + c_0 + \frac{c_{-1}}{\delta} + \cdots + \frac{c_{-m_\mu}}{\delta^{m_\mu}}$$
$$\beta(x) = \frac{b_{n_\beta}N(x)^{n_\beta} + \cdots + b_1 N(x)}{b_{n_\beta} + \cdots + b_1 + b_0}$$

and require the coefficients a_i, c_i and b_i to be non-negative, b_0 to be non zero, and at least one of the a_i for $i < -1$ and one of the b_i with $i > 0$ to be non zero. In alternative, all parameters can be required to be positive.

These definitions satisfy the requirements on the functions λ, μ and β because they are positive, continuous, $\beta(x) < 1$, and $\int_0^{\frac{1}{2}} \lambda(x)\,\mathrm{d}x$ diverges because of the terms $\frac{a_{-i}}{\delta^i}$ for $i > 1$. The divergence of the integrals $\int_{N^{-1}(\tilde{\delta})}^{\infty} x\beta(x)\,\mathrm{d}x$ and $\int_{N^{-1}(\tilde{\delta})}^{\infty} x(1-\beta(x))\,\mathrm{d}x$ is guaranteed by the fact that $\lim_{x\to\infty}\beta(x) \in]0, 1[$.

In this way, the calibration algorithm in point (ii) can be implemented as a least squares, i.e. a minimization algorithm on the target function defined as the sum of squared differences between market and model delta volatilities evaluated at market delta points $\{(\delta_i, \sigma_i)\}_i$. In particular, the minimization depends on the positive parameters a_i, c_i and b_i. The `scipy` library of Python can easily perform this kind of calibrations via the function `optimize.least_squares`.

5.3. Examples of smiles in Σ_{WA}

5.3.1. Bounded smiles. Let us look at the requirement on β detailed in Section 5.1.1. It has been shown that $\beta(x) = 1 - \frac{c_\beta}{x}$ with $c_\beta \le N^{-1}(\tilde{\delta})$ satisfies conditions of Theorem 5.2. An interesting consequence of this example is that the limit of $\sigma(\delta)$ in 1 is finite in this case. Indeed, for $\delta > \tilde{\delta}$, it holds

$$\sigma(\delta)\sqrt{T} = N^{-1}(\delta) - \sqrt{N^{-1}(\delta)\left(N^{-1}(\delta) - 2c_\beta\right) + d}$$

where $d = N^{-1}(\tilde{\delta})^2(2c_\beta - 1)$ is a constant term. In turn, the latter expression coincides with

$$\frac{2c_\beta N^{-1}(\delta) - d}{N^{-1}(\delta) + \sqrt{N^{-1}(\delta)\left(N^{-1}(\delta) - 2c_\beta\right) + d}}$$

which converges to c_β as δ goes to 1. This means that it is possible to obtain bounded smiles on the right appropriately choosing the function $\beta(x)$ in the parametrization of Theorem 5.2.

Similarly, it is possible to have bounded wings on the left choosing a suitable $\lambda(\delta)$ function. For example, we can define $\lambda(\delta) = \frac{c_\lambda}{n(N^{-1}(\delta))}$ to have the convergence of the smile to c_λ on the left.

Figure 1 shows a skew-shaped smile, which notably has a bounded left wing of the smile in delta. The function λ is defined as above with $c_\lambda = 0.1$. In order to guarantee continuity of the derivative and a nice shape of the smile, the other parameters have been chosen as

$$\mu(\delta) = \frac{c_\lambda}{n(0)}\frac{\tilde{\delta} - \delta}{\tilde{\delta} - \frac{1}{2}}$$

$$\beta(x) = \begin{cases} \dfrac{n(x)}{x}\mu(2\tilde{\delta} - N(x)) & \text{if } x < N^{-1}(\hat{\delta}) \\ \dfrac{n(N^{-1}(\hat{\delta}))}{N^{-1}(\hat{\delta})}\mu\left(\dfrac{1}{2}\right) & \text{if } x \geq N^{-1}(\hat{\delta}) \end{cases} \tag{20}$$

where $\hat{\delta} = 2\tilde{\delta} - \frac{1}{2}$ and $\tilde{\delta} = 0.7$. As a consequence, $\tilde{k} = k(\tilde{\delta}) \approx -0.137$ and $k\left(\frac{1}{2}\right) \approx 0.025$.

This example can be further pushed to obtain a flat smile. Indeed, if we want a flat total implied volatility $\sigma(\delta)\sqrt{T}$ at a level c, we can define

$$\lambda(\delta) = \frac{c}{n(N^{-1}(\delta))} \quad \mu(\delta) = \frac{c - N^{-1}(\delta)}{n(N^{-1}(\delta))} \quad \beta(x) = 1 - \frac{c}{x}$$

and $\tilde{\delta} = N(c)$.

Remark 5.3 Smiles of the form in Theorem 5.2 allow for bounded wings and for flat shapes.

5.3.2. W-shaped smile.

The parametrization in Theorem 5.2 can be used to model very different kind of smiles, and also unusual ones. For example, we can model 'sad smiles' setting $\tilde{\delta} = 0.7$ and

$$\lambda(\delta) = \begin{cases} \dfrac{\delta_1^2 c}{\delta^2} & \text{if } \delta < \delta_1 \\ c & \text{if } \delta \geq \delta_1 \end{cases} \quad \mu(\delta) = c - \frac{N^{-1}(\delta)}{n(N^{-1}(\delta))}$$

$$\beta(x) = \begin{cases} 1 - c\dfrac{n(x)}{x} & \text{if } x < N^{-1}(\delta_2) \\ 1 - c\dfrac{n(N^{-1}(\delta_2))}{N^{-1}(\delta_2)} & \text{if } x \geq N^{-1}(\delta_2) \end{cases} \tag{21}$$

where $c = \frac{N^{-1}(\tilde{\delta})}{n(N^{-1}(\tilde{\delta}))}$, $\delta_1 = 0.02$ and $\delta_2 = 0.9$.

With these parameters, all conditions of Theorem 5.2 are satisfied and the resulting smile in delta $\sigma(\delta)$ has a W-shape as in figure 2.

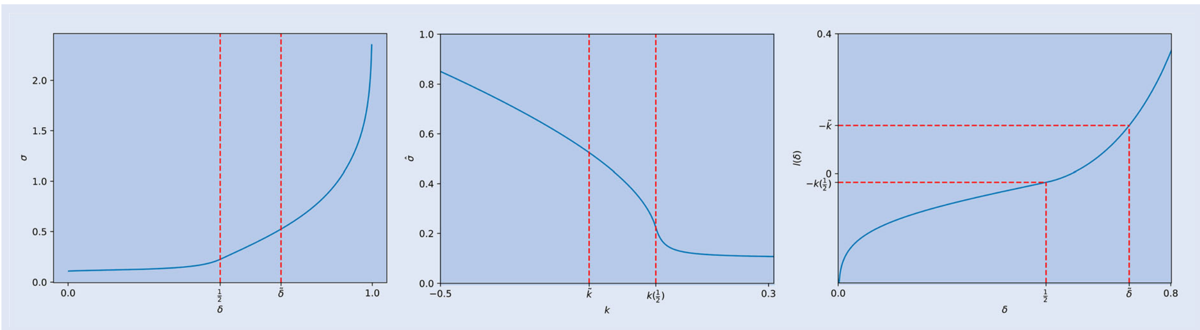

Figure 1. Skew shaped smile with bounded left wing in delta (left) and bounded right wing in log-forward moneyness (center) obtained with parameters as in equation (20). On the right, the corresponding function $l(\delta) = -k(\delta)$ for $\delta < 0.8$.

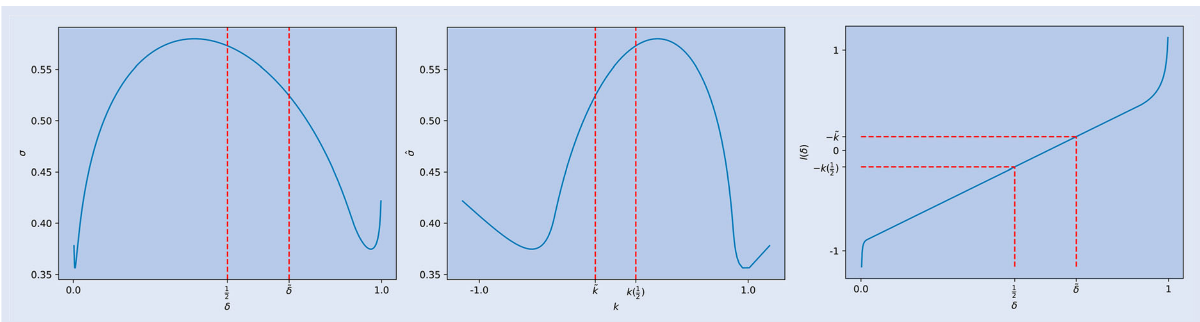

Figure 2. W-shaped smile in delta (left) and in log-forward moneyness (center) obtained with parameters as in equation (21). On the right, the corresponding function $l(\delta) = -k(\delta)$.

It is easy to show that the left and right limits of the smile (both in delta and in strike) are infinite. Choosing different values for δ_1 and δ_2 allows to move the location of the two minima and the maxima of the smile. In this way, it is possible to obtain smiles with W-shapes that have been described in the log-normal mixture framework by Glasserman and Pirjol (2021) and have been seen, for example, for the Amazon stock on the 26 of April 2018 for options with expiry 27 April 2018, before to the first quarter earnings announcement.

5.3.3. SVI. The SVI model has been introduced by Gatheral at the Global Derivatives conference in Madrid in 2004 Gatheral (2004). It is a model for the implied total variance $\hat{\omega}(k) = \hat{\sigma}(k)^2 T$ as a function of the log-forward moneyness k and it is defined as

$$\hat{\omega}(k) = a + b\left(\rho(k-m) + \sqrt{(k-m)^2 + \bar{\sigma}^2}\right).$$

We suppose that the SVI parameters under study satisfy **P1**, **P2**, **F1** and **F2**. These conditions have been explicited in Martini and Mingone (2022). In such way, the corresponding smile in delta obtained through the definition $\sigma(\delta) = \hat{\sigma}(k(\delta))$ where $k(\delta) = N^{-1}(d_1^{-1}(\delta))$, belongs to Σ_{WA}. In the following, we do not compute explicitly the whole delta parametrization of SVI smiles and we just compute the quantity $\tilde{\delta}$.

By Lemma 5.1, there exists a unique $\tilde{\delta}$ such that $l(\tilde{\delta}) = \frac{N^{-1}(\tilde{\delta})^2}{2}$. In the strike notation, this is equivalent to say that there exists a unique $\tilde{k}$ such that $-\tilde{k} = \frac{d_1(\tilde{k})^2}{2}$, or simplifying $\sigma(\tilde{k})^2 T = -2\tilde{k}$. We now calculate such $\tilde{k}$.

We need to look at the solutions of

$$a + b(\rho(k-m) + \sqrt{(k-m)^2 + \bar{\sigma}^2}) = -2k,$$

or equivalently

$$-(2+b\rho)k + b\rho m - a = b\sqrt{(k-m)^2 + \bar{\sigma}^2}.$$

Under the Lee moment formula for sufficiently large δ (larger than $\frac{1}{2}$), or sufficiently large $-k$, it holds $b(1-\rho) < 2$, so $2 + b\rho > 2 + b\rho - b > 0$. Then, the above condition is never satisfied if $k \geq \frac{b\rho m - a}{2+b\rho} := E$. Otherwise, we can take the square and simplifying, one recovers a second-degree equation of the form $Ak^2 + Bk + C = 0$ where

$$\begin{aligned}
A &:= (2+b\rho-b)(2+b\rho+b)\\
B &:= 2\left(a(2+b\rho) - m\left(b^2(1-\rho^2) - 2b\rho\right)\right)\\
C &:= (b\rho m - a)^2 - b^2(m^2 + \bar{\sigma}^2) = 0.
\end{aligned}$$

The leading coefficient A is positive, and the Delta of such equation is $\Delta = 4b^2((a+2m)^2 + \bar{\sigma}^2(2+b\rho-b)(2+b\rho+b))$, which is also positive since both terms are positive. Let us call k_+ and k_- the two possible solutions, with $k_- < k_+$. They are acceptable if and only if they are smaller than E, or if and only if $\pm\sqrt{\Delta} < 2AE + B$ respectively. The RHS is $\frac{2b^2(a+2m)}{2+b\rho}$, which is positive if and only if $a > -2m$. In such case, the $+$ solution is acceptable if and only if

$$\begin{aligned}
0 < (2AE+B)^2 - \Delta &= 2A(2AE^2 - EB + 2C)\\
&= -\frac{4b^2 A}{(2+b\rho)^2}\left((a+2m)^2 + \bar{\sigma}^2(2+b\rho)^2\right)
\end{aligned}$$

which is not possible. On the other hand, the $-$ solution is acceptable if and only if $a > -2m$ or $\Delta - (2AE+B)^2 > 0$, which is always verified as proved above.

In particular,

$$\tilde{k} = \frac{bm(2\rho - b(1-\rho^2)) - a(2+b\rho) - b \times\sqrt{(a+2m)^2 + \bar{\sigma}^2(2+b(1+\rho))(2-b(1-\rho))}}{(2+b(1+\rho))(2-b(1-\rho))}.$$

The SVI model has given birth to other sub-models, obtained reducing the original 5 parameters model to a model with fewer parameters. Among them, the SSVI model by Gatheral and Jacquier (2014) has been largely used in industry. It has the form

$$\hat{\omega}(k) = \frac{\theta}{2}\left(1 + \rho\varphi k + \sqrt{(\varphi k + \rho)^2 + (1-\rho^2)}\right).$$

where the parameters are defined from the SVI ones as $\varphi = \frac{\sqrt{1-\rho^2}}{\sigma}$ and $\theta = \frac{2b\sigma}{\sqrt{1-\rho^2}}$.

In the case of SSVI, the expression for $\tilde{k}$ is easier. Indeed

$$\tilde{k} = -\frac{2\theta}{R}$$

where

$$R = \left(2 + \frac{\theta\varphi}{2}(1+\rho)\right)\left(2 - \frac{\theta\varphi}{2}(1-\rho)\right)$$

is positive because of the Lee bounds, which require $\frac{\theta\varphi}{2}(1+|\rho|) < 2$. The corresponding delta is $\tilde{\delta} = N\left(4\sqrt{\frac{\theta}{R}}\right)$.

6. Conclusion

The possibility to pass from a smile in delta to a smile in strike and vice versa has been characterized requiring that the d_1 function of the Black–Scholes formula has to be decreasing and surjective. This condition is one of the two necessary requirements for the absence of butterfly arbitrage obtained by Fukasawa. The requirement that the d_2 function is decreasing too ensures that also the symmetric smile can be transformed into the delta space.

The requirements that the d_1 and d_2 functions have to be decreasing can be translated into the delta space with specific conditions. These conditions identify a characterization of the set of smiles in delta satisfying the weak no butterfly arbitrage requirements and allow to parametrize such set. As a consequence, we have obtained a parametrization depending on one real number and three positive functions which guarantees that the resulting smiles in delta satisfy the weak no arbitrage conditions identified by Fukasawa.

Practitioners who use smiles in delta could use those parametrizations to ensure at least weak no butterfly arbitrage. An open challenging task is to characterize the subfamily of no butterfly arbitrage smiles in delta. We recall that the task of characterizing the set of butterfly arbitrage-free smiles is open in both delta and strike spaces. The results in the present article give hope of achieving the characterization of such set using the delta parametrization.

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Appendices

Appendix 1. Proof of Proposition 5.3

Let $\sigma(\delta)$ be a smile in Σ_{WA}, then it is a.s. differentiable. Then, for Theorem 5.2 it has a parametrization as in equation (15), and for Theorem 2.2 of Lucic (2021) the corresponding smile $\hat{\sigma}(k)$ has the parametrization of equation (17) since it has decreasing functions $d_1(k,\hat{\sigma}(k))$ and $d_2(k,\hat{\sigma}(k))$. From the proof of Theorem 5.2, the function $l(\delta)$ defined as in equation (18) is strictly increasing and surjective, so that it has an inverse. Given the parametrization of $\sigma(\delta)$, it holds $l(\delta)=\left(N^{-1}(\delta)-\frac{\sigma(\delta)\sqrt{T}}{2}\right)\sigma(\delta)\sqrt{T}$. Since $N^{-1}(\delta(k))=d_1(k,\hat{\sigma}(k))$ and $\sigma(\delta(k))=\hat{\sigma}(k)$, it follows $l(\delta(k))=-k$, or $\delta(k)=l^{-1}(-k)$. In particular, $N^{-1}(l^{-1}(-k))=d_1(k,\hat{\sigma}(k))$ and $\phi(k)=-d_1(k,\hat{\sigma}(k))|d_1(k,\hat{\sigma}(k))|$. This function coincides with the one given in proof of Theorem 2.2 of Lucic (2021), so that the definition of $\phi(k)$ is correct. In particular, $\delta(k^*)$ coincides with $N(d_1(k^*,\hat{\sigma}(k^*)))=N(0)$, i.e. $\delta(k^*)=\frac{1}{2}$, and $l(\delta(k_*))=-k_*$ which is equivalent to $-\frac{\phi(k_*)}{2}$ and, from the definition of $\phi(k)$, to $\frac{N^{-1}(\delta(k_*))^2}{2}$, i.e. $\delta(k_*)=\tilde{\delta}$. The bounds on the slope of $\phi'(k)$ coincide with the requirement that the function $m(\delta)$ is increasing. Indeed, $\phi(k)=-N^{-1}(l^{-1}(-k))^2$ for $k\le k^*$ so its derivative is

$$\phi'(k)=2\frac{N^{-1}(l^{-1}(-k))}{n(N^{-1}(l^{-1}(-k)))}\frac{1}{l'(l^{-1}(-k))}$$

or equivalently, substituting k with $k(\delta)$ for $\delta\ge\frac{1}{2}$,

$$\phi'(k(\delta))=2\frac{N^{-1}(\delta)}{n(N^{-1}(\delta))}\frac{1}{l'(\delta)}.$$

This quantity is larger than 2 for $k<k_*$ and smaller than 2 for $k_*<k\le k^*$ iff the quantity $l'(\delta)$ is smaller than $\frac{N^{-1}(\delta)}{n(N^{-1}(\delta))}$ for $\delta>\tilde{\delta}$ and larger than $\frac{N^{-1}(\delta)}{n(N^{-1}(\delta))}$ for $\frac{1}{2}\le\delta<\tilde{\delta}$. This corresponds to equation (16), i.e. to the fact that $m(\delta)$ is increasing.

Vice versa, let $\hat{\sigma}(k)$ be a smile with parametrization equation (17). Since $\phi(k)$ is surjective and increasing, it is a.s. differentiable, and so is $\hat{\sigma}(k)$. For Theorem 2.2 of Lucic (2021), $\hat{\sigma}(k)$ has strictly decreasing functions $d_1(k,\hat{\sigma}(k))$ and $d_2(k,\hat{\sigma}(k))$. Furthermore, in the proof of Theorem 2.2, Lucic shows that $d_1(k,\hat{\sigma}(k))=\pm\sqrt{\mp\phi(k)}$ for $k\lesseqgtr k^*$ respectively, and

$$d_2(k,\hat{\sigma}(k))=\begin{cases}\sqrt{2k-\phi(k)} & \text{if } k\le k_*\\ -\sqrt{2k-\phi(k)} & \text{if } k_*<k\le k^*\\ -\sqrt{2k+\phi(k)} & \text{if } k>k^*.\end{cases}$$

As a consequence, since $\phi(k)$ is required to be surjective, $d_1(k,\hat{\sigma}(k))$ is also surjective and $d_2(k,\hat{\sigma}(k))$ has infinite right limit. Furthermore, since for hypothesis $2k-\phi(k)$ explodes at $-\infty$, also $d_2(k,\hat{\sigma}(k))$ does. Then, the smile $\hat{\sigma}(k)$ admits a delta smile $\sigma(\delta)$ which lives in Σ_{WA} and it has a parametrization as in equation (15) for Theorem 5.2. It holds $d_1(k(\delta),\sigma(\delta))=N^{-1}(\delta)$, so that $N^{-1}(\delta)=\mp\sqrt{\pm\phi(k(\delta))}$ for $\delta\lesseqgtr\frac{1}{2}$ or equivalently $k(\delta)=\phi^{-1}(\pm N^{-1}(\delta)^2)$ for $\delta\lesseqgtr\frac{1}{2}$. Then $k\left(\frac{1}{2}\right)=\phi^{-1}(0)=k^*$ and $\phi(k(\tilde{\delta}))=-N^{-1}(\tilde{\delta})^2$ which coincides with $-2l(\tilde{\delta})$ and in turn with $2k(\tilde{\delta})$, so $k(\tilde{\delta}) = k_*$. From the definition of $\mu(\delta)$ in equation (19) and from the fact that

$$\frac{N^{-1}(\tilde{\delta})^2}{2} + \phi^{-1}(-N^{-1}(\tilde{\delta})^2) = -\frac{\phi(k(\tilde{\delta}))}{2} + k(\tilde{\delta}) = 0,$$

it is easy to see that

$$\int_\delta^{\tilde{\delta}} \mu(x)\,\mathrm{d}x = \frac{N^{-1}(\delta)^2}{2} + \phi^{-1}(-N^{-1}(\delta)^2).$$

It holds $\sigma(\delta)\sqrt{T} = \hat{\sigma}(k(\delta))\sqrt{T}$, which has the form $\sqrt{2k(\delta) - \phi(k(\delta))} + \sqrt{-\phi(k(\delta))}$ for $\frac{1}{2} < \delta \le \tilde{\delta}$. Substituting with the expression of $k(\delta)$, it follows $\sigma(\delta)\sqrt{T} = N^{-1}(\delta) + \sqrt{2\phi^{-1}(-N^{-1}(\delta)^2) + N^{-1}(\delta)^2}$, or $\sigma(\delta)\sqrt{T} = N^{-1}(\delta) + \sqrt{2\int_\delta^{\tilde{\delta}} \mu(x)\,\mathrm{d}x}$. This is what we looked for, so the definition of $\mu(\delta)$ is correct. The proof is similar for $\lambda(\delta)$ and $\beta(x)$. The conditions on parameters $\lambda(\delta)$, $\mu(\delta)$ and $\beta(x)$ in Theorem 5.2 hold true because $\sigma(\delta)$ lives in Σ_{WA}.

Appendix 2. Relations of the parameters with *l* and *m*

In this appendix we study the relation between the parameters $\tilde{\delta}$, λ, μ and β and the two functions $l(\delta)$ and $m(\delta)$.

Given a function $\delta \to \sigma(\delta)$ in Σ_{WA}, the point $\tilde{\delta}$ is the only solution (which will be automatically greater than $\frac{1}{2}$) to $l(\delta) = \frac{N^{-1}(\delta)^2}{2}$. Equivalently, the point $\tilde{\delta}$ is the only solution to $m(\delta) = 0$.

The function μ can be recovered from

$$\sigma(\delta)\sqrt{T} = N^{-1}(\delta) + \sqrt{2\int_\delta^{\tilde{\delta}} \mu(x)\,\mathrm{d}x}$$

for $\delta \in \left[\frac{1}{2}, \tilde{\delta}\right[$. In particular, $\int_\delta^{\tilde{\delta}} \mu(x)\,\mathrm{d}x = \frac{(\sigma(\delta)\sqrt{T} - N^{-1}(\delta))^2}{2} = \frac{m(\delta)^2}{2}$, and deriving one finds

$$\mu(\delta) = -m(\delta)m'(\delta).$$

In the proof of Theorem 5.2, we showed $\lambda(\delta) = l'(\delta)$ for $\delta \le \frac{1}{2}$.

Finally, consider $\delta > \tilde{\delta}$. Then

$$\sigma(\delta)\sqrt{T} = N^{-1}(\delta) - \sqrt{2\int_{N^{-1}(\tilde{\delta})}^{N^{-1}(\delta)} x\beta(x)\,\mathrm{d}x}$$

and similarly as before

$$\beta(x) = \frac{n(x)}{x} m(N(x))m'(N(x)).$$