# BST 6200 Spatial Statistics and Disease Mapping

Steven E. Rigdon



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# Chapter 6: Point Pattern Analysis in R

A **point process** is a set of point locations $(x_1, y_1),\ (x_2, y_2),\ \ldots,\ (x_n, y_n)$ in some domain.

A **marked point process** is a set of point locations $(x_1, y_1),\ (x_2, y_2),\ \ldots,\ (x_n, y_n)$ in some domain together with some "marked" variable $z_i$.

# 6.3 Kernel Density Estimation

Objective: estimate the probability density of points.

$$\hat{f}(x,y) = \frac{1}{n\ h_x h_y} \sum_{i=1}^{n} k\left(\frac{x - x_i}{h_x}, \frac{y - y_i}{h_y}\right)$$

$k$ is called the **kernel**. Often $k$ is the PDF of the bivariate normal distribution with mean $\boldsymbol{\mu}[0,0]$ and covariance matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Other choices: UNIF$(-1,1)$, $t(1)$.

# Multivariate Normal Distribution

$\boldsymbol{X} = [X_1, X_2, \ldots, X_p]^t$. $X \sim \mathrm{N}(\boldsymbol{\mu}, \Sigma)$ means that $\boldsymbol{X}$ has a

multivariate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\Sigma$ with PDF

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu})^t \Sigma^{-1} (\boldsymbol{x} - \boldsymbol{\mu})\right)$$

# Choice of $h_x$ and $h_y$

Usually $h_x = h_y$.

Larger $h_x$ and $h_y$ lead to smoother density estimates.

Smaller $h_x$ and $h_y$ lead to "choppier" density estimates.

Scott's (1997) rule:

$$h_x = \sigma_x \left(\frac{2}{3n}\right)^{1/6}$$

$$h_y = \sigma_y \left(\frac{2}{3n}\right)^{1/6}$$

Trial and error can be used to get appropriate $h_x$ and $h_y$.

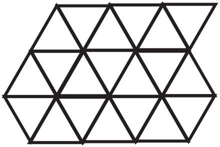
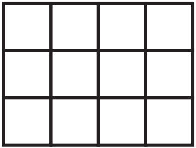
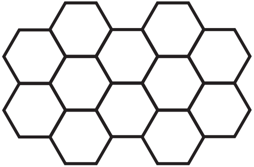
# Using R for KDE

BC_Chapter6.R

# Tesselations (or Tiling the Plane)

A tesselation or a tiling of a plane is a set of geometric objects that cover the plane with no gaps and no overlap.
A regular tesselation or tiling of a plane is a set of identical (congruent) regular polygons that cover the plane with no gaps and no overlap.
There are only three regular tesselatioins of the plane:

1. triangles
2. squares
3. hexagons

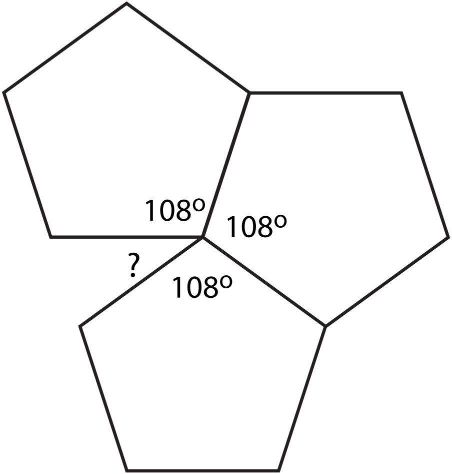




# Why are there only three?

| | | | If it is a Regular Polygon (all sides are equal, all angles are equal) | |
|---|---|---|---|---|
| **Shape** | **Sides** | **Sum of Interior Angles** | **Shape** | **Each Angle** |
| Triangle | 3 | 180° |  | 60° |
| Quadrilateral | 4 | 360° |  | 90° |
| Pentagon | 5 | 540° |  | 108° |
| Hexagon | 6 | 720° |  | 120° |
| Heptagon *(or Septagon)* | 7 | 900° |  | 128.57...° |
| Octagon | 8 | 1080° |  | 135° |
| ... | ... | .. | ... | ... |
| Any Polygon | n | (n-2) × 180° | n | (n-2) × 180° / n |

Source: https://www.dominatethegmat.com/2013/11/gmat-geometry-polygons/

# Why Pentagonal Tiling Won't Work



# Hexagonal Binning

BC_Chapter6.R

# 6.5 Second-Order Analysis of Point Patterns

$N_d$ = number of events within a distance of $d$
from a randomly selected event point

Ripley's $K$ function

$$K(d) = \frac{E(N_d)}{\lambda}, \qquad d > 0$$

# Complete Spatial Randomness (CSR)

The point process exhibits **complete spatial randomness** if the joint (bivariate) distribution of event locations is uniform across the domain.

Book's definition (equivalent to above only in the case of a rectangular domain). The point process exhibits **complete spatial randomness** if the $x$ and $y$ components are independent and the marginal distributions are uniform across the domain.

See BC_Chapter6.R

CSR is also called a **Poisson process** because the number of events that fall in a region $A$ has a Poisson distribution with mean equal to $\lambda|A|$, where $|A|$ denotes the area of $A$.

# $K$ Function under CSR

If we have complete spatial randomness, then

$$K_{\text{CSR}}(d) = \frac{E(N_d)}{\lambda} = \frac{\lambda \times \text{Area(circle of rad. } d\text{)}}{\lambda} = \pi d^2$$

If $K(d) > K_{\text{CSR}}(d)$ for some $d$ this "suggests that there is an excess of nearby points – or to put it another way, there is clustering at the spatial scale associated with the distance $d$.

If $K(d) < K_{\text{CSR}}(d)$ for some $d$ this "suggests spatial dispersion at this scale – the presence of one point suggests other points are less likely to appear nearby ..."

# Estimating $K$

Let $d_{ij}$ denote the distance between points $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$.

Let $\hat{\lambda} = \frac{n}{|A|}$.

$$\hat{K}(d) = \frac{1}{\hat{\lambda}} \sum_{i=1}^{n} \sum_{j=1; j \neq i}^{n} \frac{I(d_{ij} < d)}{n(n-1)}$$

Here $I(\texttt{logical}) = 1$ if `logical` is true and 0 if `logical` is false. This is called an indicator function.

# Edge Effects

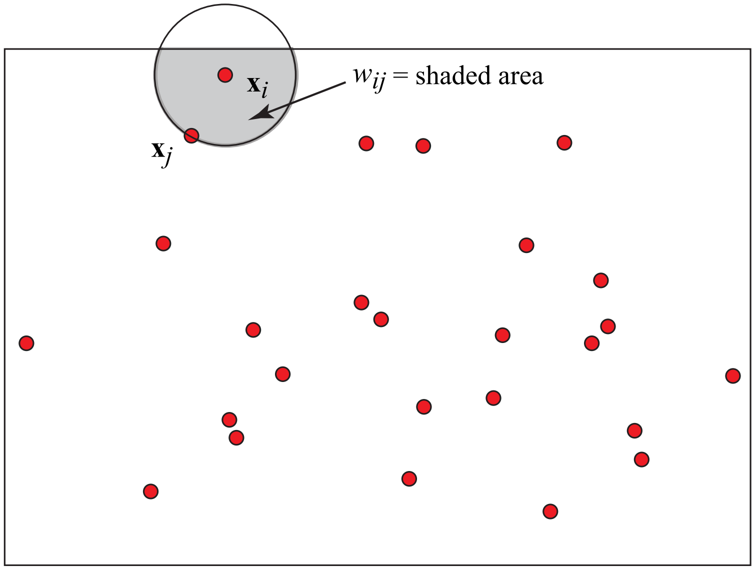
If points are close to the edge of the region $A$ part of the circle of radius $d$ may be outside of $A$, so fewer points should be expected within $d$.

Modified estimator:

$$\hat{K}(d) = \frac{1}{\hat{\lambda}} \sum_{i=1}^{n} \sum_{j=1;j\neq i}^{n} \frac{2I(d_{ij} < d)}{n(n-1)w_{ij}}$$

where $w_{ij}$ is the area of that part of the circle centered at $\boldsymbol{x}_i$ passing through $\boldsymbol{x}_j$ that lies within the region $A$.

# Explanation of $w_{ij}$

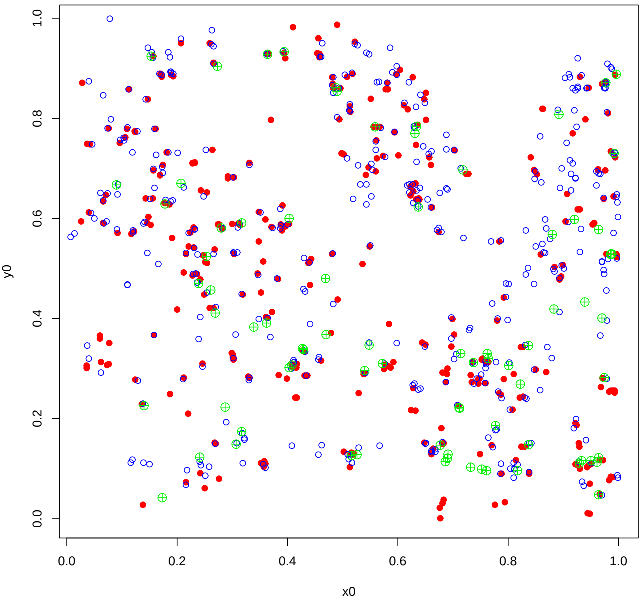


# Is there evidence that the process is anything but CSR (Poisson process)

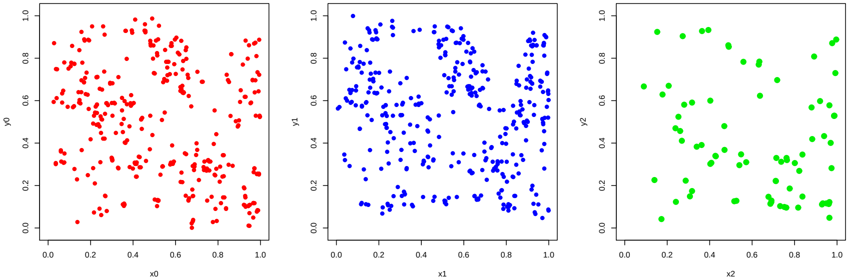
To test this, follow these steps

1. Simulate CSR on the same region $A$ with the same number of points.
2. Compute $\hat{K}(d)$ for the simulated data set.
3. Repeat steps 1 and 2 many (say 100) times.
4. Plot the lower and upper (2.5% and 97.5%) percentiles; this is called the envelope.
5. Plot $\hat{K}(d)$ for the actual data on the same graph and see where it is not contained in the envelope.

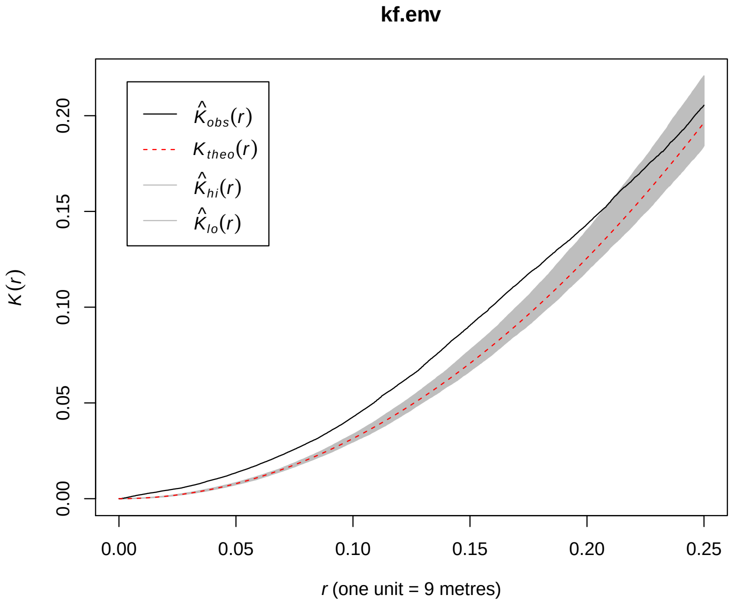
# Bramblecane Data



# Bramblecane Data



# K Function Estimate with Envelope



# Multiple Tests

If we compare $\hat{K}(d)$ with the envelope obtained by simulation, we are doing multiple tests. Thus the overall $\alpha$ (the probability of rejecting at least one test given CSR) is unknown.

Method 1 to compare $\hat{K}(d)$ with that expected under CSR ($\pi d^2$)

$$MAD = \max_d |\hat{K}(d) - K_{\mathrm{CSR}}(d)| = \max_d |\hat{K}(d) - \pi d^2|$$

Method 2 finds the average squared deviation between $\hat{K}(d)$ and $K_{\mathrm{CSR}}(d)$. This is called the dclf test (Diggle, Cressie, Loosmore, Ford).

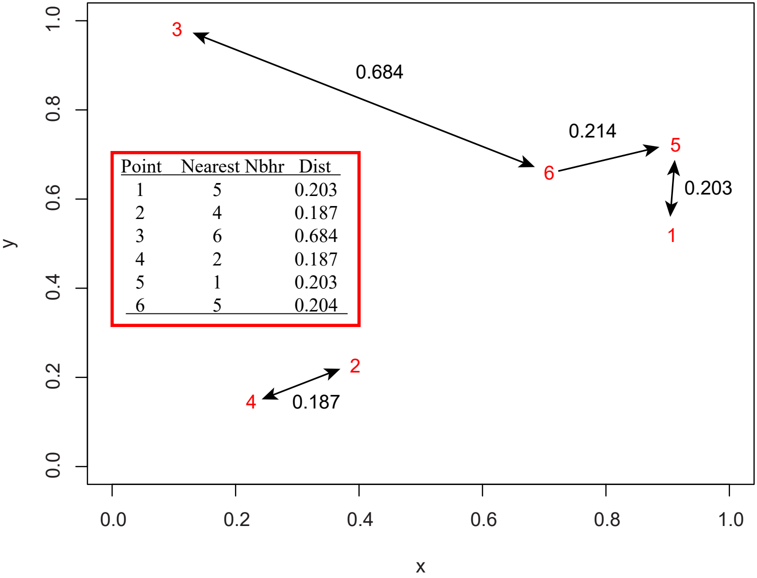
# The $L$ Function

$$L(d) = \sqrt{\frac{K(d)}{\pi}}$$

This is just a transformation of the $K(d)$ so it seems to offer no new information, but under CSR

$$L_{\mathsf{CSR}}(d) = \sqrt{\frac{K_{\mathsf{CSR}}(d)}{\pi}} = \sqrt{\frac{\pi d^2}{\pi}} = d$$

# The $G$ Function




| Point | Nearest Nbhr | Dist |
|---|---|---|
| 1 | 5 | 0.203 |
| 2 | 4 | 0.187 |
| 3 | 6 | 0.684 |
| 4 | 2 | 0.187 |
| 5 | 1 | 0.203 |
| 6 | 5 | 0.204 |

# The $G$ Function