Signal and System fiche

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Contents

Markdown version on *github*

General Stuff

- **Euler formula**

$$\cos(x) = \text{Re}(e^{ix}) = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin(x) = \text{Im}(e^{ix}) = \frac{e^{ix} - e^{-ix}}{2i}$$

$$e^{ix} = \cos(x) + i\sin(x)$$

$$e^{-ix} = \cos(-x) + i\sin(-x) = \cos(x) - i\sin(x)$$

  - Impulse response $= h(t)$
  - Transfer function $= H(s)$
  - Frequency response $= H(j\omega)$

Signals and Systems

Signals

- A continuous-time signal $x(t)$ is called **periodic** with priod T if for all times t we have : $x(t) = x(t+T)$ (idem for discrete time)
- The **Energy** of a signal:
  - Continuous signal: $\mathcal{E} = \int_{-\infty}^{\infty} |x(t)|^2 dt$
  - Discrete signal: $\mathcal{E} = \sum_{n=-\infty}^{\infty} |x[n]|^2$

- The **Power** of a signal:
  - Continuous time: $\mathcal{P} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$
  - Discrete time: $\mathcal{P} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$

Systems

- A **System** takes a signal as input and outputs a new signal. It is express as : $y(t) = \mathcal{H}\{x(t)\}$ or $y[n] = \mathcal{H}\{x[n]\}$
- Properties:
  - **Linearity**: $\mathcal{H}\{a_1x_1(t) + a_2x_2(t)\} = a_1\mathcal{H}\{x_1(t)\} + a_2\mathcal{H}\{x_2(t)\}$ (idem for discrete time)
  - **Time Invariance**: *if* system input $x(t)$ procuces system output $y(t)$ *then* system input $x(t-\tau)$ produces system output $y(t-\tau)$ (idem for discrete time)
  - **Memory**: The system output only depends on the current system input (idem for discrete time)
  - **Invertibility**: A system is called invertible if distinct inputs lead to distinct outputs (idem for discrete time)
  - **Causality**: A System is causal if its output signal only depends on present and past inputs, but not on future inputs (idem for discrete time)
  - **Stability**: A system $\mathcal{H}$ is *stable* if for all *bounded* input signals $x(t)$, the corresponding output signal $y(t) = \mathcal{H}\{x(t)\}$ is also bounded. (idem for discrete time)

Linear Time Invariant Systems (LTI)

- *Kronecker-delta fucntion*:

$$\delta[n] = \begin{cases} 1, & \text{for } n = 0, \\ 0, & \text{otherwise} \end{cases}$$

- **Impulse Response**: The fundamental upshot is that any LTI system os uniquely characterized by its impulse response.
  - Discrete time: $h(n) = \mathcal{H}\{\delta[n]\}$ is simply the system repsonse when the input is Kronecker-delta fucntion $\delta[n]$. The signal $h[n]$ is called the *inpulse response* of the system $\mathcal{H}\{\cdot\}$. We can characterize the system output signal as:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

  - Continuous time: $h(t) = \mathcal{H}\{\delta(t)\}$ is simply the system repsonse when the input is Dirac delta fucntion $\delta(t)$. The signal $h(t)$ is called the *inpulse response* of the system $\mathcal{H}\{\cdot\}$. We can characterize the system output signal as:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

- **Convolution operation**: The output signal is simply ginven by the convolution of the input signal with the impulse response.
  - Discrete time: $[x * h](n) = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$
  - Continuous time: $(x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$
  - Properties:
    - *Commutative*: $(x * h)(t) = (h * x)(t)$ (idem for discrete time)
    - *Distributive*: $(x * (h_1 + h_2))(t) = (x * h_1)(t) + (x * h_2)(t)$ (idem for dicrete time)

    * *Associative*: $((x * h_1) * h_2) = (x * (h_1 * h_2))$ (idem for discrete time)

- **Composition** of LTI systems
  - Parallel: $y(t) = \mathcal{G}\{x(t)\} = \mathcal{H}_1\{x(t)\} + \mathcal{H}_2\{x(t)\}$. If both $\mathcal{H}_1$ and $\mathcal{H}_2$ are LTI system $\mathcal{G}$ is also an LTI system and its impulse response $g(t)$ is given by: $g(t) = h_1(t) + h_2(t)$ (idem for discrete time)
  - Serie: $y(t) = \mathcal{G} = \mathcal{H}_2\{\mathcal{H}_1\{x(t)\}\}$. If both $\mathcal{H}_1$ and $\mathcal{H}_2$ are LTI system $\mathcal{G}$ is also an LTI system and its impulse response $g(t)$ is given by: $g(t) = \int_{-\infty}^{\infty} h_1(\tau)h_2(t-\tau)d\tau$ or $g[n] = \sum_{k=-\infty}^{\infty} h_1[k]h_2[n-k]$

- **Properties**:
  - **Memory**: An LTI system is *memoryless* if and only if, for some constant a we have: $y(t) = ax(t)$ (idem for discrete time) (idem for discrete time)
  - **Invertibility** An LTI system with impulse response $h(t)$ si *invertible* if and only if there exists a function $g(t)$ such that $(g * h)(t) = \delta(t)$. (idem for discrete time)
  - **Causality**: An LTI system is *causal* if and only if the impulse response function is indetically zero for negative lags: $h(t) = 0$ for $t < 0$ (idem for discrete time)
  - **Stability**: An LTI system is *stable* if and only if the impulse response function absolutely integrable (or summable), i.e., if and only if: $\int_{-\infty}^{\infty} |h(t)|dt < \infty$ or $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$

Fourier Methods for Stable LTI Systems

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Frequency Response of Stable LTI Systems

- **Frequency response** Let us suppose that the input to our stable LTI system is given by $x(t) = e^{jw_0 t}$ or $x[n] = e^{jw_0 t}$. Then the output is given by :
  - Continuous:

$$y(t) = \int_{-\infty}^{\infty} e^{jw_0(t-\tau)}h(\tau)d\tau = H(w_0)e^{jw_0 t}$$

  - Discrete:

$$y[n] = \sum_{k=-\infty}^{\infty} e^{jw_0(n-k)}h[k] = H(e^{jw_0})e^{jw_0 n}$$

  We call $H(w_0)$ the *frequency response* of out LTI system at frequency w_0

- **Properties**
  - $x(t) = e^{-jw_0 t} = \cos(-w_0 t) + j\sin(-w_0 t)$ (idem for discrete time)
  - When the impulse response $h(t)$ of the system is *real-valued*, the frequency response satifies : $H(w_0) = H^*(-w_0)$. Where $*$ denotes the complex conjugate. One often syas that in thsi case the frequency response is*conjugate-sysmmetric* (idem for discrete time)

- **Convolution** Let us consider two systems with frequency responses $H_1(w)$ and $H_2(w)$ respectively:
  - Parallel: The overall system has frequency response: $G(w) = H_1(w) + H_2(w)$ (idem for dicrete time)
  - Serie: The overall system has frequency response: $G(w) = H_1(w)H_2(w)$ (idem for discrete time)

- **Sampling** $x_p(t) = x(t)p(t)$ where $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nY)$. Combining the two first result we have :

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

$$X_p(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s)$$

- *Sampling theorem*: Let $x(t)$ be a band-limited signal with $X(w) = 0$ for $|w| > w_m$. Then $x(t)$ is uniquely determined by its samples $x(nT), n = 0, \pm 1, \pm 2, \dots,$ if $w_s > 2w_M$ where $w_s = \frac{2\pi}{T}$. The frequency $2w_M$ is commonly referred as the *Nyquist rate* (The frequency w_M correponding to one-half the Nyquiest rate if often referred to as the *Nyquist frequency*)

- The **reconstrcution** in the time domain becomes :

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT) \frac{w_c T}{\pi} \frac{\sin_{w_c}(t - nT)}{w_c(t - nT)}$$

$$X_r(\omega) = X_p(\omega) H(\omega)$$

The Transfer Function and The Z-Transform

- We define the Z-transform as :

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- The signal is *causal* (that is right-sided) it the ROC extends indefinitely outwards
- The signal is *anti-causal* (that is left-sided) if the ROC includes the origin
- The time-domain signal is *stable* (that is absolutely summable) if and only if the ROC includes the *unit-circle. Whenever the ROC inlcudes the unit circle, thsi implies that the discrete-time Fourier transform of the time-domain signal also exists.
- To be a valid ROC we must have:
  - The ROC is either a circle or an annulus (possibly spreading indefinitely) centered at the origin of the z-plane.
  - The ROC is bounded by poles or extends to inifinity. It connot contain any poles of $H(z)$
  - The ROC inlcudes the unit circle, then the system is stable
- **Composition** Same as above

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Transfer Fucntion an The Laplace Transform

- **The Transfer Fucntion** Let us suppoe thath input to out LTI system is gievn by $x(t) = e^{st}$ for an arbitrary *complex-valued* constant s. Then, the output is givent by :

$$y(t) = \int_{-\infty}^{\infty} e^{s(t-\tau)} h(\tau) d\tau = H(s) e^{st}$$

- **The Laplace Transform** For a time-domain signal $x(t)$, the Laplace transform is defined as :

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

We observe that by only considering s of the form $s = jw$, that is, by evaluating the Laplace Transform only along the imaginary axis in the complex s-plane, we obtain exactly the Fourier transform. In this sense, the Lapalce Trasnform is a strcit generalization of the Fourier transform

- The signal is **causal** (that is rigth sided) if the ROC extends indefinitely to the rigth
- The signal is **anti-causal** (that is left sided) if the ROC extends indefinitely to the left
- The time-domain signal is **stable** (that is absolutely integrable) if and only if ROC includes the *imaginary axis*. Whenever the ROC included the imaginary axis, this implies that Fourier transform of the time-domain signal also exists
- TO be a valid ROC we must have:
  - The ROC consists of strips parallel to the jw-axis in the s-plane
  - The ROC in bounded by poles or extends to infinity. It cannot contain any poles.
  - If the ROC includes the imaginary axis, then the signal is stable
- **Composition** same as above

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Example

- **Non time invariant** :

$$y[n] = \mathcal{H}(x[n]) = x[x]\cos(\omega_0 n) \implies y[n-n_0] = x[n-n_0]\cos(\omega_0(n-n_0))$$
$$\mathcal{H}(x[n-n_0]) = x[n-n_0]\cos(\omega_0 n)$$