

# Probalities and Statistics

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Markdown version on *github*

## General Stuff

- Given  $n$  distinct objects, the number of different **permutation** (without repetition) of length  $r \leq n$  is :

$$n(n-1)(n-2)\dots(n-r+1) = \frac{n!}{(n-r)!}$$

- Binomial coefficient** (The number of ways of distributing a set of  $r$  objects from a set of  $n$  distinct objects without repetition):

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

- Geometric Series :**

$$\sum_{i=0}^n a\theta^i = \begin{cases} a \frac{1-\theta^{n+1}}{1-\theta}, & \theta \neq 1 \\ a(n+1), & \theta = 1 \end{cases}$$

## Probability

- $P(A^c) = 1 - P(A)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A | B) = \frac{P(A \cap B)}{P(B)}$
- Bayes' Theorem** :  $P(A) = \sum_{i=1}^{\infty} P(A \cap B_i) = \sum_{i=1}^{\infty} P(A | B_i)P(B_i)$
- $P(A_1 \cap A_2 \cap A_3) = P(A_3 | A_1 \cap A_2)P(A_2 | A_1)P(A_1)$
- $A$  and  $B$  are **independant** iff  $P(A \cap B) = P(A)P(B)$

## Random Variable

- A random variable that takes onmy the values 0 and 1 is called an **indicator variable** or a **Bernouilli random variable**, or a **bernouilli trial**.

- **Probability mass function** (PMF) is  $f_X(x) = P(X = x) = P(A_x)$
- **Binomial** random variable is used when we are considering the number of 'sucesses' of a trial which is independently repeated a fixed number of times, and where each trial has the same probability of sucecess.  $X$  has PMF :

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \dots, n, \quad n \in \mathbb{N}, \quad 0 \leq p \leq 1$$

$$- E(X | N = n) = np$$

$$- var(X | N = n) = np(1-p)$$

- **Geometric** random variable models the waiting time until a first event, in a series of independant trials having the same sucess porbability.  $X$  has PMF :

$$f_X(x) = p(1-p)^{x-1}, \quad x = 1, 2, \dots, \quad 0 \leq p \leq 1$$

We write  $X \sim Geom(p)$   $E(X) = \frac{1}{p}$  and  $var(X) = \frac{1-p}{p^2}$

- **Negative Binomial** random variable models the waiting time until the  $n$ th sucess in a series of independant trials having the same sucess probability.  $X$  has PMF :

$$f_X(x) = \binom{x-1}{n-1} p^n (1-p)^{x-n}, \quad x = n, n+1, n+2, \dots, \quad 0 \leq p \leq 1$$

- **Hypergeometric** random variable, independant draw, without replacement from a finite population of size  $N$  We draw a sample of  $m$  balls without replacement from an urn containing  $w$  white balls and  $b$  black balls. Let  $X$  be the number of white balls drawn :

$$f_X(x) = \frac{\binom{w}{x} \binom{b}{m-x}}{\binom{w+b}{m}}, \quad x = \max(0, m-b), \dots, \min(w, m)$$

- **Poisson** random variable appears often as a model for counts, or for number of rare events. It also provides approximation to probabilities, for example for random permutations or the binomial distribution.

$$f_X(x) = \frac{\lambda^x}{x!} e^{-\lambda}, \quad x = 0, 1, \dots, \quad \lambda > 0$$

Is  $X$  based on independant trials (0/1) with a same propability  $p$ , or on draws from a finite population, with replacement

- if **Yes**, is the total number of trials  $n$  fixed, so  $X \in \{1, \dots, n\}$  ?
  - if **Yes**: use the *binomial* distribution,  $X \sim B(n, p)$  (and use the *bernouilli* distribution is  $n = 1$ ).
  - ▷ if  $n \approx \infty$  or  $n \gg np$ , we can use the *poisson* distribution  $X \sim P(np)$
  - if **No**, then  $X \in \{n, n+1, \dots\}$ , and we use the *geometric* (if  $X$  is the number of trials until one sucesses) or *negative binomial* (if  $X$  is the number of trials until the last of several sucesses) ditribution.
- if **No**, then if the draw is independant but without replacement from a finite population, then  $X \sim hypergeometric$  distribution.
- **Cumulative distribution function** (CDF) of a random variable is :  $F_X(x) = P(X \leq x)$ ,  $x \in \mathbb{R}$  if  $X$  is discrete we can write

$$F_X(x) = \sum_{\{x_i \in D_X : x_i \leq x\}} P(P = x_i)$$

- The **Expectation** (or **expected value** or **mean**) of  $X$  is

$$E\{g(X)\} = \sum_{x \in D_X} g(x)P(X = x) = \sum_{x \in D_X} g(x)f_X(x)$$

- 1)  $E(\cdot)$  is linear operator,  $E(aX + b) = aE(X) + b$
  - 2)  $E\{g(X) + h(X)\} = E\{g(X)\} + E\{h(X)\}$
  - 3) if  $P(X = b) = 1$ , then  $E(X) = b$
  - 4) if  $P(a < X \leq b) = 1$ , then  $a < E(X) \leq b$
  - 5)  $\{E(X)\}^2 \leq E(X^2)$
  - 6)  $E(X) = \sum_{i=1}^{\infty} E(X | B_i)P(B_i)$
  - 7)  $E(X) = E_Y\{E(X | Y = y)\}$
  - 8)  $E\{g(X, Y)\} = E_X[E\{g(X, Y) | X = x\}]$
- The **Variance** of  $X$  is  $var(X) = E[\{X - E(X)\}^2]$ 
    - 1)  $var(X) = E(X^2) - E(X)^2 = E\{X(X - 1)\} + E(X) - E(X)^2$
    - 2)  $var(aX + b) = a^2 var(X)$
    - 3)  $var(X) = 0 \implies X$  is constant with probability 1.
    - 4)  $var\{g(X, Y)\} = E_X[var\{g(X, Y) | X = x\}] + var_X[E\{g(X, Y) | X = x\}]$
  - The **Standard deviation** of  $X$  is defined as  $\sqrt{var(X)}$
  - **Law of small numbers** Let  $X_n \sim B(n, p_n)$ , and suppose that  $np_n \rightarrow \lambda > 0$  when  $n \rightarrow \infty$ . Then  $X_n \xrightarrow{D} X$ , where  $X \sim P(\lambda)$

## Continuous Random Variable

- A random variable  $X$  is **continuous** if there exist a function  $f(x)$ , called the **probability density function** (PDF) of  $X$ , such that :

$$P(X \leq x) = F(x) = \int_{-\infty}^x f(u)du, \quad x \in \mathbb{R}$$

The properties of  $F$  imply that (i)  $f(x) \geq 0$  and (ii)  $\int_{-\infty}^{\infty} f(x)dx = 1$

- **Uniform distribution.** The random variable  $U$  having density :

$$f(u) = \begin{cases} \frac{1}{b-a}, & a \leq u \leq b \\ 0, & \text{otherwise} \end{cases}$$

and :

$$F(u) = \begin{cases} \frac{u-a}{b-a}, & a \leq u \leq b \\ 0, & \text{otherwise} \end{cases}$$

is called a **uniform random variable**. We write  $U \sim U(a, b)$ .

- **Exponential random variable.** The random variable  $X$  having density :

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

and :

$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

is called and **exponential random variable** with parameter  $\lambda > 0$ .

- We write  $X \sim \exp(\lambda)$
- We  $E(x) = \frac{1}{\lambda}$
- **Gamma distribution.** The random variable  $X$  having density !

$$f(x) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

is called a **gamma random variable** with parameters  $\alpha, \lambda > 0$ . We write  $X \sim \text{Gamma}(\alpha, \lambda)$

- $E(X) = \frac{\alpha}{\beta}$
- $\text{var}(X) = \frac{\alpha}{\beta^2}$
- **Laplace.** The random variable  $X$  having density :

$$f(x) = \frac{\lambda}{2} e^{-\lambda|x-\eta|}, \quad x \in \mathbb{R}, \quad \eta \in \mathbb{R}, \lambda > 0$$

is called a **Laplace random variable**.  $\eta$  is the *median* of the distribution.

- **Pareto.** The random variable  $X$  with cumulative distribution function :

$$F(x) = \begin{cases} 0, & x < \beta \\ 1 - \left(\frac{\beta}{x}\right)^\alpha, & x \geq \beta, \end{cases}$$

is called a **Pareto random variable**

- We define the **expectation** of  $g(X)$  to be :

$$E\{g(X)\} = \int_{-\infty}^{\infty} g(x)f(x)dx$$

- The **variance** of  $X$  is :

$$\text{var}(X) = \int_{-\infty}^{\infty} \{x - E(X)\}^2 f(x)dx = E(X^2) - E(X)^2$$

.	Discrete	Continuous
Support $D_X$	countable	contains an interval $(x_-, x_+) \subset \mathbb{R}$
$f_X$	mass function dimensionless	density function units $[x]^{-1}$
$F_X(a) = P(X \leq a)$	$0 \leq f_X(x) \leq A \sum_{x \in \mathbb{R}} f_X(x) = 1$	$0 \leq f_X(x) \int_{-\infty}^{\infty} f_X(x)dx = 1$
$P(X \in A)$	$\sum_{x \leq a} f_X(x)$	$\int_{-\infty}^a f_X(x)dx$
$P(a < X \leq b)$	$\sum_{x \in A} f_X(x)$	$\int_A f_X(x)dx$
$P(X = a)$	$\sum_{\{x: a < x \leq b\}} f_X(x)$	$\int_a^b f_X(x)dx$
$E\{g(X)\}$ (if well defined)	$f_X(a) \geq 0$	$\int_a^b f_X(x)dx = 0$
	$\sum_{x \in \mathbb{R}} g(x)f_X(x)$	$\int_{-\infty}^{\infty} g(x)f_X(x)dx$

- Let  $0 < p < 1$ . We define the  $p$  **quantile** of the cumulative distribution function  $F(x)$  to be :

$$x_p = \inf\{x : F(x) \geq p\}$$

For most continuous random variable,  $x_p$  is unique and equals  $x_p = F^{-1}(p)$ , where  $F^{-1}$  is the inverse function  $F$ . Then  $x_p$  is the value for which  $P(X \leq x_p) = p$ .

The  $p$  quantile of  $Y$  is the value  $y_p$  that solves  $F_Y(y_p) = p$ .

- A random variable  $X$  having density :

$$f(x) = \frac{1}{(2\pi)^{1/2}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}, \quad x \in \mathbb{R}, \mu \in \mathbb{R}, \sigma > 0$$

is a **normal random variable** with the expectation  $\mu$  and variance  $\sigma^2$ : we write  $X \sim N(\mu, \sigma^2)$ . When  $\mu = 0, \sigma^2 = 1$  the corresponding variable  $Z$  is **standard normal**,  $Z \sim n(0, 1)$  with density :

$$\phi(z) = (2\pi)^{-1/2} e^{-z^2/2}, \quad z \in \mathbb{R}$$

Then :

$$F_Z(x) = P(Z \leq x) = \Phi(x) = \int_{-\infty}^x \phi(z) dz = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^x e^{-z^2/2} dz$$

The **normal distribution** is often called the **Gaussian distribution**.

$Z \sim N(0, 1)$  :

- 1) Then density is symmetric with respect to  $z = 0$  i.e.  $\phi(z) = \phi(-z)$
- 2)  $P(Z \leq z) = \Phi(z) = 1 - \Phi(-z) = 1 - P(Z \geq -z)$
- 3) We have :

$$\phi'(z) = -z\phi(z), \quad \phi''(z) = (z^2 - 1)\phi(z), \quad \phi'''(z) = -(z^3 - 3z)\phi(z), \dots$$

This implies that  $E(Z) = 0, \text{var}(Z) = 1, E(Z^3) = 0$  etc.

- 4) If  $X \sim N(\mu, \sigma^2)$  then  $Z = (X - \mu)/\sigma \sim N(0, 1)$

MGF of Normal law :  $e^{\mu t + \sigma^2 t^2/2}$

- **Moivre-Laplace** Let  $X_n \sim B(n, p)$ , let :

$$\mu_n = E(X_n) = np, \quad \sigma_n^2 = \text{var}(X_n) = np(1-p)$$

When  $n \rightarrow \infty$  we can approximate :  $X_n \sim N\{np, np(1-p)\}$

This gives us :

$$P(X_n \leq r) = P\left(\frac{X_n - \mu_n}{\sigma_n} \leq \frac{r - \mu_n}{\sigma_n}\right) = \Phi\left(\frac{r - \mu_n}{\sigma_n}\right)$$

A better approximation of  $P(X_n \leq r)$  is given by replacing  $r$  by  $r + \frac{1}{2}$ . The  $\frac{1}{2}$  is called the *continuity correction*

Which density ?

- **Uniform** variables lie in a finite interval, and give equal probability to each part of the interval
- **Exponential** and **Gamma** variables lie in  $(0, \infty)$ , and are often used to model waiting times and other positive quantities,
  - The gamma has two parameters and is more flexible, but the exponential is simpler and has some elegant properties
- **Pareto** variables lie in the interval  $(\beta, \infty)$ , so are not appropriate for arbitrary positive quantities (which could be smaller than  $\beta$ ) but are often used to model financial losses over some threshold  $\beta$
- **Normal** variables lie in  $\mathbb{R}$  and are used to model quantities that arise (or might arise) through averaging of many small effects (e.g. height and weight, which are influenced by many genetic factors), or where measurements are subject to error.
- **Laplace** variables lie in  $\mathbb{R}$ . the Laplace distribution can be used in place of the normal in situations where outliers might be present.

## Several random variable

- Let  $(X, Y)$  be a discrete random variable: the set :

$$D = \{(x, y) \in \mathbb{R}^2 : P\{(X, Y) = (x, z)\} > 0\}$$

is countable. The **joint probability mass function** of  $(X, Y)$  is

$$f_{X,Y}(x, y) = P\{(X, Y) = (x, y)\}, \quad (x, y) \in \mathbb{R}^2$$

and the **joint cumulative distribution function** of  $(X, Y)$  is

$$F_{X,Y}(x, y) = P(X \leq x, Y \leq y), \quad (x, y) \in \mathbb{R}^2$$

if then random variable is *continuous* then the **joint cumulative function** of  $(X, Y)$  can be written :

$$F_{X,Y}(x, y) = P(X \leq x, Y \leq y) = \int_{-\infty}^x \int_{-\infty}^y f_{X,Y}(u, v) du dv \quad (x, y) \in \mathbb{R}^2$$

this implies :

$$f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y)$$

- The **Marginal probability mass/density function** of  $X$  is :

$$f_X(x) = \begin{cases} \sum_y f_{X,Y}(x, y) & \text{discrete case} \\ \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy, & \text{continuous case} \end{cases}$$

- Let  $X, Y$  be random variables of density  $f_{X,Y}(x, y)$ . Then if  $E\{|g(X, Y)|\} < \infty$ , we can define the **expectation** of  $g(X, Y)$  to be :

$$E\{g(X, Y)\} = \begin{cases} \sum_{x,y} g(x, y) f_{X,Y}(x, y), & \text{discrete case} \\ \int \int g(x, y) f_{X,Y}(x, y) dx dy, & \text{continuous case.} \end{cases}$$

- We define the **covariance** of  $X$  and  $Y$  as :

$$\text{cov}(X, Y) = E(XY) - E(X)E(Y)$$

- $\text{cov}(X, X) = \text{var}(X)$
- $\text{cov}(a, X) = 0$
- $\text{cov}(X, Y) = \text{cov}(Y, X)$  (*symetry*)
- $\text{cov}(a + bX + cY, Z) = b\text{cov}(X, Z) + c\text{cov}(Y, Z)$  (*bilinearity*)
- $\text{cov}(a + bX, c + dY) = bdcov(X, Y)$
- $\text{var}(a + bX + cY) = b^2\text{var}(X) + 2bccov(X, Y) + c^2\text{var}(Y)$
- $\text{cov}(X, Y)^2 \leq \text{var}(X)\text{var}(Y)$  (*Cauchy-Schwarz inequality*)
- $X, Y$  independant  $\implies \text{cov}(X, Y) = 0$ . However, the converse is false
- The **correlation** of  $X$  and  $Y$  is :

$$\text{corr}(X, Y) = \frac{\text{cov}(X, Y)}{\{\text{var}(X)\text{var}(Y)\}^{1/2}}$$

Let  $\rho = \text{corr}(X, Y)$ . Then :

- $-1 \leq \rho \leq 1$
- If  $\rho = \pm 1$  then there exist  $a, b, c \in \mathbb{R}$  such that :  $aX + bY + c = 0$ . With probability 1 ( $X$  and  $Y$  are then linearly independant)
- If  $X$  and  $Y$  are independent then  $\text{corr}(X, Y) = 0$
- The effect of the transformation  $(X, Y) \mapsto (a + bX, c + dY)$  is  $\text{corr}(X, Y) \mapsto \text{sign}(bd)\text{corr}(X, Y)$
- We define the **moment-generating function** (MGF) of a random variable  $X$  by :

$$M_X(t) = E(e^{tX})$$

- $M_X(t)$  is also called the **Laplace tranform** of  $f_X(x)$
- The MGF is useful as s summary of all properties of  $X$ , we can write :

$$M_X(t) = E(e^{tX}) = E\left(\sum_{r=0}^{\infty} \frac{t^r X^r}{r!}\right) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r)$$

- $M_X(0) = 1$
- $M_{a+bX+cY}(t) = e^{at} M_X(bt) M_Y(ct)$
- $E(X^r) = \frac{\partial M_X(t)}{\partial t^r} \Big|_{t=0}$
- $E(X) = M'_X(0)$
- $\text{var}(X) = M''_X(0) - M'_X(0)^2$
- We say that the random variables  $\{X_n\}$  **converge in distribution** to  $X$ , if, for all  $x \in \mathbb{R}$  where  $F$  is continuous :

$$F_n(x) \rightarrow F(x), \quad n \rightarrow \infty$$

- The cumulative case :

$$M_X(t) = E(e^{t^T X}) = E(e^{\sum_{r=1}^p t_r X_r}), \quad t \in \mathcal{T}$$

- The **cumulant-generating function** (CGF) of  $X$  is  $K_X(t) = \log M_X(t)$ . The cumulants  $\kappa_r$  of  $X$  are defined by :

$$K_X(t) = \sum_{r=1}^{\infty} \frac{t^r}{r!} \kappa_r, \quad \kappa_r = \left. \frac{d^r K_X(t)}{dt^r} \right|_{t=0}$$

It is easy to verify that  $E(X) = \kappa_1$  and  $\text{var}(X) = \kappa_2$ . The CGF is equivalent to MGF, and so shares its properties, but it is often easier to work with the CGF.

- The random vector  $X = (X_1, \dots, X_p)^T$  has a **multivariate normal distribution** if there exist a  $p \times 1$  vector  $\mu = (\mu_1, \dots, \mu_p)^T \in \mathbb{R}^p$  and a  $p \times p$  symmetric matrix  $\Omega$  with elements  $w_{ij}$  such that :

$$u^T X \sim \mathcal{N}(u^T \mu, u^T \Omega u), \quad u \in \mathbb{R}^p$$

then we write  $X \sim \mathcal{N}(\mu, \Omega)$ .

- We have :

$$E(X_j) = \mu_j, \quad \text{var}(X_j) = w_{jj}, \quad \text{cov}(X_j, X_k) = w_{jk}, \quad j \neq k$$

so  $\mu$  and  $\Omega$  are called the *mean vector* and *covariance matrix* of  $X$ .

- The moment-generating function of  $X$  is  $M_X(u) = \exp(u^T \mu + \frac{1}{2} u^T \Omega u)$ , for  $u \in \mathbb{R}^p$ .
- If  $\mathcal{A}, \mathcal{B} \subset \{1, \dots, p\}$ , and  $\mathcal{A} \cap \mathcal{B} = \emptyset$  then

$$X_{\mathcal{A}} \perp\!\!\!\perp X_{\mathcal{B}} \Leftrightarrow \Omega_{\mathcal{A}, \mathcal{B}} = 0$$

- If  $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} \mathcal{N}(\mu, \sigma^2)$ , then  $X_{n \times 1} = (X_1, \dots, X_n)^T \sim \mathcal{N}(\mu 1_n, \sigma^2 I_n)$
- Linear combination of normal variables are normal :

$$a_{r \times 1} + B_{r \times p} X \sim \mathcal{N}_r(a + B\mu, B\Omega B^T)$$

.

- The random vector  $X \sim \mathcal{N}(\mu, \Omega)$ , has density function on  $\mathbb{R}^p$  if and only if  $\Omega$  is positive definite, i.e.,  $\Omega$  has rank  $p$ . If so, then density function is :

$$f(x; \mu, \Omega) = \frac{1}{(2\pi)^{p/2} |\Omega|^{1/2}} \exp\left\{-\frac{1}{2}(x - \mu)^T \Omega^{-1} (x - \mu)\right\}, \quad x \in \mathbb{R}^p$$

- **Marginal and conditional distributions.** Let  $X \sim \mathcal{N}(\mu_{p \times 1}, \Omega_{p \times p})$ , where  $|\Omega| > 0$ , and let  $\mathcal{A}, \mathcal{B} \subset \{1, \dots, p\}$  with  $|\mathcal{A}| = q < p$ ,  $|\mathcal{B}| = r < p$  and  $\mathcal{A} \cap \mathcal{B} = \emptyset$ . Let  $\mu_{\mathcal{A}}, \Omega_{\mathcal{A}}$  and  $\Omega_{\mathcal{A}\mathcal{B}}$  be respectively the  $q \times 1$  subvector of  $\mu$ ,  $q \times q$  and  $q \times r$  submatrices of  $\Omega$  conformable with  $\mathcal{A}, \mathcal{A} \times \mathcal{A}$  and  $\mathcal{A} \times \mathcal{B}$ . Then :

- The marginal distribution of  $X_{\mathcal{A}}$  is normal,

$$X_{\mathcal{A}} \sim \mathcal{N}(\mu_{\mathcal{A}}, \Omega_{\mathcal{A}})$$

- The conditional distribution of  $X_{\mathcal{A}}$  given  $X_{\mathcal{B}} = x_{\mathcal{B}}$  is normal,

$$X_{\mathcal{A}} | X_{\mathcal{B}} = x_{\mathcal{B}} \sim \mathcal{N}\{\mu_{\mathcal{A}} + \Omega_{\mathcal{A}\mathcal{B}} \Omega_{\mathcal{B}}^{-1} (x_{\mathcal{B}} - \mu_{\mathcal{B}}), \Omega_{\mathcal{A}} - \Omega_{\mathcal{A}\mathcal{B}} \Omega_{\mathcal{B}}^{-1} \Omega_{\mathcal{B}\mathcal{A}}\}$$

This has two important implications:



- \* Implies that any subvector of  $X$  also has a multivariate normal distribution.
- \* Implies that two components of  $X_{\mathcal{A}}$  are conditionally independent given  $X_{\mathcal{B}}$  if and only if the corresponding off-diagonal element of  $\Omega_{\mathcal{A}} - \Omega_{\mathcal{A}\mathcal{B}}\Omega_{\mathcal{B}}^{-1}\Omega_{\mathcal{B}\mathcal{A}}$  equals zero.

**Reminder :** Transformation of random variables

We often want to calculate the distributions of random variables based on other random variables

- \* Let  $Y = g(X)$ , where the function  $g$  is known. We want to obtain  $F_Y$  and  $f_Y$  from  $F_X$  and  $f_X$ .
- \* Let  $g : \mathbb{R} \mapsto \mathbb{R}$ ,  $\mathcal{B} \subset \mathbb{R}$ , and  $g^{-1}(\mathcal{B}) \subset \mathbb{R}$  be the set for which  $g\{g^{-1}(\mathcal{B})\} = \mathcal{B}$ . Then

$$P(Y \in \mathcal{B}) = P\{g(X) \in \mathcal{B}\} = P\{X \in g^{-1}(\mathcal{B})\}.$$

Since  $X \in g^{-1}(\mathcal{B})$  iff  $g(X) = Y \in \mathcal{B}$ . \* To find  $F_Y(y)$ , we take  $\mathcal{B} = (-\infty, y]$ , giving

$$F_Y(y) = P(Y \leq y) = P\{g(X) \in \mathcal{B}_y\} = P\{X \in g^{-1}(\mathcal{B}_y)\}$$

- \* If the function  $g$  is monotonic increasing with (monotonic increasing) inverse  $g^{-1}$ , then

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{dX\{g^{-1}(y)\}}{dy} = f_X\{g^{-1}(y)\} \times \left| \frac{dg^{-1}(y)}{dy} \right|$$

where the  $|\cdot|$  ensures that the same formula holds with monotonic decreasing  $g$ .

- **Sum of independent variables** If  $X, Y$  are independent random variables, then the PDF of their sum  $S = X + Y$  is the convolution  $f_X * f_Y$  of the PDFs  $f_X, f_Y$ :

$$f_S(s) = f_X * f_Y(s) = \begin{cases} \int_{-\infty}^{\infty} f_X(x)f_Y(s-x)dx, & X, Y \text{ continuous} \\ \sum_x f_X(x)f_Y(s-x), & X, Y \text{ discrete.} \end{cases}$$

- The **order statistics** of the rv's  $X_1, \dots, X_n$  are the ordered values

$$X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$$

If the  $X_1, \dots, X_n$  are continuous, then no two of the  $X_j$  can be equal, i.e.,

$$X_{(1)} < X_{(2)} < \dots < X_{(n)}$$

In particular, the *minimum* is  $X_{(1)}$ , then *maximum* is  $X_{(n)}$ , and the *median* is

$$X_{(m+1)} \quad (n = 2m + 1, \text{ odd}), \quad \frac{1}{2}(X_{(m)} + X_{(m+1)}) \quad (n = 2m, \text{ even})$$

The median is the central value of  $X_1, \dots, X_n$ .

- Let  $X_1, \dots, X_n \stackrel{iid}{\sim} F$ , from a continuous distribution with density  $f$ , then:

- \*  $P(X_{(n)} \leq x) = F(x)^n$
- \*  $P(X_{(1)} \leq x) = 1 - \{1 - F(x)\}^n$
- \*

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} F(x)^{r-1} f(x) \{1 - F(x)\}^{n-r}, \quad r = 1, \dots, n$$

## Approximation and Convergence

- Inequalities

If  $X$  is a random variable,  $a > 0$  a constant,  $h$  a non-negative function and  $g$  a convex function, then

- $P\{h(X) \geq a\} \leq E\{h(X)\}/a$ , (basic inequality)
- $P(|X| \geq a) \leq E(|X|)/a$ , (Markov's inequality)
- $P(|X| \geq a) \leq E(X^2)/a^2$ , (Chebyshev's inequality)
- $E\{g(X)\} \geq g\{E(X)\}$ . (Jensen's inequality)

On replacing  $X$  by  $X - E(X)$ , Chebyshev's inequality gives :

$$P\{|X - E(X)| \geq a\} \leq \text{var}(X)/a^2$$

- **Hoeffding's inequality** Let  $Z_1, \dots, Z_n$  be independent random variables such that  $E(Z_i) = 0$  and  $a_i \leq Z_i \leq b_i$  for constants  $a_i \leq b_i$ . If  $\epsilon > 0$ , then for all  $t > 0$ ,

$$P\left(\sum_{i=1}^n Z_i \geq \epsilon\right) \leq e^{-t\epsilon} \prod_{i=1}^n e^{t^2(b_i - a_i)^2/8}$$

- Convergence

Let  $X_1, X_2, \dots$  be random variable with cumulative distribution function  $F, F_1, F_2, \dots$ . Then :

- $X_n$  converges to  $X$  *almost surely*,  $X_n \xrightarrow{a.s.} X$ , if

$$P\left(\lim_{n \rightarrow \infty} X_n = X\right) = 1$$

- $X_n$  converges to  $X$  *in mean square*,  $X_n \xrightarrow{2} X$ , if

$$\lim_{n \rightarrow \infty} E\{(X_n - X)^2\} = 0, \quad \text{where } E(X_n^2), E(X^2) < \infty$$

- $X_n$  converges to  $X$  *in probability*,  $X_n \xrightarrow{P} X$ , if for all  $\epsilon > 0$ ,

$$\lim_{n \rightarrow \infty} P(|X_n - X| > \epsilon) = 0$$

- $X_n$  converges to  $X$  *in distribution*,  $X_n \xrightarrow{D} X$ , if

$$\lim_{n \rightarrow \infty} F_n(x) = F(x) \quad \text{at each point } x \text{ where } F(X) \text{ is continuous.}$$

- If  $X_n \xrightarrow{a.s.} X$ ,  $X_n \xrightarrow{2} X$ ,  $X_n \xrightarrow{P} X$ , then  $X_1, X_2, \dots, X$  must all be defined with respect to only one probability space. This is not the case for  $X_n \xrightarrow{D} X$ , which only concerns the probabilities. This last is thus weaker than the other.
- This mode of convergence are related to one another in the following way :

$$X_n \xrightarrow{a.s.} X \implies X_n \xrightarrow{P} X \implies X_n \xrightarrow{D} X$$

$$X_n \xrightarrow{2} X \implies X_n \xrightarrow{P} X \implies X_n \xrightarrow{D} X$$

- **Continuity Theorem** Let  $\{X_n\}$ ,  $X$  be random variables with cumulative distribution functions  $\{F_n\}$ ,  $F$ , whose MGFs  $M_n(t)$ ,  $M(t)$  exist for  $0 \leq |t| < b$ . If there exists  $a < b$  such that  $M_n(t) \leftarrow M(t)$  for  $|t| \leq a$  when  $n \rightarrow \infty$ , then  $X_n \xrightarrow{D} X$ , that is to say,  $F_n(x) \leftarrow F(x)$  at each  $x \in \mathbb{R}$  where  $F$  is continuous.
  - We could replace  $M_n(t)$  and  $M(t)$  by the cumulant-generating functions  $K_n(t) = \log M_n(t)$  and  $K(t) = \log M(t)$
- Let  $x_0, y_0$  be constants,  $X, Y, \{X_n\}, \{Y_n\}$  random variables, and  $h$  a function continuous at  $x_0$ . Then

$$X_n \xrightarrow{D} x_0 \Rightarrow X_n \xrightarrow{P} x_0$$

$$X_n \xrightarrow{P} x_0 \Rightarrow h(X_n) \xrightarrow{P} h(x_0)$$

$$X_n \xrightarrow{D} X \text{ and } Y_n \xrightarrow{P} y_0 \Rightarrow X_n + Y_n \xrightarrow{D} X + y_0, X_n Y_n \xrightarrow{D} X y_0$$

The third line is known as **Slutsky's lemma**. (very useful)

- **Laws of Large Number** Let  $X_1, X_2, \dots$  be a sequence of independent identically distributed random variables with finite expectation  $\mu$ , and write their average as

$$\bar{X} = n^{-1}(X_1 + \dots + X_n)$$

Then  $\bar{X} \xrightarrow{P} \mu$ ; i.e., for all  $\epsilon > 0$ ,

$$P(|\bar{X} - \mu| > \epsilon) \rightarrow 0, \quad n \rightarrow \infty$$

- **Strong law of large number** Under the conditions of the last theorem,  $\bar{X} \xrightarrow{a.s.} \mu$ :

$$P(\lim_{n \rightarrow \infty} \bar{X} = \mu) = 1$$

- **Central limit Theorem (CLT)** Let  $X_1, X_2, \dots$  be independent random variables with expectation  $\mu$  and variance  $0 < \sigma^2 < \infty$ . Then

$$Z_n = \frac{n^{1/2}(\bar{X} - \mu)}{\sigma} \xrightarrow{D} Z, \quad n \rightarrow \infty$$

Where  $Z \sim N(0, 1)$

Thus

$$P\left\{\frac{n^{1/2}(\bar{X} - \mu)}{\sigma} \leq z\right\} \doteq P(Z \leq z) = \Phi(z)$$

for large  $n$ .

- **Delta Model** Let  $X_1, X_2, \dots$  be independent random variable with expectation  $\mu$  and variance  $0 < \sigma^2 < \infty$  and let  $g'(\mu) \neq 0$ , where  $g'$  is the derivative of  $g$ . Then

$$\frac{g(\bar{X}) - g(\mu)}{\{g'(\mu)^2 \sigma^2 / 2\}^{1/2}} \xrightarrow{D} N(0, 1), \quad n \rightarrow \infty$$

This implies that for large  $n$ , we have  $g(\bar{X}) \sim N\{g(\mu), g'(\mu)^2 \sigma^2/n\}$  Combined with Slutsky's lemma, we have :

$$g(\bar{X}) \sim N\{g(\mu), g'(\bar{X})^2 S^2/n\}, \quad S^2 = \frac{1}{n-1} \sum_{j=1}^n (X_j - \bar{X})^2$$

- Might need to add Sample quantiles

## Statistical Inference

- The **Method of moments estimate** of a parameters  $\theta$  is the value  $\theta$  that matches the theoretical and empirical moments.

For a model with  $p$  unknown parameters, we set the theoretical moments of the population equal to the empirical moments of the sample  $y_1, \dots, y_n$  and solve the resulting equations :

$$E(Y^r) = \int y^r f(y; \theta) dy = \frac{1}{n} \sum_{j=1}^n y_j^r, \quad r = 1, \dots, p$$

- **Maximum likelihood estimation** If  $y_1, \dots, y_n$  is a random sample from the density  $f(y, \theta)$ , then the likelihood for  $\theta$  is :

$$L(\theta) = f(y_1, \dots, y_n; \theta) = f(y_1; \theta) \times f(y_2; \theta) \times \dots \times f(y_n; \theta)$$

The data are treated as fixed, and the likelihood  $L(\theta)$  is regarded as a function of  $\theta$

The *maximum likelihood estimate* (MLE)  $\hat{\theta}$  of a parameter  $\theta$  is the value that gives the observed data the highest likelihood. Thus :

$$L(\hat{\theta}) \geq L(\theta) \text{ for each } \theta$$

We simplify the calculations by maximising  $l(\theta) = \log(L(\theta))$  rather than  $L(\theta)$

- Calculate the log-likelihood  $l(\theta)$
- Find the value  $\hat{\theta}$  maximising  $l(\theta)$ , which often satisfies  $dl(\hat{\theta})/d\theta = 0$
- Check that  $\hat{\theta}$  gives a maximum, often by checking that  $d^2l(\hat{\theta})/d\theta^2 < 0$
- **Mean square error** of the estimator  $\hat{\theta}$  of  $\theta$  is :

$$MSE(\hat{\theta}) = E\{(\hat{\theta} - \theta)^2\} = \dots = \text{var}(\hat{\theta}) + b(\theta)^2$$

- **M-estimation** This generalises maximum likelihood estimation. We maximise a function of the form :

$$\rho(\theta; Y) = \sum_{j=1}^n \rho(\theta; Y_j)$$

Where  $\rho(\theta; y)$  is concave as a function of  $\theta$  and for all  $y$ . Equivalently we minimise  $-\rho(\theta; Y)$

- Let  $Y = (Y_1, \dots, Y_n)$  be sampled from a distribution  $F$  with parameter  $\theta$ . Then a **pivot** is a function  $Q = q(Y, \theta)$  of the data and the parameters  $\theta$ , where the distribution of  $Q$  is known and does not depend on  $\theta$ . We say that  $Q$  is **pivotal**.

- **Confidence intervals** Let  $Y = (Y_1, \dots, Y_n)$  be the data from a parametric statistical model with scalar parameter  $\theta$ . A *confidence interval* (CI)  $(C, L)$  for  $\theta$  with lower bound  $L$  and upper bound  $U$  is a random interval that contains  $\theta$  with specified probability, called the (confidence) level of the interval.

- $L = l(Y)$  and  $U = u(Y)$  are statistics that can be computed from the data  $Y_1, \dots, Y_n$ . They do not depend in  $\theta$ .
- In a continuous setting (so  $<$  gives the sample probabilities as  $\leq$ ), and if we write the probabilities that  $\theta$  lies below and above the interval as :

$$P(\theta < L) = \alpha_L, \quad P(U < \theta) = \alpha_U$$

then  $(L, U)$  has confidence level

$$P(L \leq \theta \leq U) = 1 - P(\theta < L) - P(U < \theta) = 1 - \alpha_L - \alpha_U$$

- Often we seek an interval with equal probabilities of not containing  $\theta$  at each end, with  $\alpha_L = \alpha_U = \alpha/2$ , giving an *qui-tailed*  $(1 - \alpha) \times 100$  *confidence interval*
- We usually take standard values of  $\alpha$ , such that  $1 - \alpha = 0.9, 0.95, 0.99, \dots$
- If  $Y_1, \dots, Y_n \stackrel{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$ , then

$$Z = \frac{\bar{Y} - \mu}{\sqrt{\sigma^2/n}} \sim \mathcal{N}(0, 1)$$

is a pivot that provides an exact  $(1 - \alpha_L - \alpha_U)$  confidence interval for  $\mu$ , of the form

$$(L, U) = \left( \bar{Y} - \frac{\sigma}{\sqrt{n}} z_{1-\alpha_L}, \bar{Y} - \frac{\sigma}{\sqrt{n}} z_{\alpha_U} \right)$$

where  $z_p$  denotes the  $p$  quantile of the standard normal distribution.

- In application  $\sigma^2$  is usually unknown. Then we have :

$$\frac{\bar{Y} - \mu}{\sqrt{S^2/n}} \sim t_{n-1}, \quad \frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2$$

are pivots that provide confidence intervals for  $\mu$  and  $\sigma^2$ , respectively,

$$(L, U) = \left( \bar{Y} - \frac{S}{\sqrt{n}} t_{n-1}(1 - \alpha_L), \bar{Y} - \frac{S}{\sqrt{n}} t_{n-1}(\alpha_U) \right)$$

$$(L, U) = \left( \frac{(n-1)S^2}{\chi_{n-1}^2(1 - \alpha_L)}, \frac{(n-1)S^2}{\chi_{n-1}^2(\alpha_U)} \right)$$

where :

- \*  $t_v(p)$  is the  $p$  quantile of the *Student t distribution with  $v$  degrees of freedom*
- \*  $\chi_v^2(p)$  is the  $p$  quantile of the *chi-square distribution with  $v$  degrees of freedom*

- For symmetric densities such as the normal and the Student  $t$ , the quantiles satisfy :

$$z_p = -z_{1-p}, \quad t_v(p) = -t_v(1-p)$$

so equi-tailed  $(1 - \alpha) \times 100$  CIs have the forms :

$$\bar{Y} \pm n^{-1/2} \sigma z_{1-\alpha/2}, \quad \bar{Y} \pm n^{-1/2} S t_{n-1}(1 - \alpha/2)$$

8.4 pas fait