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# A theoretical generalization of the Markowitz model incorporating skewness and kurtosis

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This paper proposes a generalization of Markowitz model that incorporates skewness and kurtosis into the classical mean–variance allocation framework. The principal appeal of the present approach is that it provides the closed-form solution of the optimization problem. The four moments optimal portfolio is then decomposed into the sum of three portfolios: the mean–variance optimal portfolio plus two self-financing portfolios, respectively, accounting for skewness and kurtosis. Theoretical properties of the optimal solution are discussed together with the economic interpretation. Finally, an empirical exercise on real financial data shows the contribution of the two portfolios accounting for skewness and kurtosis when financial returns depart from Normal distribution.

*Keywords*: Portfolio optimization; Higher moments; Skewness; Kurtosis; Mathematical programming

## 1. Introduction

The seminal mean–variance approach developed by Markowitz (1952) is the starting point of modern portfolio theory. Such an approach is based on the hypotheses that financial returns follow a Normal distribution and/or, alternatively, that the individual utility function is quadratic, i.e. individual preferences exclusively depend on the first two moments of returns distribution. When the normality assumption is relaxed, higher moments, typically skewness and kurtosis, need to be incorporated both in the optimal allocation problem and in the individual utility function. For this reason, the argument that the higher moments of returns distribution are relevant to the investor's decision cannot be neglected, see for example Müller and Machina (1987), Jurczenko and Maillet (2006), Liechty and Sağlam (2017) and Stöckl and Kaiser (2021). The necessity to consider the non-Normality of asset returns is determined by the need to reconcile the classical theoretical approach with the empirical evidence that financial returns often and strongly depart from a Normal distribution see, for example, Chen and Zhou (2018), Clark (1973), Harvey and Siddique (2000), Richardson and Smith (1993), Xu (2007), Blau (2017), Finta and Aboura (2020) and Karoglou (2010). Moreover, investors care about the higher moments of their investment portfolios, see Scott and Horvath (1980), Harvey and Siddique (2000) and Ang *et al.* (2006). Important streams of research focus on non-Gaussianity of returns through the use of Taylor expansion of the utility function, see, among others, Jondeau and Rockinger (2006), Guidolin and Timmermann (2008), Zakamouline and Koekebakker (2009) and Martellini and Ziemann (2010), or through the use of Gram–Charlier expansion of downside risk measures, see Favre and Galeano (2002), Leon and Moreno (2017), Zoia *et al.* (2018) and Lassance and Vrins (2021).

In the out-of-sample framework; however, optimal portfolio strategies might under-perform when compared to heuristic approaches as shown in DeMiguel *et al.* (2009). This phenomenon has been extensively discussed in the literature and can be determined by model uncertainty, see Pflug *et al.* (2012). One further stream of research relates the difficult implementation of optimal approaches to the numerical instability of the solution, see Torrente and Uberti (2021). In other words, the in-sample mean–variance frontier is a biased estimator of the real efficient frontier, see Kan and Smith (2008). Many authors, see for instance Best and Grauer (1991) and Kan and Zhou (2007), consider the estimation uncertainty as the principal cause of instability in the model.

A bunch of different papers proposes alternative solutions to incorporate skewness and kurtosis into the classical portfolio optimization scheme, see among the others Aksarayli and Pala (2018), Jean (1971), Jondeau and Rockinger (2006),

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Kon (2012), Saranya and Prasanna (2014) and Chen and Zhou (2018). The evidence is that both skewness and kurtosis significantly impact the optimal allocation, see Xiong and Idzorek (2011) and Wilcox (2020). In particular, skewness introduces the asymmetry in the model while kurtosis accounts for fat tails, potentially reducing the effects of extreme events. Even if some limited empirical evidence suggests that mean–variance criterion could be good also when returns are non normal, see Levy and Markowitz (1979), Pulley (1981) and Kroll *et al.* (1984), probably depending on the fact that returns could be driven by an elliptical distribution, for which the mean–variance approximation of the expected utility remains good for all utility functions Chamberlain (1983), on the opposite, under large departures from normality, in particular when the distribution is severely asymmetric, Chunhachinda *et al.* (1997), Athayde and Flôres (2004) and Jondeau and Rockinger (2006) show that the mean–variance criterion can lead to unsatisfactory results. In such a case, a three or four moments optimization strategy can improve the portfolio performances.

In practical applications, as highlighted in Lassance and Vrins (2021), higher moments need to be incorporated into the allocation model taking into account the extra parameters that need to be estimated.

In the present paper, a four moments asset allocation scheme is studied. The proposal is an extension up to the fourth moment of the results obtained in Gamba and Rossi (1998a) for three moments case. Some assumptions are made in order to simplify the optimization model and to obtain the solution in a closed-form. The key assumption concerns the representation of skewness and kurtosis, which is consistent with the approach proposed by Ingersoll (1987). This hypothesis permits also to reduce the number of parameters without considering the co-moments for skewness and kurtosis, resulting in a parsimonious approach in terms of the number of parameters to be estimated. The proposal consists in a generalization of Markowitz model. The optimization problem minimizes portfolio kurtosis subject to portfolio expected return, variance, skewness and the standard budget constraint. The optimal portfolio can be written as the sum of three portfolios: the classical mean–variance optimal portfolio plus two self-financing portfolios accounting, respectively for skewness and kurtosis. When kurtosis is equal to 3, i.e. there is no kurtosis in excess with respect to the Gaussian case, the solution boils down to the mean–variance-skewness optimal portfolio obtained in Gamba and Rossi (1998a). Moreover, when skewness is null and kurtosis is equal to 3, the optimal solution coincides with the classical mean–variance optimal portfolio.

The general idea is that, when skewness and kurtosis are taken into account in the allocation model, the optimal portfolio is characterized by a higher variance compared to the standard mean–variance optimal portfolio. This variance in excess is counterbalanced in terms of utility by the preferences of the investor for higher-order moments. It is assumed that investors like return and skewness, the odd moments, whereas they dislike even moments, variance and kurtosis, see Horvath and Scott (1980) and Menegatti (2015). No further assumptions on the shape of the individual utility function are made.

An empirical application on real financial data shows the effectiveness of the proposal. The model is performed by substituting the theoretical moments with the respective empirical moments calculated on the basis of the historical data. The application highlights that the self-financing portfolios added to the mean–variance optimal one contribute, respectively to increase skewness and reduce kurtosis, in accordance to the investor preferences for higher-order moments.

The paper is organized as follows: Section 2 presents the theoretical formalization of the proposal and it is divided into three subsections containing, respectively, the assumptions of the model, the mathematical formalization of the optimization problem and its solution in the case of $n$ risky assets and in the case of $n$ risky asset and one riskless asset; Section 3 contains an empirical application on real financial data while Section 4 ends the paper.

## 2. Theoretical formalization

### 2.1. Assumptions

In this section, the hypotheses of the proposal are summarized. Assume that $n$ risky assets are available and that:

- $r$ is the column vector of returns in $\mathbf{R}^n$
- $y$ is a random variable such that $E(y) = 0$, $E(y^2) = \sigma_y^2 = 1$, $E(y^3) = \xi_y^3 \neq 0$ and $E(y^4) = k_y^4 = 3$;
- $z$ is a random variable such that $E(z) = 0$, $E(z^2) = \sigma_z^2 = 1$, $E(z^3) = \xi_z^3 = 0$ and $E(z^4) = k_z^4 \neq 3$;
- $\epsilon$ is a column vector in $\mathbf{R}^n$ random variable such that $\epsilon \mid y, z$ is Gaussian with zero mean and covariance matrix $C = E[\epsilon\epsilon' \mid y, z]$;
- the variables $y$ and $z$ are independent;
- $b$, $t$ are two column vectors in $\mathbf{R}^n$;
- the vectors $\mu$, $\mathbf{1}$, $b$ and $t$ are assumed to be linearly independent, where $\mathbf{1}$ is the unitary vector in $\mathbf{R}^n$;
- the vector of the risky returns $r$ can be written as follows:

$$r = \mu + \epsilon + by + tz \tag{1}$$

Equation (1), considering both skewness and kurtosis, is a generalization of returns representation proposed in Ingersoll (1987). In equation (1), $\epsilon$ describes the Gaussian component of the returns, while the variables $y$ and $z$ add separately asymmetry and fat tails to the returns distribution. The variables $y$ and $z$ are instrumental variables with the exclusive technical role to introduce skewness and kurtosis into the model. Thanks to the independence assumption, the covariance matrix of the returns is

$$D = [E(\epsilon\epsilon'|y, z) + bb'\sigma_y^2 + tt'\sigma_z^2]. \tag{2}$$

The matrix $D$ is assumed to be positive definite and, consequently, non singular. The co-skewness $\xi_{i,j,l}$ between assets $i$, $j$ and $l$ and co-kurtosis $k_{i,j,l,m}$ between the assets $i$, $j$, $l$ and $m$ are defined as

$$\xi_{i,j,l} = E\left[(r_i - \mu_i)(r_j - \mu_j)(r_l - \mu_l)\right] = b_i b_j b_l \xi_y^3, \tag{3}$$

$$\begin{aligned} k_{i,j,l,m} &= E\left[(r_i - \mu_i)(r_j - \mu_j)(r_l - \mu_l)(r_m - \mu_m)\right] \\ &= t_i t_j t_l t_m k_z^4, \end{aligned} \tag{4}$$

where the subscripts indicate the entries of the vectors $r$, $\mu$, $b$ and $t$. Note that at least three and four assets are needed to define, respectively, skewness and kurtosis. Moreover, note that equations (3) and (4) are linear functions of $\xi_y^3$ and $k_z^4$.

In particular, due to the assumptions on the structure of returns, the moments of a portfolio $p$ can be defined as

$$\mu_p = x'\mu, \quad \sigma_p^2 = x'Dx, \quad s_p^3 = x'b\xi_y, \quad k_p^4 = x'tk_z,$$

where $x \in \mathbf{R}^n$ is the column vector of portfolio weights. While the first two moments are computed as in the mean–variance framework, skewness and kurtosis are linear functions, respectively, of the skewness of $y$ and the kurtosis of $z$. This particular structure simplifies the functional form of portfolio skewness and kurtosis since the co-moments are not considered. The principal advantages of the approach are that it permits to calculate the optimal portfolio in a closed-form, as shown in the next section, and that it drastically reduces the number of parameters to be estimated for the implementation of the model.

In general, it is assumed that investors like skewness while dislike kurtosis. The E-V-S-K (Expected return-Variance-Skewness-Kurtosis) dominance rule is defined as follows.

DEFINITION 2.1 *Given two portfolios A and B identified by the weights $x_A$ and $x_B$, A E-V-S-K dominates B if $\mu_A \geq \mu_B$ and $\sigma_B^2 \geq \sigma_A^2$ and $\xi_A^3 \geq \xi_B^3$ and $k_B^4 \geq k_A^4$, with at least one of the previous inequalities holding strictly.*

Definition 2.1 naturally generalizes the mean–variance dominance rule for which one asset/portfolio is preferred if it provides a higher expected return for a given level of variance or a lower variance for a given expected return. The E-V-S-K dominance rule is in accordance to the general result that 'under usual hypotheses on the utility function, it is sufficient to assume that the generic $n$th-order derivative is either negative when n is even or positive when n is odd to get that the utility function exhibits the property that all its even derivatives until order n are negative and all its odd derivatives until order n are positive', see Menegatti (2015). For the purpose of the present research, no further discussion on the shape of the individual utility function is necessary.

### 2.2. *Four moments optimal portfolio*

In the four moments framework, the optimal portfolio is the one that minimizes kurtosis subject to expected return, variance, skewness and budget constraints. In order to simplify the calculations, the optimization problem is written as a maximization problem. Negative portfolio weights have the usual interpretation of short sellings.

### 2.3. *No riskless asset*

A portfolio composed by $n \geq 4$ risky assets is considered. The investor chooses a portfolio in accordance to the E-V-S-K dominance rule as expressed in Definition 2.1. The objective function is:

$$F(x) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{m=1}^{n} x_i x_j x_l x_m k_{i,j,l,m}. \tag{5}$$

Thanks to the assumptions, see Section 2.1, the objective function in equation (5) can be expressed in an equivalent form as

$$F(x) = (x't)^4 k_z^4.$$

The optimization problem, given portfolio expected return $\mu_p$, variance $\sigma_p^2$ and skewness $\xi_p^3$ is

$$\max_x \quad -x'tk_z \tag{6}$$

$$\text{s.t.} \quad x'Dx = \sigma_p^2 \tag{7}$$

$$x'\mu = \mu_p \tag{8}$$

$$x'\mathbf{1} = 1 \tag{9}$$

$$x'b = \frac{\xi_p}{\xi_y}. \tag{10}$$

The optimization problem presents a linear objective function, three linear constraints and a quadratic constraint. Equations (7) and (8) set the desired level of portfolio variance and expected return. Equation (9) is the usual budget constraint requiring that the totality of initial wealth is invested in the portfolio. Equation (10) is the constraint on portfolio skewness; note that it is possible to divide by the skewness of $y$ due to the assumption $\xi_y \neq 0$.

The matrix $P$ is defined as

$$P = M'D^{-1}M = \begin{bmatrix} a & c & f & p \\ c & d & g & q \\ f & g & e & r \\ p & q & r & s \end{bmatrix} \tag{11}$$

where $M = [\mu, \mathbf{1}, b, t]$. The matrix $M$ is a full rank matrix thanks to the assumptions, see subsection 2.1. The matrix $P$ is symmetric by construction and its entries are:

$$a = \mu'D^{-1}\mu, \quad c = \mu'D^{-1}\mathbf{1}, \quad f = \mu'D^{-1}b,$$
$$p = \mu'D^{-1}t, \quad d = \mathbf{1}'D^{-1}\mathbf{1},$$
$$g = \mathbf{1}'D^{-1}b, \quad q = \mathbf{1}'D^{-1}t, \quad e = b'D^{-1}b,$$
$$r = b'D^{-1}t, \quad s = t'D^{-1}t.$$

The sub-matrices $A$ and $P_2$ of $P$ are defined as:

$$P_2 = \begin{bmatrix} a & c & f \\ c & d & g \\ f & g & e \end{bmatrix} \quad A = \begin{bmatrix} a & c \\ c & d \end{bmatrix}.$$

The matrices $A$ and $P_2$ coincide, respectively, with the ones defined in Constantinides and Malliaris (1995) and Gamba and Rossi (1998a) to write the mean–variance and the mean–variance–skewness optimal portfolios in a closed-form. Then, the quantities $\psi' = [p \;\; q \;\; r]$ and $H = \psi' P_2^{-1}\psi$ are defined in order to simplify the calculations. The matrix $P$ can be written as:

$$P = \begin{bmatrix} P_2 & \psi \\ \psi' & s \end{bmatrix}.$$

The next lemma provides a technical result that will be useful for the solution of the optimization problem.

LEMMA 2.2 *If* $\operatorname{rank}(M) = 4$, *then* $s - H = \dfrac{\det P}{\det P_2} > 0$.

*Proof* See Appendix. ■

Let us define $\sigma_{P_2}^2 = \beta' P_2^{-1} \beta$, where $\beta = [\mu_p \;\; 1 \;\; \frac{\xi_p}{\xi_y}]'$. The quantity $\sigma_{P_2}^2$ can be interpreted as a variance, considering that it is positive thanks to the positive definiteness of matrix $P_2^{-1}$.

The following Lemma provides the optimal solution of problem (6)–(10).

LEMMA 2.3 *Given* $\mu_P$, $\sigma_P^2 \geq \sigma_{P_2}^2$, $\xi_P > 0$ *and* $k_z < 0$ ($k_z > 0$), *the unique optimal portfolio-solving problem* (6)–(10) *is*

$$x^* = D^{-1}[\mu \quad \mathbf{1} \quad b] P_2^{-1} \beta + (-)\sqrt{\frac{\sigma_P^2 - \sigma_{P_2}^2}{s - H}} \left(D^{-1} t - D^{-1}[\mu \quad \mathbf{1} \quad b] P_2^{-1} \psi\right) \quad (12)$$

*with the associated optimal kurtosis* $k^*$ *equal to*

$$k^* = k_z \left(\psi' P_2^{-1} \beta + \sqrt{s - H} \sqrt{\sigma_p^2 - \sigma_{P_2}^2}\right). \quad (13)$$

*Proof* See Appendix. ■

To obtain an interesting interpretation of the optimal portfolio represented in equation (12), it is useful to decompose it into the sum of three portfolios: the optimal mean–variance portfolio plus two self-financing portfolios accounting, respectively, for skewness and kurtosis.

The following corollary of Lemma 2.2 provides a technical result that is useful for the decomposition of the optimal portfolio and provides the fundamental economic intuition behind the proposal.

COROLLARY 2.4 *The following equation holds:*

$$\gamma' P \gamma = \sigma_P^2 > \sigma_{P_2}^2 = [\gamma_1 \quad \gamma_2 \quad \gamma_3] P_2 \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix}$$

*Proof* Trivial thanks Lemma 2.2. ■

A simple economic interpretation of the meaning of Corollary 2.4 is that an investor with a given structure of preferences for higher moments (see E-V-S-K dominance rule as defined in 2.1) is willing to choose portfolios with higher variance compared to the mean–variance efficient ones. In other words, such an investor balances the higher portfolio variance with a larger skewness and/or a smaller kurtosis with respect to her/his preferences toward higher-order moments.

It is finally possible to express the optimal solution of problem (6)–(10) as a function of matrix $A$. This allows to directly compare the four moments optimal portfolio obtained in the present paper with the three moments optimal portfolio as defined in Gamba and Rossi (1998a). Let define the quantity $\sigma_A^2$ as

$$\sigma_A^2 = [\mu_p \quad 1 \quad \gamma_4 p \quad \gamma_4 q] T \begin{bmatrix} \mu_p \\ 1 \\ \gamma_4 p \\ \gamma_4 q \end{bmatrix} \quad \text{with}$$

$$T = \begin{bmatrix} A^{-1} & -A^{-1} \\ -A^{-1} & A^{-1} \end{bmatrix}.$$

PROPOSITION 2.5 *Let* $\sigma_P^2 \geq \sigma_{P_2}^2 \geq \sigma_A^2$, *then the optimal portfolio for problem* (6)–(10) *can be equivalently written as:*

$$x^* = D^{-1}[\mu \quad \mathbf{1}] A^{-1} \begin{bmatrix} \mu_p \\ 1 \end{bmatrix} + \left(D^{-1} b - D^{-1}[\mu \quad \mathbf{1}] A^{-1} \begin{bmatrix} f \\ g \end{bmatrix}\right) \times \sqrt{\frac{\sigma_{P_2}^2 - \sigma_A^2}{e - h}} + \left(D^{-1} t - D^{-1}[\mu \quad \mathbf{1}] A^{-1} \begin{bmatrix} p \\ q \end{bmatrix}\right) \times \sqrt{\frac{\sigma_P^2 - \sigma_{P_2}^2}{s - H}}, \quad (14)$$

*where* $e - h = \frac{\det P_2}{\det A}$, $e$, $f$, $g$ *and* $s$ *are the entries of* $P$ *as defined in* (11) *and* $h = [f \quad g] A^{-1} [f \quad g]'$.

*Proof* See Appendix. ■

Starting from equation (14), the optimal portfolio can be decomposed as

$$x^* = x_{mv} + x_{sk} + x_k,$$

where

$$x_{mv} = D^{-1}[\mu \quad \mathbf{1}] A^{-1} \begin{bmatrix} \mu_p \\ 1 \end{bmatrix}, \quad (15)$$

$$x_{sk} = \left(D^{-1} b - D^{-1}[\mu \quad \mathbf{1}] A^{-1} \begin{bmatrix} f \\ g \end{bmatrix}\right) \sqrt{\frac{\sigma_{P_2}^2 - \sigma_A^2}{e - h}}, \quad (16)$$

$$x_k = \left(D^{-1} t - D^{-1}[\mu \quad \mathbf{1}] A^{-1} \begin{bmatrix} p \\ q \end{bmatrix}\right) \sqrt{\frac{\sigma_P^2 - \sigma_{P_2}^2}{s - H}}. \quad (17)$$

Portfolio $x_{mv}$ is the minimum variance portfolio, given the expected return $\mu_p$, see Constantinides and Malliaris (1995). Therefore $x_{mv}$ is such that $\mathbf{1}' x_{mv} = 1$, $\mu' x_{mv} = \mu_p$ and $x_{mv}' V x_{mv} = \sigma_A^2$.

Portfolio $x_{mv} + x_{sk}$ is the one that maximizes the skewness given the expected return $\mu_p$ and the variance $\sigma_{P_2}^2$ as obtained in Gamba and Rossi (1998a). Therefore $x_{mv} + x_{sk}$ is such that $\mathbf{1}'(x_{mv} + x_{sk}) = 1$, $\mu'(x_{mv} + x_{sk}) = \mu_p$ and $(x_{mv} + x_{sk})' V (x_{mv} + x_{sk}) = \sigma_{P_2}^2 \geq \sigma_A^2$.

Note that, by construction, portfolio $x_{sk}$ displays the property $\mathbf{1}' x_{sk} = 0$, i.e. it is a self-financing portfolio. Moreover, $x_{sk}$ is such that $\mu' x_{sk} = 0$. In words, $x_{sk}$ is neutral in terms of expected return not impacting the overall expected return. Portfolio $x_{sk}$ adds variance to the optimal mean–variance portfolio following the individual preferences for skewness and is null if and only if $\sigma_{P_2}^2 = \sigma_A^2$.

Similar comments are valid for $x_k$. Also $x_k$ is a self-financing portfolio, $\boldsymbol{1}'x_k = 0$, with a null expected return, $\mu' x_k = 0$, that introduces a further additional amount of variance following the individual preferences toward kurtosis ($x_k$ is null if and only if $\sigma_P^2 = \sigma_{P_2}^2$).

### 2.4. With a riskless asset

In this section, the four moment optimal portfolio in the case of $n$ risky assets and one riskless asset is derived. Let us consider the case where the assumptions in Section 2.1 are integrated with the presence of the risk-free asset with deterministic return $\mu_0$. The optimization problem can be written as:

$$\max_x \quad -x'tk_z \tag{18}$$

$$\text{s.t.} \quad x'Dx = \sigma_p^2 \tag{19}$$

$$x'(\mu - \mu_0 \boldsymbol{1}) = \mu_p - \mu_0 \tag{20}$$

$$x'b = \frac{\xi_p}{\xi_y} \tag{21}$$

With respect to the case of $n$ risky assets, the budget constraint is integrated into the restriction on portfolio expected return, see equation (20). As a consequence, the optimization problem has only two linear constraints beyond the quadratic restriction on the variance. The allocation on the riskless asset is calculated as the complement to one of the allocation on the $n$ risky assets.

Let us define the matrix $Q$ as

$$Q = N'D^{-1}N = \begin{bmatrix} m_1 & l_1 & f_1 \\ l_1 & e & g_1 \\ f_1 & g_1 & s \end{bmatrix}$$

where the $(n, 3)$ matrix $N$ is given by $N = [(\mu - \mu_0 \boldsymbol{1}), b, t]$. The matrix $Q$ is assumed to have full column rank (see Section 2.1) and is symmetric by construction. Let also define the quantities $\iota' = [f_1 \quad g_1]$ and $H_1 = \iota' Q_2^{-1} \iota$ and the matrix $Q_2$ as a sub-matrix of $Q$:

$$Q_2 = \begin{bmatrix} m_1 & l_1 \\ l_1 & e \end{bmatrix}$$

Note that the coefficient $m_1$ and the matrix $Q_2$ coincide, respectively, with the ones defined in Constantinides and Malliaris (1995) and Gamba and Rossi (1998a) to write the mean–variance and the mean–variance–skewness optimal portfolio in a closed-form.

Lemma 2.6 *If* $\operatorname{rank}(N) = 3$, *then* $s - H_1 = \dfrac{\det Q}{\det Q_2} > 0$.

*Proof* See Appendix. ■

The quadratic form $\sigma_{Q_2}^2 = \beta' Q_2^{-1} \beta$, where $\beta = [\mu_p \quad 1 \quad \frac{\xi_p}{\xi_y}]'$, is positive since the matrix $Q_2^{-1}$ is positive definite. The optimal portfolio for problem (18)–(21) is calculated in the following lemma.

Lemma 2.7 *Given* $\mu_p$, $\sigma_p^2 \geq \sigma_{Q_2}^2$, $\xi_p > 0$ *and* $k_z < 0 (k_z > 0)$ *the optimal portfolio for problem* (18)–(21) *is*

$$\begin{aligned} x^* &= D^{-1}[(\mu - \mu_0 \boldsymbol{1}) \quad b] Q_2^{-1} \begin{bmatrix} \mu_p - \mu_0 \\ \frac{\xi_p}{\xi_y} \end{bmatrix} \\ &\quad + (-)\sqrt{\frac{\sigma_p^2 - \sigma_{Q_2}^2}{s - H_1}} \\ &\quad \times \left( D^{-1}t - D^{-1}[(\mu - \mu_0 \boldsymbol{1}) \quad b] Q_2^{-1} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} \right) \end{aligned} \tag{22}$$

*and the optimal kurtosis* $k^*$ *is*

$$k^* = k_z \left( [f_1 \quad g_1] Q_2^{-1} \begin{bmatrix} \mu_p - \mu_0 \\ \frac{\xi_p}{\xi_y} \end{bmatrix} + \sqrt{s - H_1} \sqrt{\sigma_p^2 - \sigma_{Q_2}^2} \right). \tag{23}$$

*Proof* See Appendix. ■

As in the case of $n$ risky assets, after defining $\sigma_B^2 = -2\delta_3(\mu_p - \mu_0)\frac{f_1}{m_1} + \frac{(\mu_p - \mu_0)^2}{m_1} + \frac{f_1^2}{m_1}\delta_3^2$ (see the Appendix for more details), it is possible to obtain a decomposition of $x^*$ which highlights the contribution of skewness and kurtosis to the optimal mean–variance allocation.

Proposition 2.8 *Let* $\sigma_p^2 \geq \sigma_{Q_2}^2 \geq \sigma_B^2$, *then the optimal portfolio for problem* (18)–(21) *can equivalently be written as:*

$$\begin{aligned} x^* &= D^{-1}(\mu - \mu_0 \boldsymbol{1}) \frac{(\mu_p - \mu_0)}{m_1} \\ &\quad + \left[ D^{-1}b + D^{-1}(\mu - \mu_0 \boldsymbol{1}) \frac{l_1}{m_1} \right] \sqrt{\frac{\sigma_{Q_2}^2 - \sigma_B^2}{e - k}} \\ &\quad + \left[ D^{-1}t + D^{-1}(\mu - \mu_0 \boldsymbol{1}) \frac{f_1}{m_1} \right] \sqrt{\frac{\sigma_p^2 - \sigma_{Q_2}^2}{s - H_1}} \end{aligned} \tag{24}$$

*Proof* See Appendix. ■

Similarly to the case of $n$ risky assets, the optimal portfolio in equation (22) is decomposed in the sum of three components

$$x^* = x_{mv} + x_{sk} + x_k,$$

where

$$x_{mv} = D^{-1}(\mu - \mu_0 \boldsymbol{1}) \frac{(\mu_p - \mu_0)}{m_1} \tag{25}$$

$$x_{sk} = \left[ D^{-1}b + D^{-1}(\mu - \mu_0 \boldsymbol{1}) \frac{l_1}{m_1} \right] \sqrt{\frac{\sigma_{Q_2}^2 - \sigma_B^2}{e - k}} \tag{26}$$

$$x_k = \left[ D^{-1}t + D^{-1}(\mu - \mu_0 \boldsymbol{1}) \frac{f_1}{m_1} \right] \sqrt{\frac{\sigma_P^2 - \sigma_{Q_2}^2}{s - H_1}} \tag{27}$$

Portfolio $x_{mv}$ is the minimum variance portfolio, given the expected return $\mu_p$, see Constantinides and Malliaris (1995); its properties are: $\boldsymbol{1}'x_{mv} = 1$, $\mu' x_{mv} = \mu_p$ and $x_{mv}' V x_{mv} = \sigma_B^2$.

Portfolio $x_{mv}+x_{sk}$ is the portfolio that maximizes the skewness given the expected return $\mu_p$ and the variance $\sigma_B^2$ as obtained in Gamba and Rossi (1998a). Therefore $x_{sk}$ verifies the following equations: $\mathbf{1}'x_{sk}=0$, $\mu'x_{sk}=0$ and $(x_{mv}+x_{sk})'V(x_{mv}+x_{sk})=\sigma^2_{Q_2}>\sigma^2_B$. Finally $x_k$ is such that $\mathbf{1}'x_k=0$, $\mu'x_k=0$ and $(x_{mv}+x_{sk}+x_k)'V(x_{mv}+x_{sk}+x_k)=\sigma^2_Q>\sigma^2_{Q_2}$. Further comments are redundant considering the logical similarity with the case of $n$ risky assets.

The natural extension of the optimal portfolio model presented in this section is the generalization of the CAPM to the four moments framework. This would lead to a generalization of the three moment CAPM as introduced in Gamba and Rossi (1998b), incorporating in the pricing model a kurtosis premia, besides variance premia and skewness discounts, with the potential appeal of measuring deviations from standard (or multi-factors) CAPM. A study of the four moment CAPM extension is left to future research since it requires additional assumptions with respect to the ones in Section 2.1 and a more detailed discussion on the individual utility function as showed in Gamba and Rossi (1998b).

## 3. Empirical results

In this section, an example of application of the proposed allocation model in the case of $n$ risky assets is provided. Despite the fact that the empirical application is conducted on real financial data, it has to be interpreted as a toy example to show the effectiveness of the allocation procedure from a qualitative point of view. The parameters of the model, the vector of expected returns $\mu$, the covariance matrix $D$ and the vectors that introduce skewness and kurtosis in the returns, respectively $b$ and $t$, are estimated on the basis of the historical returns. After the proposal of Markowitz model and its various generalizations, an entire branch of the literature focused on the problem of the estimation of the parameters, highlighting its prominent role in producing good performance in practical applications. In particular, it is known how the optimal portfolios are highly sensitive to small perturbations of the parameters; the phenomenon is called estimation risk and can depend on the choice of poor probability models and/or on ignoring parameter uncertainty. Estimation risk is present in the mean–variance framework and is exacerbated when higher moments are taken into account due to the increasing number of parameters. For a detailed discussion of the topic in the context of three moments asset allocation and for a comprehensive review of the literature see Harvey *et al.* (2010).

Since the paper is mainly focused on the formalization of a novel theoretical model, the application described in the present section plays the fundamental role to explain how to implement the model in practice. A more detailed analysis of the empirical performances of the proposed model together with the discussion on the issue of parameters estimation is left to future research designed to deepen the empirical and computational aspects of the present approach.

The dataset is composed by $T=8125$ daily returns from 1-3-2000 to 23-2-2021 of the ten sectors portfolios of the S&P index obtained using the Global Industry Classification Standard (GICS): Energy (ENRS), Material (MATR), Industrials (INDU), Consumer-Discretionary (COND), Consumer-Staples (CONS), Healthcare (HLTH), Financials (FINL), Information-Technology (INFT), Telecommunications (TELS), and Utilities (UTIL).†

Table 1. First four moments of the returns distribution of the $n$ = 10 considered asset classes.

| | Mean | Variance | Skewness | Kurtosis |
|---|---|---|---|---|
| INFT* | 0.0005784 | 0.000275 | 0.239913 | 9.888727 |
| HLTH* | 0.0004256 | 0.000135 | − 0.02677 | 9.829053 |
| ENRS* | 0.0002914 | 0.000247 | − 0.14724 | 18.8766 |
| UTIL* | 0.0001949 | 0.000124 | 0.189206 | 18.9683 |
| FINL* | 0.0003885 | 0.000294 | 0.320646 | 20.48587 |
| CONS* | 0.000337 | 0.00009 | 0.008364 | 13.11328 |
| COND* | 0.0004414 | 0.000162 | − 0.07459 | 11.608772 |
| INDU* | 0.0003646 | 0.000156 | − 0.22611 | 11.794582 |
| TELS* | 0.0001911 | 0.000172 | 0.139834 | 11.026998 |
| MATR* | 0.0003176 | 0.000191 | − 0.10692 | 11.401184 |

Table 1 contains the sample estimations of the first four moments of the returns distribution for each asset class. One first evidence is that the asset classes show high levels of kurtosis testifying the presence of fat tails with respect to the Normal distribution. A general comment on skewness is more difficult since both positive and negative values are observed. A standard Jarque Bera test (see Jarque and Bera (1987)) has been performed on the returns of the $n=10$ asset classes: the Normality hypothesis for the totality of the considered assets is rejected at a significance level of 1%. Since the Jarque Bera test considers jointly skewness and kurtosis, it is impossible to discriminate if non-normality depends on asymmetry, fat tails or both of them. The non-Normal asset classes are identified with a star in table 1.

In order to implement the model, according to equation 1, the vectors $b$ and $t$ need to be estimated. Therefore two vectors $\hat{y}$ and $\hat{z}$ with $T=8125$ entries have been sampled independently, respectively, from a skew-normal distribution and from a $T$-student. The choice of the two distributions is in accordance to the assumptions on $y$ and $z$, see Section 2.1. The first four moments of $\hat{y}$ and $\hat{z}$ are collected in table 2; it is evident how the two variables add separately skewness and kurtosis to the distribution of returns. It is then possible to estimate $\hat{\mu}$, $\hat{b}$ and $\hat{t}$ simply applying the ordinary least squares to equation 1; the estimated values are reported in table 3. The covariance matrix $\hat{D}$ is finally calculated after the computation of the residuals $\hat{\epsilon}$.

The optimal portfolio in equation (12) describes the whole efficient frontier when $\mu_p$, $\sigma_P^2$ and $\xi_p^3$ vary. The application provides an intuitive and effective example on how the arbitrage portfolios $x_{sk}$ and $x_k$ correct the mean–variance optimal allocation $x_{mv}$ accounting for skewness and kurtosis. As shown in Proposition 2.5, the corrections for skewness and

† A non-Normal distribution of assets' returns is the necessary condition to apply the proposed four moments optimization model obtaining a solution that differs from the standard mean variance solution. Considering that monthly returns calculated on equity indexes can show limited deviations from the Normal distribution, it has been chosen to use daily returns to have data characterized by a more significant deviation from the Normal distribution.

Table 2. Empirical first four moments of $\hat{y}$ and $\hat{z}$.

| | Mean | Variance | Skewness | Kurtosis |
|---|---|---|---|---|
| $\hat{y}$ | 3.077D − 09 | 1 | − 0.3022332 | 0.0161179 |
| $\hat{z}$ | 3.175D − 08 | 1.0001231 | − 9.524D − 08 | 2.8242246 |

Table 3. Estimated values for $\hat{\mu}$, $\hat{b}$ and $\hat{t}$.

| | $\hat{\mu}$ | $\hat{b}$ | $\hat{t}$ |
|---|---|---|---|
| INFT | 0.0005784 | 0.0000067 | 0.0002425 |
| HLTH | 0.0004256 | − 0.0000450 | 0.0001342 |
| ENRS | 0.0002914 | − 0.0001031 | 0.0002852 |
| UTIL | 0.0001949 | − 0.0000578 | 0.0001888 |
| FINL | 0.0003885 | 0.0000012 | 0.0002002 |
| CONS | 0.0003370 | 0.0000962 | 0.0000947 |
| COND | 0.0004414 | 0.0000969 | 0.0002096 |
| INDU | 0.0003646 | 0.0000009 | 0.0002163 |
| TELS | 0.0001911 | 0.0000837 | 0.0000121 |
| MATR | 0.0003176 | − 0.0000194 | 0.0001123 |

kurtosis depend on the quantity of extra variance the investor is willing to tolerate with respect to the variance of $x_{mv}$ to modify her/his allocation. Let define the variance in excess to take position on skewness and kurtosis as $\Delta = \sigma^2_{P_2} - \sigma^2_A = \sigma^2_P - \sigma^2_{P_2}$. Note that in this simplified example the extra variance needed to take position with respect to skewness is set equal to the one with respect to kurtosis. In general, an investor can decide how to split the extra variance in accordance with her/his individual preferences towards skewness and kurtosis. Figure 1 shows the effect on the first four moments of the optimal portfolio when $x_{sk}$ and $x_k$ are added to $x_{mv}$. On the horizontal axis of each sub-figure there is $\Delta$. To draw Figure 1, $x_{mv}$ is set equal to the global minimum variance portfolio of the mean–variance efficient frontier; the arbitrariness of the choice of one point $x_{mv}$ on the mean–variance efficient frontier does not qualitatively affect the results since all the efficient mean variance portfolios show the same behavior depicted in figure 1 when corrected for skewness and kurtosis.

The following comments on figure 1 are evident. First, as described in the theoretical part of the paper, portfolios $x_{sk}$ and $x_k$ do not impact the expected return of the optimal portfolio, see the sub-figure on the top left. Second, the introduction of higher moments impacts the variance of the optimal portfolio; the more the investor is willing to take position with respect to higher moments, the higher the variance of the optimal portfolio, see the sub-figure on the top right. Moreover, the extra variance with respect to $x_{mv}$ of the four moments optimal portfolios is counterbalanced by a lower value of kurtosis and a higher value of skewness, see the sub-figures on the bottom left and right. It is finally interesting to notice the monotone behavior of all the functions drawn in figure 1.

The results of the application support the main intuition behind the theoretical proposal: according to the values of skewness and kurtosis of the assets, the two portfolios $x_{sk}$ and $x_k$ correct the mean–variance optimal allocation, respectively,

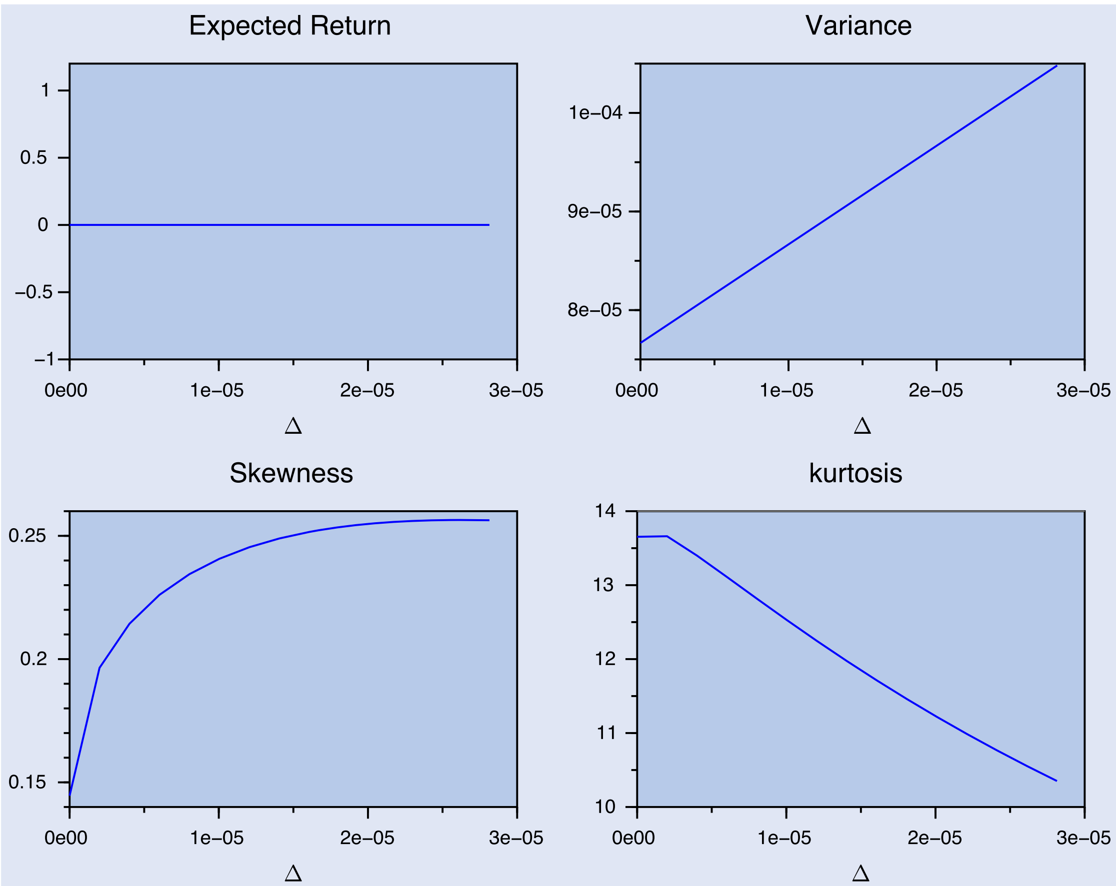


Figure 1. Variations of the first four moments of the optimal portfolio with respect to $\Delta$.

reducing its kurtosis and increasing its skewness. This evidence is clear despite the simplified structure of the higher-order moments where the co-moments are not taken into account.

## 4. Conclusions

The paper proposes an approach to optimal portfolio selection that incorporates expected return, variance, skewness and kurtosis in a unique theoretical framework. A closed-form solution for the optimization problem is obtained. The decomposition of the optimal solution in the sum of three portfolios highlights that the proposal is a generalization of Markowitz model. The self-financing portfolios added to the mean-variance optimal portfolio correct the allocation, increasing skewness and reducing kurtosis. This theoretical evidence is also confirmed from an empirical point of view. Moreover, the empirical section shows how to simply apply the theoretical model on real financial data. Future research will focus on a possible further generalization of the proposed model able to introduce in the allocation scheme the moments up to degree $n$, together with an extensive empirical study to strongly validate the present approach in practice. Controlling both skewness and kurtosis is expected to significantly reduce portfolio draw-downs during periods of financial crisis and distress.

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## Appendix

This is a technical appendix giving the complete proofs of the results stated in the paper.

*Proof of Lemma 2.2.*

*Proof* Let $y = Mx$, by substitution, it is possible to write

$$x'Px = x'M'D^{-1}Mx = y'D^{-1}y.$$

$D$ is positive definite, then $P$ is positive definite, i.e. $\det(P) > 0$ and $\det(P_2) > 0$. Moreover, since $(s-H)\det(P_2) = \det(P)$, then $s - H > 0$. ■

*Proof of Lemma 2.3.*

*Proof* The Lagrangian function for problem (6)–(10) is

$$\begin{aligned} L(x,\lambda) &= -x'tk_z - \lambda_1(x'Dx - \sigma_p^2) - \lambda_2(x'\mu - \mu_p) \\ &\quad - \lambda_3(x'\mathbf{1} - 1) - \lambda_4(x'b\xi_y - \xi_p), \end{aligned}$$

with first-order conditions

$$\begin{aligned} \frac{\partial L}{\partial x} &= -tk_z - 2\lambda_1 Dx - \lambda_2\mu - \lambda_3\mathbf{1} - \lambda_4 b\xi_y = \mathbf{0} \\ \frac{\partial L}{\partial \lambda_1} &= x'Dx - \sigma_p^2 = 0 \\ \frac{\partial L}{\partial \lambda_2} &= x'\mu - \mu_p = 0 \\ \frac{\partial L}{\partial \lambda_3} &= x'\mathbf{1} - 1 = 0 \\ \frac{\partial L}{\partial \lambda_4} &= x'b\xi_y - \xi_p = 0 \end{aligned} \tag{A1}$$

If $\lambda_1 \neq 0$, that is equivalent to require the quadratic restriction to hold, from (A1), the optimal portfolio can be written as

$$x^* = -\frac{\lambda_4}{2\lambda_1}\xi_y D^{-1}b - \frac{\lambda_2}{2\lambda_1}D^{-1}\mu - \frac{\lambda_3}{2\lambda_1}D^{-1}\mathbf{1} - \frac{k_z}{2\lambda_1}D^{-1}t. \tag{A2}$$

The vector $\gamma$ is defined as $\gamma' = [\gamma_1 \ \ \gamma_2 \ \ \gamma_3 \ \ \gamma_4]$ where

$$\begin{aligned} \gamma_1 &= -\frac{\lambda_2}{2\lambda_1}, \qquad \gamma_2 = -\frac{\lambda_3}{2\lambda_1}, \\ \gamma_3 &= -\frac{\lambda_4}{2\lambda_1}\xi_y, \qquad \gamma_4 = -\frac{k_z}{2\lambda_1}. \end{aligned} \tag{A3}$$

Equation (A2) can be rewritten in terms of $\gamma$ as $x^* = D^{-1}M\gamma$. Plugging $x^*$ into the constraints (7), (8), (9) and (10), the following equations hold

$$x'Dx = \gamma'M'D^{-1}DD^{-1}M\gamma = \gamma'P\gamma = \sigma_p^2 \tag{A4}$$

$$\mu'x = \mu D^{-1}M\gamma = \mu_p \tag{A5}$$

$$\mathbf{1}'x = \mathbf{1}'D^{-1}M\gamma = 1 \tag{A6}$$

$$b'x = b'D^{-1}M\gamma = \frac{\xi_p}{\xi_y}. \tag{A7}$$

From equations (A5), (A6) and (A7), $\gamma_1$, $\gamma_2$ and $\gamma_3$ are expressed as functions of $\gamma_4$

$$\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} = P_2^{-1}\beta - P_2^{-1}\psi\gamma_4.$$

By substituting in (A4)

$$\gamma_4 = \pm\sqrt{\frac{\sigma_p^2 - \sigma_{P_2}^2}{s-H}}. \tag{A8}$$

and, from equation (A3) $\lambda_1 = -\frac{k_z}{2\gamma_4}$.

The second-order condition for problem (6)–(10) is $H_x(L) = \frac{\partial^2 L}{\partial x^2} = -2\lambda_1 D$, where $H_x$ is the Hessian matrix, with respect to $x$, of the Lagrangian function. $D$ is positive definite, therefore the second-order condition depends on the sign of $\lambda_1$. Hence, the first-order conditions are sufficient for problem (6)–(10) if $\lambda_1 > 0$:

$$\begin{cases} \gamma_4 = +\sqrt{\dfrac{\sigma_p^2 - \sigma_{P_2}^2}{s-H}} & \text{if } k_z < 0 \\ \gamma_4 = -\sqrt{\dfrac{\sigma_p^2 - \sigma_{P_2}^2}{s-H}} & \text{if } k_z > 0 \end{cases}$$

The optimal kurtosis $k^*$ associated with the optimal portfolio $x^*$ can be obtained from $k = x'tk_z$ and it is equal to

$$k^* = k_z\left(\psi'D^{-1}\beta + \sqrt{s-H}\sqrt{\sigma_p^2 - \sigma_{P_2}^2}\right). \qquad ■$$

*Proof of Proposition 2.5.*

*Proof* To decompose the optimal portfolio $\gamma$ is written as a function of $A$ and $[\gamma_1 \ \ \gamma_2]$ as a function of $\gamma_3$ and $\gamma_4$ using equations (A5) and (A6):

$$\begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} = A^{-1}\begin{bmatrix} \mu_p \\ 1 \end{bmatrix} - A^{-1}\begin{bmatrix} f \\ g \end{bmatrix}\gamma_3 - A^{-1}\begin{bmatrix} p \\ q \end{bmatrix}\gamma_4. \tag{A9}$$

Plugging equation (A9) into (A4), $\gamma_3$ can be written as

$$\gamma_3^2(e-h) = \sigma_{P_2}^2 + 2\gamma_4[p \quad q]A^{-1}\begin{bmatrix} \mu_p \\ 1 \end{bmatrix} - [\mu_p \quad 1]A^{-1}\begin{bmatrix} \mu_p \\ 1 \end{bmatrix}$$

$$- \gamma_4^2 [p \quad q] A^{-1} \begin{bmatrix} p \\ q \end{bmatrix}$$

and, therefore, $\gamma_3^2(e-h) = \sigma_{P_2}^2 - \sigma_A^2$ where

$$\sigma_A^2 = \begin{bmatrix} \mu_p & 1 & \gamma_4 p & \gamma_4 q \end{bmatrix} T \begin{bmatrix} \mu_p \\ 1 \\ \gamma_4 p \\ \gamma_4 q \end{bmatrix} \quad \text{with}$$

$$T = \begin{bmatrix} A^{-1} & -A^{-1} \\ -A^{-1} & A^{-1} \end{bmatrix}.$$

Note that $\sigma_P^2 - \sigma_{P_2}^2 > 0$ thanks to Corollary 2.4 and that the quadratic form that defines $\sigma_A^2$ is semi-positive definite. In fact, being $A$ positive definite, then $A^{-1}$ is positive definite. The matrix $T$ has two positive eigenvalues and two null eigenvalues by construction. Then $\gamma_3 = \sqrt{\frac{\sigma_{P_2}^2 - \sigma_A^2}{(e-h)}}$; from this relation and from equation (A8), the result directly follows. ■

*Proof of Lemma 2.6.*

*Proof* Let $y = Nx$, by substitution it is possible to write

$$x'Qx = x'N'D^{-1}Nx = y'D^{-1}y.$$

$D$ is positive definite, then also $Q$ is positive definite, i.e. $\det(Q) > 0$ and $\det(Q_2) > 0$. Moreover, being $(s - H_1)\det(Q_2) = \det(Q)$, then $s - H_1 > 0$. ■

*Proof of Lemma 2.7.*

*Proof* The Lagrangian function for problem (18)–(21) is

$$L(x, \lambda) = -x'tk_z - \lambda_1(x'Dx - \sigma_p^2) - \lambda_2[x'(\mu - \mu_0 \boldsymbol{1}) - (\mu_p - \mu_0)] - \lambda_3(x'b\xi_y - \xi_p).$$

The partial derivatives of the Lagrangian function $L(x, \lambda)$ are:

$$\frac{\partial L}{\partial x} = -tk_z - 2\lambda_1 Dx - \lambda_2(\mu - \mu_0 \boldsymbol{1}) - \lambda_3 b\xi_y = \boldsymbol{0}$$

$$\frac{\partial L}{\partial \lambda_1} = x'Dx - \sigma_p^2 = 0$$

$$\frac{\partial L}{\partial \lambda_2} = x'(\mu - \mu_0 \boldsymbol{1}) - (\mu_p - \mu_0) = 0$$

$$\frac{\partial L}{\partial \lambda_3} = x'b\xi_y = \xi_p \tag{A10}$$

From (A10), assuming $\lambda_1 \neq 0$, the optimal portfolio can be written as

$$x^* = -\frac{k_z}{2\lambda_1} D^{-1} t - \frac{\lambda_2}{2\lambda_1} D^{-1}(\mu - \mu_0 \boldsymbol{1}) + \frac{\lambda_3}{2\lambda_1} D^{-1} \xi_y b \tag{A11}$$

Define the vector $\delta' = [\delta_1 \ \ \delta_2 \ \ \delta_3]$ where

$$\delta_1 = -\frac{\lambda_2}{2\lambda_1}, \quad \delta_2 = -\frac{\lambda_3}{2\lambda_1}\xi_y, \quad \delta_3 = -\frac{k_z}{2\lambda_1}. \tag{A12}$$

Equation (A11) can be rewritten in terms of $\delta$ as $x^* = D^{-1}N\delta$. Substituting the optimal $x^*$ into the constraints (19), (20) and (21), the following equations hold

$$x'Dx = \delta' N' D^{-1} D D^{-1} N \delta = \delta' Q \delta = \sigma_p^2 \tag{A13}$$

$$(\mu - \mu_0 \boldsymbol{1})'x = (\mu - \mu_0 \boldsymbol{1})' D^{-1} N \delta = \mu_p - \mu_0 \tag{A14}$$

$$\xi_y b'x = \xi_y b' D^{-1} N \delta = \xi_p \quad \Rightarrow \quad b' D^{-1} N \delta = \frac{\xi_p}{\xi_y}. \tag{A15}$$

From (A14) and (A15), it is possible to obtain $\delta_1$ and $\delta_2$ as functions of $\delta_3$:

$$\begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = Q_2^{-1} \begin{bmatrix} \mu_p - \mu_0 \\ \frac{\xi_p}{\xi_y} \end{bmatrix} - Q_2^{-1} \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} \delta_3.$$

By substitution in (A13), it is possible to solve for $\delta_3$:

$$\delta_3 = \pm \sqrt{\frac{\sigma_p^2 - \sigma_{Q_2}^2}{s - H_1}}. \tag{A16}$$

From equation (A12) $\lambda_1 = -\frac{k_z}{2\delta_3}$.

The second-order condition for problem (18)–(21) is $H_x(L) = \frac{\partial^2 L}{\partial x^2} = -2\lambda_1 D$, where $H_x$ is the Hessian matrix, with respect to $x$, of the Lagrangian function. $D$ is positive definite, therefore the second-order conditions depend on the sign of $\lambda_1$. Hence, the first-order conditions are sufficient for problem (18)–(21) if $\lambda_1 > 0$:

$$\text{if } k_z < 0, \quad \text{then} \quad \delta_3 = +\sqrt{\frac{\sigma_p^2 - \sigma_{Q_2}^2}{s - H - 1}}, \quad \text{if } k_z > 0,$$

$$\text{then} \quad \delta_3 = -\sqrt{\frac{\sigma_p^2 - \sigma_{Q_2}^2}{s - H_1}}.$$

The optimal kurtosis $k^*$ associated with the optimal portfolio $x^*$ can be obtained from $k = x'tk_z$:

$$k^* = k_z \left( \begin{bmatrix} f_1 & g_1 \end{bmatrix} Q_2^{-1} \begin{bmatrix} \mu_p - \mu_0 \\ \frac{\xi_p}{\xi_y} \end{bmatrix} + \sqrt{\sigma_p^2 - \sigma_{Q_2}^2}\sqrt{s - H_1} \right).$$

■

*Proof of Proposition 2.8.*

*Proof* To decompose the optimal portfolio, it is necessary to write $\delta$ as a function of $m_1$ and $\delta_1$ as a function of $\delta_2$ and $\delta_3$ using equation (A14):

$$\delta_1 = \frac{\mu_p - \mu_0}{m_1} - \frac{l_1}{m_1}\delta_2 - \frac{f_1}{m_1}\delta_3 \tag{A17}$$

The following equation holds as a trivial corollary of Lemma 2.6

$$\delta' Q \delta = \sigma_P^2 > \sigma_{Q_2}^2 = [\delta_1, \delta_2] Q_2 \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} \tag{A18}$$

Lets evaluate the value of $\delta_2$ plugging equation (A17) into equation (A13). The value of $\delta_2$ can be written as

$$\delta_2^2 \left( e - \frac{l_1^2}{m_1} \right) = \sigma_{Q_2}^2 + 2\delta_3(\mu_p - \mu_0)\frac{f_1}{m_1} \tag{A19}$$

$$- \frac{(\mu_p - \mu_0)^2}{m1} - \frac{f_1^2}{m_1}\delta_3^2 \tag{A20}$$

Let rewrite equation (A20) in a more useful way

$$\delta_2^2(e - k) = \sigma_Q^2 - \sigma_B^2 \tag{A21}$$

where

$$k = \frac{l_1^2}{m_1}$$

$$\sigma_B^2 = -2\delta_3(\mu_p - \mu_0)\frac{f_1}{m_1} + \frac{(\mu_p - \mu_0)^2}{m_1} + \frac{f_1^2}{m_1}\delta_3^2$$

Note that the quantity $\sigma_B^2$ is non negative by construction being

$$\frac{[(\mu_p - \mu_0) - f_1\delta_3]^2}{m_1} \quad \text{and} \quad m_1 > 0.$$

The result follows. ■