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Dynamic currency hedging with non-Gaussianity and ambiguity

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This paper introduces a non-Gaussian dynamic currency hedging strategy for globally diversified investors with ambiguity. It provides theoretical and empirical evidence that, under the stylized fact of non-Gaussianity of financial returns and for a given optimal portfolio, the investor-specific ambiguity can be estimated from historical asset returns without the need for additional exogenous information. Acknowledging non-Gaussianity, we compute an optimal ambiguity-adjusted mean-variance (dynamic) currency allocation. Next, we propose an extended filtered historical simulation that combines Monte Carlo simulation based on volatility clustering patterns with the semi-parametric non-normal return distribution from historical data. This simulation allows us to incorporate investor's ambiguity into a dynamic currency hedging strategy algorithm that can numerically optimize an arbitrary risk measure, such as the expected shortfall. The out-of-sample backtest demonstrates that, for globally diversified investors, the derived non-Gaussian dynamic currency hedging strategy is stable, robust, and highly risk reductive. It outperforms the benchmarks of constant hedging as well as static/dynamic hedging approaches with Gaussianity in terms of lower maximum drawdown and higher Sharpe and Sortino ratios, net of transaction costs.

Keywords: Currency hedging; Non-Gaussianity; Ambiguity; Filtered historical simulation; Expected shortfall; Currency risk management

JEL Classifications: C53, C58, F31, G11, G15

1. Introduction

Efficient international asset allocation lies at the core of quantitative risk management practices. To enhance diversification, investors hold global portfolios with exposure to different asset classes and underlying currencies. Investing internationally reduces the exposure to systematic domestic market risk and offers an opportunity for enhanced portfolio growth and improved risk-adjusted portfolio performance. On the other hand, investing in foreign markets creates exposure to the currency exchange rate variation. Portfolio losses driven by adverse exchange rate movements represent one of the major risks for market participants with multi-currency portfolios, such as pension funds, insurance companies, banks, multinational firms, and other financial intermediaries. Consequently, understanding and managing the currency risk of international portfolios is essential from both theoretical and empirical perspectives.

The currency allocation decision is a special case of the general portfolio theory for international investors. As in any portfolio optimization problem, investors inevitably face parameter and model uncertainty. To integrate uncertainty into the portfolio optimization process, methods based on Bayesian portfolio analysis (see Black and Litterman 1992, Pástor 2000, Tu and Zhou 2010) and ambiguity-adjusted preferences (see Schmeidler 1989, Hansen and Sargent 2001, Klibanoff *et al.* 2005, Maccheroni *et al.* 2013) have been developed. The former methods incorporate investors' prior information arising from news, macroeconomic data, and asset pricing models, which is otherwise ignored in the classical statistical analysis. The latter methods differentiate between the 'non-probabilized' uncertainty, also known as ambiguity, contrary to the 'probabilized' uncertainty, widely regarded as risk. They deviate from the

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traditional rational expectations paradigm and the corresponding maximization of expected utility by modeling ambiguity (i.e. uncertainty about the assets and currencies return distribution) directly in the objective function. We build upon the literature on ambiguity-adjusted preferences by studying the relation between ambiguity and non-Gaussianity. We employ a flexible non-Gaussian returns model in the assumed dynamics of asset and currency returns and show that it parametrizes the robust mean-variance portfolio optimization model from Maccheroni *et al.* (2013) to market returns data. This approach allows us to account for ambiguity in the construction of optimal currency allocation under non-Gaussianity and estimate the investor-specific ambiguity from market data. Moreover, we show that the continuous mixing random variable of the assumed non-Gaussian return dynamics is analogous to employing a continuum of possible models in the ambiguity-adjusted preferences. We apply the derived concept to solve the problem of optimal currency allocation in international portfolios and propose an ambiguity-adjusted dynamic currency hedging strategy.

Managing the currency risk of an international portfolio is performed via currency hedging. Different derivative instruments, such as forwards, futures, swaps, and options, can be utilized to perform hedging. The primary purpose of hedging is offsetting the change in the value of the target being hedged. In currency risk management, the hedging portfolio is set up such that any decrease (increase) in the value of the portfolio in domestic currency due to a change in a specific currency exchange rate is offset by the gain (loss) on the corresponding currency forward contract. By entering into a currency forward contract, an investor locks in an exchange rate between the base and foreign currencies. Although the spot exchange rates continue to fluctuate, the investor has fixed the exchange rate over the life of the entered currency forward contract. At maturity, another contract is entered, and the process continues over time. The investor's main objective is to determine the optimal notional value of the used hedging instrument to enter, or equivalently, the relative amount of implicit foreign currency exposure to hedge.

Many researchers have studied the problem of managing currency risk in international portfolios, and a number of currency hedging strategies have been presented in the existing literature. In the beginning, opinions among different authors were divided. Perold and Schulman (1988) proposed 100% hedging as the optimal strategy, whereas Froot (1993) concluded that full hedging could actually increase risk without an adequate return compensation in the long run. On the other hand, Black (1989) derived a model in which, under strong assumptions, all investors apply a universal hedging policy, irrespective of the portfolio composition and the reference currency. Glen and Jorion (1993) showed that international diversification decreases portfolio risk irrespective of assets being hedged or not. With time researchers started to agree on the optimal hedging practice. Especially after Solnik (1993) demonstrated that, in the short term, the optimal currency hedging is specific for each investor, and it depends on the portfolio structure.

A broad consensus that currency hedging tends to lower portfolio volatility has been formed. Haefliger *et al.* (2002) proposed full hedging of fixed-income portfolios and partial hedging for the equity portfolios, depending on the underlying correlation structure between equity and currency returns. Schmittmann (2010) analyzed the performance of static variance minimizing hedging ratios obtained with ordinary least squares in comparison to constant hedging. Campbell *et al.* (2010) proved that it is possible to find optimal hedging ratios that minimize volatility for arbitrary portfolios. Moreover, the authors showed that the US dollar, the euro, and the Swiss franc moved against world equity markets and should therefore be appealing to risk-minimizing international equity investors irrespective of their low average returns. de Boer *et al.* (2020) confirmed most of these results in the extended sample period and showed that the role of the euro as a reserve currency vanishes during the financial crisis. Moreover, Cho *et al.* (2016) showed that capital tends to move out of emerging into developed countries in global down markets, leading to depreciation (appreciation) of emerging (developed) currencies. Boudoukh *et al.* (2019) provided a decomposition of a currency overlay portfolio in a mean-variance framework consisting of a hedge sub-portfolio and an alpha-seeking currency sub-portfolio. Ulrych and Vasiljević (2020) generalized this concept to the risk and ambiguity aversion and showed that investors' dislike for model uncertainty induces stronger currency hedging demand.

All of the works mentioned above study currency risk management from the perspective of minimizing the volatility of portfolio returns. However, there are different ways to measure the risk of a portfolio. In recent years, the expected shortfall has gained prominence as a coherent measure of downside risk. This measure represents the expected loss given that a value-at-risk threshold is breached. One of the important features of the expected shortfall is that it takes into account the tail of the loss distribution, which is what investors are predominantly concerned about. Harris and Shen (2006) show that although currency hedging based on minimum variance reduces the volatility of portfolio returns, it can increase both negative skewness and excess kurtosis. The authors study the hedging effectiveness in terms of value-at-risk and expected shortfall. On a similar note, Guo and Ryan (2018) use the expected shortfall optimization framework to more accurately account for tail risk with non-Gaussian returns. A similar analysis is also performed in Álvarez-Díez *et al.* (2016). Simulation via principal component analysis and subsequent optimization of the expected shortfall is presented in Topaloglou *et al.* (2002).

Many efforts have also been performed on the econometric modeling of asset and currency returns that drive the optimal hedging decision. Another strand of literature studying optimal currency hedging moves away from the so-called static modeling, where independent and identically distributed asset and currency returns are assumed. It employs multivariate GARCH (Generalized AutoRegressive Conditional Heteroskedasticity) time-series models and, given the time-dependent properties of such approaches, calls the corresponding hedging strategies dynamic.† For example, De

† Note that such approaches still consider a single period optimization framework and should be differentiated from dynamic programming approaches solving recursive multi-period optimization problems.

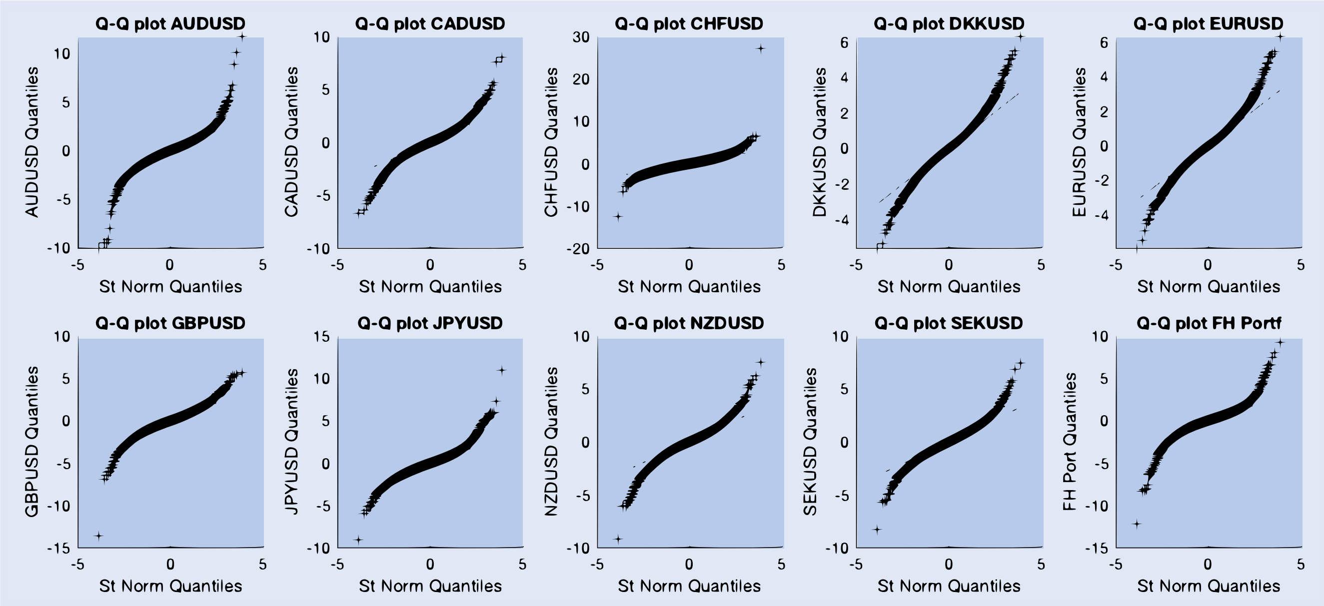

Figure 1. Quantile-quantile plot of the normalized currency excess returns (with respect to the USD) and the fully hedged equally-weighted portfolio returns versus the standard normal distribution. Observe that the empirical sample of currency excess returns and fully hedged portfolio returns exhibit heavier tails compared to the Gaussian distribution.

Roon *et al.* (2003) show that dynamic hedges conditional on the interest rate spread outperform static currency hedging approaches and provide significant improvements in portfolio performance. Similar findings are, among others, also presented in Tong (1996), Brown *et al.* (2012), Caporin *et al.* (2014), and Cho *et al.* (2020). Different authors presented improvements to the dynamic currency hedging strategies by studying various generalizations of those models. For instance, regime-switching dynamic hedging approaches are investigated in Lee and Yoder (2007a, 2007b). Moreover, Hsu *et al.* (2008) show that copula-based GARCH models perform more effectively compared to other dynamic hedging models (see also Paolella and Polak 2015a, 2018 for applications of copula models in portfolio optimization). Wang *et al.* (2015) comprehensively investigate the hedging performance of the constant (full) hedging strategy in comparison to the static as well as dynamic minimum variance-based strategies in different futures markets, including commodities, currencies, and stock indices. Recent literature focuses on exploring the time-series predictability of exchange rate returns via the use of currency risk factors in dynamic currency hedges. Opie and Riddiough (2020) show that exploiting a forecastable component in global factor returns leads to a hedging strategy that outperforms other dynamic hedging benchmarks. On a similar note, Filipozzi and Harkmann (2020) show that carry trades are a part of an optimal portfolio. Joint optimization of assets and currencies is investigated in Burkhardt and Ulrych (2023) and Barroso *et al.* (2023).

The well-known stylized fact of the heavy-tailed distribution of the equity returns is also present in currency returns as well as in the fully hedged portfolio returns comprised of equities and bonds. Figure 1 depicts the quantile-quantile plot of the normalized empirical sample of the currency exchange rates (with the USD base currency) and the equally-weighted fully hedged portfolio returns from our empirical analysis. The non-Gaussianity is clearly represented in all the plots, and these observations motivate our heavy-tailed hedging strategy presented in the theoretical part of this paper.

In order to capture the aforementioned empirical stylized facts, we extend the existing currency hedging approaches by modeling a set of asset and currency returns via a COMFORT model, as introduced in Paolella and Polak (2015b). By enhancing Bollerslev (1990) CCC-GARCH model in several ways, our approach allows for the incorporation of volatility clustering, non-normality (i.e. excess kurtosis and asymmetry, see also table 1 below), and dynamics in the dependency between assets and currencies over time. A fast expectation-maximization (EM) algorithm is employed for estimation. This makes our proposed dynamic non-Gaussian hedging strategy applicable in a multivariate setting for a potentially large number of foreign currency exposures. Moreover, we establish a relation between the investor's ambiguity and the employed non-Gaussian returns model. We show that the continuous normal mean-variance mixture representation of the COMFORT model parametrizes the ambiguity-adjusted preferences from Maccheroni *et al.* (2013).

The class of COMFORT models is a general modeling framework that incorporates a conditional multivariate continuous normal mean-variance mixture distribution with GARCH dynamics. What distinguishes these models from Gaussian-based multivariate GARCH models is the use of a mixing random variable. Paolella *et al.* (2021) show that Alexander (2002) O-GARCH model enriched with COMFORT structure provides smooth portfolios that perform well out of sample and after the transaction costs even with daily rebalancing. Paolella *et al.* (2019) incorporate regime-switching structure into the model and demonstrate that, compared with similar Gaussian constructions, more robust and systematic regimes can be captured with the COMFORT model.

The main contribution of this paper lies in deriving the relation between the non-Gaussianity of financial returns and

Table 1. This table presents the full sample unconditional annualized mean, annualized volatility, skewness, and kurtosis of local equity, local bond, and currency excess returns (against the USD) for ten different economies.

Summary Statistics										
	AUD	CAD	CHF	DKK	EUR	GBP	JPY	NZD	SEK	USD
Panel A: Equities										
Mean	10.29%	9.85%	9.13%	11.60%	8.84%	8.76%	3.54%	9.02%	12.36%	11.31%
Vol	14.99%	15.18%	15.85%	17.06%	18.21%	16.36%	19.69%	12.76%	21.98%	17.62%
Skew	− 0.49	− 0.69	− 0.35	− 0.30	− 0.33	− 0.20	− 0.02	− 0.30	0.11	− 0.25
Kurt	10.55	21.20	10.63	9.49	11.65	11.04	9.33	18.63	8.15	13.95
Panel B: Bonds										
Mean	6.30%	5.56%	3.41%	4.72%	4.31%	5.60%	2.84%	6.19%	5.49%	4.71%
Vol	4.94%	4.56%	3.64%	4.40%	3.88%	6.60%	2.54%	3.86%	4.12%	4.55%
Skew	− 0.13	− 0.10	0.14	0.00	− 0.08	0.81	− 0.27	− 0.08	0.75	− 0.11
Kurt	6.05	5.53	13.43	10.80	7.63	23.77	8.21	11.07	28.37	6.11
Panel C: Currencies										
Mean	1.86%	0.23%	0.50%	0.27%	0.01%	0.51%	− 1.58%	3.02%	− 0.56%	·
Vol	11.59%	7.63%	10.99%	9.67%	9.57%	9.42%	10.47%	11.36%	11.50%	·
Skew	− 0.20	− 0.06	2.21	0.06	0.05	− 0.46	0.41	− 0.24	− 0.05	·
Kurt	14.02	6.91	68.81	5.04	5.08	9.85	8.92	7.01	6.09	·

the ambiguity of an individual investor. We show that the investor-specific ambiguity can be approximated from historical asset returns without the need of additional exogenous information. For that purpose, we employ the COMFORT model that drives the non-Gaussianity of the joint distribution of portfolio and currency returns. Each element of the vector return at time t is endowed with a common univariate shock. This univariate shock is interpretable as a common market factor that arises from new information coming to the market. We show that the mixing random variable characterizing this shock is related to the investor's ambiguity and parametrize it from historical market returns.

Using the proposed model, we present a novel dynamic non-Gaussian currency hedging strategy that is able to optimize a currency hedge with respect to an arbitrary, e.g. tail-based, risk measure. Barone-Adesi *et al.* (1999) introduced the filtered historical simulation of GARCH processes to model the future distribution of asset returns. The simulation is based on a parametric GARCH modeling under the Gaussian distribution and a non-parametric simulation of historical portfolio returns. We extend this approach to the aforementioned continuous normal mean-variance mixture setting and present a semi-parametric extended filtered historical simulation method to model the future distribution of asset and currency returns. Such a method is consistent with the non-Gaussian time-series returns model employed in our study. This tractable approach enables us to numerically optimize a dynamic currency hedge with respect to an arbitrary risk measure for multiple periods ahead.

We empirically validate our extended filtered historical simulation and the proposed dynamic non-Gaussian currency hedging strategy via an extensive out-of-sample backtesting exercise. The results show that our method yields a robust and highly risk-reductive hedging strategy for both optimizing the ambiguity-adjusted mean-variance as well as the expected shortfall. The proposed hedging strategy outperforms the benchmarks of constant hedging, static hedging (based on mean and variance shrinkage estimation), and dynamic hedging approaches (based on Gaussian-GARCH modeling) net of transaction costs. Taking into account the non-Gaussianity and ambiguity when making currency hedging decisions is the main driver of the outperformance of the dynamic currency hedging strategy presented in this paper.

The remainder of the paper is organized as follows. The theoretical model and its characteristics are established in Section 2. Section 3 demonstrates the empirical performance of the model. Section 4 provides some concluding remarks, and the Appendix gathers additional technical results and figures.

2. Model

In this section, the theoretical model is presented. First, following Ulrych and Vasiljević (2020), a general framework for portfolio optimization in an international context is introduced. Next, we briefly recap the common market factor non-Gaussian returns (COMFORT) model, introduced by Paolella and Polak (2015b), which is used for modeling the asset and currency returns. We then show how the utilized non-Gaussian returns model can parametrize the investor's ambiguity from Maccheroni *et al.* (2013). Given this parametrization, we perform an analysis of optimal currency exposure in the theoretically tractable robust mean-variance model and derive a closed-form expression for the optimal dynamic mean-variance currency exposure with ambiguity. Last, we propose a generalized filtered historical simulation algorithm utilized to numerically optimize a currency exposure with respect to an arbitrary risk measure.

2.1. *Portfolio return with currency hedging*

We start by presenting a framework for modeling portfolio returns in an international setting. Thereby, investment in assets denominated in arbitrary currencies is possible. We assume hedging is performed with currency forward contracts and provide expressions for hedged portfolio returns.

Denote with $P_{i,t}$ a price of an asset i at time t, expressed in some local currency (LC). The corresponding simple return of this asset over the time period from t to $t+1$ is given by $R_{i,t+1}$. As we are working with portfolio returns in a cross-section of underlying assets, we, throughout the paper, use simple, and not logarithmic, returns. Assume a domestic currency is prespecified and denote with $S_{c_i,t}$ the spot currency exchange rate in the domestic currency per unit of foreign currency at time t, where c_i denotes the local currency in which asset i is denominated. The corresponding spot currency exchange rate return from t to $t+1$ is denoted by $e_{c_i,t+1}$. Assume that an investor does not rebalance her position in asset i between t and $t+1$. Then, the unhedged return of an asset i expressed in domestic currency $\tilde{R}^u_{i,t+1}$ admits the following decomposition

$$\begin{aligned}\tilde{R}^u_{i,t+1} &= \frac{P_{i,t+1}S_{c_i,t+1}}{P_{i,t}S_{c_i,t}} - 1 \\ &= \underbrace{R_{i,t+1}}_{LC\ asset\ return} + \underbrace{e_{c_i,t+1}}_{FX\ spot\ return} + \underbrace{R_{i,t+1}e_{c_i,t+1}}_{cross-product}.\end{aligned}$$

The single asset return from above motivates an equivalent analysis in a portfolio context. Consider an international investor with an arbitrary domestic currency who is invested in a portfolio $\mathcal{P}$ consisting of N assets. The portfolio weights of assets $i = 1, 2, \ldots, N$ at time t are denoted by $x_{i,t}$. The return of a portfolio is then given by

$$\tilde{R}^u_{\mathcal{P},t+1} = \sum_{i=1}^{N} x_{i,t}\tilde{R}^u_{i,t+1},$$

with $\sum_{i=1}^{N} x_{i,t} = 1$, for every t. We allow the portfolio to have a direct exposure to $K \leq N$ foreign currencies. Take $c = 1$ for the domestic currency and denote by $c = 2, 3, \ldots, K+1$ the foreign currencies. Since multiple assets can be denominated in the same currency, we can simplify the return of the unhedged international portfolio by grouping the currency exchange rate returns. Denote the set of assets denominated in a particular currency c held in the portfolio at time t as $\mathcal{A}_{c,t}$. Then, the fraction of portfolio wealth exposed to currency c at time t is characterized by $w_{c,t} := \sum_{j\in\mathcal{A}_{c,t}} x_{j,t}$. Using this notation, we can express the unhedged return on portfolio $\mathcal{P}$ in domestic currency as

$$\tilde{R}^u_{\mathcal{P},t+1} = \underbrace{\sum_{i=1}^{N} x_{i,t}R_{i,t+1}}_{LC\ asset\ returns} + \underbrace{\sum_{c=2}^{K+1} w_{c,t}e_{c,t+1}}_{FX\ spot\ returns} + \underbrace{\sum_{i=1}^{N} x_{i,t}R_{i,t+1}e_{c_i,t+1}}_{cross-products}. \tag{1}$$

Next, we extend the current framework by introducing hedging with currency forward contracts. Assume that the price of a currency contract in currency c is zero at inception and denote by $F_{c,t}$ the forward exchange rate in domestic currency per unit of foreign currency c at time t. The maturity of the forward contract is at time $t+1$. The forward premium is defined as $f_{c,t} := (F_{c,t} - S_{c,t})/S_{c,t}$ and is a deterministic quantity known at time t. It can be understood as an interest rate component within the FX return, as reflected in the forward discounts. Note that for the domestic currency $S_{1,t} = F_{1,t} = 1$ and $e_{1,t} = f_{1,t} = 0$ trivially hold for all t.

Denote by $\phi_{c,t}$ the relative notional value of a forward currency exchange contract for currency c at time t, denominated in domestic currency and expressed as a fraction of total portfolio value. Hedging can be realized by selling a forward contract (i.e. shorting foreign bonds and holding domestic bonds), which we indicate by $\phi_{c,t} > 0$. A hedged portfolio return is then equal to

$$\tilde{R}^h_{\mathcal{P},t+1} = \tilde{R}^u_{\mathcal{P},t+1} + \sum_{c=2}^{K+1} \phi_{c,t}(f_{c,t} - e_{c,t+1}), \tag{2}$$

where $f_{c,t} - e_{c,t+1} = (F_{c,t} - S_{c,t+1})/S_{c,t}$ expresses the payoff of a short forward contract (for $\phi_{c,t} > 0$) on currency c at time $t+1$ denominated in domestic currency. Note that the investment universe could be extended by assuming that the total number of foreign currencies available on the market is greater than K, see Ulrych and Vasiljević (2020).

The choice of $\phi_{1,t}$ is arbitrary, and we set it to $\phi_{1,t} = 1 - \sum_{c=2}^{K+1} \phi_{c,t}$ such that the currency portfolio is a zero investment portfolio, also known as the hedging portfolio. Since $w_{c,t} \neq 0$, we can define the hedge ratio as $h_{c,t} := \phi_{c,t}/w_{c,t}$. This implies that assets are unhedged if $\phi_{c,t} = h_{c,t} = 0$, and fully hedged if $\phi_{c,t} = w_{c,t}$ or $h_{c,t} = 1$. Using $\phi_{c,t} = w_{c,t}$, fully hedged portfolio return can be expressed as

$$\tilde{R}^{fh}_{\mathcal{P},t+1} = \underbrace{\sum_{i=1}^{N} x_{i,t}R_{i,t+1}}_{LC\ asset\ returns} + \underbrace{\sum_{c=2}^{K+1} w_{c,t}f_{c,t}}_{FX\ forward\ premia} + \underbrace{\sum_{i=1}^{N} x_{i,t}R_{i,t+1}e_{c_i,t+1}}_{cross-products}.$$

In comparison to the unhedged portfolio return from equation (1), observe that stochastic FX return is replaced by the deterministic forward premia in the case of full currency hedging. This is how hedging eliminates the risk arising from currency exchange rate fluctuations. Since the exact currency exposure at time $t+1$ is unknown at time t, 'perfect' full hedging is impossible, and the second-order cross-terms remain. Over shorter time horizons (e.g.monthly or quarterly), and in most market environments, the cross-terms are negligible in practice. Nevertheless, over longer investment horizons or sudden large market movements, these terms can exhibit a significant impact on portfolio returns. Hence, we leave them in our calculations.

To increase the intuition of what the magnitude of $\phi_{c,t}$ means, we study the currency exposures $\psi_{c,t}$. We define the net exposure to currency c as $\psi_{c,t} := w_{c,t} - \phi_{c,t}$, where $w_{c,t}$ represents the direct currency exposure and $\phi_{c,t}$ reflects the currency hedging position. This notation allows us to characterize the cases of (i) no hedging: $\psi_{c,t} = w_{c,t}$, (ii) full hedging: $\psi_{c,t} = 0$, (iii) partial hedging: $0 < \psi_{c,t} \leq w_{c,t}$, and iv) over and under hedging: $\psi_{c,t} < 0$ and $\psi_{c,t} > w_{c,t}$, respectively.

The expression for hedged portfolio return from equation (2) can now be interpreted also as

$$\tilde{R}^h_{\mathcal{P},t+1} = \tilde{R}^{fh}_{\mathcal{P},t+1} + \sum_{c=2}^{K+1} \psi_{c,t}(e_{c,t+1} - f_{c,t}), \tag{3}$$

where the exposure to the domestic currency is, equivalently to $\phi_{1,t}$, computed as $\psi_{1,t} = -\sum_{c=2}^{K+1} \psi_{c,t}$. This shows that the individual currency exposures add to zero, and the currency portfolio is indeed a zero investment portfolio. Equation (3) is useful as it provides a decomposition of the hedged portfolio return into a component of fully hedged asset returns, which are as close as possible to being orthogonal to currencies, and a component of net currency exposures. Since forward premium $f_{c,t}$ can be interpreted as a riskless return attained by entering a forward contract on currency c, we refer to $e_{c,t+1} - f_{c,t}$ as the currency excess return on currency c. This currency excess return reflects the full FX return, encompassing both the spot component $e_{c,t+1}$ and the forward discount (interest rate) component $f_{c,t}$.

2.2. *Non-Gaussian returns model*

The hedged portfolio returns from equation (3) are computed in a model-free setting, using only the definitions of an asset return and a payoff of a currency forward contract. We can specify an arbitrary econometric model for the asset and currency returns and utilize the expressions derived above. Our modeling choice is a flexible non-Gaussian returns model that takes into account the primary stylized facts of financial asset returns, such as volatility clustering, non-normality, and dynamics in the dependency between assets and currencies over time, and presents a generalization to the classical multivariate Gaussian-GARCH models.

We denote the $(M \times 1)$ return vector at time t by $\mathbf{Y}_t$. Its conditional time-varying distribution is assumed to be multivariate asymmetric variance-gamma (MVG), which is a special case of the multivariate generalized hyperbolic (MGHyp). It is a semi-heavy-tailed distribution for which, as opposed to the Lévy alpha-stable class, all the moments are finite, and the corresponding probability density function is given in closed form. Because of its tail properties and closed-form portfolio distribution, the MGHyp and its special cases are often employed in derivatives pricing, portfolio optimization, and risk estimation, see Eberlein and Prause (2002). The MGHyp is a very general class of distributions initially introduced by Barndorff-Nielsen (1977). Stochastic processes based on these distributions were first applied in finance by Eberlein and Keller (1995). As reported by Prause (1999) and confirmed by Protassov (2004), the general MGHyp has a flat likelihood problem. Paolella and Polak (2015c) show that all the aforementioned special and limiting cases of MGHyp distribution achieve similar out-of-sample portfolio performance, and, more recently, Paolella and Polak (2023) draw similar conclusions for the daily risk prediction of a portfolio of US equities. In one or two dimensions, these distributions have a similar shape of the probability density function; and in larger dimensions, and after a finite-sample fit, their out-of-sample performance is very similar. What is important is that they all account for the non-Gaussianity present in the data. Therefore, in our empirical analysis, we work with only one limiting case—the MVG.

Variance Gamma (VG) distribution is an infinitely divisible distribution, which is a property inherited from the MGHyp class, and it is often used for the modeling of financial returns. Because of its infinite divisibility and relation to Lévy processes, it has been used by Madan and Seneta (1990) and Madan *et al.* (1998) in the context of option pricing. In the multivariate case, similarly to all the MGHyp distributions, the corresponding portfolio admits a closed-form expression. Consequently, the VG distribution has been used in portfolio optimization and risk estimation in Eberlein and Prause (2002), Protassov (2004), and Finlay (2009). Using the mixture representation, see, for example, McNeil *et al.* (2015) for details, we can express the return vector as

$$\mathbf{Y}_t = \boldsymbol{\mu} + \boldsymbol{\gamma} G_t + \boldsymbol{\epsilon}_t, \quad \text{with } \boldsymbol{\epsilon}_t = \mathbf{H}_t^{\frac{1}{2}} \sqrt{G_t} \mathbf{Z}_t, \tag{4}$$

where $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}$ are $(M \times 1)$ vectors, $\mathbf{H}_t$ is a positive definite, symmetric, conditional $(M \times M)$ dispersion matrix, $\mathbf{Z}_t$ is a sequence of independent and identically distributed standard normal random $(M \times 1)$ vectors, and $(G_t \mid \mathcal{F}_{t-1}) \sim GIG(a, b, p)$ is a scalar mixing random variable, independent of $\mathbf{Z}_t$, with the generalized inverse Gaussian (GIG) density, given by

$$f_{G_t|\mathcal{F}_{t-1}}(x) = \frac{(a/b)^{p/2}}{2K_p(\sqrt{ab})} x^{p-1} e^{-(ax+b/x)/2}, \quad x > 0, \tag{5}$$

with K_p denoting the modified Bessel function of the second kind with real parameters $a > 0$, $b > 0$, and p; and where the information set at time t is given as the sigma-algebra generated by the history of returns $\mathcal{F}_t = \sigma(\{\mathbf{Y}_1, \dots, \mathbf{Y}_t\})$. MVG is a limiting case of MGHyp with two GIG parameters fixed at $a = 2$, $b = 0$, and a positive shape parameter $p > 0$ estimated from the data. The conditional $(M \times M)$ dispersion matrix H_t is decomposed as the product of scale terms and a conditional dependency matrix via

$$\mathbf{H}_t = \mathbf{S}_t \Gamma \mathbf{S}_t, \tag{6}$$

where $\mathbf{S}_t$ is an $(M \times M)$ diagonal matrix with conditional scale terms $s_{m,t} > 0$, for $m = 1, 2, \dots, M$, and Γ is a time-invariant and symmetric (with ones on the main diagonal) conditional $(M \times M)$ dependency matrix, such that $\mathbf{H}_t$ is positive definite. Note that Γ is a correlation matrix only conditionally on the realization of the mixing process. Therefore, we call Γ the dependency matrix. The univariate scale terms $s_{m,t}$ are modeled by a GARCH(1,1) process

$$s_{m,t}^2 = \omega_m + \alpha_m \epsilon_{m,t-1}^2 + \beta_m s_{m,t-1}^2, \tag{7}$$

where $\epsilon_{m,t} = Y_{m,t} - \mu_m - \gamma_m G_t$ is the mth element of $\boldsymbol{\epsilon}_t$, and $\omega_m \geq 0$, $\alpha_m \geq 0$, $\beta_m \geq 0$, for $m = 1, 2, \dots, M$.

In the model specified in equation (4), $\boldsymbol{\mu}$ is the location vector, $\boldsymbol{\gamma}$ is the asymmetry vector, and $\mathbf{H}_t$ is the dispersion matrix of the conditional distribution of $\mathbf{Y}_t$, while the conditional mean and covariance matrix are, by the law of iterated expectations, and the law of total variance, respectively, given by

$$\begin{aligned} \mathrm{E}[\mathbf{Y}_t \mid \mathcal{F}_{t-1}] &= \boldsymbol{\mu} + \mathrm{E}[G_t \mid \mathcal{F}_{t-1}]\boldsymbol{\gamma}, \quad \text{and} \\ \mathrm{Var}(\mathbf{Y}_t \mid \mathcal{F}_{t-1}) &= \mathrm{E}[G_t \mid \mathcal{F}_{t-1}]\mathbf{H}_t + \mathrm{Var}(G_t \mid \mathcal{F}_{t-1})\boldsymbol{\gamma}\boldsymbol{\gamma}', \end{aligned} \tag{8}$$

where $\mathrm{Var}(G_t \mid \mathcal{F}_{t-1}) = \mathrm{E}[G_t^2 \mid \mathcal{F}_{t-1}] - (\mathrm{E}[G_t \mid \mathcal{F}_{t-1}])^2$.

The generalization of the non-Gaussian COMFORT model compared to the Gaussian-GARCH approaches is achieved by introducing the mixing random variable G_t. The latter class of models can be thought of as the COMFORT model with constant G_t, for every t. The mixing random variable can be interpreted as a common market factor as it accounts for information arrivals and jumps in such a way that, conditional on it, the returns distribution is Gaussian. It drives the dynamics of conditional (co-)moments of the returns and results in an enhanced ability for risk management and asset (currency) allocation. Because of the asset-specific conditional asymmetry coefficient $\boldsymbol{\gamma}$, the impact of G_t varies across assets. The maximum likelihood estimation of the COMFORT model is feasible and computationally inexpensive via the use of the EM algorithm, see Paolella and Polak (2015b) for details.

2.3. *Ambiguity and optimizing currency exposure*

This section utilizes the hedged portfolio returns that follow the non-Gaussian returns model presented above. In Section 2.3.1, we derive a closed-form optimal currency exposure in a robust mean-variance setting. Moreover, we demonstrate that investor's ambiguity can be parametrized from market data via the continuous mean-variance mixture representation of the employed non-Gaussian returns model. Section 2.3.2 generalizes the robust mean-variance framework by proposing a generalized filtered historical simulation algorithm that numerically optimizes a currency hedge with respect to an arbitrary risk measure.

2.3.1. Ambiguity-adjusted mean-variance currency exposure. We start the analysis with an ambiguity-adjusted mean-variance investor who is optimizing the currency exposure of her international portfolio. We assume that the asset (portfolio) weights are predetermined, and the investor is optimizing only the currency exposure, which is altered by taking positions in currency forward contracts. Such hedging (or currency overlay) strategies are prevalent in the asset and wealth management industry, where management of currency risk is treated in isolation from the asset allocation, see Kim and Chance (2018). To account for ambiguity, we employ the robust mean-variance model from Maccheroni *et al.* (2013). The authors consider a space Δ of possible models (i.e. forecasts) $\mathcal{Q}$ that represent investor's ambiguity. An agent's prior over all probability measures $\mathbb{Q}$ corresponding to the models in $\mathcal{Q}$ is given by η. The reduced probability is then expressed as $\bar{\mathbb{Q}} := \int_\Delta \mathbb{Q}\,\mathrm{d}\eta(\mathbb{Q})$. Note that the prior probability measure η is unobserved and, in general, can be agent/portfolio specific. Nevertheless, we can, given an individual portfolio, attempt to estimate it from historical returns data. The continuous normal mean-variance mixture distribution is a good candidate for such a parametrization since it preserves the mixing structure from the Maccheroni *et al.* (2013) model. A detailed derivation of the relation between the smooth model of decision-making under ambiguity from Klibanoff *et al.* (2005), the corresponding ambiguity-averse portfolio optimization from Maccheroni *et al.* (2013), and our parametrization using the normal mean-variance mixture distribution for a general portfolio optimization problem is provided in the Appendix.

In addition to the employed continuous normal mean-variance mixture distribution, the COMFORT model enables to isolate the persistence of the volatility dynamics from the estimated ambiguity due to the GARCH dynamics imposed on the dispersion matrix $\mathbf{H}_t$ and not on the covariance matrix of the returns. Note that $\mathbf{H}_t$ is proportional to the covariance of the returns only conditionally on the realization of the mixing random variable, i.e. $\mathrm{Cov}(\mathbf{Y}_t \mid G_t = g, \mathcal{F}_{t-1}) = g\mathbf{H}_t$. Hence, the COMFORT model allows us to parametrize the individual investor's measure of latent ambiguity from Maccheroni *et al.* (2013) in the conditional dynamics. The probability density function of a normal mean-variance mixture $f_{\mathbf{Y}_t|\mathcal{F}_{t-1}}(x)$ with mixing probability density $f_{G_t}(g)$, as defined in equation (4), is given by

$$f_{\mathbf{Y}_t|\mathcal{F}_{t-1}}(x) = \int_0^\infty \mathcal{N}_{\mathbf{Y}_t|G_t,\mathcal{F}_{t-1}}(x \mid \boldsymbol{\mu} + \boldsymbol{\gamma} g, g\mathbf{H}_t) f_{G_t}(g)\,\mathrm{d}g, \quad (9)$$

where $\mathcal{N}_{\mathbf{Y}_t|G_t,\mathcal{F}_{t-1}}(\cdot \mid \cdot,\cdot)$ is a probability density function of a multivariate Gaussian distribution and $f_{G_t}(g)$ is given in equation (5). This representation displays a direct link between the Maccheroni *et al.* (2013) ambiguity model and the normal mean-variance mixture model employed in the COMFORT setting. Thereby, it holds that $\mathbb{Q} \equiv \mathcal{N}_{\mathbf{Y}_t|G_t=g,\mathcal{F}_{t-1}}(\cdot \mid \boldsymbol{\mu} + \boldsymbol{\gamma} g, g\mathbf{H}_t)$, $\eta \equiv G_t$, and $\bar{\mathbb{Q}} \equiv \mathbf{Y}_t \mid \mathcal{F}_{t-1}$. Probability measure $\mathbb{Q}$ that corresponds to a single model from $\mathcal{Q}$ is in the COMFORT setting given by a Gaussian distribution. These models are mixed according to the investor's prior η, which is parametrized by the mixing random variable G_t in the COMFORT model. Finally, the reduced probability $\bar{\mathbb{Q}}$ is the normal mean-variance mixture given by the MGHyp distribution, as shown in equation (9), and the corresponding continuous mixing random variable G_t enables us to parametrize and estimate investor's ambiguity (see the Appendix for the detailed derivation and Section 3.2 for an empirical example).

Consider a hedged portfolio return $\tilde{R}^h_{\mathcal{P},t+1}$ from equation (3). As proposed in Maccheroni *et al.* (2013), the objective of the investor with ambiguity-adjusted mean-variance preferences, given the risk- and ambiguity-aversion coefficients $\lambda \geq 0$ and $\theta \geq 0$, respectively, is to maximize

$$\max_{\Psi_t} \left\{ \mathrm{E}_{\bar{\mathbb{Q}}}[\tilde{R}^h_{\mathcal{P},t+1}] - \frac{\lambda}{2}\mathrm{Var}_{\bar{\mathbb{Q}}}(\tilde{R}^h_{\mathcal{P},t+1}) \right. \\ \left. - \frac{\theta}{2}\mathrm{Var}_\eta(\mathrm{E}_{\mathbb{Q}}[\tilde{R}^h_{\mathcal{P},t+1}]) \right\}, \quad (10)$$

where $\Psi_t = (\psi_{2,t}, \ldots, \psi_{K+1,t})'$ denotes a $(K \times 1)$-dimensional vector of foreign currency exposures. The argument Ψ_t^*, which maximizes the robust mean-variance objective from equation (10), is the optimal currency exposure for a risk-and-ambiguity-averse international investor. It is derived in Ulrych and Vasiljević (2020) and is, in closed form, given by

$$\Psi_t^* = -\Big(\lambda \mathrm{Var}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1} - \mathbf{f}_t] + \theta \mathrm{Var}_\eta[\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1} - \mathbf{f}_t]]\Big)^{-1} \\ \times \Big(\lambda \mathrm{Cov}_{\bar{\mathbb{Q}}}[\tilde{R}^{fh}_{\mathcal{P},t+1}, \mathbf{e}_{t+1} - \mathbf{f}_t] + \theta \mathrm{Cov}_\eta \\ [\mathrm{E}_{\mathbb{Q}}[\tilde{R}^{fh}_{\mathcal{P},t+1}], \mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1} - \mathbf{f}_t]] - \mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1} - \mathbf{f}_t]\Big), \quad (11)$$

where $\mathrm{Var}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1} - \mathbf{f}_t]$ and $\mathrm{Var}_\eta[\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1} - \mathbf{f}_t]]$ denote the $(K \times K)$-dimensional covariance matrices of the random vector $(\mathbf{e}_{t+1} - \mathbf{f}_t)$ under $\bar{\mathbb{Q}}$ and η, respectively, $\mathrm{Cov}_{\bar{\mathbb{Q}}}[\tilde{R}^{fh}_{\mathcal{P},t+1}, \mathbf{e}_{t+1} - \mathbf{f}_t]$ and $\mathrm{Cov}_\eta[\mathrm{E}_{\mathbb{Q}}[\tilde{R}^{fh}_{\mathcal{P},t+1}], \mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1} - \mathbf{f}_t]]$ denote the $(K \times 1)$-dimensional vectors of covariances between $\tilde{R}^{fh}_{\mathcal{P},t+1}$ and $e_{i,t+1} - f_{i,t}$, for $i = 1, \ldots, K$, under $\bar{\mathbb{Q}}$ and η, respectively, and $\mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1} - \mathbf{f}_t]$ denotes the expectation of the random vector $(\mathbf{e}_{t+1} - \mathbf{f}_t)$ under $\bar{\mathbb{Q}}$. The presented expression allows for i) over-hedging: shorting foreign currency in excess of what would be required to fully hedge the implicit currency exposure, and ii) under-hedging: holding foreign currency in excess to the current exposure of $w_{c,t}$. To avoid over- or under-hedging, for example, because of regulatory constraints, the same optimization problem can be cast in a quadratic programming format. There, equations (10) and (11) determine the objective function, and arbitrary linear constraints on optimal exposures can be prescribed. The solution is then obtained numerically; for more details, see Ulrych and Vasiljević (2020).

Next, we derive the optimal ambiguity-adjusted mean-variance currency exposure given the dynamics of the non-Gaussian returns model presented in Section 2.2. Using the notation from equation (4), we model the $(K + 1 \times 1)$-dimensional random vector as $\mathbf{Y}_{t+1} = (\tilde{R}^{fh}_{\mathcal{P},t+1}, e_{2,t+1} - f_{2,t}, \ldots, e_{K+1,t+1} - f_{K+1,t})$. Given the conditional dispersion matrix $\mathbf{H}_t$ defined in equation (6), denote with $\mathbf{H}^c_t$ the $(K \times K)$-dimensional matrix without the first row and the first column of $\mathbf{H}_t$, hence corresponding to the conditional dispersion only among the currency excess returns. In a similar fashion, denote with $\boldsymbol{h}^c_t$ the $(K \times 1)$-dimensional vector of the first column (without the first element) of matrix $\mathbf{H}_t$. Hence, $\boldsymbol{h}^c_t$ corresponds to the conditional dispersion between the fully hedged portfolio return and the currency excess returns. Similarly, define the $(K \times 1)$-dimensional vectors $\boldsymbol{\mu}_c$ and $\boldsymbol{\gamma}_c$, and a scalar γ_1, where $\boldsymbol{\gamma} = (\gamma_1, \boldsymbol{\gamma}'_c)'$. Taking into account $\mathbb{Q} \equiv \mathcal{N}_{\mathbf{Y}_t|G_t,\mathcal{F}_{t-1}}(\cdot \mid \boldsymbol{\mu} + \boldsymbol{\gamma} g, g\mathbf{H}_t)$, $\eta \equiv G_t$, and $\bar{\mathbb{Q}} \equiv \mathbf{Y}_t \mid \mathcal{F}_{t-1}$ and utilizing the expressions from equation (8), we have

$$
\begin{aligned}
&\mathrm{Var}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1} - \mathbf{f}_t] \\
&= \mathrm{E}[G_{t+1} \mid \mathcal{F}_t]\, \mathbf{H}^c_{t+1|\mathcal{F}_t} + \mathrm{Var}(G_{t+1} \mid \mathcal{F}_t)\, \boldsymbol{\gamma_c}\boldsymbol{\gamma_c}', \\
&\mathrm{Cov}_{\bar{\mathbb{Q}}}[\tilde{R}^{fh}_{\mathcal{P},t+1}, \mathbf{e}_{t+1} - \mathbf{f}_t] \\
&= \mathrm{E}[G_{t+1} \mid \mathcal{F}_t]\, \mathbf{h}^c_{t+1|\mathcal{F}_t} + \mathrm{Var}(G_{t+1} \mid \mathcal{F}_t)\gamma_1\boldsymbol{\gamma_c}, \\
&\mathrm{Var}_\eta[\mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1} - \mathbf{f}_t]] \\
&= \mathrm{Var}(G_{t+1} \mid \mathcal{F}_t)\, \boldsymbol{\gamma_c}\boldsymbol{\gamma_c}', \\
&\mathrm{Cov}_\eta[\mathrm{E}_{\mathbb{Q}}[\tilde{R}^{fh}_{\mathcal{P},t+1}], \mathrm{E}_{\mathbb{Q}}[\mathbf{e}_{t+1} - \mathbf{f}_t]] \\
&= \mathrm{Var}(G_{t+1} \mid \mathcal{F}_t)\gamma_1\boldsymbol{\gamma_c}, \quad \text{and} \\
&\mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{e}_{t+1} - \mathbf{f}_t] \\
&= \boldsymbol{\mu_c} + \mathrm{E}[G_{t+1} \mid \mathcal{F}_t]\, \boldsymbol{\gamma_c}.
\end{aligned}
\tag{12}
$$

Plugging the terms from equation (12) into the model-free exposure from equation (11), we obtain the optimal COMFORT-based ambiguity-adjusted mean-variance currency exposure as

$$
\begin{aligned}
\Psi^*_{t,COM} = &-\big[\lambda \mathrm{E}[G_{t+1} \mid \mathcal{F}_t]\, \mathbf{H}^c_{t+1|\mathcal{F}_t} \\
&+(\lambda + \theta)\mathrm{Var}(G_{t+1} \mid \mathcal{F}_t)\, \boldsymbol{\gamma_c}\boldsymbol{\gamma_c}'\big]^{-1} \\
&\times \big[\lambda \mathrm{E}[G_{t+1} \mid \mathcal{F}_t]\, \mathbf{h}^c_{t+1|\mathcal{F}_t} \\
&+(\lambda + \theta)\mathrm{Var}(G_{t+1} \mid \mathcal{F}_t)\gamma_1\boldsymbol{\gamma_c} \\
&- (\boldsymbol{\mu_c} + \mathrm{E}[G_{t+1} \mid \mathcal{F}_t]\, \boldsymbol{\gamma_c})\big].
\end{aligned}
\tag{13}
$$

Even though there is a distinct term characterizing ambiguity in the robust mean-variance model from equation (10), we see that ambiguity-like behavior in the COMFORT-based optimal currency exposure from equation (13) persists even in the case when $\theta \to 0$. The reason for this lies in the presence of the mixing random variable G_t. Only when $\mathrm{Var}(G_{t+1} \mid \mathcal{F}_t) \to 0$, the modeled returns become conditionally Gaussian, and the ambiguity about the return distribution vanishes (i.e.there is no uncertainty about the future market shocks). Simultaneously, the COMFORT model converges to the standard constant conditional correlation (CCC) Gaussian-GARCH type of dynamics, which is the prevailing modeling choice present in the existing dynamic currency hedging literature. However, when $\mathrm{Var}(G_{t+1} \mid \mathcal{F}_t) > 0$, the ambiguity is present and its effect on the optimal currency allocation gets amplified via the parameter of ambiguity aversion θ, given the optimized robust mean-variance model. Since we are, in this section, studying the optimal robust mean-variance currency exposure, only the first two moments affect the corresponding return ambiguity. The larger the uncertainty around the possible market shock, the larger the effect on the optimal ambiguity-adjusted mean-variance currency exposure in comparison to the Gaussian-GARCH case. The direction of the effect of the aggregated market ambiguity on the optimal currency allocation is driven by the asset-specific conditional asymmetry vector $\boldsymbol{\gamma}$ and hence varies across currencies. It can also be understood as the ambiguity-induced shrinkage of the optimal currency allocation.

In general, using our framework, an ambiguity-averse investor can also use an arbitrary risk measure (e.g. expected shortfall) to optimize her portfolio. Analogously to equation (10), one can introduce the general ambiguity-adjusted mean-expected-shortfall portfolio objective function. Given the risk and ambiguity aversion coefficients $\lambda \geq 0$ and $\theta \geq 0$, respectively, the investor seeks to maximize

$$
\max_{\Psi_t} \left\{ \mathrm{E}_{\bar{\mathbb{Q}}}[\tilde{R}^h_{\mathcal{P},t+1}] - \frac{\lambda}{2}\mathrm{ES}_{\bar{\mathbb{Q}}}(\tilde{R}^h_{\mathcal{P},t+1}) - \frac{\theta}{2}\mathrm{Var}_\eta(\mathrm{E}_{\mathbb{Q}}[\tilde{R}^h_{\mathcal{P},t+1}]) \right\},
\tag{14}
$$

where $\Psi_t = (\psi_{2,t}, \ldots, \psi_{K+1,t})'$ denotes a $(K \times 1)$-dimensional vector of foreign currency exposures. Equation (14) differs from equation (10) only in the middle term that corresponds to the portfolio tail-based risk measure. The last term has to be symmetric because investors are model uncertain for arbitrary changes in the parameters. In the next section, we derive a numerical method for optimizing objective functions like equation (14) under non-Gaussian dynamics.

2.3.2. General dynamic optimal currency exposure.

Working with the ambiguity-adjusted mean-variance model is useful as it yields a closed-form solution and thereby enhances the understanding of the estimation of the optimal currency exposure. In this section, we propose a numerical algorithm that extends the computation of the optimal currency exposure to a general, e.g. tail-based, risk measure. Furthermore, all equations introduced so far dealt with returns and the corresponding currency exposure measured over a single time period starting at time t and ending at $t+1$, where $t+1$ denotes the maturity of the hedge. A usual practical choice for the maturity of a currency hedge is, for example, a quarter (a month or a half-year). However, the time-series models, such as Gaussian-GARCH and COMFORT, are commonly calibrated using daily returns. In such a case, one needs a conditional distribution of the multi-period returns, where the period length depends on the hedge maturity. To circumvent multi-period modeling, one could employ coarser sampling; however, such a method is inefficient since it discards a lot of data and information. Alternatively, computing with overlapping quarterly (or monthly) returns is unsuitable either since it induces an artificial serial dependence leading to inefficient and biased estimates. Another possibility would be utilizing the daily time frequency and scaling the one-step-ahead conditional distribution estimates. This method exaggerates volatility of volatility and is, therefore, also not appropriate. In this section, we present an approach that circumvents such modeling issues. It uses daily modeling frequency and applies either a parametric bootstrap or its non-parametric equivalent called filtered historical simulation in order to build up a cumulative (i.e.multi-period) conditional distribution of the random vector of returns $\mathbf{Y}_{t+1}$. Additionally, this approach enables us to numerically optimize a general risk measure—for which closed-form solutions necessarily do not exist—such as expected shortfall.

We start with the parametric version of the return accumulation algorithm. Consider the COMFORT dynamics with a daily modeling frequency. We aim to build up a conditional distribution of h-step ahead cumulative returns $\mathbf{Y}_{T+h|\mathcal{F}_T}(h)$ for $h>1$ corresponding to the hedge maturity (i.e. expressed in terms of days). To model the conditional dispersion matrix $\mathbf{H}_{T+h|\mathcal{F}_T}$, we need an expression for the univariate scale terms given in equation (7). Assuming a parametric distributional structure of the COMFORT model, we prove, in the Appendix, that the forecasted scale terms $\hat{s}^2_{k,T+h|\mathcal{F}_T} := \mathrm{E}[s^2_{k,T+h} \mid \mathcal{F}_T]$ are given by

$$\hat{s}^2_{k,T+h|\mathcal{F}_T} = \begin{cases} \hat{\omega}_k + \hat{\alpha}_k \left(Y_{k,T} - \hat{\mu}_k - \hat{\gamma}_k \mathrm{E}[G_T \mid \mathcal{F}_T]\right)^2 \\ \quad + \hat{\beta}_k \hat{s}^2_{k,T}, & \text{for } h = 1, \\ \hat{\omega}_k + \hat{s}^2_{k,T+h-1|\mathcal{F}_T} \\ \quad \left(\hat{\alpha}_k \mathrm{E}[G_{T+h-1} \mid \mathcal{F}_T] + \hat{\beta}_k\right), & \text{for } h > 1, \end{cases} \tag{15}$$

where $\mathrm{E}[G_T \mid \mathcal{F}_T]$ is imputed from the E-step of the EM-algorithm; and $\mathrm{E}[G_{T+h} \mid \mathcal{F}_T] = \mathrm{E}[G_T]$ for any $h>1$ because of the assumed independence from previous data and iid structure of the mixing random variable G_t. The optimal prediction is obtained in the L^2 sense. It differs from a standard Gaussian-GARCH case, as in Francq and Zakoian (2019), because of the additional mixing random variable factor for $h>1$.

Equation (15) enables us to construct the daily conditional dispersion matrix $\mathbf{H}_{T+h|\mathcal{F}_T}$ for any $h>1$. Employing Monte-Carlo sampling, we can repeatedly draw sequences of consecutive returns

$$\mathbf{Y}^{(i)}_{T+1|\mathcal{F}_T} \to \mathbf{Y}^{(i)}_{T+2|\mathcal{F}_T} \to \cdots \to \mathbf{Y}^{(i)}_{T+h|\mathcal{F}_T}, \quad i = 1, 2, \ldots B, \tag{16}$$

where B denotes the number of simulations (i.e. bootstraps). To simulate these single-period returns, one uses the representation of the model as given in equation (4) and repeatedly samples from the corresponding multivariate standard normal and generalized inverse Gaussian distributions, as estimated at time T. Utilizing these distributions is the reason that the presented approach is called parametric. After simulating the single-period returns, one simply needs to aggregate them into the h-step conditional cumulative return distribution $Y^{(i)}_{T+h|\mathcal{F}_T}(h)$ via

$$\mathbf{Y}^{(i)}_{T+h|\mathcal{F}_T}(h) = (1 + \mathbf{Y}^{(i)}_{T+1|\mathcal{F}_T})(1 + \mathbf{Y}^{(i)}_{T+2|\mathcal{F}_T}) \cdots (1 + \mathbf{Y}^{(i)}_{T+h|\mathcal{F}_T}) - 1, \quad i = 1, 2, \ldots B, \tag{17}$$

where the multiplication is performed on a component-by-component basis.

Now we present the non-parametric version of the described algorithm. The foundation of our procedure is the well-known filtered historical simulation (FHS) approach, as presented in Barone-Adesi *et al.* (1999). The authors introduced a simulation model that does not impose any theoretical distribution on the data. It uses the historical distribution of the return series solely. The procedure is utilized via a GARCH filter that aims to remove the serial correlation and volatility clusters present in the data set. The filtered returns are rendered identically and independently distributed, and the non-parametric bootstrapping can hence be applied. We extend this approach to the COMFORT model that, in addition to the GARCH component, also employs a mixing random variable.

Consider the COMFORT model specification as given in equation (4). Denote an arbitrary asset (currency) in the model with k and assume we work with the data series originating at times $t = 1, \ldots T$. The standardized residuals of the asset k at times $t = 1, \ldots T$ are given by

$$\tilde{e}^{(t)}_k = \frac{Y_{k,t} - \hat{\mu}_k - \hat{\gamma}_k \mathrm{E}[G_t \mid \mathcal{F}_t]}{\hat{s}_{k,t}\sqrt{\mathrm{E}[G_t \mid \mathcal{F}_t]}},$$

where the term $\mathrm{E}[G_t \mid \mathcal{F}_t]$ corresponds to the imputed mixing random variable from the expectation step of the EM-algorithm used for the estimation of the model parameters.

Under the COMFORT model specification, the standardized residuals are independent and identically distributed by definition and hence suitable for historical simulation – parametric approach. Empirical observations might not exactly satisfy this. However, filtered historical innovations can be drawn randomly with replacement and used as innovations to generate pathways of future returns consistent with the estimated time-series model. By doing so, filtered historical simulation can be seen as a non-parametric bootstrap for time-series models. We denote with $\tilde{\mathbf{Z}}$ the $(K \times T)$ matrix of standardized residuals, where $[\tilde{\mathbf{Z}}]_{k,t} = \tilde{e}^{(t)}_k$.

In addition to the standardized residuals, we also need to standardize the corresponding filtered mixing random variables that are utilized in the filtered historical simulation. Consider a sample of $t = 1, \dots, T$ filtered variables $\mathrm{E}[G_t \mid \mathcal{F}_t]$. We employ a kernel cumulative distribution function (cdf) estimation. This is a non-parametric way to estimate the cdf of a random variable. Kernel cdf estimation is a fundamental data smoothing problem where inferences about the population are made based on a finite data sample. For us, the sample consists of the imputed mixing random variables $\mathrm{E}[G_t \mid \mathcal{F}_t]$, for $t = 1, \dots, T$. A kernel cdf over this sample is estimated and denoted by $K_{cdf}(\cdot)$. Then, $K_{cdf}(\mathrm{E}[G_t \mid \mathcal{F}_t])$ represents a realization of an empirical kernel cumulative distribution function of the filtered mixing random variable, for each $t = 1, \dots, T$.

By construction of the COMFORT model, we have $(G_{T+1} \mid \mathcal{F}_T) \sim GIG(a, b, p)$. Denote with $\mathrm{GIG}_{\mathcal{F}_T}(\cdot)$ the corresponding cumulative distribution function. One then, for each $t = 1, \dots T$, obtains standardized conditional (on the information $\mathcal{F}_T$ available at time T) mixing random variables as

$$\tilde{G}^{(t)} = \mathrm{GIG}_{\mathcal{F}_T}^{-1}\left(K_{cdf}\left(\mathrm{E}[G_t \mid \mathcal{F}_t]\right)\right),$$

where $\mathrm{GIG}_{\mathcal{F}_T}^{-1}(\cdot)$ denotes the (generalized) inverse of $\mathrm{GIG}_{\mathcal{F}_T}(\cdot)$ with the parameters estimated from the data. A $(T \times 1)$-dimensional vector $\tilde{\mathbf{G}}$ is a vector of standardized mixing random variables, where each vector component t, for $t = 1, \dots, T$, corresponds to the equivalent entry in the matrix of standardized residuals $\tilde{\mathbf{Z}}$. Note that consistent sampling of $\tilde{\mathbf{G}}$ and $\tilde{\mathbf{Z}}$ is essential in order to preserve the implicit conditional dependence structure and the stability of the simulated dynamics.

For $T + 1$, the filtered historical simulation of $\mathbf{Y}^{FHS,(i)}_{T+1|\mathcal{F}_T}$ is carried out by

$$\mathbf{Y}^{FHS,(i)}_{T+1|\mathcal{F}_T} = \hat{\boldsymbol{\mu}} + \hat{\boldsymbol{\gamma}}\tilde{G}_{T+1|\mathcal{F}_T} + \sqrt{\tilde{G}_{T+1|\mathcal{F}_T}}\hat{\mathbf{S}}_{T+1|\mathcal{F}_T}\tilde{\mathbf{Z}}_{T+1|\mathcal{F}_T}, \quad i = 1, 2, \dots B,$$

where a scalar $\tilde{G}_{T+1|\mathcal{F}_T}$ and a $(K \times 1)$-dimensional vector $\tilde{\mathbf{Z}}_{T+1|\mathcal{F}_T}$ are realizations of $\tilde{\mathbf{G}}$ and $\tilde{\mathbf{Z}}$, for a randomly chosen t, consistent among both variables. Moreover, $\hat{\mathbf{S}}_{T+1|\mathcal{F}_T}$ is a $(K \times K)$-dimensional diagonal matrix of strictly positive conditional scale terms $s_{k,T+1|\mathcal{F}_T}$, where

$$\hat{s}^2_{k,T+1|\mathcal{F}_T} = \hat{\omega}_k + \hat{\alpha}_k(Y_{k,T} - \hat{\mu}_k - \hat{\gamma}_k\mathrm{E}[G_T \mid \mathcal{F}_T])^2 + \hat{\beta}_k\hat{s}^2_{k,T}.$$

The h-step prediction, where $h > 1$, is recursively obtained through

$$\mathbf{Y}^{FHS,(i)}_{T+h|\mathcal{F}_T} = \hat{\boldsymbol{\mu}} + \hat{\boldsymbol{\gamma}}\tilde{G}_{T+h|\mathcal{F}_T} + \sqrt{\tilde{G}_{T+h|\mathcal{F}_T}}\hat{\mathbf{S}}_{T+h|\mathcal{F}_T}\tilde{\mathbf{Z}}_{T+h|\mathcal{F}_T}, \quad i = 1, 2, \dots B,$$

where $\tilde{G}_{T+h|\mathcal{F}_T}$ and $\tilde{\mathbf{Z}}_{T+h|\mathcal{F}_T}$ are randomly and consistently sampled from $\tilde{\mathbf{G}}$ and $\tilde{\mathbf{Z}}$, and $\hat{\mathbf{S}}_{T+h|\mathcal{F}_T}$ is constructed via

$$\hat{s}^{2,(i)}_{k,T+h|\mathcal{F}_T} = \hat{\omega}_k + \hat{\alpha}_k(Y^{FHS,(i)}_{k,T+h-1|\mathcal{F}_T} - \hat{\mu}_k - \hat{\gamma}_k\tilde{G}_{T+h-1|\mathcal{F}_T})^2 + \hat{\beta}_k\hat{s}^{2,(i)}_{k,T+h-1|\mathcal{F}_T}, \quad i = 1, 2, \dots B.$$

On every step of the described process, a random date t (among $1, 2, \dots, T$) is selected and applied to the associated set of standardized residuals and corresponding standardized mixing random variables. In such a way, the dependence structure between the modeled multivariate return series is preserved. Note that the matrix $\tilde{\mathbf{Z}}$ is constructed in a non-parametric manner, whereas the vector $\tilde{\mathbf{G}}$ is established via the parametric assumption of the $GIG(\cdot)$ distribution. This combination of a parametric and a non-parametric part of the procedure renders our approach semi-parametric, conditional on the COMFORT dynamics.

The construction of the h-period cumulative filtered return $\mathbf{Y}^{FHS,(i)}_{T+h|\mathcal{F}_T}(h)$ is exactly the same as in the parametric case described in equation (16) and (17). At first, the single-period returns are obtained recursively. Next, they are aggregated to cumulative h-period returns. The procedure is repeated B times to obtain the predicted FHS distribution of the multivariate return vector. This computationally efficient numerical algorithm makes our model both general and tractable.

For a large enough number of simulations B, it is now possible to numerically optimize an arbitrary risk measure/objective function. For example, the extended filtered historical simulation can be used to generate a forward-looking sample of multivariate returns. Then, the currency exposure can be optimized with respect to the expected shortfall by formulating the optimization problem in the linear programming format, such as presented in Rockafellar and Uryasev (2000).

2.4. Currency hedging strategy

Here, we propose a dynamic currency hedging strategy. This is a fixed plan that performs trading (i.e. hedging) decisions through time, applying the previously discussed estimation, simulation, and optimization approaches. The main idea is to run the currency exposure optimization procedure for every pre-specified rolling window (e.g. quarterly or monthly) and enter the computed positions in the currency forward contracts for each currency in the portfolio over time. Below, the currency hedging strategy algorithm is specified.

The estimation part of the algorithm depends on the chosen underlying model, e.g. the Gaussian-GARCH or the non-Gaussian returns model we propose. Then, the (extended) filtered historical simulation is performed in order to construct the forward-looking cumulative returns consistent with the chosen time-series model. The optimization part depends on the chosen risk measure. The unconstrained robust mean-variance optimization can be performed with the analytically tractable procedure, while for general risk measures, such as the constrained expected shortfall, numerical optimization techniques are employed. Given the optimized values of the currency forward notionals, the corresponding currency forward contracts are entered. At maturity, the forward payoffs occur, get reinvested, and the whole process repeats over time (i.e. for each rolling window). We test the out-of-sample performance of the presented currency hedging strategy in the next section.

Algorithm 1 Dynamic currency hedging strategy algorithm

Input: Initial portfolio (number of shares of each stock, or an equivalent number of fixed-income instruments, commodity contracts, or other investment positions), a base currency, a risk measure to optimize, constraints on the currency exposure, and a maturity of the hedge.

Output: Optimal (constrained) currency exposure given a chosen risk measure over time.

for `each rolling window` **do**

1) ESTIMATION: Calculate the current asset weights and estimate the chosen multivariate non-Gaussian time-series model.
2) FILTERED HISTORICAL SIMULATION: Perform the extended filtered historical simulation to construct forward-looking cumulative returns.
3) OPTIMIZATION: Given the chosen risk measure and constructed cumulative returns, run the numerical optimization of the (possibly constrained) optimal currency exposure.
4) HEDGE: Enter into the computed amount of currency forward contracts for each currency in the portfolio. At maturity, the payoffs of the held currency forward contracts occur and are reinvested.

end for

3. Empirical analysis

In this section, we empirically investigate the dynamic currency hedging strategy presented above. An out-of-sample backtest is conducted and the performance of the proposed dynamic non-Gaussian hedging strategy is compared to the benchmarks of constant hedging, static Gaussian hedging, and dynamic Gaussian-GARCH-induced hedging, net of transaction costs.

3.1. Data

The empirical analysis covers 10 developed economies: Australia, Canada, Switzerland, Denmark, the Eurozone, the United Kingdom, Japan, New Zealand, Sweden, and the United States. The time-series data of spot and forward currency exchange rates, equity market total return indices, and government bond total return indices (covering all maturities combined) are employed. The data series are available at a daily frequency and are obtained from Refinitiv Datastream. The sample period starts on 05/25/1990 and ends on 08/01/2023, covering a history where data across all studied economies is consistent and available without interruptions or missing data points.

The data enables us to test the out-of-sample performance of the hedging strategy over some major events that happened in the world financial markets and had a profound effect on the currency exchange rates, such as the global financial crisis of 2008–2009, the European sovereign debt crisis of 2009–2011, the Swiss franc unpeg of 2015, the Brexit referendum of 2016, to more recent events like the COVID-19 pandemic. We investigate the performance of the dynamic non-Gaussian currency hedging strategy and compare it to constant hedging benchmarks and static/dynamic Gaussian approaches presented in the existing literature.

Table 1 reports the full sample annualized mean, annualized volatility, skewness, and kurtosis of local equity (Panel A), local bond (Panel B), and currency excess returns with respect to the USD base currency (Panel C). Notably, average returns for both equity and bonds are consistently positive across all countries. However, it is interesting to observe that JPY and SEK attain negative average currency excess returns against the USD. Additionally, among the currencies, NZD and AUD have the highest positive average currency excess returns, with 3.02% and 1.86%, respectively. Volatilities are the lowest for bonds, followed by currencies, and are the largest for equities. While equity returns generally exhibit negative skewness, bond and currency return skewness changes the sign across different countries. On average, equity and currency returns exhibit larger excess kurtosis compared to bond returns. Those summary statistics are in line with the commonly known empirical stylized facts of financial asset returns.

Figure 2 compares the cumulative return distribution of the USD equity total return index as constructed from both the parametric simulation and the filtered historical simulation (FHS) given the assumed COMFORT dynamics. Both approaches yield similar distributions, showing that the model is well-parametrized. However, one can observe that the forecasted distribution arising from FHS is more negatively skewed and heavy-tailed, which is more in line with what can be observed empirically. Moreover, FHS is also a distribution-free method. Therefore, it is more flexible, and it should be preferred in practice. Hence, in the out-of-sample analysis below, we use the FHS approach. The results for the parametric simulation are similar and available on request.

3.2. Out-of-Sample backtest

The backtest analysis presented in this section is performed with hedging on a quarterly time horizon. It is conducted for international portfolios comprised of three asset classes: equity, bonds, and cash. The analyzed portfolio is comprised of 70% equities, 20% bonds, and 10% cash, and equally weighted between the 10 different economies (meaning $w_{c,0} = 0.1$ for each currency c). For practical reasons, such as regulatory constraints, we constrain the optimal currency exposure to lie on the interval $[-2w_{c,0}, 3w_{c,0}]$ (expressed in terms of relative value compared to the total portfolio value) for each currency c over the whole backtest. This choice reflects a symmetric treatment of over- and under-hedging and prohibits extreme currency positions. Such a constraint can also be interpreted as a form of shrinkage, see, for example, Jagannathan and Ma (2003). Furthermore, we investigate various choices of investor's base currency. All results analyzed in this section are presented net of transaction costs, which are assumed to be five basis points relative to the notional of the entered currency forward contract for each foreign currency present in the portfolio.

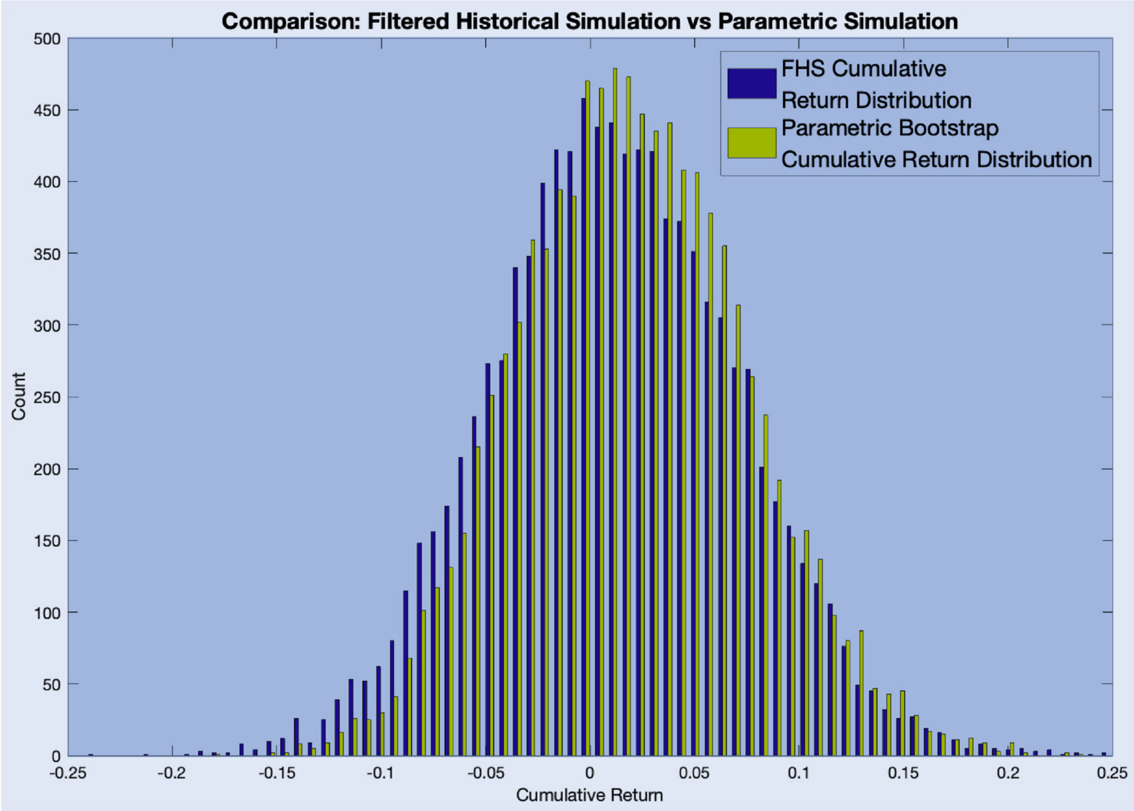

Figure 2. Comparison between the cumulative return distribution constructed from parametric simulation and filtered historical simulation, given the COMFORT model dynamics.

In practice, a currency forward contract is constructed with two different types of transactions: an FX spot and an FX swap. The spot transaction offsets the first leg of the FX swap, and what one is left with is equivalent to the currency forward. Note that transaction costs are paid for both spot and swap transactions. However, when an investor is rolling over currency forward contracts over time (e.g. as in our currency hedging setting), the initial spot transaction can be avoided, and only a new FX swap can be entered at the maturity of the old one. This is important because transaction costs on FX swaps are considerably lower compared to an FX spot or FX forward transactions, see Ackermann *et al.* (2017). In light of falling transaction costs in the last years, Jahan-Parvar and Zikes (2019) show that the effective spread on the most traded foreign exchange rates is below one basis point, meaning that the transaction costs on the FX swaps are even lower. Therefore, our choice of transaction costs of five basis points can be interpreted as conservative.

In addition to rolling over the currency forward contracts quarterly, we also rebalance the asset positions quarterly. The payoffs from the currency forward contracts are reinvested into assets such that the equal portfolio weighting between the analyzed countries is preserved over the backtest. We assume that there are no margin requirements for entering the currency forward contracts and that the asset positions are rebalanced costless since we aim to test the performance of different currency hedging strategies in isolation from any other effects on portfolio performance—it is a standard approach in this type of analysis; among others, see De Roon *et al.* (2003) and Cho *et al.* (2020). On the other hand, equity rebalancing costs and margin requirements are investigated in Barroso *et al.* (2022). Note that the 10% exposure to cash in our portfolios at least partially captures the opportunity costs due to margin requirements. Moreover, all models are calibrated with a rolling window of two-year daily historical data. Hence, the first two years of the sample are employed for the starting calibration, and the rest of the sample (i.e. around 31 years) corresponds to the out-of-sample empirical backtesting exercise.

We start the empirical analysis by investigating the optimal currency exposure estimated from different models and for different risk measures. We compare investors who manage their currency exposure with respect to (i) the robust mean-variance (MV) and (ii) the minimum expected shortfall (ES) at the confidence level $\zeta = 0.85$. To illustrate the performance of the FHS algorithm from Section 2.3.2, we work with the minimum expected shortfall portfolio optimization, which is a limiting case of equation (14) with $\lambda \to \infty$. The general case of equation (14) is a simple extension using the moments given in equation (12). Note that, when $\lambda \to \infty$, at least for the mean-variance portfolio and for all levels of ambiguity aversion $\theta \geq 0$, we can see from equation (13), that our non-Gaussian returns model still induces optimal currency allocations adjusted for non-Gaussianity, through the first two moments of the mixing random variable.

We investigate currency hedging strategies based on static and dynamic models. The benchmark static shrunk (SS) hedging model is based on the non-linear covariance shrinkage from Ledoit and Wolf (2020) and the mean shrinkage

Table 2. Observe the realized annualized portfolio return volatility, mean, Sharpe and Sortino ratios, expected shortfall (ES) (for the confidence level of $\zeta = 0.95$), maximum drawdown, and turnover of different model-based and constant hedging strategies obtained in the out-of-sample backtest.

Hedging strategy out-of-sample backtest results							
	No Hedge	Full Hedge	MV-SS	MV-MN	ES-MN	MV-COM	ES-COM
Base: CHF							
Volatility	12.04%	8.04%	8.70%	8.63%	7.26%	**7.25%**	7.30%
Mean	6.23%	5.86%	6.13%	**6.43%**	4.81%	5.53%	5.68%
Sharpe Ratio	0.52	0.73	0.70	0.75	0.66	0.76	**0.78**
Sortino Ratio	0.71	1.00	0.98	1.05	0.93	1.07	**1.10**
Exp Shortfall	1.83%	1.24%	1.31%	1.29%	1.08%	**1.07%**	**1.07%**
Max Drawdown	48.49%	38.35%	37.25%	39.17%	32.21%	30.37%	**30.24%**
Turnover	0	0.90	1.87	1.84	1.37	1.47	1.46
Base: EUR							
Volatility	9.84%	8.05%	8.44%	8.32%	7.30%	**7.11%**	7.13%
Mean	**8.07%**	7.07%	7.66%	8.03%	6.40%	7.04%	7.07%
Sharpe Ratio	0.82	0.88	0.91	0.97	0.88	**0.99**	**0.99**
Sortino Ratio	1.14	1.21	1.27	1.36	1.23	**1.40**	1.39
Exp Shortfall	1.50%	1.24%	1.27%	1.23%	1.09%	**1.05**	**1.05%**
Max Drawdown	40.87%	37.32%	34.77%	35.35%	27.70%	27.48%	**26.35%**
Turnover	0	0.90	1.85	1.82	1.31	1.38	1.45
Base: GBP							
Volatility	10.23%	8.05%	8.59%	8.52%	7.30%	**7.10%**	7.32%
Mean	8.79%	7.94%	**8.82%**	8.80%	7.37%	7.97%	7.80%
Sharpe Ratio	0.86	0.99	1.03	1.03	1.01	**1.12**	1.07
Sortino Ratio	1.23	1.37	1.45	1.45	1.44	**1.60**	1.52
Exp Shortfall	1.49%	1.24%	1.28%	1.26%	1.06%	**1.04%**	1.07%
Max Drawdown	29.20%	36.60%	31.87%	34.19%	26.13%	**25.68%**	30.10%
Turnover	0	0.90	1.85	1.81	1.31	1.42	1.51
Base: JPY							
Volatility	14.48%	8.04%	8.82%	8.94%	7.36%	**7.27%**	7.36%
Mean	**8.31%**	5.27%	6.03%	6.92%	4.55%	4.82%	4.95%
Sharpe Ratio	0.57	0.66	0.68	**0.77**	0.62	0.66	0.67
Sortino Ratio	0.78	0.90	0.95	**1.08**	0.86	0.93	0.94
Exp Shortfall	2.26%	1.24%	1.35%	1.35%	1.10%	**1.07**	1.09%
Max Drawdown	56.23%	38.71%	39.76%	41.42%	33.60%	**32.93%**	34.92%
Turnover	0	0.90	1.80	1.76	1.34	1.44	1.52
Base: USD							
Volatility	11.56%	8.05%	8.84%	8.65%	7.28%	**7.13%**	7.20%
Mean	7.82%	7.44%	8.20%	**8.58%**	6.20%	7.06%	7.02%
Sharpe Ratio	0.68	0.92	0.93	**0.99**	0.85	**0.99**	0.98
Sortino Ratio	0.95	1.28	1.30	**1.40**	1.21	**1.40**	1.38
Exp Shortfall	1.72%	1.24%	1.34%	1.29%	1.07%	**1.05%**	1.06%
Max Drawdown	48.03%	37.50%	35.83%	33.60%	30.10%	**28.04%**	29.77%
Turnover	0	0.90	1.86	1.83	1.33	1.43	1.44

Notes: We present the results for the quarterly rebalanced portfolio comprised of 70% equity, 20% bonds, and 10% cash, equally weighted between 10 different economies in dependence of different base currencies. We analyze a static shrunk (SS) hedging model based on non-linear covariance shrinkage from Ledoit and Wolf (2020) and two dynamic hedging models: (i) multivariate normal (MN) model based on the CCC-GARCH time-series structure and (ii) the proposed non-Gaussian COMFORT (COM) time-series model. Currency hedging based on optimizing the robust mean-variance (MV) and expected shortfall (ES) risk measures is investigated and compared to the constant hedging benchmarks of zero and full hedging. The results for different choices of the base currency are presented net of transaction (i.e. hedging) costs, which are assumed to be five basis points per notional of each entered currency forward contract.

from Jorion (1986). The dynamic models are driven by a CCC-GARCH-based multivariate normal (MN) model and by the non-Gaussian (COMFORT) time-series model, as presented in previous sections.

Table 2 presents the results of the out-of-sample backtest comparing the performance of model-based currency hedging to the benchmarks of zero and full hedging. We investigate the performance of SS, MN, and COMFORT-based hedging. The optimization is conducted for MV and ES risk measures. The 70/20/10 portfolio comprised of 10 different countries is analyzed for different choices of the base currency, and all results are presented net of transaction costs. The results for the remaining base currencies are given in the Appendix; see table A1. Moreover, the results are shown for the risk and ambiguity aversion parameters $\lambda = 3$ and $\theta = 3$, respectively. Further analysis around the choice of λ and θ is presented in figure 3. Additionally, detailed insights into the optimal currency exposures generated by the investigated models can be found in the Appendix; see figure A2. This plot illustrates how the COMFORT model results in more stable and distinct optimized currency exposures over time compared to the Gaussian models.

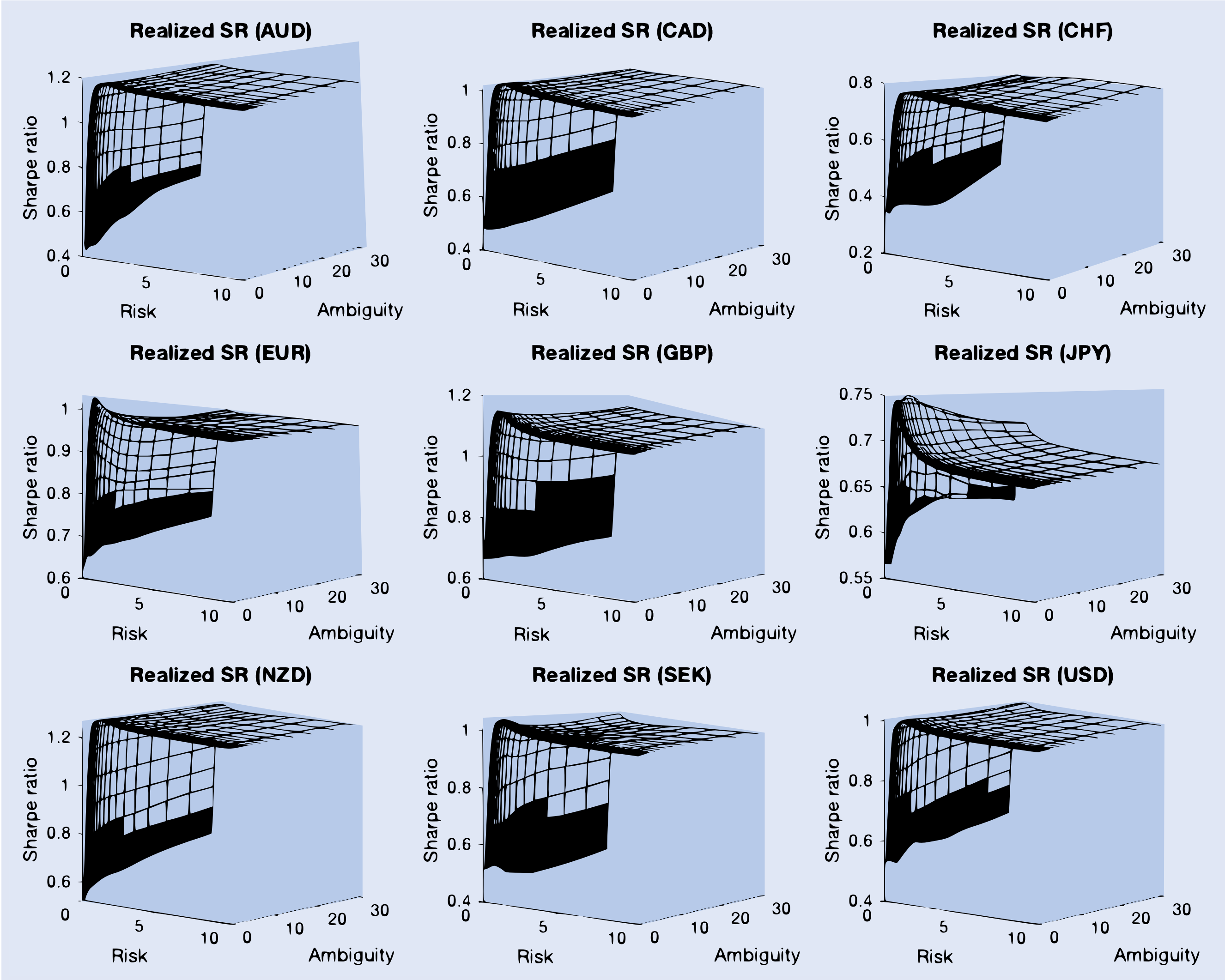

Figure 3. Out-of-sample realized Sharpe ratio surfaces from a rolling window exercise for different risk and ambiguity aversion parameters λ and θ, respectively, for the ambiguity-adjusted mean-variance model from equation (10). The subplots represent different base currencies given the analyzed 70/20/10 portfolio.

Observe that all model-based hedging strategies effectively reduce portfolio risk compared to constant (i.e. zero and full) hedging. This risk reduction is expressed through lower realized volatility, expected shortfall, and maximum drawdown. Notably, the dynamic non-Gaussian model, whether optimizing for mean-variance (MV) or expected shortfall (ES), consistently achieves the most substantial risk reduction across all analyzed base currencies.

While the annualized mean of portfolio returns is highest when employing the Gaussian model or no hedging, it is essential to emphasize that model-induced hedging consistently outperforms the constant hedging benchmarks in terms of risk-adjusted returns, as measured by the Sharpe and Sortino ratios.† Furthermore, dynamic approaches, both Gaussian and non-Gaussian, consistently outperform their static counterparts, including the strategy based on covariance matrix shrinkage from Ledoit and Wolf (2020) and mean shrinkage from Jorion (1986). This finding aligns with previous research, demonstrating that dynamic hedging tends to outperform static currency hedging approaches, see, for example, Brown *et al.* (2012) and Cho *et al.* (2020). Similarly to the risk reduction case, the largest risk-adjusted returns are consistently attained by the dynamic non-Gaussian returns model.

We express the average (i.e. quarterly) hedging turnover $\overline{\mathrm{HT}^s}$ of a currency hedging strategy s as

$$\overline{\mathrm{HT}^s} = \frac{1}{T}\sum_{t=1}^{T}\sum_{c=2}^{K+1}|\phi^s_{c,t}|, \tag{18}$$

where T is the number of trading instances, K is the number of foreign currency exposures, and $\phi^s_{c,t}$ is the notional of a currency forward contract used for hedging currency c at time t. The notional is expressed in relative terms compared to the total portfolio value. An average amount of trading for a hedging strategy s computed over the whole backtest is indicated by $\overline{\mathrm{HT}^s}$. As shown in table 2, the dynamic MV models exhibit lower turnover in comparison

† Sortino ratio penalizes negatively skewed portfolio return distributions by computing the volatility of returns only below the risk-free rate, hence accounting for the downside risk.

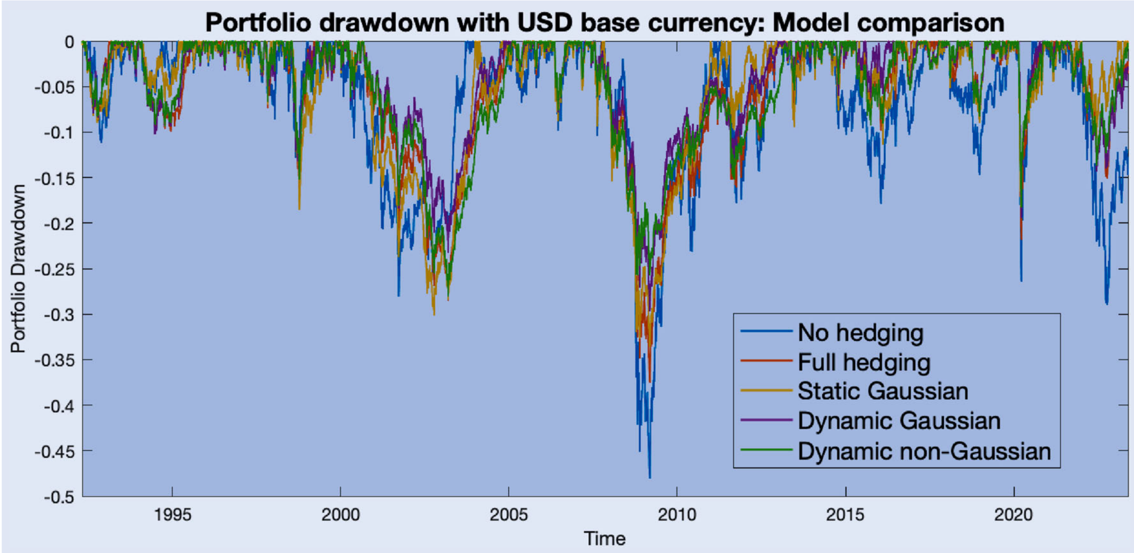

Figure 4. This plot shows a drawdown of the analyzed 70/20/10 portfolio for a USD-based investor. Zero hedging, full hedging, and mean-variance-based hedging as computed by the static MV-SS, the dynamic multivariate normal MV-MN, and the dynamic non-Gaussian MV-COM-based currency hedging strategies are presented.

to the static MV approach. Specifically, the MV-COM model, benefitting from the increased stability of the non-Gaussian framework, achieves the lowest turnover among the MV models. Conversely, in the case of ES optimization, the dynamic non-Gaussian currency hedging strategy experiences higher hedging turnover than its Gaussian counterpart. Moreover, given the regulatory constraints or the investor preferences, the turnover of the hedging strategy can be additionally controlled by specifying the applicable currency exposure constraints in the optimization process.

The out-of-sample backtest demonstrates the outperformance of the non-Gaussian currency hedging strategy over its static shrunk and dynamic Gaussian counterparts, even when accounting for transaction costs. Across different base currencies, the proposed non-Gaussian returns model, in both the MV-COM and ES-COM cases, achieves the highest realized Sharpe and Sortino ratios while simultaneously minimizing realized volatility, expected shortfall, and maximum drawdown. These superior results hold true not only when considering transaction costs but also in gross terms (i.e.without transaction costs). This outperformance can be attributed to the inherent non-Gaussian characteristics of the employed COMFORT model, characterized by asymmetry and heavy tails, along with its dynamic forecasting of cumulative returns. By leveraging the additional information provided by the mixing (i.e. ambiguity) and asymmetry parameters, the model enhances the predictive accuracy of the cumulative return distribution, resulting in superior out-of-sample performance compared to Gaussian models and constant hedging benchmarks.

Figure 3 depicts the out-of-sample realized Sharpe ratio for the ambiguity-adjusted mean-variance model given various base currencies in dependence of risk and ambiguity aversion parameters λ and θ, respectively. Observe that the parameter of risk aversion has a larger impact on the realized Sharpe ratio compared to the parameter of ambiguity aversion. This is also expected given the derived optimal exposure from equation (13). Moreover, ambiguity aversion is more relevant when an investor exhibits low risk aversion—the slope in the ambiguity aversion direction of the Sharpe ratio surface is larger for small risk aversion than for large risk aversion. When investors admit low risk aversion, it is beneficial to be more ambiguity averse, as illustrated in the increased realized Sharpe ratio. This shows that ambiguity becomes an essential component in portfolio optimization when investors are taking more risks. On the other hand, for larger values of risk aversion, the portfolio performance stabilizes and converges towards the minimum variance-based Sharpe ratio.

Figure 4 shows the portfolio drawdown of the USD-based portfolio from table 2. One can observe that model-induced hedging substantially reduces the portfolio drawdown over the whole period of the backtest. This is the most prominent in the time of the global financial crisis of 2008–2009 and the COVID-19 pandemic of 2020–2022. Many investors who are, for example, withdrawing retirement money from their pension funds are mostly concerned about drawdowns. Consequently, large drawdowns can be extremely problematic not only for retirees but also for asset managers. We show that when investors employ the non-Gaussian returns model, they can significantly reduce portfolio drawdowns and achieve higher risk-adjusted portfolio returns net of transaction costs.

Portfolio drawdowns can also be driven by specific events occurring in financial markets that have profound effects on currency exchange rates. A prominent example of such an event is the decision of the Swiss National Bank to scrap its currency peg of 1.20 to the euro on January 15, 2015. The Swiss franc immediately appreciated against almost all other currencies, most notably around 20% against the euro. Another example of an event that had a large effect on currency markets is the Brexit referendum, where voters in the United Kingdom, on June 23, 2016, decided to withdraw from the European Union. Consequently, the British pound

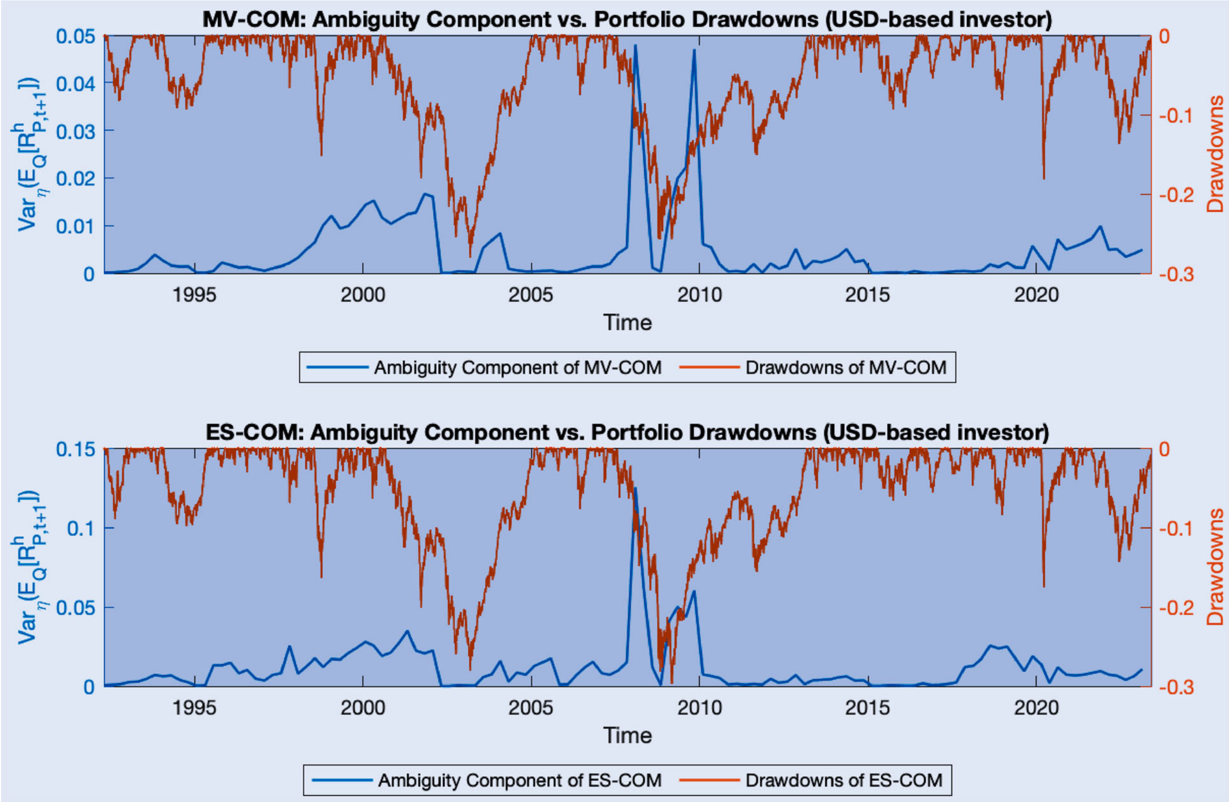

Figure 5. Rolling window forecasted values of investor's ambiguity component $\mathrm{Var}_\eta(\mathrm{E}_\mathbb{Q}[\tilde{R}^h_{\mathcal{P},t+1}])$ for a USD-based investor using MV–COMFORT (top panel) and ES-COMFORT (bottom panel) models, respectively. Along with the investor's ambiguity component, the drawdowns of the corresponding portfolios are depicted.

depreciated against almost all major currencies. In figure A3 in the Appendix, we study the effect of currency hedging around these two events on portfolio performance.

There are different ways to measure market-implied ambiguity. Some studies use dispersions in forecasts made by professional market experts (e.g.Anderson *et al.* 2009, Drechsler 2013), newspaper-based measures of economic policy uncertainty (e.g. Baker *et al.* 2016); and market-based measures such as the VIX and the VVIX which reflect investors perception of future risks and market uncertainty (e.g.Bali and Zhou 2016, Baltussen *et al.* 2018, Hollstein and Prokopczuk 2018). In particular, the VVIX resembles the second-order beliefs, which, according to different theoretical models (e.g.Segal 1987, Klibanoff *et al.* 2005, Nau 2006), are appropriate for capturing ambiguity; see also Kostopoulos *et al.* (2022) for the corresponding empirical verification utilizing V-VSTOXX—European markets equivalent of VVIX.

The ambiguity in our currency hedging problem is dominated by uncertainty in a global portfolio of assets from different economies and corresponding currencies. Therefore, we cannot use the aforementioned market- and asset-class-specific measures of aggregated ambiguity as a benchmark for our approach. Moreover, a market level of aggregated ambiguity does not have to reflect the level of ambiguity associated with a given portfolio of a specific investor. By its definition, ambiguity is attributed to the given portfolio and not the market. Nevertheless, what is common to all these measures, independently of their origin, is that investors' ambiguity is greater during the largest market movements associated with big drawdowns and rapid recoveries (in the market or for a specific portfolio).

Figure 5 plots the predictions of the ambiguity component $\mathrm{Var}_\eta(\mathrm{E}_\mathbb{Q}[\tilde{R}^h_{\mathcal{P},t+1}])$ and the corresponding portfolio drawdowns for the USD-based investor with MV-COM and ES-COM ambiguity-robust portfolio objective functions. Observe that the largest ambiguity levels according to the measure induced by our dynamic non-Gaussian model are during the formation and recovery from the largest drawdowns, especially during the global financial crisis. Intuitively, these are the periods when the uncertainty about the mean of the returns is the largest. Our conditional model accounts for the dynamics in the volatility, so the $\mathbb{Q}$ measure in equations (10) and (14) is conditioned on past returns. Even after accounting for this, $\mathrm{Var}_\eta(\mathrm{E}_\mathbb{Q}[\tilde{R}^h_{\mathcal{P},t+1}])$ still substantially increases during the formations and the recoveries of the investor's portfolio drawdowns, and, according to our model and the derivations above, this is due to the increased ambiguity of the investor. All in all, modeling and exploiting the non-Gaussianity and ambiguity in currencies are the key drivers of the outperformance of the dynamic non-Gaussian currency hedging strategy proposed in this paper.

4. Conclusions

The main goal of this paper is to develop a flexible dynamic currency hedging strategy that accounts for the stylized facts of financial returns and allows for a linkage to investor's ambiguity. The derived hedging strategy enhances the existing approaches by modeling a set of asset and currency returns via a conditional multivariate continuous normal mean-variance mixture distribution with GARCH dynamics in the scale term. We show that the proposed non-Gaussian model can, using market data, parametrize the smooth model of ambiguity from Klibanoff *et al.* (2005), using the robust mean-variance model from Maccheroni *et al.* (2013).

Moreover, we derive a semi-parametric extended filtered historical simulation method to model the future distribution of asset and currency returns. This method combines the parametric non-Gaussian time-series model employed in our study and a non-parametric simulation of historical portfolio returns. With the use of this tractable method, we can simulate a forecasted conditional cumulative distribution of portfolio returns and numerically optimize a currency hedge with respect to an arbitrary risk measure. Utilizing this procedure, we propose an algorithm for a dynamic non-Gaussian currency hedging strategy that can optimize a general risk measure in a multi-step-ahead forecast.

In the empirical part of the paper, we demonstrate the performance of the proposed non-Gaussian currency hedging strategy. An out-of-sample backtest on historical market data of 10 developed economies over the period from June 1990 to August 2023 is performed. The empirical results reveal that the proposed dynamic non-Gaussian method yields a robust, stable, and highly risk-reductive hedging strategy. It outperforms the benchmarks of constant hedging, static hedging, as well as dynamic hedging approaches based on Gaussian-GARCH modeling in terms of risk-adjusted returns and portfolio drawdown net of transaction costs. This outperformance is driven by the asymmetries and heavy tails induced by the mixing representation in the employed non-Gaussian returns model. Moreover, we show that the investor-specific ambiguity about portfolio mean returns can be estimated from historical asset returns via the parametrization through the employed mixing random variable. Modeling and exploiting such information from market data is the key driver of the outperformance of the proposed dynamic non-Gaussian currency hedging strategy.

Our work allows for several theoretical and empirical extensions. Since currency forward contracts employed for hedging in the current setup are linear instruments, one could study hedging with currency options, especially to investigate the effect of mitigating the currency downside risk. In such a case, the setup would become non-linear, which would require an enhanced theoretical framework. Moreover, one could extend the dynamic currency hedging approach by allowing for currency re-hedging. Thereby, the optimization algorithm is run every day (or week), and an investor compares the currently hedged amount of foreign currency exposure with the newly computed optimal currency allocation taking into account the most recent market data. Then, depending on a prespecified re-hedging threshold governing the frequency of re-hedging, an investor decides whether a re-hedge of the foreign currency exposure is required. Furthermore, one could investigate the currency hedging of portfolios exposed to emerging market economies. Trading emerging market currencies exhibits considerably higher transaction costs compared to trading developed market currencies. In connection with Brodie *et al.* (2009), hedging currencies with higher transaction costs could potentially be directly modeled by including an L^1 penalty term to emerging market currencies in the optimization problem, yielding a potentially sparse currency hedge.

To summarize, an important result that emerges from our analysis is that accounting for non-Gaussianity in the econometric modeling of portfolio return dynamics significantly improves the performance of the dynamic currency hedging strategy net of transaction costs. Furthermore, we show that investor-specific ambiguity can be parametrized and estimated from historical asset returns using the non-Gaussian model based on the continuous mean-variance mixture representation. In addition to an academic interest in studying the problem of optimal currency hedging, this research topic is also practically relevant and widely discussed in the financial services industry, especially in the areas of strategic asset allocation and wealth management (see Chang 2009, Bender *et al.* 2012, de Boer 2016).

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Appendix

A.1. *Parametrization of investor's ambiguity using COMFORT model*

The proposed parametrization of investor ambiguity is not restricted to the currency hedging problem considered in this paper. Hence, in the derivations below, we consider a general portfolio optimization problem for a vector of asset returns **R** and portfolio weights **w**. The currency hedging problem can then be recovered as a special case of the general portfolio optimization problem. Furthermore, in order to simplify the explanation, in the following derivations, we drop the conditioning on past returns $\mathcal{F}_{t-1}$.

Following Maccheroni *et al.* (2013), we consider a portfolio problem with ambiguity aversion

$$\max_{\mathbf{w}} \left\{ \mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{R}(\mathbf{w})] - \frac{\lambda}{2}\mathrm{Var}_{\bar{\mathbb{Q}}}(\mathbf{R}(\mathbf{w})) - \frac{\theta}{2}\mathrm{Var}_{\eta}(\mathrm{E}_{\mathbb{Q}}[\mathbf{R}(\mathbf{w})]) \right\}. \quad \text{(A1)}$$

Here, in addition to the risk aversion term with the corresponding risk-aversion coefficient λ, we have the last term that captures the investor's aversion against the uncertainty about the true mean of the returns, and θ is the corresponding ambiguity-aversion coefficient.

In the equation above, $\bar{\mathbb{Q}}$ is the measure that integrates over all possible distributions of the returns $\mathbb{Q}$ considered by the investor, i.e. $\bar{\mathbb{Q}} = \int \mathbb{Q}\ d\eta(\mathbb{Q})$, where η is the prior belief of the investor about these distributions. Hence,

$$\mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{R}(\mathbf{w})] = \int \mathbf{R}(\mathbf{w})\,\mathrm{d}\bar{\mathbb{Q}} = \int\int \mathbf{R}(\mathbf{w})\,\mathrm{d}\mathbb{Q}\,\mathrm{d}\eta(\mathbb{Q}), \quad \text{(A2)}$$

$$\begin{aligned}\mathrm{Var}_{\bar{\mathbb{Q}}}(\mathbf{R}(\mathbf{w})) &= \int \mathbf{R}^2(\mathbf{w})\,\mathrm{d}\bar{\mathbb{Q}} - \left[\int \mathbf{R}(\mathbf{w})\,\mathrm{d}\bar{\mathbb{Q}}\right]^2 \\ &= \int\int \mathbf{R}^2(\mathbf{w})\,\mathrm{d}\mathbb{Q}\,\mathrm{d}\eta(\mathbb{Q}) - \left[\int\int \mathbf{R}(\mathbf{w})\,\mathrm{d}\mathbb{Q}\,\mathrm{d}\eta(\mathbb{Q})\right]^2,\end{aligned} \quad \text{(A3)}$$

where all integrals are evaluated over the corresponding sample spaces.

Our non-Gaussian returns model has the following distributional assumption for the returns

$$f_{\mathbf{R}(\mathbf{w})}(x) = \int_0^\infty \mathcal{N}_{\mathbf{R}(\mathbf{w})|G=g}(x \mid \boldsymbol{\mu}'\mathbf{w} + \boldsymbol{\gamma}'\mathbf{w}g, g\mathbf{w}'\mathbf{H}\mathbf{w}) f_G(g)\,\mathrm{d}g, \quad \text{(A4)}$$

where $\mathcal{N}_{\mathbf{R}(\mathbf{w})|G=g}(x \mid \boldsymbol{\mu}'\mathbf{w} + \boldsymbol{\gamma}'\mathbf{w}g, g\mathbf{w}'\mathbf{H}\mathbf{w})$ denotes the multivariate normal probability density function of **R** given $G = g$, with mean $\boldsymbol{\mu}'\mathbf{w} + \boldsymbol{\gamma}'\mathbf{w}g$, variance $g\mathbf{w}'\mathbf{H}\mathbf{w}$, and evaluated at the point x.

Based on our model assumptions $\mathrm{d}\mathbb{Q} = \mathcal{N}_{\mathbf{R}(\mathbf{w})|G}(x \mid \boldsymbol{\mu}'\mathbf{w} + \boldsymbol{\gamma}'\mathbf{w}G, G\mathbf{w}'\mathbf{H}\mathbf{w})\,\mathrm{d}x$. Hence,

$$\begin{aligned}E_{\mathbb{Q}}[\mathbf{R}(\mathbf{w})] &= \int \mathbf{R}(\mathbf{w})\,\mathrm{d}\mathbb{Q} \\ &= \int x\mathcal{N}_{\mathbf{R}(\mathbf{w})|G}(x \mid \boldsymbol{\mu}'\mathbf{w} + \boldsymbol{\gamma}'\mathbf{w}G, G\mathbf{w}'\mathbf{H}\mathbf{w})\,\mathrm{d}x \\ &= \boldsymbol{\mu}'\mathbf{w} + \boldsymbol{\gamma}'\mathbf{w}G.\end{aligned} \quad \text{(A5)}$$

Next, note that in our model $\mathrm{d}\bar{\mathbb{Q}} = f_{\mathbf{R}(\mathbf{w})}(x)\,\mathrm{d}x$. Therefore, the mean and the variance of the returns are given by

$$\begin{aligned}&E_{\bar{\mathbb{Q}}}[\mathbf{R}(\mathbf{w})] \\ &= \int \mathbf{R}(\mathbf{w})\,\mathrm{d}\bar{\mathbb{Q}} \\ &= \int x f_{\mathbf{R}(\mathbf{w})}(x)\,\mathrm{d}x \\ &= \int\int_0^\infty x\,\mathcal{N}_{\mathbf{R}(\mathbf{w})|G=g}(x \mid \boldsymbol{\mu}'\mathbf{w} + \boldsymbol{\gamma}'\mathbf{w}g, g\mathbf{w}'\mathbf{H}\mathbf{w}) f_G(g)\,\mathrm{d}g\,\mathrm{d}x \\ &= \int_0^\infty \int x\,\mathcal{N}_{\mathbf{R}(\mathbf{w})|G=g}(x \mid \boldsymbol{\mu}'\mathbf{w} + \boldsymbol{\gamma}'\mathbf{w}g, g\mathbf{w}'\mathbf{H}\mathbf{w}) dx\, f_G(g)\,\mathrm{d}g \\ &= \int_0^\infty (\boldsymbol{\mu}'\mathbf{w} + \boldsymbol{\gamma}'\mathbf{w}g) f_G(g)\,\mathrm{d}g \\ &= \boldsymbol{\mu}'\mathbf{w} + \boldsymbol{\gamma}'\mathbf{w}E_\eta[G],\end{aligned} \quad \text{(A6)}$$

$$\begin{aligned}&\mathrm{Var}_{\bar{\mathbb{Q}}}(\mathbf{R}(\mathbf{w})) \\ &= \int \mathbf{R}^2(\mathbf{w})\,\mathrm{d}\bar{\mathbb{Q}} - \left[\int \mathbf{R}(\mathbf{w})\,\mathrm{d}\bar{\mathbb{Q}}\right]^2 \\ &= \int_0^\infty \int x^2\,\mathcal{N}_{\mathbf{R}(\mathbf{w})|G=g}(x \mid \boldsymbol{\mu}'\mathbf{w} + \boldsymbol{\gamma}'\mathbf{w}g, g\mathbf{w}'\mathbf{H}\mathbf{w})\,\mathrm{d}x\, f_G(g)\,\mathrm{d}g \\ &\quad - \left[\int_0^\infty \int x\,\mathcal{N}_{\mathbf{R}(\mathbf{w})|G=g}(x \mid \boldsymbol{\mu}'\mathbf{w} + \boldsymbol{\gamma}'\mathbf{w}g, g\mathbf{w}'\mathbf{H}\mathbf{w})\,\mathrm{d}x\, f_G(g)\,\mathrm{d}g\right]^2 \\ &= E_\eta(G)\mathbf{w}'\mathbf{H}\mathbf{w} + \mathrm{Var}_\eta(G)\mathbf{w}'\boldsymbol{\gamma}\boldsymbol{\gamma}'\mathbf{w}.\end{aligned} \quad \text{(A7)}$$

By matching the moments in the equations from the original Maccheroni *et al.* (2013) model given in equations (A2) and (A3) above with the moments from our model given in equations (A6) and (A7), we obtain

$$\begin{aligned}\mathrm{d}\eta(\mathbb{Q}) &= f_G(g)\,\mathrm{d}g, \\ \mathrm{d}\mathbb{Q} &= \mathcal{N}_{\mathbf{R}(\mathbf{w})|G=g}(x \mid \boldsymbol{\mu}'\mathbf{w} + \boldsymbol{\gamma}'\mathbf{w}g, g\mathbf{w}'\mathbf{H}\mathbf{w})\,\mathrm{d}x, \\ \mathrm{d}\bar{\mathbb{Q}} &= f_{\mathbf{R}(\mathbf{w})}(x)dx.\end{aligned} \quad \text{(A8)}$$

Hence, the last term in equation (A1) that represents the ambiguity component of the investor's objective function can be, using our model, written as

$$\mathrm{Var}_\eta(E_{\mathbb{Q}}[\mathbf{R}(\mathbf{w})])$$

$$= \int \left(E_{\mathbb{Q}}[\mathbf{R}(\mathbf{w})]\right)^2 \, d\eta(\mathbb{Q}) - \left[\int E_{\mathbb{Q}}[\mathbf{R}(\mathbf{w})] \, d\eta(\mathbb{Q})\right]^2$$
$$= \int \left(E_{\mathbb{Q}}[\mathbf{R}(\mathbf{w})]\right)^2 f_G(g) \, dg - \left[\int E_{\mathbb{Q}}[\mathbf{R}(\mathbf{w})] f_G(g) \, dg\right]^2$$
$$= \int \left(\boldsymbol{\mu}'\mathbf{w} + \boldsymbol{\gamma}'\mathbf{w}g\right)^2 f_G(g) \, dg - \left[\int \boldsymbol{\mu}'\mathbf{w} + \boldsymbol{\gamma}'\mathbf{w}g f_G(g) \, dg\right]^2$$
$$= E_\eta\left[(\boldsymbol{\mu}'\mathbf{w} + \boldsymbol{\gamma}'\mathbf{w}G)^2\right] - \left[E_\eta(\boldsymbol{\mu}'\mathbf{w} + \boldsymbol{\gamma}'\mathbf{w}G)\right]^2$$
$$= \mathrm{Var}_\eta(\boldsymbol{\mu}'\mathbf{w} + \boldsymbol{\gamma}'\mathbf{w}G)$$
$$= \mathrm{Var}_\eta(G)\mathbf{w}'\boldsymbol{\gamma}\boldsymbol{\gamma}'\mathbf{w}. \tag{A9}$$

Since the moments in the two problems match, we conclude that our model parametrizes the ambiguity-averse investor from Maccheroni *et al.* (2013) as expressed in equations (A1) to (A3) above. In particular, our mixing random variable G corresponds to η—the probability measure over possible returns models representing investor's ambiguity—and allows for estimation from market data.

The resulting closed-form expressions for the moments are used in our mean-variance-based currency hedging strategy, and the distribution from equation (A4) is used in the minimum expected shortfall case together with the filtered historical simulations for the multiple-step-ahead forecast.

Under the assumption of normally distributed returns $\mathbf{R} \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{H})$, see, e.g. Ingersoll (1987, Chapter 4.7), the maximization of the expected exponential utility function is equivalent to the maximization of the mean-variance portfolio problem, i.e.

$$\max_{\mathbf{w}:\mathbf{w}'\mathbf{1}=1} u_{exp}(\mathbf{R}, \mathbf{w}) := 1 - e^{-\lambda \mathbf{R}'\mathbf{w}} \iff \max_{\mathbf{w}:\mathbf{w}'\mathbf{1}=1} \boldsymbol{\mu}'\mathbf{w} - \frac{\lambda}{2}\mathbf{w}'\mathbf{H}\mathbf{w}. \tag{A10}$$

However, it is a well-known stylized fact of financial returns observed on all the markets and assets (e.g. equities and currencies) that they are not multivariate Gaussian. In particular, they exhibit heavier than Gaussian tails, and one possible and more accurate distributional assumption is the multivariate generalized hyperbolic distribution considered in our paper.

On the other hand, the portfolio problem from equation (A1) extends the equivalence above. Maccheroni *et al.* (2013) study a mean-variance optimization problem with the aversion of the investor against uncertainty about portfolio expected returns. For that purpose, the authors consider an investor who selects a portfolio according to the functional of the double expectational form, which was first introduced in the classical paper by Klibanoff *et al.* (2005). Namely,

$$\max_{\mathbf{w}:\mathbf{w}'\mathbf{1}=1} V(\mathbf{R}, \mathbf{w}) := \int \xi\left(\int u(\mathbf{R}(\mathbf{w})) \, d\mathbb{Q}\right) d\eta(\mathbb{Q}), \tag{A11}$$

where u denotes the investor's utility function and function ξ captures the attitude of the investor towards ambiguity (e.g. $\xi(x) = x$ corresponds to ambiguity neutral investor).

Then, either by assuming exponential utility function $u_{exp}(\mathbf{R}, \mathbf{w}) := 1 - e^{-\lambda \mathbf{R}'\mathbf{w}}$ or more generally for arbitrary u and by using the quadratic approximation for the certainty equivalent, they show that the above problem is equivalent or can be approximated, respectively, by

$$\max_{\mathbf{w}:\mathbf{w}'\mathbf{1}=1} \left\{ \mathrm{E}_{\bar{\mathbb{Q}}}[\mathbf{R}(\mathbf{w})] - \frac{\lambda}{2}\mathrm{Var}_{\bar{\mathbb{Q}}}(\mathbf{R}(\mathbf{w})) - \frac{\theta}{2}\mathrm{Var}_\eta(\mathrm{E}_{\mathbb{Q}}[\mathbf{R}(\mathbf{w})]) \right\}. \tag{A12}$$

Derivations in equation (A6) to (A8) show that this general relation between investor's preferences characterized by (A11) and portfolio optimization with ambiguity aversion in the objective function can be parametrized using the continuous mean-variance mixture of normal distributions and used on real financial data. In particular, we parametrize $d\eta(\mathbb{Q}) = f_G(g) \, dg$ and measure it from the returns data.

The Gaussianity assumption implies the equivalence in equation (A10), we show that the non-Gaussianity of the returns can be used in support of the more general equivalence between equations (A11) and (A12) derived in Maccheroni *et al.* (2013). Stochastic volatility models, GARCH models, and/or non-Gaussian returns distribution all induce market incompleteness. Similarly to various explanations for market-incompleteness, as opposed to complete market models that require iid Gaussian (or binomial) returns, there are many economic explanations and corresponding models that can capture investor's ambiguity. Our parametrization, for a given utility function u and ambiguity attitude function ξ, allows us to estimate the investor's ambiguity and to construct $V(\mathbf{R}, \mathbf{w})$ from Klibanoff *et al.* (2005) with η estimated from the market returns.

A.2. Additional technical results

Here, we provide the derivation of equation (15). This derivation applies the econometric model presented in Section 2.2. Using equation (4) we see that $\mathbf{Y}_t - \boldsymbol{\mu} - \boldsymbol{\gamma} G_t = \mathbf{H}_t^{\frac{1}{2}} \sqrt{G_t}\mathbf{Z}_t$, and by equation (6) it holds that

$$\left(Y_{m,t} - \mu_m - \gamma_m G_t\right)^2 = G_t s_{m,t}^2 \tilde{Z}_{m,t}^2,$$

for $m = 1, 2, \ldots, M$, where we define $\tilde{\mathbf{Z}}_t \overset{iid}{\sim} \mathcal{N}(\mathbf{0}, \Gamma)$. Utilizing equation (7) we can express

$$\begin{aligned} s_{m,t}^2 &= \omega_m + \alpha_m G_{t-1} s_{m,t-1}^2 \tilde{Z}_{m,t-1}^2 + \beta_m s_{m,t-1}^2 \\ &= \omega_m + \alpha_m s_{m,t-1}^2 \left(G_{t-1}\tilde{Z}_{m,t-1}^2 - G_{t-1}\right) \\ &\quad + \beta_m s_{m,t-1}^2 + \alpha_m s_{m,t-1}^2 G_{t-1} \\ &= \omega_m + \alpha_m s_{m,t-1}^2 G_{t-1}\left(\tilde{Z}_{m,t-1}^2 - 1\right) \\ &\quad + \alpha_m s_{m,t-1}^2 \left(G_{t-1} - \mathrm{E}[G_{t-1}]\right) \\ &\quad + \beta_m s_{m,t-1}^2 + \alpha_m s_{m,t-1}^2 \mathrm{E}[G_{t-1}]. \end{aligned}$$

Define $\hat{s}_{m,t+h|\mathcal{F}_t}^2 := \mathrm{E}[s_{m,t+h}^2 \mid \mathcal{F}_t]$ as the optimal prediction in the L^2 sense of $s_{m,t+h|\mathcal{F}_t}^2$. Since $\mathrm{E}[\tilde{Z}_{m,t+h}^2 - 1 \mid \mathcal{F}_t] = 0$ and $\mathrm{E}[G_{t+h} - \mathrm{E}[G_{t+h}] \mid \mathcal{F}_t] = 0$ for all $h > 0$, then

$$\hat{s}_{m,t+h|\mathcal{F}_t}^2 = \begin{cases} \hat{\omega}_m + \hat{\alpha}_m\left(Y_{m,t} - \hat{\mu}_m - \hat{\gamma}_m \mathrm{E}[G_t \mid \mathcal{F}_t]\right)^2 & \\ \quad +\hat{\beta}_m \hat{s}_{m,t}^2, & \text{for } h = 1, \\ \hat{\omega}_m + \hat{s}_{m,t+h-1|\mathcal{F}_t}^2 & \\ \quad \left(\hat{\alpha}_m \mathrm{E}[G_{t+h-1} \mid \mathcal{F}_t] + \hat{\beta}_m\right), & \text{for } h > 1, \end{cases}$$

which concludes the derivation of equation (15).

A.3. Additional empirical results

Figure A1 shows a heat plot of country pairwise unconditional (i.e. full sample) correlations between equity, government bond, and currency exchange rate returns, considering the US dollar as the base currency. Different asset classes can be recognized by larger pairwise correlations. Moreover, the unconditional correlations between equities and bonds are stable across different countries and generally slightly negative. Also, the correlations between bonds and currencies are stable across different countries and take values around zero–this is the rationale for the optimality of full currency hedging in bond-only portfolios. The largest variation in pairwise correlations among different countries is exhibited among equities and currencies. Note that such correlations change over time, whereas here, only the sample unconditional correlation is displayed. This motivates the use of the multivariate non-Gaussian returns model that captures some of the dynamics in the dependency between assets and currencies over time.

Table A1 presents the results of the out-of-sample backtest for the remaining base currencies not covered in table 2. The performance

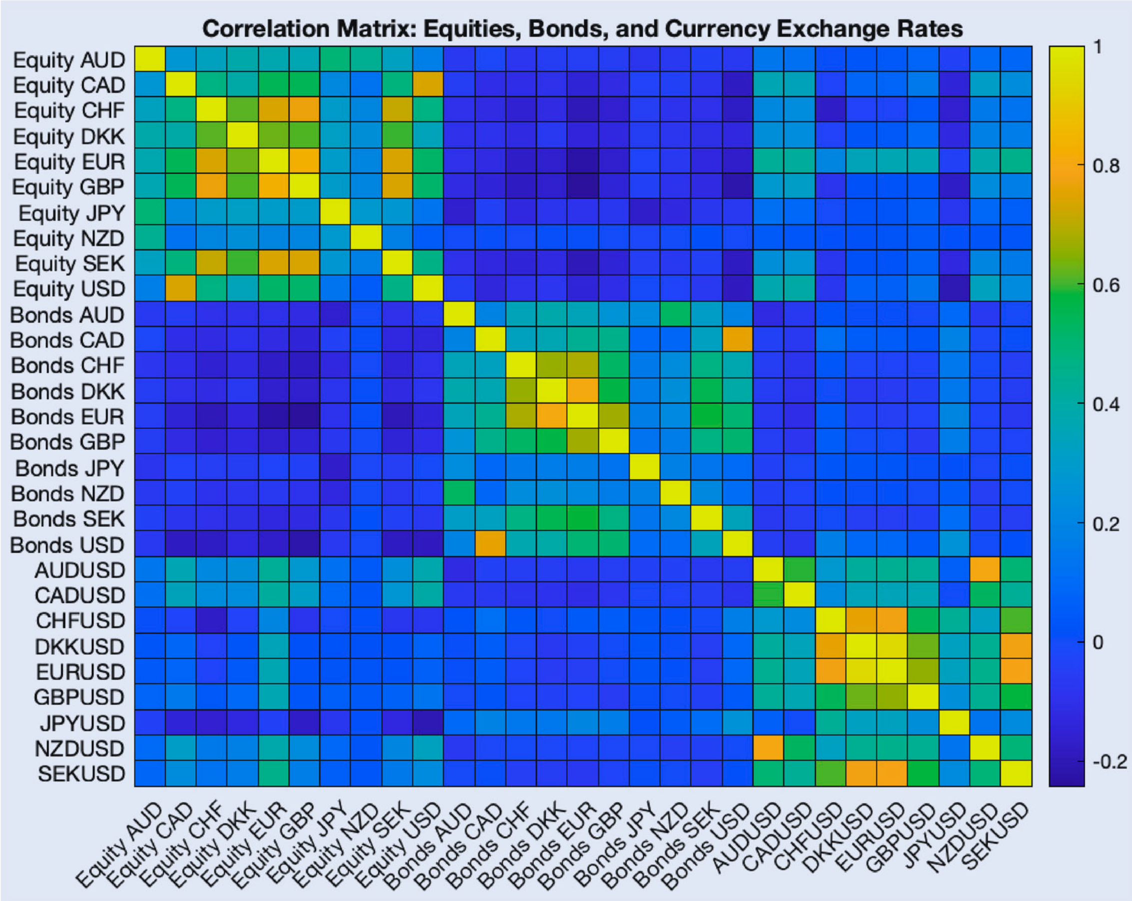

Figure A1. A heat map of the unconditional pairwise correlations between equity, bond, and currency exchange rate returns (with the USD base currency) for the 10 analyzed economies is plotted here. Notice that the correlations within each asset class are generally larger in comparison to the cross-correlations between assets from different asset classes. One can also observe larger correlations between countries located geographically close to one another, reflecting a larger integration of such markets.

of SS, MN, and COMFORT-based hedging, optimizing MV and ES risk measures, is assessed against zero and full hedging benchmarks. All presented results are given net of transaction costs.

In essence, the findings in table A1 align with those in table 2. Model-based hedging strategies consistently succeed in reducing portfolio risk, as evidenced by lower realized volatility, expected shortfall, and maximum drawdown compared to constant hedging (zero and full). Notably, the dynamic non-Gaussian model, whether optimizing for MV or ES, consistently achieves the most substantial risk reduction across all analyzed base currencies.

Moreover, model-induced hedging consistently outperforms the constant hedging benchmarks in terms of risk-adjusted returns, measured by the Sharpe and Sortino ratios. Dynamic approaches, both Gaussian and non-Gaussian, consistently outperform their static counterparts, including the strategy based on covariance matrix shrinkage from Ledoit and Wolf (2020) and mean shrinkage from Jorion (1986). Similarly to the risk reduction case, the largest risk-adjusted returns are consistently attained by the dynamic non-Gaussian returns model.

The out-of-sample backtest illustrates the superiority of the non-Gaussian currency hedging strategy over its static shrunk and dynamic Gaussian counterparts, even after accounting for transaction costs. Across various base currencies, the proposed non-Gaussian returns model, in both the MV-COM and ES-COM cases, attains the highest realized Sharpe and Sortino ratios while simultaneously minimizing realized volatility, expected shortfall, and maximum drawdown.

Figure A2 depicts the optimal exposure in EUR for a USD-based investor for SS, MN, and COMFORT models. In particular, subplot (a) presents the optimal MV exposure, and subplot (b) the optimal ES exposure. The optimal exposure is, as defined in Section 2.1, expressed in relative terms compared to the total portfolio value and, as described above, constraint to lie on the interval $[-2w_{c,0}, 3w_{c,0}]$ for each foreign currency c. Even though the shrinkage of mean and variance is applied in the static model, the dynamic models, especially the non-Gaussian model, generally produce more stable exposures over time. The asymmetries captured in the COMFORT model play a large role in the computed optimal currency exposure. There, the changing market conditions and ambiguity are captured through mixing in the non-Gaussian model, yielding more distinct and stable optimized currency exposures compared to the Gaussian models. Note that the analyzed portfolio consists of 10 different currency exposures, whereas, for the sake of clarity and interpretability, only a single one is plotted here.

In figure A3, we study the effect of the Swiss franc unpeg and the Brexit referendum on portfolio performance. We make use of the portfolio return decomposition as given in equation (3), where hedged portfolio return is presented as a sum of two components: (i) fully hedged asset return and (ii) net currency exposure return. We observe the effect of the proposed dynamic currency hedging strategy on the currency component of the hedged portfolio return. Subplot (a) presents the decomposed cumulative return plot around the period of the Swiss franc unpeg for a CHF-based investor. Since other currencies depreciated against the CHF, one can observe a large drop in the currency component (i.e.summed over all foreign currencies present in the portfolio) for the no-hedging benchmark. Full hedging straightforwardly represents a currency component that is through the whole backtest close to zero. Minuscule deviations are present because of changing interest rates and the fact that 'perfect' full hedging is impossible in practice. More importantly,

Table A1. Observe the realized annualized portfolio return volatility, mean, Sharpe and Sortino ratios, expected shortfall (ES) (for the confidence level of $\zeta = 0.95$), maximum drawdown, and turnover of different model-based and constant hedging strategies obtained in the out-of-sample backtest.

Hedging strategy out-of-sample backtest results II							
	No Hedge	Full Hedge	MV-SS	MV-MN	ES-MN	MV-COM	ES-COM
Base: AUD							
Volatility	9.48%	8.07%	8.33%	8.19%	7.34%	7.18%	**7.17%**
Mean	8.05%	8.53%	9.19%	**9.26%**	7.75%	8.57%	8.56%
Sharpe Ratio	0.85	1.06	1.10	1.13	1.06	**1.19**	**1.19**
Sortino Ratio	1.22	1.46	1.55	1.59	1.51	**1.71**	**1.71**
Exp Shortfall	1.34%	1.25%	1.25%	1.22%	1.07%	1.05%	**1.04%**
Max Drawdown	27.04%	36.66%	32.68%	33.67%	27.97%	**24.84%**	26.05%
Turnover	0	0.90	1.81	1.80	1.20	1.32	1.38
Base: CAD							
Volatility	9.36%	8.06%	8.48%	8.30%	7.27%	**7.05%**	7.09%
Mean	8.01%	7.33%	7.85%	**8.31%**	7.26%	7.21%	7.20%
Sharpe Ratio	0.86	0.91	0.93	1.00	1.00	**1.02**	**1.02**
Sortino Ratio	1.23	1.26	1.30	1.41	1.42	**1.45**	**1.45**
Exp Shortfall	1.31%	1.25%	1.27%	1.23%	1.07%	**1.03%**	1.04%
Max Drawdown	34.45%	38.44%	34.95%	35.51%	28.04%	**27.62%**	28.81%
Turnover	0	0.90	1.77	1.74	1.25	1.31	1.31
Base: DKK							
Volatility	9.92%	8.05%	8.45%	8.34%	7.35%	**7.11%**	7.14%
Mean	7.88%	7.14%	7.68%	**8.01%**	6.23%	6.90%	6.81%
Sharpe Ratio	0.79	0.89	0.91	0.96	0.85	**0.97**	0.95
Sortino Ratio	1.10	1.23	1.27	1.34	1.19	**1.38**	1.34
Exp Shortfall	1.51%	1.24%	1.27%	1.24%	1.09%	**1.05%**	**1.05%**
Max Drawdown	40.78%	37.10%	34.27%	35.03%	**26.44%**	26.82%	26.74%
Turnover	0	0.90	1.87	1.85	1.35	1.39	1.44
Base: NZD							
Volatility	9.92%	8.06%	8.39%	8.34%	7.27%	**7.04%**	7.07%
Mean	7.22%	9.16%	**9.90%**	9.86%	8.29%	8.97%	8.88%
Sharpe Ratio	0.73	1.14	1.18	1.18	1.14	**1.27**	1.26
Sortino Ratio	1.05	1.58	1.67	1.67	1.62	**1.83**	1.81
Exp Shortfall	1.38%	1.24%	1.25%	1.24%	1.07%	**1.03%**	**1.03%**
Max Drawdown	37.17%	34.91%	31.29%	34.04%	27.20%	23.28%	**22.89%**
Turnover	0	0.90	1.90	1.87	1.34	1.42	1.44
Base: SEK							
Volatility	9.59%	8.06%	8.46%	8.43%	7.34%	**7.19%**	7.33%
Mean	**9.42%**	7.31%	7.96%	8.36%	7.19%	7.60%	7.35%
Sharpe Ratio	0.98	0.91	0.94	0.99	0.98	**1.06**	1.00
Sortino Ratio	1.43	1.25	1.32	1.39	1.38	**1.50**	1.42
Exp Shortfall	1.34%	1.25%	1.27%	1.25%	1.09%	**1.06%**	1.08%
Max Drawdown	29.73%	37.77%	32.65%	34.75%	30.21%	**27.94%**	31.30%
Turnover	0	0.90	1.74	1.72	1.25	1.30	1.35

Notes: We present the results for the quarterly rebalanced portfolio comprised of 70% equity, 20% bonds, and 10% cash, equally weighted between 10 different economies. We analyze a static shrunk (SS) hedging model based on non-linear covariance shrinkage from Ledoit and Wolf (2020) and two dynamic hedging models: (i) multivariate normal (MN) model based on the CCC-GARCH time-series structure and (ii) the proposed non-Gaussian COMFORT (COM) time-series model. Currency hedging based on optimizing the robust mean-variance (MV) and expected shortfall (ES) risk measures is investigated and compared to the constant hedging benchmarks of zero and full hedging. The results for different choices of the base currency are presented net of transaction (i.e. hedging) costs, which are assumed to be five basis points per notional of each entered currency forward contract.

observe the large positive outperformance of the currency component of the portfolio managed using the proposed non-Gaussian currency hedging strategy for both MV- and ES-based optimization. A similar pattern can also be recognized in subplot (b). There, a net currency exposure in GBP (i.e. in isolation from other currency exposures) is plotted for a USD-based investor. One can study the effect of the Brexit referendum on the unfavorable performance of the GBP currency component in the portfolio without currency hedging. On the other hand, the non-Gaussian currency hedging strategy again prevents the portfolio drawdown and at the same time also provides an outperformance in comparison to full hedging.

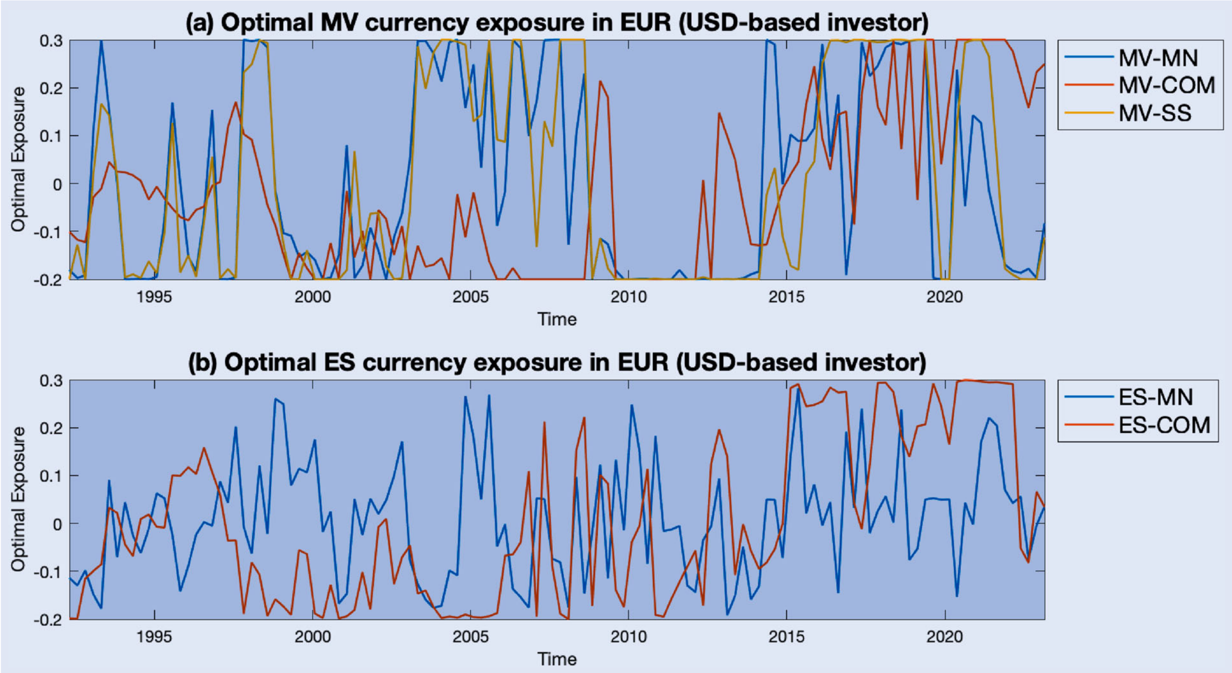

Figure A2. Subplot (a) depicts the optimal robust mean-variance (MV) exposure in EUR for a USD-based investor as computed by the static shrunk (SS) model and two dynamic models – multivariate normal (MN) and non-Gaussian (COMFORT). Subplot (b) shows optimal expected shortfall (ES) exposure in EUR for a USD-based investor as computed by dynamic multivariate normal and non-Gaussian models. Optimal exposure is expressed in relative terms compared to the total portfolio value.

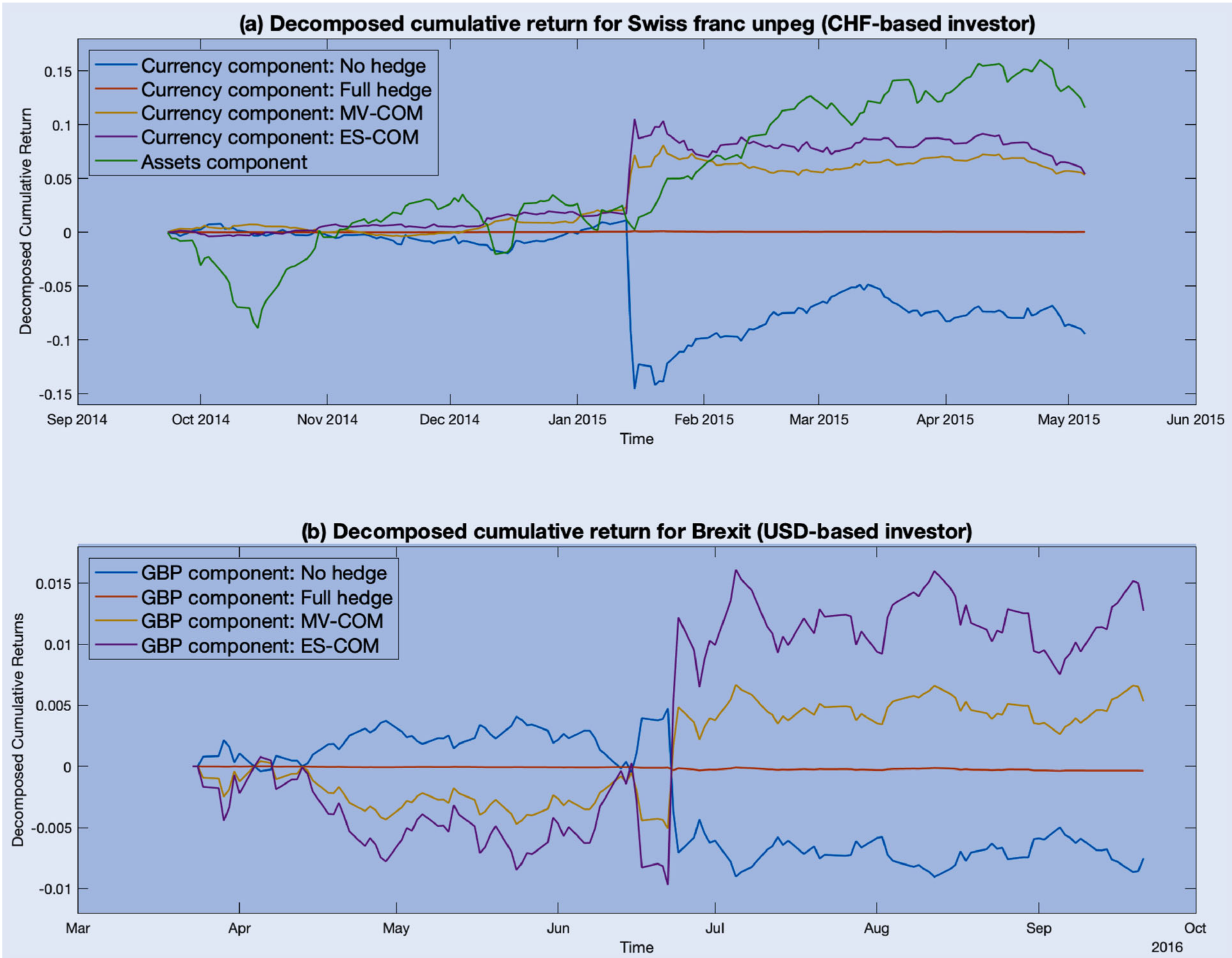

Figure A3. Subplot (a) illustrates the cumulative performance decomposition of fully hedged assets and net currency exposures for different currency hedging strategies and a CHF-based investor around the Swiss franc unpeg. Subplot (b) depicts the cumulative performance of the GBP net currency exposure for different currency hedging strategies and a USD-based investor around the Brexit referendum.