Aumann and Game Theory

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Abstract

This talk, given October 11, 2021, at the *Colloquium in Honor of Robert Aumann*, organized by the Université Panthéon–Assas, analyzes several aspects of Aumann's contribution to game theory.

It is both a pleasure and an honor to present here some comments on "Professor Aumann and Game Theory." I will divide my contribution into three parts:

(1) Aumann's achievements,

(2) Aumann's incentives,

(3) Aumann's legacy.

1 Aumann's achievements

To describe Robert Aumann's achievements is a difficult and easy task. Difficult, because of the large amount and huge variety of material to handle: two volumes of *Collected Papers* [2000], 800 pages each, which cover works until 1995, and 32 publications since. Easy, because one can rely on a number

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of previous presentations provided by eminent scholars on several occasions: prizes, conferences, anniversaries, etc. see, e.g., Hart and Neyman [1995].

I will start with a section on repeated games, which is one of the main topics mentioned by the Nobel committee in awarding the Nobel memorial prize in economic sciences in 2005 to Robert Aumann (and Thomas Schelling) for "having enhanced our understanding of 'Conflict and Cooperation' through game theory analysis."

1.1 Repeated games

Following Aumann's terms, the analysis of interactive decisions and dynamics aims to account for phenomena, such as cooperation, altruism, revenge, threats—aspects which may at first seem irrational in terms of the usual selfish utility-maximizing paradigm.

We consider here relationships among rational decision-makers (players) that last for a long time. In the simple case of discrete time, it takes the form of multistage interaction. The duration allows for strategic behavior that takes into account past play and reacts to previous experience. More precisely, the study of repeated games deals with models where players face the same game at each stage.

1.1.1 Folk, perfect, strong

Let us analyze first the *complete information* case, with standard signaling, where all players know the one-shot game G that is repeatedly played, leading to the 'supergame' G^*. Thus, at each stage n, each of the players i in a finite set I, chooses a move a_n^i in a finite set A^i. The induced profile a_n of moves: (i) generates a vector of payoffs $G(a_n)$, and (ii) is announced to the players.

A basic result in this area was already known in the 1950s. It is referred to as the *Folk Theorem* and states the following:

> *The strategic (Nash) equilibrium payoffs of G^* are the feasible and individually rational payoffs of G.*

Individually rational (IR) for player i means that her payoff is at least her minmax v^i, which is what this player can ensure facing any strategy of her opponents. Three remarks are in order:

(1) This statement gives an explicit description of the equilibrium outcomes in G^*.

(2) It links non-cooperative analysis in G^* to cooperative concepts (feasibility) in G (part of Nash's program).

(3) The proof involves new tools (plans and threats) that will prove to be basic in all subsequent analysis.

The idea of the proof is rather simple: One shows that feasible payoffs in G can be realized through a pre-specified plan, which is a play in G^*. Moreover, one can show that starting from an IR payoff, the future payoff after each stage will share the same IR property. The equilibrium strategies then require the players—all along while observing the previous moves—to follow the plan of using a (joint) punishment strategy in case of a defection: namely to reduce the payoff of player i below v^i if she deviates. The IR condition shows that there are no unilateral and profitable changes of strategy. A modern exposition appears in Aumann's survey of repeated games in the *Essays in Honor of Oskar Morgenstern* [1981].

Later on, building on the notion of *perfection* introduced by Selten [1975]—which requires the punishment to be 'credible,' namely to generate itself an equilibrium in the induced subgame, after any history of play—Aumann and Shapley [1994], and independently Rubinstein [1994], achieved a refined version of the Folk Theorem:

> *The perfect equilibrium payoffs of G^* are the feasible and individually rational payoffs of G.*

The basic idea is to define the strategies inductively as follows: follow the plan and punish for a long but finite number of stages the last defector, if there is any, and then resume the initial plan.

In fact, one of the first papers by Aumann, "Acceptable Points in General Cooperative n-Person games" [1959], maybe his first pure game-theoretical paper, deals with the related notion of *strong equilibrium* (in the sense that no coalition of players can simultaneously gain by modifying their strategies) in the repeated game G^*. The result is as follows:

> *The strong equilibrium payoffs of G^* correspond to the core (more precisely the β-core) of G.*

The previous remarks (1) and (2) apply again.

Even if some ideas related to the Folk Theorem appear in several places before this paper, e.g., in Luce and Raiffa's book *Games and Decision* [1957], to obtain his result, Aumann produced the first formal and explicit definition of G^*, including a precise specification of the strategies, the induced probability distribution on plays and the corresponding payoffs. Furthermore, Aumann introduced and studied in this paper cooperative games with non-transferable utility (NTU), which would prove to be fundamental in further developments.

Recall that the Folk Theorem says that repetition *enables* cooperation, in the sense that Pareto optimal outcomes can be obtained at equilibrium. To *enforce* cooperation, instead, is an altogether different requirement leading to completely new issues. For a specific class of games, two-person games with common interest, Aumann (in a joint paper with Sorin [1989]) proved a result of this type under hypotheses involving bounded rationality in two ways: (i) bounded recall of the strategies, and (ii) perturbations (in the sense of 'trembling hand') of the type of the players.

1.1.2 Incomplete information

Still in the framework of repeated games, let us consider now the case of *incomplete information*, where some players may not possess a piece of relevant information on the actual game being played (later on called *state of nature*).

Following the formulation introduced by Harsanyi [1967], this model was formally introduced and studied by Aumann and Maschler in a series of reports that *Mathematica* (a consulting firm founded by Morgenstern and involving, among others, Debreu, Harsanyi, Kuhn, Maschler, Mayberry, Scarf, Selten, Shubik, and Stearns) prepared for the US Arms Control and Disarmament Agency in the years 1966–1968.

The repetition in this case has a complex impact on the behavior of the players since it enables them to learn from (the observation of) the actions of the others some meaningful information. In particular, (i) a player may want to conceal or reveal her private knowledge of the state by using a so-called type-dependent strategy; (ii) she faces the issue of interpreting her observations—namely deducing from the moves of her opponents revised posteriors on the unknown state (which raises the issue of bluffing or cheating).

Already in the two-person–zero-sum case where only one of the players is informed (lack of information on one side), this leads to a deep and elegant theory showing that partial revelation may be optimal and allowing to quantify precisely the optimal amount of information to transmit. The main result obtained by Aumann and Maschler states:

> *G^* has a value which is given by the concavification of the value of the one-shot nonrevealing game $\overline{G}$.*

Again, this gives a clear and complete characterization. Moreover, it depends only on a property of the one-shot game—which is not related to cooperative behavior—but involves the set of nonrevealing strategies.

Further results (by Aumann, Maschler, and Stearns) proved that with lack of information on both sides, the supergame G^* may have no value: it is always better to reveal one's private information only after exhausting the information of the opponent, and strategies sharing this property may not be equilibria. Aumann, Maschler, and Stearns also introduced fundamental tools in the non-zero-sum case: alternating sequences of revealing moves and 'jointly controlled lotteries.'

Very similar techniques have been used later by Aumann and Hart [2003] to study the model of a one-shot game preceded by a long conversation phase (cheap talk games), building on previous work by Hart [1985] on non-zero-sum–incomplete-information games and results of Aumann and Hart [1986] on bimartingales.

This theory also leads to the first study of the intricate link between the finitely repeated game and the supergame G^*. (I will come back to this point later on.)

These results, which were first published in the reports from 1966–1968, appeared later as a book, *Repeated Games with Incomplete Information* by Aumann and Maschler with the collaboration of Stearns [1995], with more than 100 pages of postscripts, which are a wonderful example of the way Aumann presents a unified and coherent view of the domain and its ramifications.

The next topic, under the title "Information and rationality," deals with interactive knowledge and extensions of the equilibrium concept.

1.2 Information and rationality

1.2.1 Correlated equilibrium

Extending on Harsanyi's view of a mixed strategy as a map from private signals to moves, in "Subjectivity and Correlation in Randomized Strategies" [1974], Aumann introduced the notion of a correlated equilibrium of a game G as a Nash equilibrium of the game extended by an initial signaling phase described by an information structure.[1] He then proved the celebrated canonical representation of correlated-equilibrium distributions (CED), which describe the law induced on vectors of moves by the equilibrium strategies:

> *CED are publicly known probabilities on the set of profiles such that each player, knowing only her own component, cannot do better than following the 'recommendation.'*

In particular, this property implies that the corresponding set is a convex polyhedron.

1.2.2 Agreeing

More generally, to study the rationality of the players, one has to define their information. Aumann introduced the right formulation through private signaling partitions, individual knowledge operators, and then mutual (public) knowledge and all the hierarchy of shared information levels leading to *common knowledge* (CK). He then achieved the famous 'agreement theorem' [1976b]:

> *If two players have the same priors and their posteriors for an event A are common knowledge, then they must be equal.*

1.2.3 Interactive knowledge

Back to correlated equilibrium, in "Correlated Equilibrium as an Expression of Bayesian Rationality" [1987a], Aumann proved a kind of universal representation of correlated equilibrium:

> *Assume that the players share a common prior. If they are Bayes rational at each state of the world, then their play is a correlated equilibrium.*

[1] Editor's note: See the contribution by Françoise Forges in this volume.

The hypothesis is equivalent to CK of rationality.

Aumann produced a detailed construction of the theory of epistemic knowledge for a course at the Cowles Foundation at Yale, in 1989, which was published much later under the title "Interactive Epistemology" [1999a,b]. In this study, Aumann compares in particular the *semantic approach* to modeling knowledge in terms of partitions of the space of states of the world and the *syntactic approach*, where a state of the world is the collection of assertions that hold at this state. He then covers both the constructions based on: (i) knowledge (partitions and operators), and (ii) probability (the Bayesian viewpoint).

Let us also mention Aumann's work on "Epistemic Conditions for Nash equilibria" (with Brandenburger [1995]), where, starting from the concept of Nash equilibrium (Nash [1950]), which requires only mutual knowledge of the strategies and that each player is rational and knows her own payoff function, they elaborate sufficient conditions for equilibria of conjectures.

All these topics are beautifully covered in a chapter on "Incomplete Information" (with Heifetz [2002]) in the *Handbook of Game Theory*.

1.2.4 More on rationality

One should also mention important works on: self-enforcing proposals [1990], a desirable property not shared by all equilibria; backward induction and CK of rationality [1995] (including the discussion of counterfactuals) by assuming rationality at nodes that are not reached under rational behavior; bounded rationality [1997] (including aspects of information, memory, recall, complexity, anticipations, etc.); 'irrationality' [1992], a framework which in particular avoids the hypothesis of Bayesian rationality at all states of the world (typically mutual knowledge of rationality holds at some state and then eventually at higher levels) and leads to the fascinating notion of the degree of irrationality.

A third area where Aumann's contribution is fundamental concerns non-atomic games and competitive markets.

1.3 Non-atomic games and competitive markets

The main issue here is the modelization of perfect competition in an exchange economy. It corresponds to situations where the individual influence of an agent on the market is negligible and hence justifies the fixed-price hypothesis. The completely new paradigm introduced by Aumann is to consider directly a continuum of participants rather than studying the limit of large games.[2]

As Aumann recalled (in the introduction to the *Collected Papers*, p. 157), this idea came from two sources: a seminar by Milnor and Shapley on Shapley values of voting in oceanic games (published much later [1978]) and a work by Debreu and Scarf [1963] on the core of markets with countably many players.

Following this approach, Aumann establishes the relation between the Walrasian equilibrium and the core of a non-atomic exchange economy.

It should be noticed that even if the notion of Walrasian equilibrium can be obtained as an equilibrium of a strategic game, by adding an agent with prices as strategies, prices do not appear in the determination of the core allocation, and the conceptual connection between the two concepts is far from obvious.

Actually, the result can be decomposed into two parts: one is the 'equivalence principle,' established in "Markets with a Continuum of Traders" [1964a]; the other deals with the existence of competitive equilibrium (where, as a consequence of the non-atomic aspect, no convexity hypothesis on the utility of the agents is needed), established in "Existence of Competitive Equilibria in Markets with a Continuum of Traders" [1966].

These advances were followed by a cascade of results. First, a similar property was obtained for the Shapley value in the TU (transferable utility) case, see the monumental book *Values of Non-Atomic Games* (with Shapley [1974]), and later on in the NTU (nontransferable utility) framework, in "Values of Markets with a Continuum of Traders" [1975]. This result was further extended to other cooperative concepts and even to an axiomatic approach. To quote Aumann: "Intuitively, the equivalence principle says that the institution of market prices arises naturally from the basic forces at work in a market, (almost) no matter what we assume about the way in which these forces work" (Aumann's article on "Game Theory" in *The New Palgrave* [1987b]. This sounds very much like a kind of Central Limit Theorem, where the normal law appears as a universal attractor as soon as randomness and independence occur.

[2]Editor's note: See Enrico Minelli's contribution in this volume.

1.4 Other game theoretical contributions

I briefly enumerate here a partial list of further topics where Aumann's contribution was substantial.

(1) Strategic games: definition and properties of almost strictly competitive games [1961], extension of Kuhn's theorem on the equivalence of mixed and behavioral strategies to infinite extensive games [1964b], comparison of equilibrium and minmax analysis, "On the Minimax Principle" (with Maschler [1972]), information structure and condition for the purification of mixed strategies (with Katznelson, Radner, Rosenthal, and Weiss [1983]), ex-ante, ex-post, and interim evaluations: rational expectations in games (with Dreze [2008]).

(2) Subjective probability and utility: a definition and construction of subjective probability (with Anscombe [1963]).

(3) Cooperative games: study of basic concepts: NTU games, von-Neumann solution, core, value, kernel, bargaining set (with, among others, Maschler, Peleg, Shapley) as well as evaluation of coalition structures (notably with Dreze and Myerson).

In fact, most of the content of volume two of the *Collected Papers* [29] is devoted to these topics, including fundamental applications to economics, such a analytical investigations of power and taxes (with Kurz and Neyman), monopolies, public goods (with Gardner and Rosenthal), satiation and fixed prices (with Dreze), etc.

The next section collects deep mathematical results.

1.5 Mathematical contributions

A large part of them deals with fundamental measurability issues, such as: "Spaces of Measurable Transformations" [1960], "Borel Structures for Function Spaces" [1961], "On Choosing a Function at Random" [1963], "Integrals of Set-Valued Functions" [1965], "Variational Problem Arising in Economics" (with Perles [1965]), "Random Measure Preserving Transformations" [1967b], "Measurable Choice Theorem" [1969], "An Elementary Proof that Integration Preserves Uppersemicontinuity" [1976a], "Bi-Convexity and Bi-Martingales" (with Hart [1986]).

Unfortunately, due to time and constraints imposed by the topic, I have to skip any comment on the deep impact and numerous consequences of these works.

2 Aumann's incentives

While various indices (citations, references, workshops and colloquiums, etc.) produce a precise evaluation of the importance of Aumann's contributions (from the consumer's point of view, say), we will try to describe here an alternative perspective, on the producer's side. Let us start with: "Aumann's view."

2.1 Aumann's view

This section deals with Aumann's own presentation and interpretation of his work and relies very much on:

(a) Aumann's own expository essay "What is Game Theory Trying to Accomplish" [1985],

(b) the numerous comments that appear in the presentation of the *Collected Papers* [2000],

(c) the post-scripts in the book with Maschler *Repeated Games with Incomplete Information* [1995], and

(d) the long, rich, and substantial interview with Hart [2005b].

In fact, several alternative trajectories through the results mentioned above are feasible, leading to new and different perspectives of the list of accomplishments. In particular, such an approach reveals structural links between most-cited papers, or well-known results, and Aumann's early attempts to solve some questions related to fundamental problems. Following this path, allows one to contemplate all the underlying work and reflexions on the way up to reaching a clear formulation. This includes sometimes experimentation in new hypothetical directions or the study of lots of specific examples with painful but unavoidable computations, i.e., in repeated games with incomplete information.

One could, for instance, focus on a classification based on the mathematical formulation of the topics, such as the following.

(1) Mathematical tools:

(1.1) Blackwell's [1956] approachability theory appears in the analysis of NTU games with vector payoffs in "Acceptable Points in General Cooperative n-Person Games" [1959] as a way to describe the payoff of a group of players within this cooperative approach. It is then used in the general framework of NTU cooperative games as a link between the $\beta-$core of G and the $\alpha-$core of G^*. Finally, this tool appears in the strategic analysis of the incomplete-information model, the components of the vector payoff being then indexed by the different states of the world (Aumann and Maschler [1995]).

(1.2) In a similar manner, measurability issues occur for defining strategies in infinite games and related concepts. This relates to a series of problems like choosing a function at random, defining a Borel structure and measurable transformations on function spaces, and eventually leads to the extension of Kuhn's theorem, or more generally, the study of games on function spaces and the expression of strategies in terms of random variables rather than distributions. On the other hand, very similar measurability concepts and results are requested to deal with non-atomic games, like the existence of a measurable selection and regularity properties of correspondences [1969]. Then, these new tools proved to be fundamental in variational problems but also utility theory [1976a].

Another trajectory into Aumann's work could be via topics, such as *interactive rationality.*

(2) Interactive rationality:

(2.1) Obviously, the first aspect of this term deals with *interaction.* But one immediately realizes that it involves two fields:

- interactive decision theory, and
- interactive information theory,

which, while deeply connected, need to be analyzed separately. The first topic deals with alternate iterations of individual dominance relation (in the spirit of rationalizability), while the second is concerned with alternate iterations of private knowledge operators. The link is obvious through the construction of strategies based on information, but it also underlines the fundamental difference between games—strategic interaction—and the one-person case, where such iterations do not exist.

(2.2) A priori, interim, and a posteriori evaluations—another significant issue. This issue emerges with the use of mixed strategies, see "Some Thoughts on the Minimax Principle" (with Maschler [1972]), but is also deeply related to the canonical representation of correlated equilibria and appears under study again, forty years later, with a new perspective in the paper with Dreze [2008] "Rational Expectation in Games."

(2.3) In the same vein one could mention the need to introduce 'irrationality' to study rational behavior (a point already touched upon by von Neumann and Morgenstern [1944]). A first step is to define 'irrational behavior,' then to study how to react to 'irrational behavior' of the opponent, the issue being how, as a rational player, to take advantage of the eventual opportunity of behaving irrationally. Obviously, thoughts in this direction directly lead to links with bounded rationality and reputation phenomena.

To go a step further, since according to Aumann, game theory is a tool for telling us where incentives will lead (see, for instance, the interview with Hart [2005b], p. 737), one could ask: What are Aumann's incentives?

2.2 What is Aumann trying to accomplish?

I will underline here four fundamental aspects of Aumann's scientific procedure.

2.2.1 A unified game theory

First, the desire to promote a *unified game theory*. "Unlike other approaches to disciplines like economics or political science, game theory does not use different, ad-hoc constructs to deal with various specific issues, such as perfect competition, monopoly, oligopoly, international trade, taxation, voting, deterrence, animal behavior, and so on. Rather, it develops methodologies that apply in principle to all interactive situations, then sees where these methodologies lead in each specific application," Aumann says in the Interview with Hart [2005b, p. 717]. In fact, a large part of the success of applications of game theory lies in its ability to represent and analyze a complex framework by identifying the relevant game-theoretical parameters: Who are the players? What are the feasible moves? Which information is available? What are the outcomes? Then, applying several solution concepts allows one to extract further insights from the initial data. Whereas using an ad-hoc procedure would produce unintelligible outcomes, such a unified approach allows one to evaluate, compare, and interpret the results of the analysis.

2.2.2 Understanding

Given that point of view, it is clear that the first objective is to look for *understanding* rather than discovering some hidden truth. "My main thesis is that a solution concept should be judged more by what it does than by what it is," Aumann says in "What Game Theory is Trying to Accomplish" (p. 5 in the *Collected Papers*). The principal aspects of the methodology are then to unify the analysis and classify the results, the main purpose being to advance the comprehension of the phenomena under study. This means that the decisive steps are to establish and develop: *relationships*.

2.2.3 Relationships

In addition to producing basic comparisons of the issues of the analysis and the associated representation, Aumann's research exhibits specific properties. Namely:

(1) *A systematic development of complementary approaches and viewpoints.* Examples could include: studying both normative and descriptive analysis (following here Morgenstern's tradition); comparing strategic and coalitional game formulations; or, as already in zero-sum games, describing the dual properties of strategies: the attempt to reach some objective or the ability to block the opponent; and similarly in incomplete-information games, where a global view takes into account the use of private information (splitting), the acquisition of information from outside (posteriors), and the prevention against bluffing (vector payoffs).

(2) *A variety of explanations and ways of using the results.* I will illustrate this by the following example of the variational problem from Aumann and Perles [1965]. Given a function $u(x,t)$, the purpose is to study the properties of the maximum of $\int_0^1 u(x(t),t)dt$, under the global constraint $\int_0^1 x(t)dt = x$. Here, as Aumann emphasizes, $t \in [0,1]$ could be time, in a one-person intertemporal utility maximization problem, ($x(t)$ being the consumption at time t); or t could represent an agent in the framework of an optimal allocation issue ($x(t)$ being the consumption of agent t) in a non-atomic population framework. Both interpretations are important and enrich each other.

(3) *The precise discussions on the hypotheses and their consequences.* For example, in the framework of the Folk Theorem, an extensive analysis will underline the importance to take into account the length of the interaction and the kind of evaluation used by the players. Similarly, the specification of the signals and information (internal and external along the play) of the players is crucial. A further step is to consider the model with a sequence of opponents and then to describe the link to non-atomic populations with random matching, where the players are not facing the same opponent, there is no memory, and no signals occur.

(4) *The various interpretations of the assumptions.* The main issue here is not to find the 'right' one, but to describe precisely the consequences and to understand how and where assumptions lead. A particularly beautiful example is given by the interpretation of the supergame G^* in Aumann and Maschler [1995, p. 136]. Here, several approaches to the model are successively proposed: repeated interaction with long, but unknown length; emergence of limit-optimal behavior (rather than focusing on the sequence of stage payoffs); example of bounded rationality (where the exact value of n, the number of repetitions, is too complex to be known); use of rules of thumb (qualitative rather than quantitative perspective, analogy); stationarity aspect (facing the same future, exhibiting a state variable); no last stage (no backward-induction effect). And they go on adding comments on the link with discounted games and a discussion of the continuous-time approach.

In "What Is Game Theory Trying to Accomplish," Aumann [1985] develops a comparison between mathematics and art. His practice is in fact reminiscent of Picasso's words: "Je ne cherche pas, je trouve." Somehow, by multiplying the approaches and viewpoints, Aumann adds new dimensions to the situation under study and generates an innovative light on its fundamentals. Obviously, this ability to establish deep and significant relationships is based on an exhaustive knowledge of the field (landscape) and very strong mathematical capacities (techniques of observation).

2.2.4 Simplicity

The fourth aspect I would like to underline is *simplicity*, which is often linked, in Aumann's case, to elegance. Basically, this property—within the precise analysis of some issue—could often be expressed as: "asking the right question." However, reaching this achievement is usually the result of a long process with a lot of preliminary partial and frustrating steps, without any a priori landmark. One has to get rid of all irrelevant hypotheses, assumptions or particularities to let emerge a more general and deep principle that reveals the underlying structure of the phenomena. "This is what theory is about: when situations that are completely paradoxical and unintelligible to anyone, even in the simplest cases, become nearly self-evident when the results are understood" (Jean-François Mertens). A beautiful typical example is the 'agreement theorem' [1976b], which is a deep result (with a lot of applications in various fields from consensus in social learning to no-trade phenomena in auctions or the value of information in economics) with an elementary proof—once the right formulation and tools are provided.

To summarize in Aumann's own words: "Game theoretic solutions concepts should be understood in terms of their applications and judged by the quantity and quality of their applications; each of them unifies a different aspect of rationality in interactive decision-making" ("What Game Theory is Trying to Accomplish," p. 38 in the *Collected Papers*).

I present now the last part, which is a very partial view of what Aumann leaves us as a heritage.

3 Aumann's legacy

In a first section, I will recall the direct impact of Aumann on the game-theory community.

3.1 Aumann's umbrella

3.1.1 Students

It seems natural to start with the list of his PhD students: 1. Bezalel Peleg, 2. David Schmeidler, 3. Shmuel Zamir, 4. Elon Kohlberg, 5. Benyamin Shitovitz, 6. Zvi Artstein, 7. Eugene Wesley, 8. Sergiu Hart, 9. Abraham Neyman, 10. Yair Tauman, 11. Dov Samet, 12. Ehud Lehrer, 13. Yossi Feinberg, 14. Itai Arieli, 15. Uri Weiss, 16. Yosef Zohar. In addition to his students, Aumann had a tremendous impact on all the members of the Israeli game-theory community, and also on this community at an international level (in particular in Belgium and France).

3.1.2 Institutions

Let me mention some of the institutions where Aumann spent a significant amount of time and had a deep influence on their members: The Hebrew University (after MIT and Princeton), Yale, Stanford, Berkeley, CORE at Louvain-la-Neuve, Stony Brook, the Center for Rationality in Jerusalem (cofounder).

3.1.3 Programs, conferences, and summer schools

Aumann played also a decisive role in programs, conferences, and summer schools. The list includes: IMSSS (Stanford), the Institute for Advanced Studies (Jerusalem), MSRI (Berkeley), the Center for Game Theory at Stony Brook (International conferences, NATO Advanced Studies Institute), the Center for Rationality (Jerusalem). At these events, especially the special programs, Aumann was not only one of the organizers, but his presence attracted lots of participants. He energized the scientific exchange and was among the most active members at all sessions, giving and organizing seminars or ad-hoc workshops. From this perspective, his personality and scientific behavior have qualified Aumann as a universal attractor for the field of game theory.

3.1.4 Surveys

Another fundamental aspect of Aumann's activity is the number of expository surveys, introductory courses, and presentations that he gave at several occasions, among which: the article on NTU cooperative games in the first 'Morgenstern Festschrift' [1967a], the article on repeated games in the second 'Morgenstern Festschrift' [1981], his exposition of the Shapley value at the International Congress of Mathematicians in Helsinki 1978, the already mentioned "What Is Game Theory Trying to Accomplish" [1985] and the course on interactive epistemology held at Yale in 1986, the N. Schwartz Memorial Lecture in 1986 on rationality and bounded rationality, the chapter on "Game Theory" in *The New Palgrave* [1987b], the article on irrationality in the 'Hahn Festschrift' [1992], and the chapter on "Incomplete Information" in the *Handbook of Game Theory* 3 (with Heifetz [2002]).

3.1.5 Societies

Aumann has been deeply involved with the *Econometric Society*, its council, its congresses, and its journal *Econometrica*. He played a crucial role in the *Game Theory Society*. Being one of its founders, in 1999, he has been a member since, was its first president (1999–2004) and has been a leading figure at all its congresses. Aumann was also active in the *Israel Mathematical Union*.

3.1.6 Journals

Another important aspect of Aumann's activity is his energy to promote and develop the impact of game theory through scientific journals. Aumann was a member of the initial editorial board of the *International Journal of Game Theory*, founded by Morgenstern (1971), and still holds this position. He was also part of the team which launched *Mathematics of Operations Research* (1976) and was its first area editor for game theory. I will not mention all the journals where he was involved, but I want to recall his fundamental editorial role and energetic editorial comments for the encyclopedic, highly cited and respected *Handbook of Game Theory*, in three volumes, co-edited with S. Hart [1992, 1994, 2002].

3.2 Aumann's breakthroughs

In this section, I will briefly mention some topics which, from my point of view, have completely changed in nature after Aumann's contribution:

(1) non-atomic models in economics and non-atomic mathematical tools: correspondences and selections,

(2) analysis of nontransferable-utility cooperative games,

(3) formalization of knowledge,

(4) value of non-atomic games (with Shapley [1974]),

(5) correlated equilibrium, and

(6) repeated games.

More specifically, in an area where I am active, zero-sum repeated games, I would like to select four fundamental advances due to Aumann's contribution:

(6.1) *Splitting lemma and $Cav\, u$ theorem.* I already briefly mentioned this item, which when understood, looks so natural, but I want to underline that this is the basic tool in all principal-agent issues involving informational analysis, or more generally signaling design and all of information economics.

(6.2) *The use of Blackwell approachability for incomplete information.* This topic also was raised before (section 2.1). Aumann's great idea is to use this approach with vector payoffs (already present in the 1959 paper "Acceptable Points") within a completely new paradigm, where the component is itself a random variable.

(6.3) *The use of recursive structure.* This approach builds on Shapley's [1953] construction for the value of stochastic games, which extends the dynamic-programming principle of Bellman. The minimax theorem applies to this framework and the stochastic state of the game, publicly known, plays the role of a state variable. This allows one to write a recursive formula satisfied by the value. In games with incomplete information, the use of the minimax theorem is to construct an equivalent game, in terms of the value—*not in terms of strategies*—where after each stage, the strategy of the opponent is revealed, then, posteriors can be computed, playing the role of state variables as in a stochastic game, and finally a recursive formula holds. An extension to a general framework, using the universal belief space (Mertens and Zamir [1985]) as state space, is described in Mertens, Sorin, and Zamir [2015].

(6.4) *Asymptotic vs uniform value*. Aumann introduces two approaches to deal with long games. The first one considers a sequence of repeated games with increasing length n and the associated sequence of value functions v_n. A corresponding limit will be an *asymptotic value*. An alternative framework focuses on the robustness of strategies (defined in the game form) in the sense of guaranteeing a certain amount for all vanishing evaluations of the stage payoffs. This gives a notion of minmax and maxmin, and of *uniform value* if they coincide. Aumann's analysis contributes to the two topics, proves that they are not equivalent and that both are important to understand how to deal with private information in long games. Moreover, these concepts apply to the general framework of zero-sum repeated games.

3.3 Aumann's imprint

In addition to the huge impact of Aumann's achievements (each of the topics that I mentioned could be the content of a one-year research course), I would like to underline specific aspects where Aumann's imprint is clear, some of them having already been mentioned:

(1) Considering game theory as a unified approach to interactive decision situations.

(2) The need to deal with important, significant, substantial topics and all their related issues, like rationality, information, bounded rationality or competition and economic mechanisms, etc.

(3) Supporting complementary approaches rather than conflicting judgments.

(4) The importance to test the robustness of the results and to analyze counterexamples. Two clear illustrations would be: (i) the various versions of the Folk Theorem: finite, discounted, undiscounted; (ii) common knowledge and approximate common knowledge (like in the centipede game).

(5) The variety in the interpretation of the results: An analysis is useful if it helps understanding a phenomenon, even if there are difficulties to interpret the result in terms of assumptions. Similarly, a property

may be considered positive or negative, depending on the viewpoint; moreover, both may be interesting. A first example would be the Folk Theorem as a clear and elegant result but with a lack of predictive power, inducing subsequent works on equilibrium selection. Another example is the paper on repetition and cooperation (Aumann and Sorin [1989]), where repetition and strategic behavior enforce cooperation, but for this purpose, bounded recall and perturbations are used and required.

(6) The importance of unexpected relationships. Let us consider the fundamental issue of learning dynamics in games, which would converge to the set of Nash equilibria. Recall that there are general impossibility results based on robust counterexamples (Hofbauer [2011], Hart and Mas-Colell [2013]). However, several learning procedures based on the no-regret rule, going back to Hannan [1957] and building again on Blackwell's approachability, have been studied. A refined version (related to calibration in statistics and requiring internal regret rather than external regret to vanish asymptotically) allows one to prove convergence in the mean of the outcomes to correlated equilibria, see, e.g., the overview of Hart [2005a]. A closer look shows that this is related to the linear structure of correlated equilibria: an elementary proof of existence (Hart and Schmeidler [1989]) is based on the minimax theorem. This leads to a completely unexpected justification of correlated equilibria and a dual approach to Aumann [1987a]. A finite set of players each using independently any procedure sharing the no-internal-regret property, but based upon the jointly induced data, namely, the payoffs that they generate through their moves, will—averaged over time—produce a sequence of distributions on profiles for which accumulation points are correlated equilibria. (Even a proof of existence of correlated equilibria follows—as a consequence of the existence of no-regret procedures).

Let us pause now and have a global look at the field and its evolution.

We met a number of concepts and results so natural that one cannot imagine a period where they were not available ...

Taking all together, Aumann has drastically shaped all developments of game theory for more than half a century and played a unique and fundamental role in our understanding of interactive decision-making. He is a leading figure among the magnificent family of scientists who contribute to realize the wish of David Hilbert:

> "Wir müssen wissen. Wir werden wissen."

Thank you.

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