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Optimal stop-loss rules in markets with long-range dependence

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Stop-loss is a common risk management tool for limiting risks and improving trading strategy performance. The effectiveness of stop-loss depends critically on asset price characteristics. This study is the first to analyze stop-loss strategy incorporating long-range dependence of asset prices through a fractional Brownian motion-based market model. It is shown that stop-loss strategy yields a positive return premium over the buy-and-hold return when asset price exhibits long-range dependence. The efficacy of stop-loss strategies and the determining criterions are investigated through both theoretical analysis and simulation studies. The performance of a stop-loss rule depends on the Hurst parameter, mean and volatility of the asset returns. The optimal stop-loss threshold model in a chosen strategy class is fitted by polynomial regression. Empirical analysis demonstrates that the class-specific optimal rules outperform stop-loss rules under alternative asset return-generating models.

Keywords: Stop-loss; Risk management; Investments; Fractional Brownian motion; Fractal markets

1. Introduction

Stop-loss rules are the most basic yet commonly adopted risk management tools, which are widely used in portfolio management strategies such as the Commodity Trading Advisors (CTAs) strategies. The stop-loss rules liquidate a position when the price of a risky asset crosses a pre-specified threshold. The purpose of setting a stop-loss is to avoid further trading losses and lock in profits when market starts to turn against the investors. Kaminski and Lo (2014) provide a framework for examining the efficacy of stop-loss rules under the classical market modelling assumptions. Following this framework, the stop-loss strategy is analyzed in a market environment driven by fractional Brownian motion (fBM). The optimal stop-loss threshold is subsequently obtained through a polynomial regression method based on simulated market data.

One strand of literature examines the efficacy of stop-loss rules as part of a trading strategy where asset prices are assumed to follow Brownian motion driven processes, autoregressive time series, or the Markov regime-switching processes (Glynn and Iglehart 1995, Imkeller and Rogers 2014, Kaminski and Lo 2014, Lo and Remorov 2017). They conclude that stop-loss rules improve the performance of buy-and-hold strategy, provided the price model of a risky asset satisfying certain sufficient conditions. Under random walk or Brownian motion price models, the optimal trailing-stop strategies are obtained (Glynn and Iglehart 1995, Imkeller and Rogers 2014). Arratia and Dorador (2019) analyze the performance of four popular stop-loss rules applied to asset price models incorporating overnight jumps. The existing literature mainly focuses the analysis on financial markets modelled by Gaussian innovation processes such as regular Brownian motions. However, the fractal behaviors of financial asset prices have long been documented (Mandelbrot 1963, Peters 1994, Corazza *et al*. 1997, Cont 2001). Accurate assessment of the value of stop-loss rules requires realistic asset price models capturing these fractal behaviors. The contributions of this work are threefold. First, the efficacy of stop-loss rules under a fBM market environment is analyzed accounting for the fractal effect of asset prices on stop-loss rules. To the best of our knowledge, this study is the first attempt in analyzing the stop-loss rules with asset price innovations driven by fBMs. Second, we characterize the conditions under which the stop-loss rules are applicable in terms of improving the performance of commonly used trading strategies (e.g. buy-and-hold). Lastly, an approximation model for obtaining the corresponding optimal stop-loss rules is proposed and validated through empirical analysis. This provides practitioners

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with guidance on how to best practice the stop-loss strategy when asset prices exhibit long-range dependence behavior. Simulation and empirical studies further demonstrate that the proposed stop-loss criterions under an fBM market model outperform those obtained under alternative market models.

Both theoretical analysis and extensive simulation studies find that the efficacy of stop-loss strategies is determined by a quantitative measure for the level of long-range dependence in asset prices, known as the Hurst parameter of fBM. The stop-loss strategy yields positive premium to a buy-and-hold strategy when the Hurst parameter is above 0.5, which indicates the existence of long-range dependence in the market. The question of whether a stop-loss strategy increases return and reduces risk is answered by comparing the expected return and return volatility of the buy-and-hold strategy with the stop-loss rule against those without stop-loss. The simulation studies show that the strategy with stop-loss outperforms the one without it.

The effectiveness of stop-loss criterions is also influenced by the expected return and price volatility of the concerned assets. The optimal stop-loss threshold is negatively correlated with the Hurst parameter and the price volatility, but positively correlated with the expected return. How to identify the most effective stop-loss strategy is always a million-dollar question for practitioners. Naturally, the optimal stop-loss threshold would be obtained by maximizing the investment objective (such as the expected return or the Sharpe ratio) subject to constraints over expected return and volatility levels. However, the analytical form of such objectives is usually not available due to the path dependence nature of the fBM-based market model. A Monte Carlo simulation approach is employed to obtain the optimal threshold and its sensitivity with respect to model parameters including the Hurst parameter, expected return and market volatility. Approximation to the functional mapping from these parameters to the optimal threshold is then established through polynomial regression.

To examine the actual performance of the proposed model with corresponding stop-loss threshold in practice, a classical moving average cross-over (MACO) strategy is applied incorporating the optimal stop-loss strategy on commodity and index futures trading in the Chinese futures markets. The MACO strategy with the proposed optimal stop-loss threshold under fBM price model outperforms the strategy without stop-loss. The profits and Sharpe ratios of the MACO strategy are improved by the optimal stop-loss rule, while the return volatilities are reduced. These findings support the perceived purpose of the stop-loss strategies in industry practice.

There has been a growing literature adopting fBM-based price models in various financial applications such as pricing options (Elliott and Chan 2004), forecasting asset prices (Garcin 2022), optimizing portfolio choices (Biagini *et al.* 2008), and trading strategies (Guasoni *et al.* 2021). We add to this literature by analyzing stop-loss rules in an fBM market environment.

The reminder of this paper is organized as follows. In section 2, an fBM-based asset price model is introduced to represent the long-range dependence characteristic found in financial markets. The theoretical analysis and simulation studies of stop-loss rules when asset prices are modelled by fGBM are carried out in section 3. Section 4 analyzes the influential factors of the optimal stop-loss thresholds through polynomial regression. Empirical studies of the optimal stop-loss rules are conducted on the trading of commodity and index futures contracts using historical data from Chinese markets. Conclusion is drawn in section 5.

2. Fractional Brownian motion-based asset price model

Motivated by the empirical evidences of long-range dependence effects and fractal characteristics of asset price, an fBM-based market price model is proposed for the purpose of analyzing the efficacy of stop-loss strategy. While Rogers (1997) and Cheridito (2003) show that trading arbitrage may exist if fBM with path-wise integral is used for modeling trading gains, Elliott and Van der Hoek (2003) demonstrate that the corresponding arbitrage issue can be resolved by modeling trading gains with fractional Wick-Ito-Skorohod integral. This section offers a brief introduction to the fractional geometric Brownian motion (fGBM) price model, estimation methods of model parameters, and empirical supports for the proposed model specification.

2.1. *Fractional geometric Brownian motion price model*

Let B_t^H denote fBM with the Hurst parameter $H \in (0, 1)$. The integral form of fBM is defined in Mandelbrot and Van Ness (1968) as follows:

$$B_t^H = k[\int_{-\infty}^{t} (t-s)^{H-\frac{1}{2}} dB_s - \int_{-\infty}^{0} (-s)^{H-\frac{1}{2}} dB_s],$$

where B_t denotes a regular Brownian motion and

$$k^{-2} = (2H)^{-1} + \int_0^{\infty} ((1+v)^{H-\frac{1}{2}} - v^{H-\frac{1}{2}})^2 dv.$$

fBM is a continuous and centered Gaussian process or a generalized Brownian motion. It reduces to a regular Brownian motion in the case of $H = 0.5$. k is a properly chosen constant for making the covariance function of fBM dependent of time and H only, which is given below:

$$R(t, s) = E(B_t^H B_s^H) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

It is straightforward to show that the increments of an fBM are positively (negatively) correlated when its $H > 0.5$ ($H < 0.5$). Thus fBMs exhibit long-range dependence effects when $H > 0.5$, and anti-persistence effects when $H < 0.5$.

Let S_t denote the price of an asset at time t. Suppose the price process $\{S_t : t \geq 0\}$ is modeled by the following stochastic differential equation (SDE) with fBM driven innovations, termed as an fGBM process:

$$dS_t = \mu S_t dt + \sigma S_t dB_t^H, \tag{1}$$

where μ, σ are constants, and B_t^H is an fBM with Hurst parameter H. There exists a solution to equation (1) under the fractional Wick-Ito-Skorohod integral framework (Biagini and Øksendal 2004, Biagini *et al.* 2008):

$$S_t = S_0 exp\{\ \mu t - H\sigma^2 t^{2H} + \sigma B_t^H\ \}\ . \tag{2}$$

To simulate sample paths of an fBM, the method for sampling a stationary Gaussian process with known covariance function is applied (Dietrich and Newsam 1997). The covariance matrix is determined by the Hurst parameter and the corresponding time intervals,

$$\Gamma = (R(i\Delta t, j\Delta t)), i, j = 1, \ldots, n.$$

The square root matrix Σ, defined by $\Sigma^2 = \Gamma$, is obtained through the Cholesky decomposition method. Then, with a vector v of n samples drawn from n independent standard Gaussian distributions, a sample path of fBM is generated by

$$u = \Sigma v.$$

Finally, plugging the simulated fBM path into equation (2) yields a sample path of the fGBM.

2.2. Estimation of model parameters

To identify parameters of the discretely observed fGBM process (1), Xiao *et al.* (2015) provide a procedure built upon the quadratic variation and the maximum likelihood approach. Specifically, the estimator of Hurst parameter from the discrete observations $\mathbf{Y} = (Y_{\Delta t}, Y_{2\Delta t}, \ldots, Y_{N\Delta t})'$ is expressed as

$$\hat{H} = \frac{1}{2} - \frac{1}{2\ln(2)} \ln \frac{\sum_{i=1}^{N-1} [\exp(Y_{(i+1)\Delta t}) - \exp(Y_{i\Delta t})]^2}{\sum_{i=1}^{N/2-1} [\exp(Y_{2(i+1)\Delta t}) - \exp(Y_{2i\Delta t})]^2},$$

where $Y_t = \log(S_t/S_0) = \mu t - H\sigma^2 t^{2H} + \sigma B_t^H$. With the estimated Hurst parameter, the estimator of σ^2 from the discrete observations is given by

$$\hat{\sigma}^2 = \frac{\sum_{i=0}^{N} [Y_{(i+1)\Delta t} - \exp Y_{i\Delta t}]^2}{N(\Delta t)^{2H}}.$$

Lastly, based on the estimators of Hurst parameter and σ^2, the exact maximum likelihood estimator of μ from the discrete observations is obtained as

$$\hat{\mu} = \frac{\sigma^2 \mathbf{t}'\Gamma^{-1}\mathbf{t}^{2H} + 2\mathbf{Y}'\Gamma^{-1}\mathbf{t}}{2\mathbf{t}'\Gamma^{-1}\mathbf{t}},$$

where Γ^{-1} is the covariance matrix mentioned in section 2.1, $\mathbf{t} = (\Delta t, 2\Delta t, \ldots, N\Delta t)'$ and $\mathbf{t}^{2H} = ((\Delta t)^{2H}, (2\Delta t)^{2H}, \ldots, (N\Delta t)^{2H})'$. The asymptotic properties of these estimators and other details are discussed in Xiao *et al.* (2015).

2.3. Empirical evidence on stylized effects of asset prices

Modelling the dynamics of asset prices by fGBMs is motivated by numerous empirical findings on asset price characteristics. For example, Lo and Mackinlay (1988) reject the random walk model using weekly stock market returns. Peters (1994) points out that the movements of the S&P 500 index and foreign exchange rates can be better explained by fractal models. Corazza *et al.* (1997) find fractal behaviors with long-range dependences in agricultural future returns.

Matteo *et al.* (2005) apply the generalized Hurst approach to model markets in different development stages. They show that long-range dependences exist in various financial markets such as foreign exchanges, equity markets and fixed income markets. Moreover, long-range dependence of asset returns can be measured by the Hurst exponent calculated using the adjusted rescaled range (R/S) analysis of Lo (1991) and the variance ratio of Lo and Mackinlay (1989). Sensoy and Hacihasanoglu (2014) show the existence of long-range dependence in energy future markets. Liu *et al.* (2019) conduct R/S analysis and variance ratio tests on a set of price series including the S&P 500 index, the Chinese composite market (SSE) index, the Heng Seng index, and the small-cap/large-cap stock indices in the USA. They conclude that return series of these financial indices have long-range dependence and similar characteristics to those found in fBMs.

As the long-range dependence effect exists broadly in financial markets, an fBM process, which is fractal and has long-range dependences (Beran 1994), can be a more realistic model for asset return processes than a regular Brownian motion. The analysis of a stop-loss strategy in a market environment modelled by an fGBM would offer investors and fund managers with greater practical values and insights.

2.4. Conditional expectation of fBM at a bounded stopping-time

Fink *et al.* (2013) provide an explicit expression for conditional expectation of fBM at a future time T with respect to filtration at any time $t < T$. We extend this result with respect to a filtration generated by a bounded stopping time.

Consider a stopping time τ, which is defined as follows:

$$\tau = \inf\{t : B_t^H = a\} \wedge T,$$

where a denotes a constant threshold and $T > 0$ is a fixed time point. For convenience, an fBM B_T^H as defined in Mandelbrot and Van Ness (1968) is expressed as

$$B_T^H = \int_0^T k_H(T, v) dB_v,$$

where $k_H(T, v) = d_H \left[(\frac{T}{v})^{H-\frac{1}{2}} (T - v) - (H - \frac{1}{2}) v^{\frac{1}{2}-H} \int_v^T z^{H-\frac{3}{2}} (z - v)^{H-\frac{1}{2}} dz \right]$ and $d_H = \sqrt{\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)}}$.

Based on the definition of Ito integral from time 0 to a bounded stopping-time τ (Schilling 2021), the conditional expectation of B_T^H with respect to the filtration $\mathscr{F}_\tau$ is given as follows:

$$\begin{aligned} E[B_T^H | \mathscr{F}_\tau] &= E\left[\int_0^T k_H(T, v) dB_v | \mathscr{F}_\tau \right] \\ &= E\left[\int_\tau^T k_H(T, v) dB_v | \mathscr{F}_\tau \right] + \int_0^\tau k_H(T, v) dB_v. \end{aligned}$$

Sottinen and Viitasaari (2017) show that $k_H(T,v)$ is square integrable as a function of v $(0 \le v \le T)$, thus making the process $\{\int_0^t k_H(T,v)dB_v, 0 \le t \le T\}$ a martingale. As τ is a non-negative bounded stopping time, $E[\int_\tau^T k_H(T,v)dB_v|\mathscr{F}_\tau] = 0$ by optional-stopping theorem (Williams 1991). Therefore, the conditional expectation of B_T^H with respect to $\mathscr{F}_\tau$ is given by $B_\tau^H + \int_0^\tau \Psi^H(T,\tau,v)dB_v^H$. Summarizing the above arguments leads to the following lemma.

Lemma 1 *Conditional expectation of fBM B_T^H at a bounded stopping-time τ $(0 < \tau \le T)$ has the following expression:*

$$\begin{aligned}E[B_T^H|\mathscr{F}_\tau] &= \int_0^\tau k_H(\tau,v)dB_v + \int_0^\tau k_H(T,v) - k_H(\tau,v)dB_v\\ &= B_\tau^H + \int_0^\tau k_H(T,v) - k_H(\tau,v)dB_v\\ &= B_\tau^H + \int_0^\tau \Psi^H(T,\tau,v)dB_v^H,\end{aligned}$$

where $\Psi^H(T,\tau,v) = \frac{\sin(\pi(H-0.5))}{\pi} v^{-(H-0.5)}(\tau - v)^{-(H-0.5)} \int_\tau^T \frac{z^{H-0.5}(z-\tau)^{H-0.5}}{z-v}dz.$

3. Stop-loss strategy under the fBM price model

3.1. Definitions

Consider a buy-and-hold strategy on a risky asset over an investment horizon of fixed length T. The asset price dynamics is modeled by SDE (1). A stop-loss policy (or, strategy) for the buy-and-hold strategy refers to the action of selling the asset whenever the cumulative investment return of buy-and-hold hits a given loss threshold γ prior to or by the horizon end along with certain market conditions being satisfied at the hitting time. The sequence of events corresponding to a buy-and-hold strategy with stop-loss is as follows. Suppose a risky asset is bought at some initial time $t_0 < T$. A stop-loss policy would be exercised at time t if the pre-specified stop-loss criterions are met at time t $(t_0 < t < T)$ and then proceeds from selling the asset are invested in a risk-free account until time T. Otherwise, the asset is held until time T.

At time t, denote $r_t = \ln(S_t)$ as the log price process of the risky asset and R_t as the cumulative return of the buy-and-hold strategy from t_0(without loss of generality, t_0 is set to 0).

$$R_T = \int_0^T dr_t = \ln(S_T/S_0).$$

Following Arratia and Dorador (2019), a continuous-time stop-loss policy is considered from here on.

Definition 1 *Given a set of stop-loss rules which includes a pre-specified return-loss level γ, a simple stop-loss policy*† *$\pi(\gamma)$ for a buy-and-hold strategy is a binary wealth allocation scheme $\{\pi_t\}$ between a risky asset and a risk-free account with return rate r_f. Following the stop-loss rules at time t, the strategy either fully invests in the risky asset ($\pi_t = 1$) or avoids the risky asset entirely ($\pi_t = 0$) and invests fully in the risk-free account.*

The above definition of a stop-loss policy disregard re-entry rules as the benefits associated with selling the risky asset to avoid further losses are the focus of ensuing analysis. The efficacy of a stop-loss strategy is assessed with two measures. The first one is the stopping premium (Kaminski and Lo 2014) which focuses on the impact of the expected return of an investment strategy. This premium is defined as follows.

Definition 2 *The stopping premium $\Delta(\gamma)$ of a stop-loss policy $\pi(\gamma)$ is the expected return difference between the buy-and-hold strategies with and without the stop-loss strategy:*

$$\begin{aligned}\Delta(\gamma) &= E[RS_T] - E[R_T]\\ &= E\left[\int_0^T \pi_t dr_t + \int_0^T (1-\pi_t)r_f dt\right]\\ &\quad - E[\int_0^T dr_t],\end{aligned}$$

where RS_T is the cumulative return of the buy-and-hold strategy with stop-loss. If the stop-loss policy is exercised at time τ, the risky asset is liquidated and the full proceeds are invested in risk-free asset. Therefore, the RS_t is exactly the same as R_t before the stopping time τ. The second measure, considering the risk-reward trade-off in the impact by stop-loss, is the difference between the Sharpe ratios of these two strategies. This risk-adjusted performance measure is defined as follows.

Definition 3 *Let the Sharpe ratio difference Δ_{SR} of a stop-loss policy $\pi(\gamma)$ be defined as*

$$\Delta_{SR} = \frac{E[RS_T]/T - r_f}{Std[RS_T]/\sqrt{T}} - \frac{E[R_T]/T - r_f}{Std[R_T]/\sqrt{T}}.$$

A desirable stop-loss policy for a buy-and-hold strategy shall yield a higher expected return or a lower volatility than the corresponding performance metrics of the otherwise identical investment strategy without exercising any stop-loss.

3.2. Analytical results

From here on, let τ specifically denote the minimum between T and the first time when the buy-and-hold return of the risky asset hits a given constant level $-\gamma$. $f(\tau)$ denotes the probability density function of τ. If the stop-loss policy is exercised at τ, the stopping premium can be expressed as

$$\Delta(\gamma) = \int_0^T [(T-\tau)r_f - E(R_T - R_\tau|\mathscr{F}_\tau)]f(\tau)d\tau.$$

Condition A Let $\mathscr{F}_\tau$ denote the filtration generated by the stopped fBM B_τ^H. Given $\mathscr{F}_\tau$,

$$\int_0^\tau \Psi^H(T,\tau,v)dB_v^H + \left(r_f\tau - \mu\tau + \frac{1}{2}\sigma^2\tau^{2H}\right)/\sigma$$

† The stop-loss policy depends on verifiable market conditions and specification of asset price dynamics, which are fully specified with respect to a chosen market model.

$$< \left(r_f T - \mu T + \frac{1}{2}\sigma^2 T^{2H} \right) / \sigma. \quad (3)$$

Note that both sides of inequality (3) are deterministic given filtration $\mathscr{F}_\tau$. Namely, (3) is verifiable given $\mathscr{F}_\tau$. If *Condition A* is satisfied, then $(T-\tau)r_f - E(R_T - R_\tau|\mathscr{F}_\tau) > 0$, which implies that the stopping premium is strictly positive. The positive stopping premium means that the stop-loss policy improves the return-performance of an investment strategy.

A simple stop-loss policy $\pi(\gamma)$ is proposed as follows:

$$\pi_t \equiv \begin{cases} 1, & \text{if } R_{t-} \geq -\gamma \text{ and } \pi_{t-} = 1 \text{ (stay in)} \\ 0, & \text{if } R_{t-} < -\gamma,\ \textit{Condition A} \text{ is satisfied, and} \\ & \pi_{t-} = 1 \text{ (exit)} \\ 0, & \text{if } \pi_{t-} = 0 \text{ (stay out)} \end{cases}.$$

Proposition 1 *Suppose the return-generating process of a risky asset is a Markov process and the expected rate of return of the asset is greater than the risk-free rate. Applying a stop-loss strategy to the buy-and-hold strategy of the risky asset does not improve the expected return of the buy-and-hold strategy.*

Proof See appendix A. ■

Kaminski and Lo (2014) analyze the stop-loss rules under the assumption that risky asset prices evolve according to random walk processes or Markovian regime-switching processes. They conclude that the simple stop-loss policy adds no value to the buy-and-hold strategy when the return-generating processes of assets are Markovian. This may explain why stop-loss rules attract little attention in the academic literature.

With the price dynamics of a risky asset being modeled by an fGBM process (1), the log-price process r_t and the cumulative return R_t of the asset are as follows:

$$dr_t = (\mu - Ht^{2H-1}\sigma^2)dt + \sigma dB_t^H,$$

$$R_t = \int_0^t dr_t = \left(\mu - \frac{1}{2}t^{2H-1}\sigma^2\right)t + \sigma B_t^H,$$

$$R_T - R_\tau = \int_\tau^T dr_t = \mu(T-\tau) - \frac{1}{2}\sigma^2(T^{2H} - \tau^{2H}) + \sigma(B_T^H - B_\tau^H).$$

The stop-loss policy being exercised at time τ implies that $R_\tau \leq -\gamma$, which is equivalent to

$$B_\tau^H \leq \left(-\gamma - \mu\tau + \frac{1}{2}\tau^{2H}\sigma^2\right)/\sigma < 0.$$

Lemma 1 implies that the conditional expectation of $R_T - R_\tau$ given $\mathscr{F}_\tau$ at stopping-time τ is tied to the realized path of $B_v^H, v \in [0, \tau]$.

Applying Lemma 1, the expected cumulative return of a stop-loss strategy occurring at stopping-time τ is equal to

$$E(R_T - R_\tau|\mathscr{F}_\tau) = \mu(T-\tau) - \frac{1}{2}\sigma^2(T^{2H} - \tau^{2H}) + \sigma\int_0^\tau \Psi^H(T,\tau,v)dB_v^H. \quad (4)$$

Therefore, the expected change in cumulative return after the occurrence of a stop-loss is computable based on the path of the fBM. Note that return-seeking investors buy a risky asset if and only if the expected return of the risky asset exceeds risk-free rate, which implies

$$\mu - \frac{1}{2}T^{2H-1}\sigma^2 > r_f.$$

Proposition 2 *Suppose long range dependence exists in the return-generating process of a risky asset (namely, $H > 0.5$) and $\mu - \frac{1}{2}T^{2H-1}\sigma^2 > r_f$. A stop-loss policy with a properly chosen stopping threshold γ yields a positive stopping premium whenever Condition A holds at stopping time τ.*

Proof See Appendix B. ■

This proposition implies that it is suboptimal to continue to hold onto a position after return-loss has accumulated up to certain level. The stop-loss policy can indeed improve the performance of the buy and hold strategy.

3.3. Numerical examples

To develop intuitions behind the theoretical results, a buy-and-hold strategy with and without a stopping threshold is respectively applied to simulated asset return paths. The parameters are $\mu = 0.12$, $\sigma = 0.2$, $r_f = 0.03$, $S_0 = 1$, and $H \in \{0.3, 0.5, 0.7\}$. The stopping threshold varies from 0.01 to 0.3. Over an investment horizon of 3 years, 5000 price paths are simulated corresponding to each combination of parameters. This approach allows for comparative studies across different market scenarios.

The panel A of table 1 presents the simulation results with Hurst parameter H being 0.3, which implies the existence of anti-persistence in asset prices. No matter what value the stopping threshold takes, the stop-loss rule fails to improve the respective return and Sharpe ratio of the simple buy-and-hold strategy. This is consistent with the intuition that the buy-and-hold strategy benefits from price reversals, and a stop-loss rule that liquidates the position after incurring certain amount of cumulative loss would miss the reversal-led gains, thus getting a lowered expected return. The same results are found when Hurst parameter is 0.5. The cumulative losses have no influences on future returns, the stop-loss rules fail to add values to the buy-and-hold strategy. When long-range dependence exists (H is 0.7), the stop-loss rule significantly improves the performance of a buy-and-hold strategy. The Sharpe ratio increases from 0.3186 to 0.4880 and the standard deviation decreases from 0.4348 to 0.2744. Adopting the stop-loss rule with a threshold of 0.01 leads to a slightly worse return than that of buy-and-hold strategy without stop-loss. The probabilities of triggering stop-loss rule are positively correlated with the stopping threshold level, regardless of the value of Hurst parameter.

To analyze the implication of *Condition A* on the generality of Proposition 2, extensive numerical simulations are conducted to characterize scenarios in which *Condition A* is not satisfied. Given a set of H, μ, and σ in their respective reasonable ranges (namely, $H \in [0.55, 0.8]$, $\mu \in [-0.1, 0.3]$, and

Table 1. Simulation analysis of stop-loss policy.

Strategy	Stopping threshold	Return	S.D.	Sharpe ratio	Prob. of stopping
Panel A: Hurst parameter is 0.3					
Buy and Hold		0.3340	0.2818	0.8659	
Stop-loss	0.01	0.0704	0.1007	− 0.1947	0.9712
	0.05	0.0711	0.1819	− 0.1039	0.8996
	0.1	0.0978	0.2698	0.0288	0.7518
	0.3	0.2914	0.3328	0.6051	0.1262
Panel B: Hurst parameter is 0.5					
Buy and Hold		0.2970	0.3506	0.5903	
Stop-loss	0.01	0.1142	0.1668	0.1450	0.9104
	0.05	0.1669	0.2745	0.2800	0.7172
	0.1	0.2168	0.3295	0.3847	0.5162
	0.3	0.2884	0.3551	0.5587	0.0938
Panel C: Hurst parameter is 0.7					
Buy and Hold		0.2285	0.4348	0.3186	
Stop-loss	0.01	0.1994	0.2744	0.3987	0.7436
	0.05	0.2506	0.3304	0.4861	0.5330
	0.1	0.2629	0.3543	0.4880	0.3988
	0.3	0.2452	0.4017	0.3863	0.1204

Note: Five thousand price paths are simulated for each Hurst parameter. Prob. of stopping denotes the probability of exercising stop-loss policy.

$\sigma \in [0.1, 0.4]$), 1000 simulated asset return paths are used for estimating the probability of *Condition A* not met at the time of hitting a stop-loss threshold. Specifically, different stop-loss thresholds for a buy-and-hold strategy are applied to these paths. For each threshold, the number of paths (out of the 1000) crossing the threshold (denoted as N_{Stop}) and the number of times that *Condition A* are satisfied (denoted as $N_{Condition}$) are recorded. The fraction of time when *Condition A* is not met is $1 - N_{Condition}/N_{Stop}$, which converges to the probability of *Condition A* not satisfied as the number of simulated paths increases to infinity. A subset of the simulation results corresponding to three different stop-loss thresholds are shown in table 2. Boldfaced numbers highlight the parameter sets for which the probability of *Condition A* not satisfied is positive. The simulation study shows that *Condition A* is almost always satisfied for reasonable choices of the stop-loss threshold, H, μ, and σ.

Remark The simulation study also indicates that the probability of *Condition A* not satisfied decreases with the stop-loss threshold level given a set of H, μ, and σ. Furthermore, the probability of *Condition A* holding true increases in both H and σ, and decreases in μ.

Table 2. Simulation analysis for Condition A.

Hurst	μ	σ	Stopping threshold	N_{Stop}	$N_{Condition}$
0.6	− 0.1	0.1	0.02	994	994
			0.05	979	979
			0.1	940	940
0.6	− 0.05	0.2	0.02	947	947
			0.05	901	901
			0.1	834	834
0.7	0.05	0.1	0.02	592	**590**
			0.05	415	415
			0.1	233	233
0.7	0.1	0.2	0.02	715	**181**
			0.05	597	**526**
			0.1	450	450
0.8	0.05	0.1	0.02	579	579
			0.05	498	498
			0.1	365	365
0.8	0.1	0.2	0.02	629	629
			0.05	534	534
			0.1	419	419

Note: One thousand paths are simulated for each combination of Hurst parameter, μ, and σ. N_{Stop} is the number of paths crossing the stop-loss threshold. $N_{Condition}$ is the number of paths in N_{Stop} on which *Condition A* is satisfied.

4. Optimal stop-loss rules

4.1. Simulation analysis

To identify the optimal stop-loss policy, a class of rules defined by a fixed stopping threshold are considered. Note that the optimal threshold stop-loss policy may still be suboptimal with respect to the entire admissible stop-loss policy space as it is optimal only in a subset (namely, the threshold policy subspace) of the admissible policy space. As explained in section 3.3, the buy-and-hold strategy with stop-loss are respectively applied to simulated paths. The corresponding returns, volatilities and Sharpe ratios of the buy-and-hold strategy with different stopping thresholds are plotted in figure 1. Apparently, the optimal stopping threshold exists under different scenarios. The investment performance is improved with stop-loss strategy, and this improvement is attributed to both increase in return and decrease in volatility. Notable, the investment performance is not sensitive to the change of stopping threshold. A slightly biased optimal threshold can also lead to a superior trading performance.

To apply the optimal stop-loss policy in financial markets, the relationship between the variables (μ, σ, and H) and optimal stopping threshold shall be thoroughly investigated. In table 3, the buy-and-hold strategy with different stopping

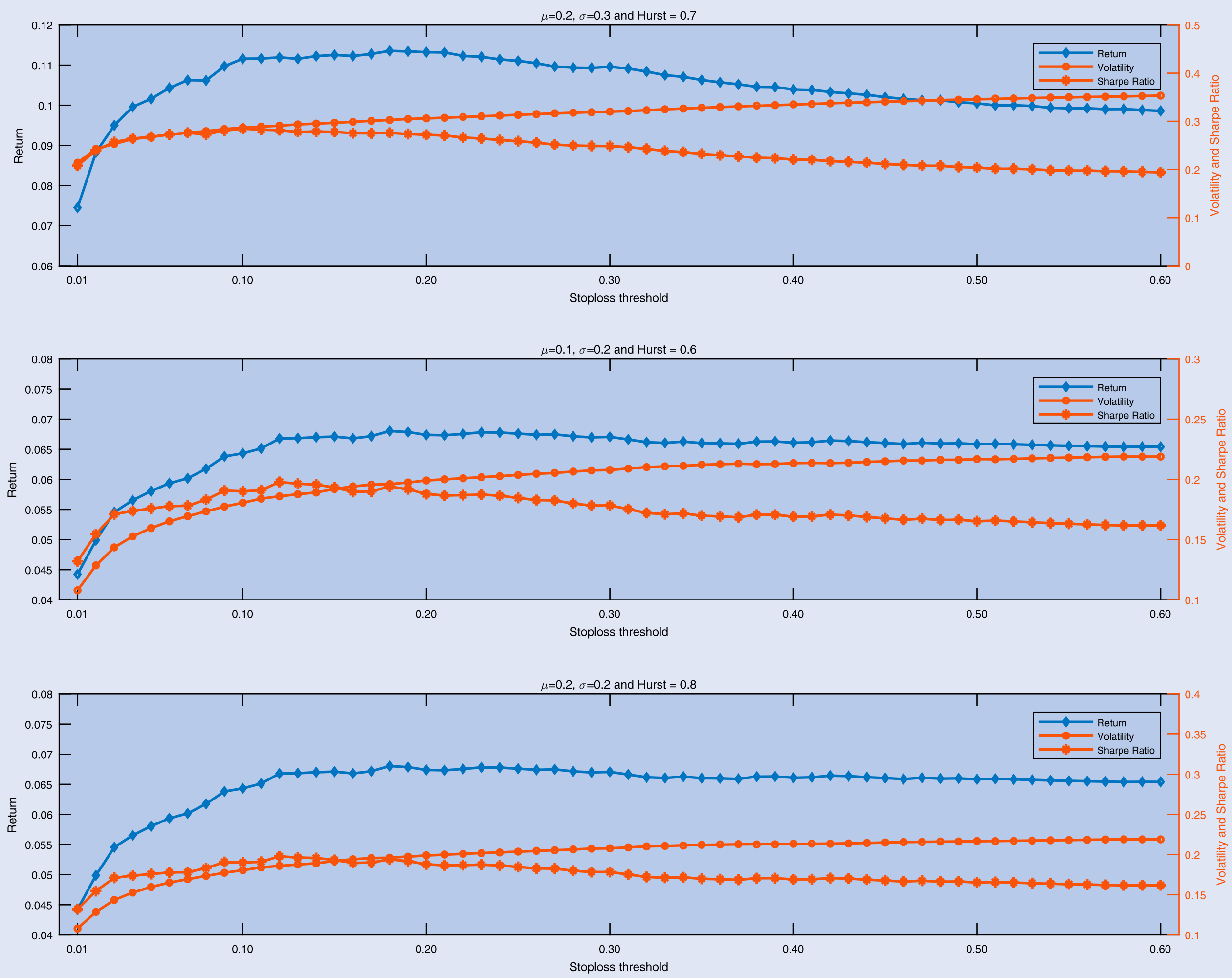

Figure 1. Trading performances of stop-loss policy under different scenarios.

thresholds are applied to 5000 paths simulated with different combinations of Hurst parameter, expected return and volatility. For each parameter set, the optimal stopping threshold is identified as the one achieving the highest Sharpe ratio.

In panel A, Hurst parameter varies from 0.55 to 0.8. A negative correlation is found between the Hurst parameter and optimal stopping threshold. Hurst parameter is positively correlated to the Sharpe ratio and probability of triggering stop-loss rule. Higher Hurst parameter indicates there exists stronger and clearer trend, therefore rational investors have quicker responses to adverse trends. This means a lower stopping threshold should be applied to the strategy. In panel B, expected return varies from 0.05 to 0.3. Expected return is positively correlated to the optimal stopping threshold, return, standard deviation, and Sharpe ratio. In panel C, volatility varies from 0.1 to 0.4. Volatility is positively correlated to the optimal stopping threshold level, return, and Sharpe ratio. In summary, these three variables all significantly influence the optimal stopping threshold.

Due to the lack of analytical characterization of first-passage time of fBMs, the explicit relationship between the optimal stopping threshold and other variables is unavailable. A polynomial regression method is used to approximate this relationship. The buy-and-hold strategy with different stopping thresholds are applied to the sample paths, which are simulated with different combinations of the Hurst parameter, expected return and volatility. Then the optimal stopping threshold is obtained through grid search corresponding to each combination of model parameters. Hurst parameter varies from 0.54 to 0.8 with an increment of 0.02; expected return varies from 0.05 to 0.3 with an increment of 0.01; and volatility varies from 0.1 to 0.5 with an increment of 0.02. Stopping threshold vary from 0.01 to 0.6 with an increment of 0.01. Finally, polynomial regression is applied to estimate a relationship between optimal stopping threshold as dependent variable and the parameters of fGBM as independent variables.

Kaminski and Lo (2014) model return-generating process as the following autoregressive model:

$$y_t = \mu + \rho y_{t-1} + \varepsilon_t,$$
$$\varepsilon_t \sim N(0, \sigma^2),$$

where ρ lies in the interval $(-1, 1)$ to ensure y_t is a stationary process. They conclude that a positive ρ provides sufficient condition for a stop-loss rule to add value for an autoregressive return-generating process. Therefore, the optimal stopping threshold is influenced by ρ. Similarly, a relationship between the optimal stopping threshold and parameters of the autoregressive model is identified by polynomial regression.

Table 3. Optimal stopping threshold under different scenarios.

Hurst	μ	σ	Optimal stopping threshold	Return	S.D.	Sharpe ratio	Prob. of stopping
Panel A: Hurst							
0.55	0.12	0.3	0.23	0.1938	0.4735	0.2193	0.4418
0.6			0.09	0.1889	0.3943	0.2509	0.6832
0.7			0.05	0.2098	0.3794	0.3156	0.7124
0.8			0.03	0.2302	0.3807	0.3683	0.7008
Panel B:μ							
0.6	0.05	0.3	0.02	0.1077	0.1984	0.0892	0.9108
	0.12		0.09	0.1889	0.3943	0.2509	0.6832
	0.2		0.32	0.3953	0.5598	0.5454	0.1850
	0.3		0.58	0.6946	0.5843	1.0346	0.0064
Panel C:σ							
0.6	0.12	0.1	0.28	0.3362	0.1951	1.2616	0.0006
		0.2	0.22	0.2699	0.3698	0.4865	0.1934
		0.3	0.09	0.1889	0.3943	0.2509	0.6832
		0.4	0.07	0.1459	0.3864	0.1447	0.8188

Note: The optimal stopping threshold for each parameter set is identified as the one achieving the highest Sharpe ratio.

Table 4 presents the results of polynomial regressions. In panel A, optimal stopping threshold is explained by the Hurst parameter, expected return and volatility of fGBM. All coefficients of these variables are statistically and economically significant. In panel B, optimal stopping threshold is explained by ρ, expected return and volatility of autoregressive model. All coefficients of these variables are significant. But the adjusted R-square of the autoregressive model is significantly less than that of the fBM model.

4.2. Case studies: future market trading

To examine the performance of optimal stop-loss policy in real markets, the classical moving average cross-over strategy with optimal stopping threshold is applied to the trading of commodity and index futures contracts in China market, such as metals (copper, aluminum, gold, and silver), agriculture products (corn and cotton), and stock index (CSI 300, a capitalization-weighted market index which consists of 300 stocks traded on the Shanghai and Shenzhen Stock Exchanges). All the daily data are downloaded from Joinquant, a quantitative trading platform in China.

Table 5 provides the summary statistics for daily returns of these future contracts, including JB statistic and Hurst exponent from rescaled adjusted range (R/S) analysis of Lo (1991). These daily returns are clearly not normally distributed as the data shows strong negative skewness and a very large kurtosis. Rare events may cause extreme returns in commodity future markets in China. The Hurst exponents of these futures' returns indicate that long-range dependence widely exists in Chinese futures markets. Therefore, modeling the return-generating processes with fBMs is imperative and empirically supported.

Simple moving average cross-over (MACO) strategy employs two moving averages: a faster (short term) moving average with a period of 20 and a slower (long term) moving average with a period of 120. This strategy opens a long position and/or closes a short position, when the faster moving average rises above the slower moving average. On the other hand, the strategy opens a short position and/or closes a long position, when the faster moving average falls below the slower moving average. The simple MACO does not use any stop-loss rule, which means that any opened positions are closed only when the two moving averages cross each other again.

To examine the optimal stop-loss policy, an optimal stopping threshold obtained under fBM is imposed on the simple MACO strategy, which is called FBM strategy. At time t, past 120 data are used to estimate the parameters of fGBM. If Hurst parameter is higher than 0.5, an optimal stopping threshold γ^*_{fBm} is obtained with the relationship identified by the polynomial regression under fBM in section 4.1. If a long (short) position is held at time t_0, close the position when $P_t < P_{t_0}(1-\gamma^*_{fBm})$ ($P_t > P_{t_0}(1+\gamma^*_{fBm})$) and invest in a risk-free asset until next crossover. If the optimal stopping threshold is obtained by assuming the return-generating process following an autoregressive model, then this strategy is named AR. When ρ is positive, an optimal stopping threshold γ^*_{AR} is obtained with the relationship identified by the polynomial regression under the autoregressive model.

Table 6 summarizes the empirical performances of simple MACO, FBM, and AR strategies. The returns of FBM strategy are significantly higher than those of AR and simple MACO strategies. The information ratios of FBM strategy are positive and greater than that of AR strategy. The performances of the optimal stop-loss rules are consistent with their intended objectives in real-world practices. These evidences illustrate the advantages of modeling the asset return-generating processes as fBMs instead of autoregressive processes. The stop-loss rules do create value in trading assets with long-range dependence effect.

5. Conclusion

Existing research shows that a stop-loss strategy adds value to the trading of risky assets if their prices evolve according to autoregressive or regime-switching processes. We extend the analysis of stop-loss strategy to a market environment in which fractal behaviors of financial asset prices such

Table 4. Polynomial regression method for identifying optimal stopping threshold.

Panel A: FBM										
Variable	Constant	Hurst	μ	σ	$Hurst^2$	μ^2	σ^2	$Hurst * \mu$	$Hurst * \sigma$	$\mu * \sigma$
Coefficient	0.70	− 1.83	4.17	− 0.53	1.12	− 0.63	− 0.43	− 4.15	0.94	− 1.00
t-statistic	10.88	− 10.20	55.62	− 11.34	8.80	− 7.07	− 12.47	− 43.88	16.07	− 20.28
Adj-R^2	0.85									
Panel B: AR										
Variable	Constant	ρ	μ	σ	ρ^2	μ^2	σ^2	$\rho * \mu$	$\rho * \sigma$	$\mu * \sigma$
Coefficient	0.10	0.28	1.52	1.90	− 0.43	− 8.82	− 2.95	3.07	− 1.92	3.87
t-statistic	14.45	15.29	31.38	61.01	− 19.56	− 74.48	− 64.98	66.97	− 67.79	59.07
Adj-R^2	0.64									

Note: The return-generating processes in panel A and panel B are fGBM and autoregressive model respectively.

Table 5. Summary statistics of commodity and index futures.

Underlying	Mean	Min	Max	S.D.	Skew	Kurt	JB	Hurst
Gold (2008/1/10–2021/2/5)	1.77E-04	− 0.0784	0.0568	0.0111	− 0.3422	4.7973	3113.4	0.5473
Copper (2005/1/4–2021/2/5)	1.82E-04	− 0.0657	0.0616	0.0152	− 0.2438	2.4914	1051.3	0.5768
Corn (2005/1/4–2021/2/5)	2.30E-04	− 0.1491	0.0828	0.0086	− 1.2589	38.6603	244843.0	0.5350
Cotton (2005/1/4–2021/2/5)	5.89E-05	− 0.1599	0.0917	0.0119	− 0.6187	13.3704	29411.3	0.5958
Aluminum (2005/1/4–2021/2/5)	− 8.16E-06	− 0.0629	0.0573	0.0103	− 0.3763	4.6813	3667.1	0.5891
CSI 300 (2010/4/19–2021/2/5)	2.04E-04	− 0.1064	0.0974	0.0163	− 0.4214	7.0720	5556.4	0.5164
Silver (2012/5/11–2021/2/5)	− 4.38E-05	− 0.1035	0.0806	0.0157	− 0.2412	5.9398	3151.9	0.5457

Note: JB denotes the statistic from Jarque–Bera test. Hurst denotes Hurst exponent from rescaled adjusted range (R/S) analysis.

as long-range dependence are explicitly modeled by fBMs. Both theoretical analysis and simulation studies demonstrate that stop-loss rules improve investment performance of the buy-and-hold strategy when return-generating processes are assumed to follow fGBMs. fBM-based asset price models lead to more accurate assessment of the efficacy of stop-loss policies. Specifically, simulation studies find that the Hurst parameter, expected return, and volatility of fGBM influence the efficacy of stop-loss rules. To achieve the best performances of stop-loss strategies, stopping threshold shall be optimized with respect to these variables. A polynomial regression method is developed to approximate the relationship between the optimal stopping threshold and these variables. Empirical studies for optimal stop-loss rules are conducted with historical market data of commodity and index futures in Chinese markets. The findings demonstrate

Table 6. Trading performances of optimal stop policy on both commodity and index futures.

Underlying	Strategy	Return	S.D.	IR	MDD	Prob. of stopping
Gold	MACO	0.0452	0.1561		0.4925	
	FBM	0.0501	0.1544	0.2173	0.4274	0.2500
	AR	0.0008	0.0802	− 0.3313	0.2134	0.5625
Copper	MACO	0.0438	0.2246		0.6487	
	FBM	0.0551	0.2166	0.1906	0.6201	0.1087
	AR	0.0494	0.1856	0.0441	0.5553	0.3696
Corn	MACO	0.0334	0.1292		0.3281	
	FBM	0.0355	0.1285	0.1532	0.3281	0.1429
	AR	0.0133	0.0607	− 0.1760	0.2168	0.7857
Cotton	MACO	0.0932	0.1906		0.4322	
	FBM	0.1015	0.1870	0.2268	0.4322	0.2500
	AR	0.0220	0.0747	− 0.4057	0.2358	0.8333
Aluminum	MACO	0.0280	0.1661		0.5743	
	FBM	0.0364	0.1597	0.1835	0.4518	0.1091
	AR	0.0251	0.1278	− 0.0272	0.3184	0.4182
CSI 300	MACO	0.0565	0.2533		0.5714	
	FBM	0.1246	0.2223	0.5629	0.4370	0.2917
	AR	0.0903	0.1771	0.1868	0.4370	0.5833
Silver	MACO	− 0.0543	0.2454		1.0348	
	FBM	− 0.0024	0.1998	0.3646	0.6518	0.3333
	AR	− 0.0030	0.1289	0.2457	0.5427	0.7576

Note: IR denotes information ratio with the benchmark strategy as MACO. MDD denotes maximum drawdown.

that commonly used trading strategies with optimal stop-loss policies outperform the ones without stop-loss in historical back testing. Theoretical studies of the efficacy of the stop-loss strategy can be extended to more generalized market settings (e.g. incorporating transaction-related costs to examine their impacts on strategy parameters) which are left for future research.

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Appendices

Appendix A: Proof of Proposition 1

If return generating process is Markovian, the expected return after a loss occurs is presented as follows:

$$E(R_T - R_\tau | \mathscr{F}_\tau) = E(R_{T-\tau}).$$

Therefore, the loss has no influence on the expected return in the future, whether or not the stop-loss policy is triggered has no informational content for the predictability of asset returns in the future. As long as the expected return is higher than risk-free rate, the investors should hold the position even when a loss occurs.

Appendix B: Proof of Proposition 2

When Hurst parameter is higher than 0.5, and the holding return R_τ touches the stopping threshold for the first time at τ, the stop-loss policy is exercised if and only if stopping premium exists, namely, $E(R_T - R_\tau | \mathscr{F}_\tau) < r_f(T - \tau)$. Applying equation (4), we get that the stopping premium exists if the following inequality hold given filtration $\mathscr{F}_\tau$:

$$\mu(T-\tau) - \frac{1}{2}\sigma^2(T^{2H} - \tau^{2H}) + \sigma \int_0^\tau \Psi^H(T,\tau,v)dB_v^H \\ < r_f(T-\tau), \tag{5}$$

where $\Psi^H(T,\tau,v) > 0$, $B_\tau^H = (-\gamma - \mu\tau + \frac{1}{2}\tau^{2H}\sigma^2)/\sigma < 0$, and $H > 0.5$. It is easy to see that (5) holds if *Condition A* is satisfied. This completes the proof of Proposition 2.

Note that in *ex ante* Condition A corresponds to the event of sum of two random variables being less than a constant. Specifically,

$$\int_0^{\tau} \Psi^H(T,\tau,v)dB_v^H + g(\tau) < g(T), \tag{6}$$

where $g(t) = (r_f t - \mu t + \frac{1}{2}\sigma^2 t^{2H})/\sigma$ and τ is the stopping time defined before. Intuitively, $\int_0^{\tau} \Psi^H(T,\tau,v)dB_v^H$ is the limit of a Riemann sum involving differences of truncated Gaussian random variables, which have negative infinity in the supports. It thus can be shown that the *ex ante* probability of the event defined by (6) is positive. Namely, *Condition A* holds with positive probability in *ex ante* (confirmed in numerical examples as well). A properly chosen stopping threshold yield a positive stopping premium with positive probability.