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Estimation risk and the implicit value of index-tracking

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We study [Roll, R., A mean/variance analysis of tracking error. *J. Portfolio Manage.*, 1992, **18**, 13–22.] conjecture that there exists an implicit value in index-tracking (IVIT) relative to forming mean-variance (MV) optimal portfolios under estimation error. We derive an analytical definition for the opportunity cost facing the MV investor who does not index-track. Our findings indicate that the opportunity cost is positive and statistically significant. The existence of an IVIT (positive opportunity cost) is strongly associated with a reduction in the portfolio's induced estimation risk under index-tracking relative to an MV-efficient portfolio of equal target mean return. Under high estimation error cases, increased IVIT translates to higher risk-adjusted returns, lower volatility, higher Sharpe-ratio, lower turnover, and larger certainty equivalent returns. Empirically, a one standard deviation increase in IVIT translates to an annual increase of 4%–5% in the out-of-sample Sharpe-ratio and a 6%–15% decrease in the monthly turnover.

Keywords: Portfolio theory; Mean-variance analysis; Shrinkage; Tracking error

JEL Code: C13, C44, C46, G11

1. Introduction

Modern portfolio theory (Markowitz 1952) prescribes that a risk-averse investor seeks an optimal allocation of wealth among portfolio assets according to an acceptable mean-variance (MV) tradeoff, known as the MV-efficient portfolio (henceforth MVEP). However, the MV model is well-known for its sensitivity to estimation error in the asset return parameters (Frankfurter *et al.* 1971, Klein and Bawa 1976, Jorion 1986). Recent debate in the portfolio literature focuses on whether investors should care for MV efficiency or simply have an equal allocation of wealth among portfolio assets (DeMiguel *et al.* 2009b), which is the so-called naïve strategy that does not depend on parameter estimation. The underlying rationale is that investors could be better-off in terms of ex-post performance using the naïve strategy, despite the portfolio being MV sub-optimal ex-ante.

Another intriguing strategy is the index-tracking portfolio, which tracks a specific benchmark (e.g. S&P 500). In terms of total assets under management, we have witnessed a steady growth of such strategies over the last two decades (Cremers *et al.* 2016). Nonetheless, such portfolios are also deemed MV sub-optimal ex-ante (Roll 1992, Edirisinghe 2013). This wide acceptance leads to the open question as to what offsetting value there might be in index-tracking despite its MV sub-optimality? It is in this context that Roll (1992) conjectures that there might exist an 'implicit value' of index-tracking in reducing the impact of estimation error in portfolio selection:

> "There remains, however, one other possible recommendation for the [index-tracking] strategy: Estimation error is severe in portfolio analysis. No one knows where the global total return efficient frontier is really located. Its position depends, inter alia, on individual asset expected returns, which can be estimated only with substantial error because of the large component of noise in observed returns. . . .[index-tracking] policy may induce portfolio managers to place less emphasis on estimates of individual expected returns. . . "

In this paper, we revisit Roll's (1992) above conjecture and investigate the out-of-sample performance of index-tracking portfolios relative to MV-optimal portfolios under estimation risk. In particular, we pursue the following question: If tracking error portfolios are associated with lower estimation risk, are investors better off with index-tracking versus MV-optimal portfolios? To quantify the ex-post portfolio performance, we employ a utility-based out-of-sample metric (Kan and Zhou 2007). We propose a metric for the value of index tracking relative to that of a corresponding MVEP, and refer to it as the Implicit Value of Index Tracking (IVIT).

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The proposed IVIT metric denotes the opportunity cost facing an MV-investor who minimizes the portfolio volatility rather than the tracking error. Hence, a positive IVIT indicates that the decision-maker is better off tracking a benchmark index.

In the absence of estimation error (i.e. the case of full information) an MV investor is always better off in terms of utility-based performance relative to an index-tracking investor. This means the implicit value of index-tracking, IVIT, is negative under full information. However, in practice, the decision-making process is based on historical data and subjected to estimation error. Using historical sample estimates under the assumption of iid normally-distributed returns, our analytical results identify contributing factors and sufficient conditions that guarantee the existence of a positive IVIT. In particular, we show that the value of index-tracking improves with the investor's risk tolerance and the number of assets, but it diminishes with market volatility and sample size. Our empirical tests provide significant evidence of the implicit value, most notably in cases associated with high estimation error.

The issues of estimation error in MV portfolios and their distributional properties have received considerable scrutiny (Okhrin and Schmid 2006, Kan and Zhou 2007, Siegel and Woodgate 2007, Kan and Smith 2008, Bodnar and Schmid 2009, Simaan *et al.* 2018). On the other hand, there is less research on the distributional properties of the tracking error portfolios.† For instance, Woodgate and Siegel (2015) consider tracking-optimal portfolio selection under estimation error and study the statistical bias in the resulting optimal allocations. They propose an unbiased estimator of the tracking portfolio weights, but without comparison with MV portfolios nor its implications on ex-post portfolio performance benefits. Our work attempts to fill this gap by considering the improvement in utility-based out-of-sample portfolio performance under estimation error. To the best of our knowledge, this is the first study to provide analytical results and empirical evidence for the implicit value in index-tracking and the aforementioned Roll's (1992) conjecture.

The sensitivity of MV optimal portfolios to small changes in the parameter values (say, due to estimation) is well-established (Michaud 1989, Best and Grauer 1991, Broadie 1993, Kolm *et al.* 2014). It has also been shown that portfolio sensitivity to estimation error in the mean vector is more significant and also more difficult to mitigate as compared to error in the second moments (Merton 1980, Chopra and Ziemba 1993). One practical remedy is to shift portfolio strategies away from the asset means and to rely only on estimates of the covariance matrix, such as the case with the global minimum-variance portfolio (henceforth GMVP).‡ Other remedies to counter estimation errors include imposing some structure on the covariance matrix, such as economic models (Black and Litterman 1992), factor models (e.g.the CAPM), or allowing for multiple priors on mean return estimates and the return-generating model itself (Garlappi *et al.* 2007). Moreover, various robust approaches have been proposed to improve the accuracy of second moments (Kan and Zhou 2007, DeMiguel and Nogales 2009), as well as covariance matrix shrinking methods (Ledoit and Wolf 2003, Jagannathan and Ma 2003, Disatnik and Benninga 2007, Ledoit and Wolf 2017).

The focus of this paper is not to eliminate the asset mean vector from portfolio construction. On the contrary, we utilize the complete first and second-order moment information of asset and benchmark index returns. We show that a 'mean-enhanced tracking efficient portfolio' (henceforth MTEP) mitigates the committed estimation risk in the portfolio and improves ex-post portfolio performance in comparison to an MVEP, constructed under the same conditions. That is, benchmark-orientation leads to reduction in the estimation risk and improved portfolio performance in the real-world, while estimation errors are known to handicap MV portfolios.

Empirical validation of our analytical conclusions is performed using Fama-French industry portfolios along with a large sample of stocks listed under the S&P 500 index. The sample is monthly-data and it dates between January 1970 and December 2015. The first empirical test is conducted using a Monte Carlo (MC) simulation to investigate the existence of the IVIT and its significance. Our results indicate that under estimation error, the IVIT can be positive and significant and is increasing in the number of assets relative to the sample size of data. A larger number of assets implies greater uncertainty in allocating wealth across the assets. A larger sample size implies more information regarding the underlying asset returns, hence, less uncertainty on return distributions. Therefore, the case of a larger number of assets with a smaller sample size is punctuated with greater estimation error, resulting in stronger IVIT. We also find that IVIT is negatively correlated with market volatility. This view is consistent with the common wisdom that an investor is better-off tracking a market index during economic expansions (since such markets have lower volatility). Additionally, our analytical findings reveal that the implicit value is evident when the investor targets mean return that exceeds a certain threshold, while holding everything else fixed.

The second empirical test conducts a back-testing procedure dating between September 1986 and December 2015. Given an initial sample, we construct MVEP and MTEP portfolios on a rolling basis and test their performance out-of-sample using common economic metrics used in the literature. While this test can be viewed as independent of the paper's theoretical propositions, the findings follow suit with analytical and Monte Carlo results. In particular, we find that in cases in which the IVIT is evident, the IVIT implies an improved risk-adjusted return, lower volatility, higher Sharpe ratio, and larger certainty equivalent return. Moreover, we show that the implicit value does not originate from excess portfolio turnover, and in fact, the MTEP is associated with a significantly-lower turnover than the MVEP.

Finally, to reconcile between the theoretical findings and the back-testing results, we conduct a cross-sectional regression that prices the IVIT in terms of mean return, volatility, Sharpe ratio, certainty equivalent return, and portfolio

† A number of approaches have been proposed in the literature to mitigate the ex-ante MV sub-optimality of the MTEP (see e.g. Jorion 2003, Alexander and Baptista 2010, Alexander *et al.* 2017, Rossbach and Karlow 2019). However, these approaches do not pursue the estimation risk investigation undertaken in this paper.
‡ Haugen and Baker 1991 compare the GMVP with capitalization-weighted and other risk-parity strategies to claim that GMVP has superior out-of-sample performance.

turnover. In particular, we regress the differential in each economic metric on the proposed theoretical measure, IVIT. We conclude that a one standard deviation increase in the IVIT measure translates to an annual increase of 4%–5% in the out-of-sample Sharpe-ratio and a 6%–15% decrease in the monthly portfolio turnover.

A possible, yet intuitive, justification of our findings is the fact that adding a constraint on the portfolio selection problem in terms of tracking error serves as a shrinkage approach. On one hand, the MTEP denotes a generalized case of the Bayes-Stein portfolio (Jorion 1986). In this case, the MTEP is an MVEP when the mean vector is shrunk with respect to the capital asset pricing model (CAPM) (Sharpe 1964, Lintner 1965). Put differently, if the decision-maker is Bayesian and holds a belief that the CAPM is true, then the tracking-error portfolio is a mean-variance portfolio that captures the investor's belief in the asset pricing model (Pástor 2000). The other possible justification is consistent with the argument that 'imposing the wrong constraints helps' by Jagannathan and Ma (2003). Consistent with this argument, we discern that solving for MTEP is consistent with solving for MVEP when the covariance matrix is shrunk using the CAPM Ledoit and Wolf (2003).

The rest of the paper is organized as follows. We start with the case of full information in section 2 and obtain the MV and index-tracking optimal portfolios. These results are then examined under parameter estimation, and their statistical properties are derived, in section 3, which summarizes the main analytical findings of the paper. Section 4 is dedicated to the empirical validation of the analytical results. In section 5, we discuss the implications of index-tracking in terms of shrinkage as well as potential extensions of our analysis. Finally, section 6 presents the concluding remarks. All mathematical proofs and additional related results are delegated to the appendix/supplement.

2. The portfolio selection problem

In this section, we discuss the portfolio problem facing the decision-maker. The discussion is formulated under the case of full information and serves as the baseline analysis of our paper.

2.1. Assumptions

Consider a risk averse agent (investor) facing a set of risky assets $\mathcal{D}$ over a fixed investment period, where $d = |\mathcal{D}|$ denotes the number of assets. The asset return vector for the period is $R \in \mathbb{R}^d$, which follows an iid multivariate normal distribution, i.e. $R \sim \mathcal{N}(\mu, \Sigma)$. The vector $\mu \in \mathbb{R}^d$ denotes mean returns and $\Sigma \in \mathbb{R}^{d\times d}$ is a positive-definite (p.d.) covariance matrix of the asset returns. The agent is evaluating two funds. The first fund depends on the information set $\mathcal{I}_A \equiv \{\mu, \Sigma\}$, whereas the second depends on an expanded information set $\mathcal{I}_B \equiv \mathcal{I}_A \cup \{\beta, \sigma_b^2\}$. Specifically, $\beta \in \mathbb{R}^d$ and σ_b denote the vector of asset betas and the index return volatility, respectively. Moreover, the return on the index denoted by $\tilde{r}_b$ also follows a normal distribution. This section considers the baseline case in which the agent has access to the full information set $\mathcal{I}_B$ (no estimation error). The analysis in section 3 relaxes this assumption and constitutes our main analytical findings. Finally, x_k^i denotes the portfolio weight that the agent allocates to asset k $(= 1, .., d)$ in fund i. Either portfolio satisfies the budget constraint: $\sum_{k=1}^d x_k^i = \mathbf{1}^\top x^i = 1$, where $\mathbf{1} \in \mathbb{R}^d$ is the vector of ones.

2.2. Fund A: mean-variance portfolio

The first fund, indexed by A, corresponds to the MVEP:

$$x^A(\kappa_A) = \alpha_0 + \kappa_A^{-1}\alpha_1, \tag{1}$$

where

$$\alpha_0 = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^\top\Sigma^{-1}\mathbf{1}}, \quad \alpha_1 = \mathbf{B}\mu, \quad \text{and } \mathbf{B} = \Sigma^{-1}\left[\mathbf{I} - \mathbf{1}\alpha_0^\top\right], \tag{2}$$

with $\kappa_A(> 0)$ denoting a free parameter. This portfolio corresponds to the solution of the following optimization problem (see, e.g. Bodnar and Schmid 2009, Simaan *et al.* 2018):

$$\max_x \left\{\mu^\top x - \frac{\kappa_A}{2}x^\top\Sigma x\right\} \quad \text{subject to } \mathbf{1}^\top x = 1. \tag{3}$$

Note that lower values of κ_A correspond to higher risk associated with the fund. The agent determines the free parameter κ_A to satisfy a specific mean target.

A number of comments are in order. First, we note that $\mathbf{B}^\top\Sigma\mathbf{B} = \mathbf{B}$ holds true. Since Σ is p.d., then $\mathbf{B}$ is p.d. too. Second, portfolio A allocates between the global minimum variance portfolio (GMVP), $\alpha_0 \in \mathbb{R}^d$, and an arbitrage portfolio, $\alpha_1 \in \mathbb{R}^d$, whose weights sum to zero. Third, the MVEP's mean and variance are given, respectively, by:

$$\mu_A(\kappa_A) = \eta_0 + \frac{1}{\kappa_A}\eta_1 \quad \text{and} \quad \sigma_A^2(\kappa_A) = \sigma_0^2 + \frac{1}{(\kappa_A)^2}\sigma_1^2, \tag{4}$$

where $\eta_0 := \mu^\top\alpha_0$, $\sigma_0^2 := \alpha_0^\top\Sigma\alpha_0$, $\eta_1 := \mu^\top\alpha_1$, and $\sigma_1^2 := \alpha_1^\top\Sigma\alpha_1$. Additionally, we note that η_0 and σ_0^2, respectively, are the mean and variance of the GMVP, whereas $\eta_1 > 0$.† Fourth, under extreme risk aversion, i.e. $\kappa_A \to \infty$, asset means are irrelevant and the MVEP $\to$ GMVP. Finally, it follows that the MV efficient frontier (henceforth MVEF) of fund A is traced for $\kappa_A \in (0, \infty)$ by the equation:

$$\mu_A = \eta_0 + \sigma_1\sqrt{\sigma_A^2 - \sigma_0^2}. \tag{5}$$

Equation (5) is the typical MVEF that represents the MV mean return, μ_A, as a function of a given level of risk σ_A.

2.3. Fund B: index-tracking portfolio

The second fund corresponds to a benchmark index-tracking portfolio. This utilizes the information set $\mathcal{I}_B$ and minimizes

† By definition, it follows that $\eta_1 = \sigma_1^2 > 0$ since $\mathbf{B}$ is positive definite.

the variance of the 'tracking error' between the portfolio return and the index return, whilst assuring a desired portfolio mean target. The mean-enhanced index-tracking problem is given by

$$\max_x \left\{\mu^\top x - \frac{\kappa_B}{2}\mathrm{Var}[R^\top x - \tilde{r}_b]\right\} \quad \text{subject to } \mathbf{1}^\top x = 1, \quad (6)$$

where the portfolio mean is traded-off with the variance of the tracking error, $R^\top x - \tilde{r}_b$, for some free parameter $\kappa_B > 0$. A sufficiently-small κ_B yields mean enhancement to the portfolio at the expense of the tracking error. Hence, the solution of (6) is referred to as 'mean-enhanced tracking-efficient portfolio' (MTEP).† Note that κ_B can be chosen such that the fund B portfolio mean coincides with that of portfolio A. Noting

$$\mathrm{Var}\left[R^\top x - \tilde{r}_b\right] = x^\top \Sigma x + \sigma_b^2 - 2\sigma_b^2 \beta^\top x, \quad (7)$$

it follows that the MTEP is the solution of

$$\max_x \left\{\mu^\top x - \frac{\kappa_B}{2}\left(x^\top \Sigma x - 2\sigma_b^2 \beta^\top x\right)\right\} \quad \text{subject to } \mathbf{1}^\top x = 1, \quad (8)$$

which is given by:

$$x^B(\kappa_B) = \alpha_0 + \frac{1}{\kappa_B}\alpha_1 + \alpha_2, \quad (9)$$

where

$$\alpha_2 = \mathbf{Bc}, \quad \mathbf{c} = \sigma_b^2 \beta, \quad (10)$$

with α_0, α_1, and $\mathbf{B}$ given in (2). Note that $\mathbf{c} \in \mathbb{R}^d$ has its i^{th} component $c_i = \mathrm{Cov}(R_i, \tilde{r}_b)$, the covariance between the returns of asset i and the index. A comparison between (1) and (9) reveals that the MVEP and MTEP differ primarily in the tracking error arbitrage portfolio α_2 whose components sum to zero.

The mean and variance of the MTEP, respectively, are given by:

$$\mu_B(\kappa_B) = \eta_0 + \frac{1}{\kappa_B}\eta_1 + \eta_2 \quad \text{and}$$
$$\sigma_B^2(\kappa_B) = \sigma_0^2 + \frac{1}{(\kappa_B)^2}\sigma_1^2 + \sigma_2^2 + \frac{2}{\kappa_B}\eta_2, \quad (11)$$

where $\sigma_2^2 = \alpha_2^\top \Sigma \alpha_2 = (\mathbf{Bc})^\top \Sigma (\mathbf{Bc}) = \mathbf{c}^\top \mathbf{Bc}$ is the volatility of the tracking error vector, and its expected return is $\eta_2 = \mu^\top \alpha_2 = \mu^\top \mathbf{Bc} = \alpha_1^\top \Sigma \alpha_2$. In fact, η_2 represents two components. On one hand, it represents the mean return on α_2. On the other hand, it denotes the return covariance between α_1 and α_2.

Although fund B is an index-tracking portfolio, its MV-based efficient frontier (MVEF) can be obtained similar to that in (5), as given by:

$$\mu_B = \eta_0 + \sigma_1 \sqrt{\left(\sigma_B^2 - \sigma_0^2\right) - \left(\sigma_2^2 - \frac{\eta_2^2}{\sigma_1^2}\right)}. \quad (12)$$

† The MVEP can be viewed as the one that maximizes the Sharpe-ratio, whereas the MTEP can be viewed as the fund that maximizes the information ratio instead of the Sharpe ratio. Hence, the latter denotes a portfolio that maximizes a *relative* rather than *absolute* performance.

Lemma 2.1 *If Σ is positive definite, then $\phi := \sigma_2^2 - \eta_2^2/\sigma_1^2 > 0$ holds. Furthermore, at a fixed target portfolio mean for MVEP and MTEP, their portfolio variances differ by $\sigma_B^2 - \sigma_A^2 = \phi > 0$, which is a constant indepedent of the target mean.*

Lemma 2.1 illustrates the level of sub-optimality of the tracking-efficient MTEP from an MV perspective. The MVEF of fund B described in (12) lies strictly below that of fund A given in (5), where $\sigma_B = \sqrt{\sigma_A^2 + \phi} > \sigma_A$ for the same mean target.‡ In other words, under full information and at any fixed portfolio target mean, fund A leads to a better mean-variance trade-off than fund B. This is consistent with the original work by Roll (1992).

2.4. The main portfolio problem

By introducing funds A and B, we formulate the the agent's main portfolio problem. First, both funds are chosen to yield the same mean return ex-ante. Second, the agent's final portfolio choice is a convex combination between the two funds. The agent's objective, therefore, is choosing the control variable $\varepsilon \in \{0, 1\}$ to determine her decision rule:

$$x_m(\varepsilon) = (1-\varepsilon)x^A(\kappa_m) + \varepsilon x^B\left(\frac{\kappa_m}{1 - l\kappa_m}\right). \quad (13)$$

The parameters κ_m and l are determined ex-ante such that both funds yield the same mean return for a predetermined mean target m. In particular, given η_0 and η_1, the value of κ_m is determined as

$$\frac{1}{\kappa_m} = \frac{m - \eta_0}{\eta_1}. \quad (14)$$

On the other hand, l is the constant that satisfies the condition that the mean return of fund B is equal to fund A, i.e.

$$l = \frac{\eta_2}{\eta_1}. \quad (15)$$

Denoting $[\eta_2]^+ := \max\{\eta_2, 0\}$, the non-negativity conditions in (14) imply that the target mean must be chosen such that:

$$m > m_{\min} := \eta_0 + [\eta_2]^+. \quad (16)$$

The portfolio choice from equation (13) can be rewritten as

$$x_m(\varepsilon) = \alpha_0 + \left[\frac{1}{\kappa_m} - \varepsilon l\right]\alpha_1 + \varepsilon\alpha_2. \quad (17)$$

We note that the portfolio selection problem in equation (17) is motivated by the two funds analysis studied by Kan and Zhou (2007), Tu and Zhou (2011), and Kan *et al.* (2021). Ex-ante, the decision maker determines the values of κ_m and l so

‡ It can also be shown that the MTEP's efficient frontier increases at a faster rate than that of the MVEP's as portfolio variance increases, i.e. $\frac{d\mu_A}{d\sigma_A}|_{\sigma_A=\bar{\sigma}} \le \frac{d\mu_B}{d\sigma_B}|_{\sigma_B=\bar{\sigma}}$, for all $\bar{\sigma}$. Moreover, as risk tolerance decreases, the two efficient frontiers converge with $\mu_B \to \mu_A$, as $\kappa_A, \kappa_B \to 0$. See Edirisinghe 2013 for an alternative exposition of these ex-ante properties.

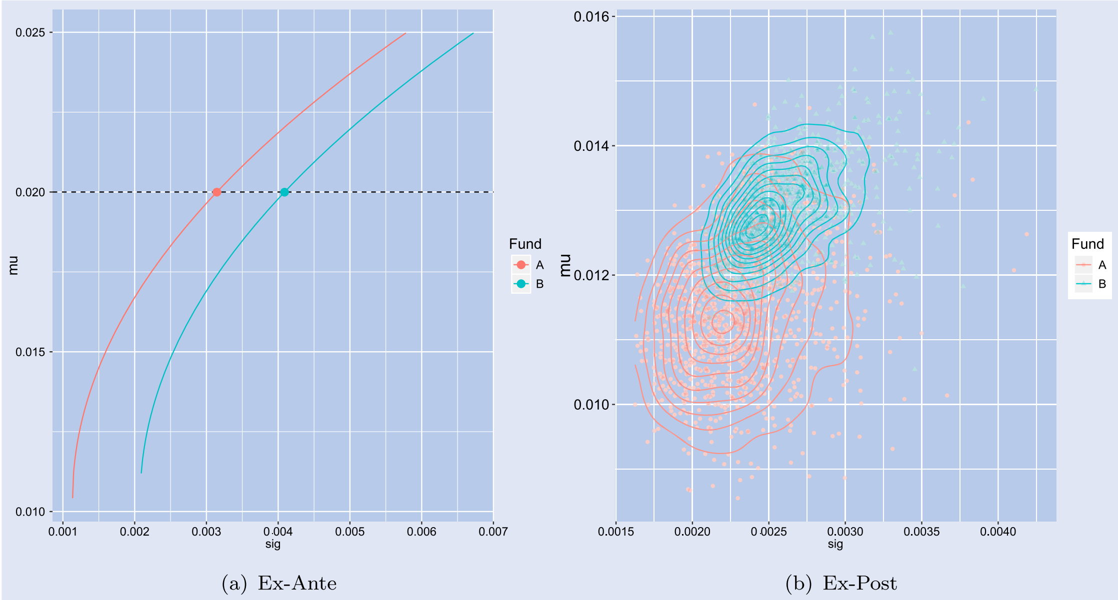

Figure 1. Mean-variance efficient frontier.
This figure illustrates both the mean-variance efficient frontier (MVEF) for fund A and fund B. Panel (a) corresponds to the MVEF under full information which is a representation of equations (5) and (12). The dashed line denotes a mean target $m = 0.02$. The red (respectively blue) line corresponds to fund A (respectively B). Panel (a) demonstrates the mean-variance inefficiency of fund B relative to fund A under full information. Panel (b), on the other hand, provides a ex-post perspective using simulated returns. Using 1000 simulations, the panel illustrates the out-of-sample mean and variance of each fund. The contours denote the distribution of each fund. Panel (b) illustrates that fund B outperforms fund A in terms of MV trade-off for a significant proportion of the simulated results. (a) Ex-Ante and (b) Ex-Post.

that each fund yields the same mean return. We consider the case where the agent chooses $\varepsilon \in \{0, 1\}$ that maximizes an ex-post objective function. A more general case is considered in Kan and Zhou (2007).†

Under full information, the objective function of the decision-maker is to choose $\varepsilon \in \{0, 1\}$ that maximizes

$$U\left(x_m(\varepsilon)\right) = x_m(\varepsilon)^\top \mu - \frac{\kappa_m}{2} x_m(\varepsilon)^\top \Sigma x_m(\varepsilon). \qquad (18)$$

By construction, as lemma 2.1 illustrates, $\varepsilon = 0$ is the optimal choce to maximize expected utility. Nonetheless, the main assumption behind the above formulation is that the decision maker knows the portfolio weights of α_0, α_1, and α_2 without any uncertainty. In practice, the choice of these vectors is subjected to estimation error. Due to estimation error, therefore, it is unclear whether this ex-ante sub-optimality of the MTEP also translates to a lower expected utility ex-post. Note that the ex-ante sub-optimality of the MTEP portfolio could be attributed to the additional constraint of tracking error. As it develops, and consistent with Jagannathan and Ma (2003), imposing the 'wrong' constraints could improve the portfolio performance ex-post.‡ We illustrate this idea in figure 1.

Panel (a) from figure 1 demonstrates the MV trade-off for each fund under full information. For the same mean target m (horizontal line), Panel (a) indicates that fund A outperforms B in terms of MV trade-off. Hence, the decision-maker is always better-off with $\varepsilon = 0$ as her optimal decision rule. In Panel (b), on the other hand, we demonstrate the out-of-sample realization of either fund using simulated returns (1000 simulations in total). The simulation highlights the sensitivity of each fund to estimation error and whether fund A is optimal out-of-sample. As Panel (b) demonstrates, there is always a positive probability that fund B dominates fund A in terms of MV trade-off. As a result, it is unclear whether the agent is better off with $\varepsilon = 0$ versus $\varepsilon = 1$ as her optimal decision rule. To determine the optimal choice of the decision-maker (i.e.ε^*), we investigate the estimation risk associated with each fund and, hence, the expected out-of-sample utility paradigm from Kan and Zhou (2007). We devote the next section to this discussion.

3. Estimation error and implicit value

Any inaccuracies in the information sets $\mathcal{I}_A$ and $\mathcal{I}_B$ induce errors in the MVEP and the MTEP, relative to the *true* portfolios, $x^A(\kappa_A) \equiv x^A(\kappa_A, \mathcal{I}_A)$ and $x^B(\kappa_B) \equiv x^B(\kappa_B, \mathcal{I}_B)$. Let $\hat{\mathcal{I}}_A = \{\hat{\mu}, \hat{\Sigma}\}$ and $\hat{\mathcal{I}}_B = \hat{\mathcal{I}}_A \cup \{\hat{\beta}, \hat{\sigma}_b^2\}$ denote the information sets possibly endowed with estimation error. In the sequel, we focus on the relative magnitude of the induced errors in $\hat{x}^A(\kappa_A) \equiv x^A(\kappa_A, \hat{\mathcal{I}}_A)$ and $\hat{x}^B(\kappa_B) \equiv x^B(\kappa_B, \hat{\mathcal{I}}_B)$, relative to the true portfolios, as well as the out-of-sample expected utility of $\hat{x}^A(\kappa_A)$ and $\hat{x}^B(\kappa_B)$.

We restrict our attention to the case where $\hat{\mathcal{I}}_A$ and $\hat{\mathcal{I}}_B$ are determined based on a time-window of historical data under sample averaging. That is, given historical T time periods, we have

$$\hat{\mu} = \frac{1}{T} \sum_{t=1}^{T} R_t, \qquad (19)$$

† As section 5 demonstrates, the more general choice $\varepsilon \in [0, 1]$ is equivalent to partial index-tracking.
‡ In section 5, we relate the portfolio problem studied in this paper in relation to partial-tracking and shrinkage.

$$\hat{\Sigma} = \frac{1}{T-1}\sum_{t=1}^{T}\left[R_t - \hat{\mu}\right]\left[R_t - \hat{\mu}\right]^\top, \tag{20}$$

and

$$\hat{\mathbf{c}} = \mathrm{Cov}\left(R, \tilde{r}_b\right) = \begin{bmatrix} \frac{1}{T-1}\sum_{t=1}^{T}\left[R_{1,t} - \hat{\mu}_1\right]\left[\tilde{r}_{b,t} - \bar{r}_b\right] \\ \vdots \\ \frac{1}{T-1}\sum_{t=1}^{T}\left[R_{d,t} - \hat{\mu}_d\right]\left[\tilde{r}_{b,t} - \bar{r}_b\right] \end{bmatrix}, \tag{21}$$

where $R_{i,t}$ denotes the return of asset $i = 1, \ldots, d$, at time t. Moreover, the average and variance of the benchmark index return over the sample period are:

$$\bar{r}_b = \frac{1}{T}\sum_{t=1}^{T}\tilde{r}_{b,t} \quad \text{and} \quad \hat{\sigma}_b^2 = \frac{1}{T-1}\sum_{t=1}^{T}(\tilde{r}_{b,t} - \bar{r}_b)^2. \tag{22}$$

Noting $\hat{\beta} = (1/\hat{\sigma}_b^2)\hat{\mathbf{c}}$, under the *estimated* information sets $\hat{\mathcal{I}}_A$ and $\hat{\mathcal{I}}_B$, we obtain

$$\hat{x}^A(\kappa_A) = \hat{\alpha}_0 + \frac{1}{\kappa_A}\hat{\alpha}_1 \tag{23}$$

and

$$\hat{x}^B(\kappa_B) = \hat{\alpha}_0 + \frac{1}{\kappa_B}\hat{\alpha}_1 + \hat{\alpha}_2, \tag{24}$$

where $\hat{\alpha}_0$, $\hat{\alpha}_1$, and $\hat{\alpha}_2$ are the estimates of the corresponding parameters in (2) and (10) under $\hat{\mathcal{I}}_A$ and $\hat{\mathcal{I}}_B$.

Given the definition of the estimated portfolio, the following section discusses the out-of-sample performance of the estimated portfolios. Specifically, we derive a closed form metric for the out-of-sample utility of $\hat{x}^A(\kappa_A)$ and $\hat{x}^B(\kappa_B)$, under the assumption that $T > d + 3$, i.e. the historical sample size is sufficiently large.

3.1. Out-of-sample utility of estimated portfolios

Although the MTEP underperforms the MVEP under the MV utility criterion when *true* parameters are known under full information (see, lemma 2.1 from section 2), it is an open question if this result holds when parameters are subject to estimation error as Panel (b) from figure 1 illustrated. To address this, we employ an ex-ante criterion referred to as the 'out-of-sample expected utility' of the estimated portfolio, which is a common metric used in the literature (Kan and Zhou 2007, Tu and Zhou 2011, Kan *et al.* 2021, Simaan and Simaan 2019).

Definition 1 (Out-of-Sample Performance) *Under true parameters μ and Σ, let the MV utility for a fixed portfolio x, under risk aversion κ, be $U_\kappa(x) = \mu^\top x - \frac{\kappa}{2}x^\top\Sigma x$. Then, the out-of-sample performance of a portfolio estimator $\hat{x}$ is measured by the MV-expected utility, as defined by:*

$$\mathbb{E}[U_m(\hat{x})] = \mathbb{E}[\hat{x}^\top\mu] - \frac{\kappa_m}{2}\mathbb{E}\left[\hat{x}^\top\Sigma\hat{x}\right], \tag{25}$$

where the expectation operator $\mathbb{E}$ is evaluated w.r.t. the sampling distribution of the estimated portfolio, $\hat{x}$.

To determine the optimal portfolio, the decision-maker evaluates the expected out-of-sample utility of its estimated portfolio from equation (17), which takes the following form

$$\hat{x}_m(\varepsilon) = \hat{\alpha}_0 + \left[\frac{1}{\kappa_m} - \varepsilon l\right]\hat{\alpha}_1 + \varepsilon\hat{\alpha}_2. \tag{26}$$

The choice of ε thus depends on the out-of-sample expected utility of this decision rule. Note that our analysis precludes estimation error associated with κ_m and l. This is due to the fact the decision-maker evaluates both funds ex-ante to yield the same mean target. Hence, estimation error in the portfolio choice originates from $\hat{\alpha}_0$, $\hat{\alpha}_1$, and $\hat{\alpha}_2$. Additionally, we note that the difference between the full information expected utility and the out-of-sample one denotes a loss function. This loss function provides the decision-maker a criterion for ranking various portfolio rules, such that the rule with the highest loss is the least preferred (see e.g. Jorion 1986). Motivated by this, we measure analytically the potential gain/loss of the MV decision maker by choosing $\varepsilon = 1$ over $\varepsilon = 0$. Specifically, this gain/loss is the basis for our definition of IVIT, which we describe in definition 2.

Definition 2 (Implicit Value of Index Tracking - IVIT) *The IVIT measure computes the difference in expected out-of-sample utility between portfolio $\hat{x}(1)$ and $\hat{x}(0)$:*

$$\Theta_m = \mathbb{E}\left[U_m\left(\hat{x}_m(1)\right)\right] - \mathbb{E}\left[U_m\left(\hat{x}_m(0)\right)\right]. \tag{27}$$

In terms of optimal choice out-of-sample, the decision maker chooses $\varepsilon = 1$ over $\varepsilon = 0$ as long as $\Theta_m > 0$. In contrast, in the absence of any estimation error, when the MVEP and the MTEP achieve the same portfolio mean, the decision-maker is better off choosing $\varepsilon = 0$ over $\varepsilon = 1$ due to lemma 2.1. To determine the optimal choice by the decision-maker, we derive the conditions under which $\Theta_m > 0$. Clearly, if $\Theta_m > 0$, then the decision-maker faces a positive opportunity cost of not index-tracking. Such positive opportunity cost denotes a positive IVIT. We utilize the following equivalent expression for this implicit value.

Proposition 3.1 *The IVIT measure defined in (27) has the equivalent expression:*

$$\Theta_m = \eta_2 F_1(m) + \frac{\kappa_m}{2}\left[g(d, T, \sigma_1^2)F_2(m) - \left(d_4\lambda + \sigma_2^2\right)\right], \tag{28}$$

where κ_m is given by (14) with

$$F_1(m) := 1 - \left(2 - \frac{\eta_2}{m - \eta_0}\right)d_3, \tag{29}$$

$$F_2(m) := \frac{\eta_2\left(2m - 2\eta_0 - \eta_2\right)}{\eta_1^2}, \tag{30}$$

$$g(d, T, \sigma_1^2) := \sigma_1^2\left(d_3^2 + d_1\right) + f(d, T, \sigma_1^2)(d-1) \geq 0, \tag{31}$$

$$d_1 = \left[\frac{T-d+1}{T-d-1}\right]d_2,$$
$$d_2 = \frac{(T-1)^2}{(T-d)(T-d-1)(T-d-3)}, \tag{32}$$
$$d_3 = (T-1)/(T-d-1),$$

$$d_4 = (d-1)/(T-d-1), \tag{33}$$

$$f(d,T,\sigma_1^2) = d_2\left[\sigma_1^2 + \frac{T-2}{T}\right], \tag{34}$$

and

$$\lambda = \sigma_b^2(1-2\beta^\top\alpha_0) - \sigma_2^2. \tag{35}$$

Since $T > d+3$, we have $d_j > 0$, $j = 1,\ldots,4$, and d_1, d_2, and $f(d,T,\sigma_1^2)$ are increasing functions in d/T, i.e. the ratio of the number of assets to the sample size. Also, notice that $f(d,T,\sigma_1^2) > 0$ and it depends on the first and second moments of asset returns, while λ depends only on the second moments of the asset returns. The sign of IVIT in (28) is determined by a number of factors, such as risk aversion, sample size, number of assets, asset return parameters, and index volatility σ_b. Section 3.2 describes conditions involving these factors such that $\Theta_m > 0$ holds.

Our conjecture posits that the existence of a positive IVIT is associated with the MTEP having less induced estimation risk compared to the MVEP. That is, an improvement in out-of-sample utility is likely accompanied by a reduction in the induced estimation risk. Before we move to section 3.2, we summarize the sensitivity of the IVIT with respect to the market volatility in corollary 3.2 below. In particular, when the covariance between the asset and the market remains fixed, the number of assets, the sample size, and the mean target determine the magnitude of the IVIT's sensitivity to market volatility.

COROLLARY 3.2 *Suppose covariance between the market index and each asset is fixed while the index volatility may vary. The relationship between the IVIT and the market volatility is negative and given by*

$$\frac{\partial\Theta_m}{\partial\sigma_b^2} = -\frac{\eta_1}{2(m-\eta_0)}d_4 < 0 \tag{36}$$

Corollary 3.2 states that the sensitivity of the IVIT with respect to the market variance (volatility) is negative. Recall that d_4 is an increasing function in d/T, i.e. when we have a larger number of assets and smaller sample size, d_4 takes larger values. Hence, we expect that such sensitivity to be more negative for larger d/T ratios. Moreover, we also note that such sensitivity is affected by the mean target of the decision-maker. A large m value implies that the decision-maker is targeting a higher mean return by tilting her portfolio towards the α_1 fund. This indicates that the tracking error fund α_2 plays a less significant role in her decision rule. As a result, we expect less sensitivity of the IVIT to the market volatility when the agent is more risk tolerant, i.e. seeking high mean return.

3.2. Existence of implicit value

The Θ_m metric denotes the implicit value of index-tracking for the decision-maker who is targeting m mean return. The choice of ε determines the final decision rule and, hence, whether the decision-maker is better off with the MVEP over the MTEP. If there is a positive opportunity cost of choosing $\varepsilon = 0$ over $\varepsilon = 1$, then the decision-maker gains higher out-of-sample utility by choosing the MTEP over the MVEP, even though the former is deemed MV inefficient under full information - as Panel (a) from figure 1 demonstrates.

To determine the sign of the Θ_m, we note two important observations. First, note that $F_1(m) < 0$ holds because $d_3 > 1$ for finite sample sizes and $m > m_{\min}$ (see, (16)). Moreover, the sign of $F_2(m)$ is the same as that of η_2. Both $F_1(m)$ and $F_2(m)$ depend on the first and second moments, while the term $(d_4\lambda + \sigma_2^2)$ in Θ_m depends only on the second moments. blackSecond, in the online appendix, we show that the λ component denotes the reduction in estimation risk. In this case, estimation risk is approximated using the portfolio mean-squared error. In particular, we find that more negative λ implies lower estimation risk associated with the MTEP over the MVEP. In other words, we expect the IVIT to be more evident when the MTEP is associated with lower estimation error.

Given equation (28), it can be shown in a straightforward manner that $\Theta_m > 0$ holds *if and only if* the chosen target m satisfies:

$$\begin{aligned}&2(m-\eta_0)\left[(1-2d_3)\eta_1\eta_2 + g(d,T,\sigma_1^2)\eta_2\right]\\&\quad > g(d,T,\sigma_1^2)\eta_2^2 + \left(d_4\lambda + \sigma_2^2\right)\eta_1^2 - 2\eta_1\eta_2^2 d_3,\end{aligned} \tag{37}$$

Equation (37) provides ranges for the target mean m, within which a positive implicit value exist. To determine these ranges, define the following target return threshold:

$$m_\theta := \eta_0 + \frac{g(d,T,\sigma_1^2)\eta_2^2 + \left(d_4\lambda + \sigma_2^2\right)\eta_1^2 - 2\eta_1\eta_2^2 d_3}{2\eta_2\left[(1-2d_3)\eta_1 + g(d,T,\sigma_1^2)\right]} \tag{38}$$

Note that m_θ is computable using the asset-index return parameters. We summarize the set of rules in corollary 3.3.

COROLLARY 3.3 *All else equal, the decision-maker faces a positive opportunity cost of not index-tracking, i.e.* $\Theta_m > 0$, *and, therefore, chooses* $\varepsilon = 1$ *when*

$$m > m_\theta \wedge \eta_2 > 0 \tag{39}$$

Note that a higher mean target, m, is associated with lower (higher) risk aversion (tolerance). Since the committed estimation risk is positively associated with risk tolerance (see e.g. Simaan *et al.* 2018), a large m denotes the case of high estimation risk. Hence, if the lower bound, m_θ, is very large, then the IVIT is only evident in cases of extreme estimation risk. The key insight from corollary 3.3 is that an investor is more likely to achieve a positive IVIT when the target return increases. Similarly, as will be shown empirically below, the value of IVIT is increasing in the target mean, m. We confirm these claims empirically under a variety of conditions in the following section.

It is worthwhile noting that when $\eta_2 < 0$, we know that the index-tracking component is associated with a negative premium by construction. For this reason, corollary 3.3 derives the condition for $\Theta_m > 0$ when $\eta_2 > 0$ alone. Nonetheless, we note that the condition $\eta_2 > 0$ by itself does not necessarily imply that the decision-maker faces a positive opportunity cost. This can be discerned from equation (28), in which for $\Theta_m > 0$ to hold true, the λ component should be sufficiently negative, all else equal.

4. Empirical investigation of the implicit value

In this section, we empirically test the analytical predictions from the above analysis. We focus our empirical tests on the existence of the IVIT in terms of $\Theta_m > 0$. We start with Monte Carlo simulations designed to assess whether a positive significant IVIT does exist in common data. We then pursue a portfolio strategy that is long in the MTEP ($\varepsilon = 1$) and short in the MVEP ($\varepsilon = 0$) in order to test the economic significance of the IVIT. Additionally, we do so under conditions where the IVIT is most evident according to our analytical findings. We assess the results using commonly used performance metrics including the Sharpe ratio, a certainty equivalence return, and portfolio turnover.

4.1. Data

We use several data sets to investigate the implications of our theoretical results. On the industry level, we consider four sets of Fama-French (FF) industry portfolios, which are drawn from the data library maintained by Kenneth R. French. The FF industry portfolios allow us to capture the size effect. In particular, we consider $d = 10$, 17, 30, and 48, each of which we denote as the d-FF-industry dataset. On the company level, we consider all companies listed on the S&P 500 between January 1970 and December 2015. In total, we identify 171 companies (using the CRSP PERMNO) that remained listed over the whole sample period. From the 171, we pick the 100 companies with the largest average market capitalization over our sample period. We split the 100 companies into two equal sized groups, where the first corresponds to the largest market capitalization stocks and the second to the next largest market capitalization stocks. This constitutes our stock-level dataset, which we refer to as the 50 S&P 500 A and B datasets, respectively. For the market benchmark, we consider the value-weighted return of all CRSP firms incorporated in the US and listed on the NYSE, AMEX, and NASDAQ. Our empirical analysis is based on monthly returns between January 1970 and December 2015, inclusive, referred to as the full-data sample.

4.2. Monte Carlo simulation

We first use Monte Carlo simulation to examine the existence of a positive IVIT. The necessary condition for such depends on η_2, λ, the ratio d/T, and the target mean m. For each data set, we estimate the model parameters using the full-data sample, and treat them as the 'true' (population) inputs. Given these inputs, we simulate $T \times d$ asset returns from an *iid* multivariate normal distribution, corresponding to T monthly returns of d assets. From the simulated asset returns, the decision-maker estimates her portfolio weights accordingly. We perform 1,000 simulations for each dataset and compute the standard errors of Θ_m for a given m. We define the normalized values of Θ_m for a given mean target m by:

$$\Theta_m^n = \frac{\Theta_m}{|\,\mathbb{E}\left[U_m\left(\hat{x}(0)\right)\right]|} \tag{40}$$

where κ_A and κ_B are determined using (14) to correspond to a mean target m (see, (16)).

We summarize the results from the Monte Carlo simulation tests in table 1. From this sample of runs, we make statistical inference on Θ_m^n. Panel (a) and Panel (b) from table 1 correspond to $T = 60$ and $T = 120$, respectively. Each row in table 1 corresponds to a fixed data set upon which the simulation study is conducted including four FF industry datasets and two sets of 50 stocks from the S&P 500 index. For each dataset, we report a number of parameters. Given that the evaluation concerns the hypothesis whether $\Theta_m > 0$, we report the mean return target m, which is determined to satisfy the condition in (16), by setting $m = 1.5 \times m_{\min}$ or $m = 2 \times m_{\min}$, where $m_{\min}$ is given in (16). We report the target mean cutoff values m_θ, above which IVIT is positive. Finally, we report Θ_m^n from (40) long with the standard errors from the 1,000 simulations for each row.

A number of comments follow from table 1. First, we observe that Θ_m^n is positive in most cases, except the cases for the $d = 10$, 17 FF industries. Moreover, IVIT becomes positive and statistically significant as the number of assets (d) increases. Second, given the insights from corollary 3.3, we observe that $\eta_2 > 0$ for all cases, except for $d = 10$. This also represents the lone case where Θ_m^n is significantly negative. However, it should be noted that as the target mean m declines and drops below the critical value m_θ, Θ_m^n becomes negative. While this not explicitly laid out in the table 1, we illustrate these insights in figure 2.

Figure 2 highlights the relationship between Θ_m and m_θ laid out in (37) and (38). Each panel generalizes a specific case from table 1. Specifically, the solid black line represents Θ_m^n from equation (40) as a function of m. The black dots represent the point m_θ from (38), which denotes the values of m above which IVIT becomes positive. The gray shaded areas represent values of $m < m_{\min}$ (see, 16), which we regard as the infeasible region (the values of m under this region correspond to negative κ_m value as defined by (14)). Panels (a)–(d) correspond to rows two and five of table 1 (e.g. the values of m_θ and $m_{\min}$ from row two of table 1 are depicted in Panel (a)).

Several insights are be drawn from figure 2. First, within the feasible region where $m > m_{\min}$, IVIT increases with m. The sharp discontinuity in Θ_m occurs at the point where $m = \eta_0$, which can be shown by taking the derivative of Θ_m with respect to m. Moreover, when the ratio d/T is high (Panels (b) and (d)), the values of m_θ are within the infeasible gray regions, meaning that a positive IVIT is attained for any feasible portfolio. As d/T declines, investors have to target increasingly high mean returns m to achieve a positive IVIT.

Overall, the evidence in table 1 and figure 2 is strongly supportive of the existence of a positive IVIT. It is also indicative of a strong positive correlation between IVIT and d/T ratio as well as m which is consistent with our argument that the positive IVIT is most pronounced in conditions where estimation error is likely to arise. For example, IVIT tends to be most positive when the ratio of d/T is greatest. Nonetheless, these results are theory-based. The important question facing the decision-maker is what is the economic value of the IVIT. We address this question in the following discussion.

Table 1. The implicit value of index tracking.

T	d	η_2	$m_{\min}$	m_θ	m	Θ_m^n	T	d	η_2	$m_{\min}$	m_θ	m	Θ_m^n
			Panel (a) $T = 60$							Panel (b) $T = 120$			
60	10	– 0.054	1.026	0.985	1.539	– 0.349*** (0.573)	120	10	– 0.054	1.026	0.954	1.539	– 1.586* (11.205)
					2.052	– 0.138** (0.244)						2.052	– 0.274* (1.537)
60	17	0.005	0.974	1.226	1.461	– 0.011 (0.243)	120	17	0.005	0.974	2.521	1.461	– 0.512 (5.149)
					1.948	0.005 (0.081)						1.948	– 0.044 (0.423)
60	30	0.055	0.972	0.947	1.458	0.181* (0.115)	120	30	0.055	0.972	1.242	1.458	0.036 (0.299)
					1.944	0.104* (0.063)						1.944	0.075 (0.146)
60	48	0.113	1.013	0.946	1.52	0.331*** (0.071)	120	48	0.113	1.013	1.096	1.52	0.197 (0.176)
					2.026	0.188*** (0.051)						2.026	0.166* (0.104)
60	50	0.183	1.225	1.133	1.838	0.409*** (0.021)	120	50	0.183	1.225	1.14	1.838	0.427*** (0.046)
					2.45	0.243*** (0.016)						2.45	0.256*** (0.030)
60	50	0.14	1.21	1.139	1.814	0.342*** (0.028)	120	50	0.14	1.21	1.158	1.814	0.348*** (0.061)
					2.419	0.197*** (0.019)						2.419	0.204*** (0.038)

Note: $^*p < 0.1$; $^{**}p < 0.05$; $^{***}p < 0.01$.
Simulation Analysis This table summarizes the values of Θ_m^n that denotes the normalized value Θ_m from equation (40), i.e. the implicit value of index-tracking (IVIT), when both agents target the same mean return, m. The metric Θ_m measures the difference in out-of-sample expected utility between agent A and B and is normalized by dividing Θ_m over the former. In either case, the mean target m is determined to satisfy the condition from (16), such that $m = 1.5 \times m_{\min}$ and $m = 2 \times m_{\min}$, which, as a result, determines κ_A and κ_B. Additionally, T and d denote the sample size and the number of assets; η_2 denotes the premium associated with the index-tracking component (see equation (14)); $m_{\min}$ is defined in (16) and represents the minimum target mean to ensure positive levels of risk aversion; m_θ represents the target mean m above which IVIT becomes positive. Standard errors (reported in brackets) along with significance levels are obtained using a Monte Carlo simulation with 1,000 runs (see section 4.2).

4.3. Portfolio performance evaluation

The preceding empirical findings on the implicit value of index-tracking corroborate with the analytical results derived in the paper. However, an important metric of IVIT from an investor's standpoint is whether or not it translates to better out-of-sample portfolio performance. We pursue this question in this section using a rolling-window framework.

We assume that the agent constructs two portfolios at time t, denoted by $\hat{x}_m(1)$ and $\hat{x}_m(0)$ from equation (26). These portfolios are constructed using a historical sample of T months. In either case, the agent determines κ_m and l based on the sample so that both portfolios yield the same mean return in-sample. For the vector of weights, the agent relies on the same sample to estimate α_i for $i = 0, 1, 2$. After realizing the period $t + 1$ asset returns, the agent rebalance both portfolios based on the revised data sample that includes the more recent T months. This results in a time series of out-of-sample portfolio return for each fund. Let the portfolio return of portfolio at month $t + 1$ be denoted by

$$\mathbf{R}_{t+1}(\varepsilon) = R_{t+1}^{act\,\top} \hat{x}_{m,t}(\varepsilon) \tag{41}$$

where $R^{act}(t + 1)$ is the vector of actual asset returns realized at $t + 1$ and $\hat{x}_{m,t}(\varepsilon)$ is the estimated portfolio from equation (26) at month t given a sample of T months. In this case, $\varepsilon = 0$ (respectively, $\varepsilon = 1$) corresponds to the MVEP (respectively MTEP).

We compare the difference of performance of the MTEP and MVEP portfolios. In this case, difference in the two portfolio returns $\mathbf{R}_{t+1}(1)$ and $\mathbf{R}_{t+1}(0)$, denoted by $\mathbf{R}_{t+1}^D$, is independent of the mean target m and is given by:

$$\begin{aligned}\mathbf{R}_{t+1}(D) \equiv \mathbf{R}_{t+1}(1) - \mathbf{R}_{t+1}(0) &= R_{t+1}^{act\,\top}[\hat{x}_{m,t}(1) - \hat{x}_{m,t}(0)] \\ &= R_{t+1}^{act\,\top}\left[\hat{\alpha}_{2,t} - l_t\hat{\alpha}_{1,t}\right]\end{aligned} \tag{42}$$

where $\hat{\alpha}(1, t)$ and $\hat{\alpha}(2, t)$ denotes the estimated vectors α_1 and α_2, respectively, at month t given a sample of T months. The l_t corresponds to the scaler l from equation (15) that the agent determines at time t such that both funds yield the same mean return m. Additionally, note that $\mathbf{R}^D$ is the return of an arbitrage portfolio that is long on MTEP and short on MVEP.

Given the time series of out-of-sample portfolio returns $\mathbf{R}_{t+1}(i)$, $i = \{0, 1, D\}$, as well as the portfolio allocations over time, we compare performance using the following common portfolio metrics, e.g. see DeMiguel *et al.* (2009b, 2013):

$$\text{Mean}_i = 12 \times \tau^{-1} \sum_{t=0}^{\tau-1} \mathbf{R}_{t+1}(i) \tag{43}$$

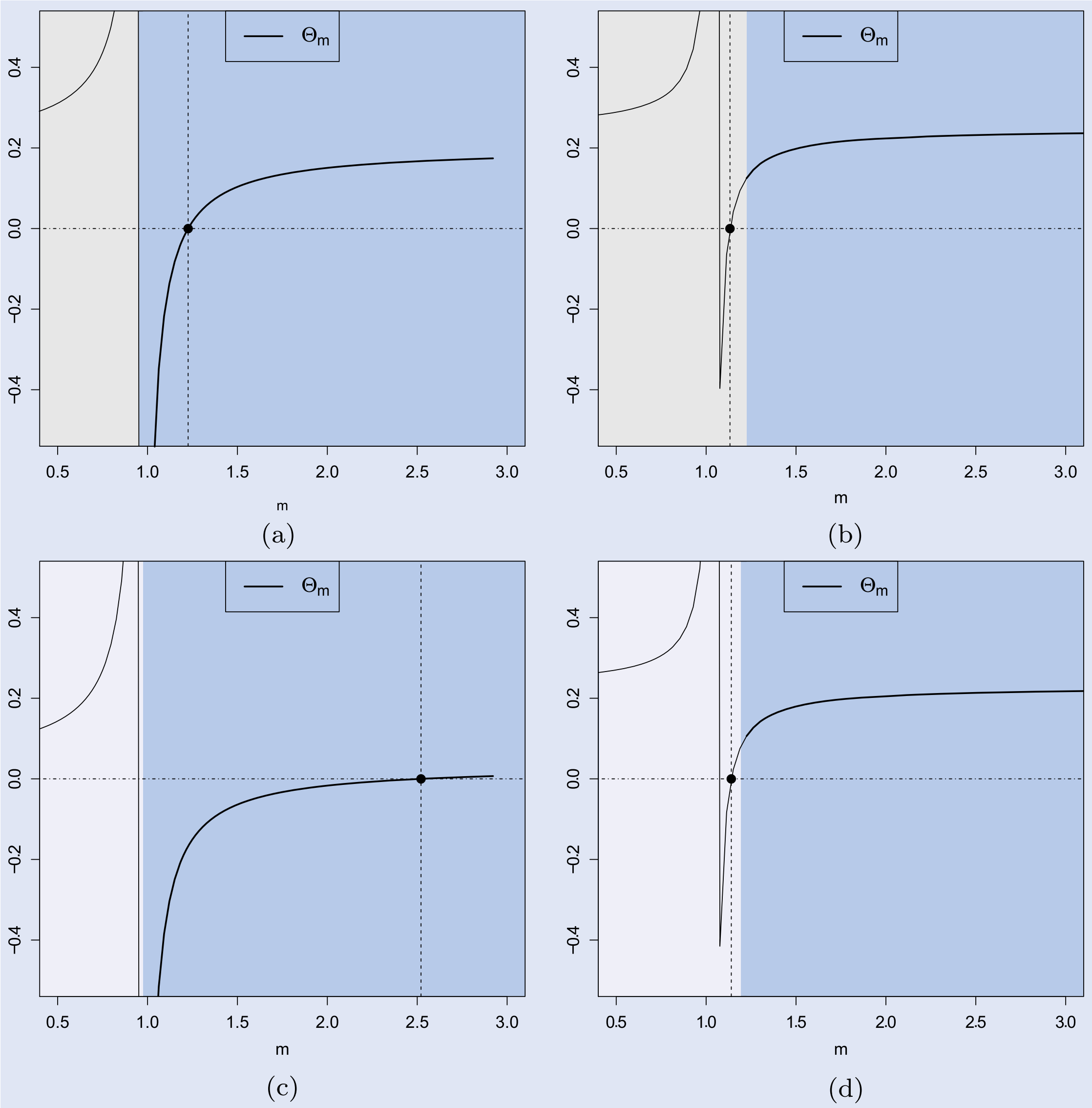

Figure 2. IVIT versus target mean.
These graphs correspond to the results in table 1. The solid black lines represent Θ_m as defined by (28) versus m. The black horizontal line represents the point m_θ as defined by (38). The black dot corresponds to the intersection of the horizontal lines with the zero level. The shaded areas on the left hand side represent values of $m < m_{\min}$ (16). Sub-figures (a)–(d) correspond to rows two and five of tables 1. In all cases, the both Θ_m is standardized across the m values to exhibit a standard deviation of 1. (a) $d = 17$, $T = 60$. (b) $d = 50$ and $T = 60$. (c) $d = 17$, $T = 120$ and (d) $d = 50$ and $T = 120$.

$$\mathrm{Std}_i = \sqrt{12} \times \sqrt{(\tau - 1)^{-1} \sum_{t=0}^{\tau-1} (\mathbf{R}_{t+1}(i) - \mathrm{Mean}_i)^2} \quad (44)$$

$$\mathrm{SR}_i = \frac{\mathrm{Mean}_i}{\mathrm{Std}_i} \quad (45)$$

$$\mathrm{CEQ}_i = \mathrm{Mean}_i - \frac{\kappa_m}{2} (\mathrm{Std}_i)^2 \quad (46)$$

$$\mathrm{TO}_i = \frac{1}{\tau} \sum_{t=0}^{\tau-1} \| \hat{x}_{m,t+1}(i) - [\hat{x}_{m,t}(i)] \circ [\mathbf{1} + \mathbf{R}_{t+1}(i)] \| \quad (47)$$

where, Mean_i, Std_i, and SR_i denote the annualized portfolio mean return, volatility, and Sharpe-ratio, respectively. Moreover, CEQ_i denotes the certainty-equivalent return of portfolio i, and TO_i is the portfolio turnover, where $\|v\|$ denotes the Euclidean-norm of vector v and $^\circ$ is the element-by-element (Hadamard) product. Note that the turnover takes into account the change in positions over time due to the return on assets.† Given the two portfolio strategies MVEP and MTEP with an equal target mean return m, the decision-maker evaluates the CEQ of each portfolio for a given level of κ_m.

We also consider risk-adjusted portfolio returns based on the 4-factor model so that

$$[\text{Mean 4F}]_i = \mathrm{Mean}_i - [\text{4F Risk Premium}]_i \quad (48)$$

† Different from DeMiguel *et al.* 2013, we control for the portfolio growth in the calculation of the portfolio turnover as an additional robustness. Nonetheless, in unreported results we find that there no significant difference between the two approaches and that our main findings are robust using either turnover definition.

where [4F Risk Premium]$_i$ denotes the total premium in portfolio i for exposure in the four risk factors, namely the market, size, and value factors (Fama and French 1993), as well as the momentum factor (Carhart 1997).

For the raw and risk-adjusted returns, we use a t-test with Newey and West (1986) standard errors (three lags) to adjust for serial correlation and heteroskedasticity. On the other hand, to compute significance levels for Std, SR, CEQ, and TO, we deploy a block re-sampling bootstrap methodology with a fixed block length and 1,000 draws. Specifically, bootstrapping is executed using the `boot` R package developed and maintained by Canty and Ripley (2017). The block re-sampling bootstrap accommodates for serial correlation as pointed out by Ledoit and Wolf (2008). Following their recommendation, we use 10 as the block length with 1,000 draws. In addition, we compute the bootstrapped p-values using the methodology suggested by Ledoit and Wolf (2008) (Remark 3.2).

4.3.1. Out-of-sample evaluation results. We first illustrate the cumulative returns of the arbitrage portfolio described in (42) in figure 3 to gain an initial insight on the relation with IVIT. This is conducted using an estimation window of 10 years, i.e. $T = 120$. Consistent with the observations from table 1, we find that the arbitrage portfolio yields a significantly positive return for the larger number of assets, as illustrated in Panel (b) from figure 3. These illustrations are consistent with the evidence of implicit value as demonstrated in table 1. Nonetheless, we witness a drop in the cumulative return during periods of high market uncertainty in all panels in figure 3, an evidence consistent with the corollary 3.2.

In table 2, we report the out-of-sample performance of the MVEP, MTEP, and the arbitrage portfolio from (42), with respect to the performance statistics in (43)–(47). For the $T = 60$ panels, we observe that the MTEP yields a lower volatility, a higher SR, a lower turnover, and a larger CEQ for cases in which the implicit value is most likely to prevail, e.g. larger d/T ratios. On the other hand, as the sample size increases, i.e. $T = 120$ panels, such evidence is weakened. From a theoretical point of view, we expect that IVIT is decreasing as the sample size increases. Hence, the decision-maker is better off with the MTEP (MVEP) for cases associated with high (low) estimation error. For instance, for the stock datasets, we observe that the MTEP outperforms the MVEP with respect to all metrics when $T = 60$, although differences in TO and Std are the only metrics that are consistently statistically significant. In particular, the MTEP yields a lower volatility, which is significant at the 1% level. In terms of relative CEQ, we observe that the difference is significant at the 5% level when $T = 60$ for the 50 S&P 500 data set A (respectively 48-FF data set) with a value of 16.11 (respectively 14.31).

The implicit value is less likely to prevail for portfolios with small number of assets or low d/T ratios. Consistent with the implicit value argument, we find weak evidence in support of index-tracking in cases associated with low d/T ratios. For instance, when $d = 17$ and $T = 120$ in Panel (a) table 2, the CEQ statistic is negative and volatility (Std) is significantly positive. This implies that the decision-maker is better off with the MVEP versus MTEP when investing among a smaller number of assets. This evidence is consistent with the findings in table 1 where we find no significant evidence in support of $\Theta_m > 0$ for small d/T ratios. Additionally, note that for cases associated with high d/T, we expect higher sensitivity to estimation error. Evidently, while the individual portfolios have high turnover with large standard deviation in this case, the improvement in the arbitrage portfolio is due to the implicit value in favor of index-tracking.

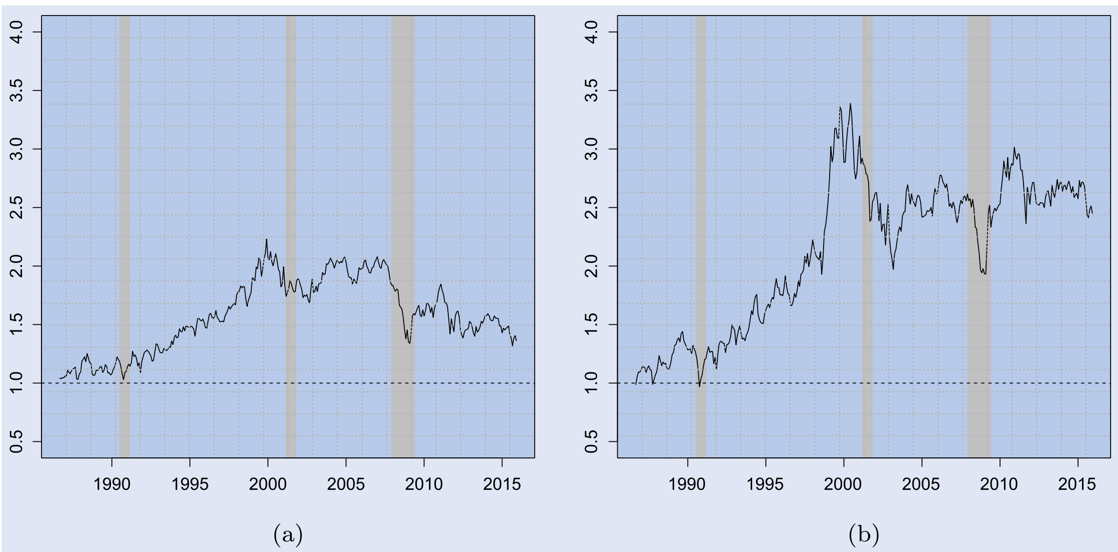

Figure 3. Out-of-sample portfolio performance.
This figure illustrates the out-of-sample performance of the arbitrage portfolio from the empirical test described in Subsection 4.3. The arbitrage portfolio goes long in the MTEP and short in the MVEP. The returns are monthly and compounded over the period between September 1986 and December 2015. d refers to the number of assets used to construct portfolios. In particular, the $d = 17$ refers to the 17 Fama-French industry portfolios, while the $d = 50$ corresponds to the largest market cap stocks listed on the S&P 500 (see the data section 4.1). The gray bars indicate recession periods identified by the National Bureau of Economic Research. In all panels, we use a sample size of $T = 120$ months and set the target mean to 12% annual, i.e. $m = 0.01$. (a) $d = 17$ and (b) $d = 50$.

Table 2. Out-of-sample portfolio performance.

	17 FF-industry data set			30 FF-industry data set			48 FF-industry data set		
	(1)	(2)	(2)–(1)	(3)	(4)	(4)–(3)	(5)	(6)	(6)–(5)
					$T = 60$				
Mean 4F	1.909	2.792***	0.883	3.795	3.511***	− 0.283	− 1.488	3.014***	4.502
Mean	6.909***	10.040***	3.132	8.242***	10.821***	2.579	3.045	10.359***	7.313
Std	13.207	15.736	2.530***	15.997	15.754	− 0.244	23.263	16.169	− 7.093***
SR	0.523	0.638	0.115	0.515	0.687	0.172	0.131	0.641	0.510**
CEQ	2.548	3.850	1.301	1.845	4.617	2.772	− 10.483	3.823	14.306***
TO	0.670	0.195	− 0.476***	1.268	0.251	− 1.018***	3.805	0.743	− 3.061***
					$T = 120$				
Mean 4F	3.942**	3.781***	− 0.161	4.233**	3.703***	− 0.530	1.927	3.788***	1.861
Mean	9.217***	10.894***	1.677	9.044***	10.701***	1.657	7.275***	10.932***	3.658
Std	12.601	15.749	3.148***	13.210	15.637	2.427**	14.313	15.729	1.416*
SR	0.731	0.692	− 0.040	0.685	0.684	− 0.000	0.508	0.695	0.187
CEQ	5.248	4.694	− 0.554	4.681	4.588	− 0.093	2.153	4.747	2.594
TO	0.347	0.105	− 0.242***	0.491	0.106	− 0.385***	0.786	0.143	− 0.642***

	50 S&P 500 data set A			50 S&P 500 data set B		
	(7)	(8)	(8)–(7)	(9)	(10)	(10)–(9)
			$T = 60$			
Mean 4F	2.931	6.128***	3.197	8.212	6.074**	− 2.138
Mean	8.328	13.184***	4.857	12.576***	13.459***	0.883
Std	27.437	17.397	− 10.040***	26.411	20.137	− 6.275***
SR	0.304	0.758	0.454	0.476	0.668	0.192
CEQ	− 10.492	5.618	16.110**	− 4.863	3.322	8.185*
TO	3.211	1.330	− 1.881***	2.904	1.638	− 1.266***
			$T = 120$			
Mean 4F	3.327	5.245***	1.918	6.262**	5.871***	− 0.391
Mean	7.877***	11.959***	4.082	11.074***	13.174***	2.101
Std	14.429	15.291	0.861	15.568	16.178	0.610
SR	0.546	0.782	0.236	0.711	0.814	0.103
CEQ	2.672	6.114	3.442	5.015	6.631	1.617
TO	0.594	0.243	− 0.351***	0.529	0.344	− 0.185***

Note:

Note : $^{*}p < 0.1$; $^{**}p < 0.05$; $^{***}p < 0.01$.
This table reports the results from the empirical test described in subsection 4.3. The sample size is denoted by T and reported in months. Note that T varies across the panels to control for the sample size impact. Odd and even columns correspond to the mean-variance efficient portfolio (MVEP) and the mean-enhanced tracking efficient portfolio (MTEP), respectively. The (even) - (odd) column reports the difference in the performance metrics between the MTEP and MVEP. In all cases, portfolios are constructed to target an average annual mean return around 12%, i.e. $m \approx 1\%$. The returns are monthly and the out-of-sample period dates between September 1986 and December 2015.

To put our results into perspective, we plot the difference in the SR and CEQ in figure 4 for the range of $T \in [60, 120]$, as well as $d = 10, 20, 30, 40$ for FF-industries and the S&P 500 data. For for the FF industry data set, we randomly sample $d = 10, 20, 30, 40$ industry portfolios from the 48 FF-industry data set. We repeat this procedure 100 times with a fixed seed to control for portfolio size. The same procedure is applied to the stock data, in which we sample randomly a similar number of stocks from the 50 S&P 500 data set A. This results in a bootstrapped performance, using which we report the average for each (d, T) unique combination. In all panels in figure 4, we observe a consistent evidence in support of the implicit value. In particular, the MTEP outperforms the MVEP in terms of SR and CEQ for cases associated with high d/T ratio (bottom right corner of each panel; darker colors represent better performance of the MTEP versus the MVEP). At the top left of figure 4, we note that the SR differential is negative, i.e. weaker evidence of IVIT.

4.4. Reconciling implicit value and out-of-Sample performance

The out-of-sample portfolio performance results in table 2 (along with figure 4) are developed independently of the (static) implicit value results in table 1. To reconcile between the two, we address the following question: what is the economic value of the IVIT? This underlines the pricing problem of the implicit value. On the one hand, the IVIT is a theoretical measure, whereas the performance metrics from the out-of-sample experiment are of higher practical appeal. To address this question, we consider the improvement in each of the five out-of-sample performance metrics from section 4.3 as a function of Θ_m.

Similar to the experiment demonstrated in figure 4, we bootstrap the differential of each performance metric 100 times. Moreover, for each iteration, we compute the corresponding Θ_m^n theoretical measure, as in table 1. In all cases, we set the mean target to be $m = 1.5 \times m_{\min}$. We regress the improvement for each out-of-sample performance metric as

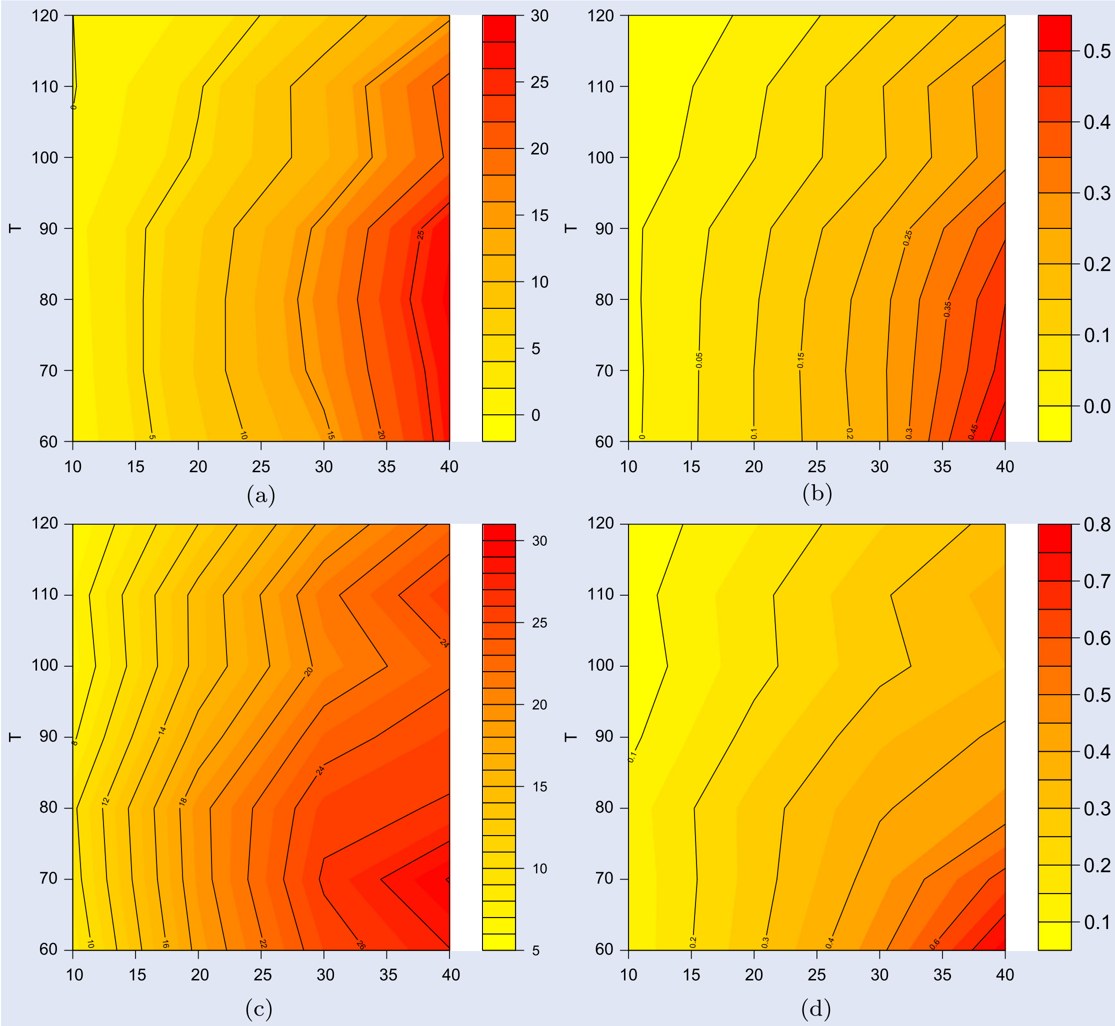

Figure 4. Economic performance: MTEP versus MVEP.
This figure demonstrates the difference in the out-of-sample Sharpe-ratio (SR) and certainty equivalent return (CEQ) between the MTEP and the MVEP in line with the empirical test conducted in section 4.3. The difference in performance is reported for d number of assets, x-axis, and T sample sizes, y-axis. Panels (a) and (b) report the results using the Fama-French industry dataset, whereas Panels (c) and (d) corresponds to the S&P 500 A data set. For for the FF industry data set, we randomly sample d = 10, 20, 30, 40 industry portfolios from the 48 FF-industry data set 100 times with a fixed seed to control for portfolio size. The same procedure is applied to the stock data, in which we sample randomly a similar number of stocks from the 50 S&P 500 data set A. Given the bootstrapped performance, we report the average for each (d, T) unique combination. The returns are monthly and the out-of-sample period dates between September 1986 and December 2015. The difference in SR is adjusted annually using a scale of $\sqrt{12}$ and reported in percentages. In all cases, the portfolios were constructed to correspond to a mean target of $m = 1.5 \times m_{\min}$. (a) FF-Industries SR. (b) FF-Industries CEQ. (c) S&P 500 SR and (d) S&P 500 CEQ.

a function with the corresponding Θ_n^m. Put formally, let $Y \in$ {Mean, Std, SR, CEQ,TO} denote a given performance metric from section 4.3, which corresponds to the annual mean return, annual volatility, annual Sharpe-ratio, annual certainty equivalent return, and monthly turnover, respectively. Hence, the dependent variable of the regression is given by $Y^B - Y^A$, i.e.the improvement in the performance metric Y for choosing the MTEP over the MVEP. On the other hand, the regressor is the standardized Θ_m^n, which has a mean value of 0 and a standard deviation of 1. We denote the latter by $\tilde{\Theta}_m^n$. We control for outliers by dropping Θ_m^n values that deviate from the first and third quartile by more than 2.5 times the interquartile.

Table 3 summarizes the regression results of $Y^B - Y^A$ over $\tilde{\Theta}_m^n$ for each given performance metric. This test reconciles the differential findings from table 2 with the simulation test from table 1 for different choices of d and T. In Panel (a) of table 3, we report the results using the Fama-French industry data sets. Panels (b), (c), and (d) correspond to the bootstrapped data. For Panel (a), this corresponds to 28 (4×7) observations, whereas for the other panels we have roughly 2800 (28×100) observations to test the link between the out-of-sample performance metrics and the IVIT.

In all panels from table 3, we find that the relationship between the IVIT and the out-performance metric is consistent with the implicit value argument. In other words, we expect a positive relationship (respectively negative) for the mean, SR, and CEQ (respectively Std and TO) performance metrics. Since the sample for the FF-industry datasets is limited to 28 observations, we focus our conclusions on the bootstrapped samples. In particular, we witness a strong statistical evidence in Panel (b) and (c). Among the 48-FF sampled industries, we find that a one standard deviation increase in

Table 3. Out-of-sample performance versus opportunity cost.

	Mean	Std	SR	CEQ	TO
			Panel (a) **Four FF-industry data sets**		
$\tilde{\Theta}_m^n$	1.806***	− 1.011**	12.076***	0.236***	− 36.951***
	(0.203)	(0.372)	(1.654)	(0.043)	(7.882)
Observations	28	28	28	28	28
Adjusted R^2	0.744	0.191	0.660	0.525	0.437
			Panel (b) **48 FF-industry data set Sampled 100 Times**		
$\tilde{\Theta}_m^n$	0.773***	− 0.428***	5.883***	0.097***	− 15.000***
	(0.029)	(0.022)	(0.195)	(0.003)	(0.503)
Observations	2,736	2,736	2,736	2,736	2,736
Adjusted R^2	0.206	0.117	0.250	0.247	0.245
			Panel (c) **50 S&P 500 data set A Sampled 100 Times**		
$\tilde{\Theta}_m^n$	0.690***	− 0.384***	4.838***	0.090***	− 6.312***
	(0.027)	(0.028)	(0.166)	(0.003)	(0.287)
Observations	2,744	2,744	2,744	2,744	2,744
Adjusted R^2	0.194	0.064	0.237	0.195	0.150
			Panel (d) **50 S&P 500 data set B Sampled 100 Times**		
$\tilde{\Theta}_m^n$	0.442***	− 0.045**	2.500***	0.041***	− 1.444***
	(0.025)	(0.022)	(0.156)	(0.003)	(0.189)
Observations	2,702	2,702	2,702	2,702	2,702
Adjusted R^2	0.106	0.001	0.086	0.068	0.021

Note: $^{*}p < 0.1$; $^{**}p < 0.05$; $^{***}p < 0.01$.
This table reports the linear regression results of the improvement in economic performance of agent B over A as a function of Θ_m^n. In all cases the independent variable is the standardized Θ_m^n, denoted by $\tilde{\Theta}_m^n$ such that it has a mean of 0 and a standard deviation of 1. Each column corresponds to a different dependent variable. For a given performance metric Y, the dependent variable is given by $Y^B - Y^A$, i.e. the improvement in the performance metric Y for choosing the MTEP over the MVEP. This is consistent with the differential column from table 2. In line with the performance metrics from section 4.3, Mean, Std, SR, CEQ, and TO refer to the annual mean return, volatility, Sharpe-ratio, certainty equivalent return, and turnover. The statistics are computed for different d and sample size $T = 60, 70, \ldots, 120$. Panel (a) reports the results using the four Fama-French industry data sets, for $d = 10, 17, 30, 48$. In Panel (b), we randomly sample $d = 10$, 20, 30, 40 industry portfolios from the 48 FF-industry data set 100 times with a fixed seed to control for portfolio size. The same procedure is applied to the stock data, in which we sample randomly a similar number of stocks from the 50 S&P 500 data set A (Panel (c)) the 50 S&P 500 data set B (Panel (d)). The returns are monthly and the out-of-sample period dates between September 1986 and December 2015. In all cases, we set $m = 1.5 \times m_{\min}$.

the IVIT is associated with a 5.8% increase (15% decrease) in the annual Sharpe-ratio (monthly portfolio turnover). For the sampled 50 S&P 500 A data, these effects are relatively smaller with 4.8% increase in the SR and 6.3% decrease in the portfolio turnover. Nevertheless, we note that the IVIT effect is less economic for the smaller large cap stocks in the sample, even though the IVIT is statistically significant for all five metrics.

It is worth discussing the more robust statistical and economic evidence in terms of reduced portfolio turnover. Such evidence should not be surprising to some extent. Note that by construction, a positive IVIT is associated with a lower estimation risk. Estimation risk denotes the deviation of the estimated portfolio from the actual portfolio. Thus, in seeking the optimal portfolio, the investor incurs higher turnover if the ex-ante constructed portfolio exhibits more significant discrepancy with the ex-post one.

5. Further implications

In this section, we discuss a number of important aspects of our analytical framework and stress its implications in terms of partial index-tracking and shrinkage.

5.1. Partial tracking

The decision-making problem from equation (26) should not be restricted to either $\varepsilon = 1$ or $\varepsilon = 0$. Given the expected out-of-sample utility framework by Kan and Zhou (2007), the optimal partial tracking ε can be attained by

$$\varepsilon^* = \operatorname{argmax} \left\{ \mathbb{E}\left[U_m\left(\hat{x}_m(\varepsilon)\right)\right]\right\}. \tag{49}$$

The optimal partial-tracking, therefore, can be represented as a function of the IVIT. That is, to what degree should the decision-maker track the index in order to maximize her expected out-of-sample utility? Nonetheless, we note that the optimal tracking level, ε^*, is an oracle one, which is infeasible in practice and is subjected to estimation error. For practical implementation, the decision-maker needs to estimate ε^* and, hence, the corresponding Θ. While the choice of $\varepsilon \in \{0, 1\}$ in our analysis seems arbitrary, our investigation refrains from estimation error arising from $\hat{\varepsilon}^*$ or $\hat{\Theta}_m$. For this reason, our analysis focuses on the out-of-sample implications given $\varepsilon = 1$ or $\varepsilon = 0$ rather than a strategy that switches between the two funds based on an estimated $\hat{\varepsilon}^*$.

The introduction of the main portfolio problem that allocates between fund A and B can be represented in a different manner. Given the idea of partial tracking, the portfolio $x_m(\varepsilon)$ from equation (17) can be attained using a mean-variance optimization problem with a constraint on the tracking error. In this regard, the optimal portfolio is the solution to the following optimization problem:

$$\max_x \quad \mu^\top x - \frac{\kappa}{2} x^\top \Sigma x$$

$$\text{s.t.} \quad \mathbf{1}^\top x = 1$$
$$\text{Var}[R^\top x - \tilde{r}_b] \le \omega, \tag{50}$$

with κ is a free parameter and ω denoting the maximum tracking error that the investor is willing to bear. By forming the Lagrangian, the optimal portfolio is the one that maximizes

$$\mu^\top x - \left(\frac{\kappa}{2} x^\top \Sigma x + \delta x^\top \Sigma x - 2\delta\sigma_b^2 \beta^\top x\right) \tag{51}$$

subject to $\mathbf{1}^\top x = 1$, with δ denoting the Lagrangian multiplier. If we let $\kappa_B/2 = \kappa/2 + \delta$, then the above objective function becomes

$$\mu^\top x - \left(\frac{\kappa_B}{2} x^\top \Sigma x - 2\delta\sigma_b^2 \beta^\top x\right), \tag{52}$$

which is a general form of the optimization problem from equation (8).

Consistent with the expected out-of-sample utility para digm by Kan and Zhou (2007), the convex combination approach studied in this paper provides a direct way to determine the optimal tracking level. For a given κ_B, the partial index-portfolio can be presented in a similar manner to the two-funds combination portfolio, $x_m(\varepsilon)$. The control variable ε denotes the degree of tracking and, hence, the Lagrangian multiplier, δ. Hence, a more general formulation allows the agent to determine the degree of index-tracking, i.e. setting the constraint for the optimization problem from equation (50).

5.2. *Index-tracking and shrinkage*

The idea of index-tracking has direct implications on shrinkage. To illustrate the shrinkage idea, consider the portfolio from equation (26) and rewrite in the following manner

$$\hat{x}_m(\varepsilon) = \hat{\alpha}_0 + \frac{1}{\kappa_m} \hat{\mathbf{B}} \left[\hat{\mu} + \kappa_m \varepsilon \left[\hat{\mathbf{c}} - l\hat{\mu}\right]\right]. \tag{53}$$

Letting $\tilde{\varepsilon} = \varepsilon\kappa_m l$, the above portfolio corresponds to

$$\hat{x}_m(\tilde{\varepsilon}) = \hat{\alpha}_0 + \frac{1}{\kappa_m} \hat{\mathbf{B}} \left[\hat{\mu}\left[1 - \tilde{\varepsilon}\right] + \frac{1}{l} \hat{\mathbf{c}} \tilde{\varepsilon}\right]. \tag{54}$$

If the decision-maker is Bayesian and holds the beliefs that the capital asset pricing model (CAPM) is correct, then portfolio $\hat{x}_m(\varepsilon)$ is an MVEP in which μ is shrunk by the CAPM. To see this, let $\kappa_m = a\hat{\mu}_b/\hat{\sigma}_b^2$ for some constant $a > 0$ and $\hat{\mu}_b$ is the estimated mean return of the index. Then the target mean vector in equation (54) denotes the one implied by the CAPM. Indeed, the result from (54) is consistent with the Bayes-Stein shrinkage approach (see e.g.Jorion 1986). Additionally, it denotes the confidence of the agent in the CAPM as a pricing model (Pástor 2000).

The MTEP provides an additional justification in the context of shrinking the covariance matrix (Ledoit and Wolf 2003). Suppose that the decision-maker is indifferent about the mean returns of the assets, i.e. $\mu = \mathbf{1}\mu_c$ for some constant μ_c. Following the formulation from equation (52), her objective function can be described by minimizing the following

$$\frac{\kappa_B}{2} x^\top \Sigma x - 2\delta\sigma_b^2 \beta^\top x \tag{55}$$

subject to $x^\top \mathbf{1} = 1$. With some adjustment, the same equation can be written as

$$x^\top \left[\frac{\kappa_B}{2} \Sigma - \delta\sigma_b^2 \left[\beta \mathbf{1}^\top + \mathbf{1}\beta^\top\right]\right] x. \tag{56}$$

Recall that Let $\kappa_B/2 = \kappa/2 + \delta$ according the partial index tracking discussion. If we assume that $\kappa/2 = 1 - \delta$, then the optimal partial tracking error portfolio is consistent with the global minimum variance that minimizes the portfolio variance when the covariance matrix is given by

$$\tilde{\Sigma} = x^\top \left[\left[1 - \delta\right] \Sigma + \delta \left[\Sigma - \sigma_b^2 \left[\beta \mathbf{1}^\top + \mathbf{1}\beta^\top\right]\right]\right] x. \tag{57}$$

In line with DeMiguel *et al.* (2009a), equation (57) underlines the fact the tracking error constraint imposed on the mean-variance optimization in equation (50) corresponds to shrinking the covariance matrix. Since the parameter δ denotes the degree of index-tracking and, hence, the control variable ε, equation (57) establishes the direct link between partial index tracking and shrinking. Additionally, such intensity can be determined based on either the expected out-of-sample utility framework by Kan and Zhou (2007) or the mean-squared error of the portfolio.

Based on the above insights, we conjecture that the IVIT is revealed in terms of shrinkage. While shrinkage techniques induce a bias in the estimates, they also trade-off bias for variance. Consistent with the literature on shrinkage-based techniques for portfolio selection, (see e.g. Ledoit and Wolf 2003, Xing *et al.* 2014), our findings confirm that the IVIT is most evident in cases associated with high estimation error.

6. Summary

Despite their ex-ante loss of utility, index-tracking optimal portfolios exhibit improved ex-post gains in performance, relative to strategies that are ex-ante MV optimal, under the conditions identified in this paper. Our theoretical and empirical findings stress the importance of index-tracking in mitigating estimation error and enhancing the portfolio out-of-sample performance. Our findings can be justified from a shrinkage point of view. Shrinkage techniques have been widely researched in portfolio selection.† Consistent with the wisdom that 'imposing the wrong constraints help' (Jagannathan and Ma 2003), imposing an index-tracking constraint helps in achieving optimal mean-variance portfolios out-of-sample. The proposed opportunity cost of not index-tracking is equivalent to the opportunity cost of not shrinking. Our analysis can be extended to find the optimal index-tracking level following the idea of *partial tracking* by Edirisinghe and Zhao (2021). In shrinkage terminology, this is referred to as the shrinkage intensity.

† See e.g. Jorion 1986, Ledoit and Wolf 2003, DeMiguel *et al.* 2009a, Xing *et al.* 2014, Li 2015, Kremer *et al.* 2020, Ledoit and Wolf 2017.

In terms of empirical stand-alone analysis, a number of potential directions could follow. First, what index should investors track? In terms of empirical asset pricing, investors may consider different indices. For instance, investors may decide to track growth versus value stocks - given the recent underperformance of the latter over the last two decades (Lev and Srivastava 2019). Second, one could determine the optimal tracking-intensity using cross-validation given a predefined objective function. For example, this could be determined using the Sharpe-ratio of the holdout sample. Different analysis could rely on a supervised machine learning, in which the investor identifies the regime in which tracking is optimal ex-ante given certain criteria. Therefore, based on a predictive model, the investor could determine the probability of index-tracking regime and, hence, the optimal intensity. We leave both ideas for future research.

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Disclosure statement

No potential conflict of interest was reported by the authors.

Supplemental data

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