A Bayesian Approach for One-Shot Device Testing With Exponential Lifetimes Under Competing Risks

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Abstract—**This paper considers a competing risk model for a one-shot device testing analysis under an accelerated life test setting. Due to the consideration of competing risks, the joint posterior distribution becomes quite complicated. The Metropolis-Hastings sampling method is used for the estimation of the posterior means of the variables of interest. A simulation study is carried out to assess the Bayesian approach with different priors, and also to compare it with the EM algorithm for maximum likelihood estimation. Finally, an example from a tumorigenicity experiment is presented.**

Index Terms—**Bayesian estimation, competing risks, exponential distribution, Metropolis-Hastings sampling, one-shot device.**

ACRONYMS AND ABBREVIATIONS

EM	expectation-maximization
MLE	maximum likelihood estimate
MSE	mean square error
LSE	least-squares estimate
ICLS	inequality constrained least-squares

NOTATION

I	number of inspection times
J	number of stress levels
R	number of competing causes (risks)
IT_i	ith inspection time
w_j	jth stress level
K_{ij}	number of devices tested at IT_i and under the jth stress level
d_{rij}	number of devices failed due to the rth cause at IT_i and under the jth stress level
S_{ij}	number of devices that survive at IT_i and under the jth stress level, $S_{ij} = K_{ij} - \sum_{r=1}^{R} d_{rij}$
a_{r0}	intercept parameter for the rth cause
a_{r1}	coefficient of the stress for the rth cause
T_{rijk}	the lifetime of rth cause of the kth item at IT_i and under the jth stress level
t_{rijk}	realization of T_{rijk}
λ_{rj}	the failure rate of rth cause under the jth stress level, $\lambda_{rj} = \alpha_{r0}\exp(\alpha_{r1}w_j)$
$F_r(\cdot\|w_j)$	cumulative distribution function of the lifetime of the rth cause under the jth stress level
Δ_{ijk}	status indicator of the kth device at IT_i and under the jth stress level
δ_{ijk}	realization of Δ_{ijk}
p_{rij}	$Pr(\Delta_{ijk} = r\|IT_i, w_j)$
p_{0i}	survival probability at IT_i under normal operating condition (reliability)
P.d1	probability of failure due to cause 1 given failure
$E(T)$	expected lifetime under normal (typical) operating condition
$E(T_r)$	expected lifetime of the rth cause under normal operating condition
Exp$(\cdot)$	estimation with an exponential prior
Norm$(\cdot)$	estimation with a normal prior
Dir$(\cdot)$	estimation with a Dirichlet prior
$\hat{p}$	prior belief on p_{rij} based on past experience
$\tilde{p}$	prior belief on p_{rij} based on data only

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I. INTRODUCTION

ONE-SHOT devices are quite common in practice; typical examples include fire extinguishers, air bags in automobiles, missiles, munitions, and packed food. They are products that will get destroyed right after use. Most of them consist of different components: fire extinguishers contain liquid carbon dioxide, valves, and safety fuses; air bags contain crash sensors, inflators, and compressed gas; packed food contains the food, package, and preservatives, and so on. If one of the components fails, the product will malfunction. But we may only know of the failure when we use the product. The lifetimes of devices observed from such a one-shot device testing will either be left-censored (if it is a failure), or right-censored (if it is a success). So, these data form an extreme case of interval censoring. Suppose, in each failed case, we know which component caused the failure. The manufacturers can then make use of this information to possibly improve their product by replacing the weakest component, or at least by improving its performance.

The accelerated life test (ALT) setting is commonly used to evaluate one-shot devices because they are often highly reliable products. Under usual conditions, the lifetime will be quite long,

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which would make the life test under this condition of a device unduly long, and expensive. For example, fire extinguishers and airbags are designed to last for several years. Applying a high stress on certain stress factors such as humidity, pressure, and temperature could significantly shorten their lifetimes. High temperature will make the chemical inside fire extinguishers and airbags deteriorate; shocks can affect the components of those devices. Thus, the test duration could be reduced significantly, and the experimenter can then make use of the information from the test at higher stress levels to extrapolate to estimate the lifetime of the device under normal operating conditions; see [1], and [2] for further details in this regard.

Several authors have studied one-shot device testing plans and associated analyses. Fan *et al.* [3] used a Bayesian approach to analyze highly reliable one-shot devices. They suggested three prior distributions for the Bayesian estimation: exponential, normal, and beta. Their simulation results show that all the priors perform similarly when the data possess enough information. However, if the data possess zero-failure cases, a normal prior distribution is suggested. Balakrishnan and Ling [4] developed the Expectation-Maximization (EM) algorithm for estimating the parameters of a one-shot device testing model under exponential lifetimes, and further extended it to the case of multiple stress levels [5]. Subsequently, they generalized their results to the cases of Weibull [6] and gamma lifetime distributions [7].

In this paper, we extend the work of Fan *et al.* [3] by incorporating a competing risk model into a one-shot device testing analysis under an accelerated life test setting, and develop a Bayesian estimation framework. We adopt the prior distributions of the parameters stated in [3], namely the exponential distribution, the normal distribution with a non-informative prior for the variance, and the Dirichlet distribution which is an extension of the beta distribution. A simulation study is then carried out to compare the performance of the developed Bayesian approach to the EM algorithm [8] for the maximum likelihood estimation. Finally, an example from a tumorigenicity experiment is presented for illustrating the results developed here.

II. Model Specification

The setting for an accelerated life test for one-shot devices considered here is as follows.

1) The tests are checked at inspection times IT_i, for $i = 1, \ldots, I$.
2) The devices are tested under different temperatures (as stress levels) w_j, for $j = 1, \ldots, J$.
3) K_{ij} devices are tested under IT_i and w_j.
4) The number of devices failed due to the rth cause under IT_i and w_j is denoted by d_{rij}, for $r = 1, \cdots, R$.
5) The number of devices that survive under IT_i and w_j is denoted by $S_{ij} = K_{ij} - \sum_{r=1}^{R} d_{rij}$.

As in [3], we assume that T_{rijk} are s-independent and follow an exponential distribution with rate parameter λ_{rj}, with p.d.f.

$$f_r(t) = \lambda_{rj} e^{-\lambda_{rj} t},\ r = 1, 2, \cdots, R.$$

TABLE I
An Example of the Failure Records Under Temperatures of 35, 45, 55, 65 (in °C) at Inspection Times 10, 20, 30 (in Hours), With 2 Competing Causes for the Failure Of Devices

		$\delta_{ijk} = 0$	$\delta_{ijk} = 1$	$\delta_{ijk} = 2$
$IT_1 = 10$	$w_1 = 35$	$S_{11} = 8$	$d_{111} = 1$	$d_{211} = 1$
	$w_2 = 45$	$S_{12} = 5$	$d_{112} = 1$	$d_{212} = 4$
	$w_3 = 55$	$S_{13} = 5$	$d_{113} = 4$	$d_{213} = 1$
	$w_4 = 65$	$S_{14} = 6$	$d_{114} = 1$	$d_{214} = 3$
$IT_2 = 20$	$w_1 = 35$	$S_{21} = 10$	$d_{121} = 0$	$d_{221} = 0$
	$w_2 = 45$	$S_{22} = 7$	$d_{122} = 2$	$d_{222} = 1$
	$w_3 = 55$	$S_{23} = 7$	$d_{123} = 2$	$d_{223} = 1$
	$w_4 = 65$	$S_{24} = 6$	$d_{124} = 3$	$d_{224} = 1$
$IT_3 = 30$	$w_1 = 35$	$S_{31} = 9$	$d_{131} = 1$	$d_{231} = 0$
	$w_2 = 45$	$S_{32} = 8$	$d_{132} = 1$	$d_{232} = 1$
	$w_3 = 55$	$S_{33} = 4$	$d_{133} = 4$	$d_{233} = 2$
	$w_4 = 65$	$S_{34} = 2$	$d_{134} = 4$	$d_{234} = 4$

The relationship between the failure rate λ_{rj} and the stress level w_j is assumed to be a log-linear function of the form

$$\lambda_{rj} = \alpha_{r0} \exp(\alpha_{r1} w_j), \quad \alpha_{r0}, \alpha_{r1}, w_j > 0. \tag{1}$$

We also denote Δ_{ijk} for the indicator of the kth device under temperature w_j at inspection time IT_i. When the product passes the test, we will set $\Delta_{ijk} = 0$; if the product fails the test, we will then investigate the cause responsible for the failure. If cause r is the cause for the failure, we will denote this event by $\Delta_{ijk} = r$, for $r = 1, 2, \cdots, R$. For simplicity, we limit the number of competing causes to be $R = 2$ even though the formulation for the general case when $R > 2$ can be provided analogously. Mathematically, the indicator Δ_{ijk} is defined as

$$\Delta_{ijk} = \begin{cases} 0 & \text{for } min(T_{1ijk}, T_{2ijk}) > IT_i; \\ 1 & \text{for } T_{1ijk} < min(T_{2ijk}, IT_i); \\ 2 & \text{for } T_{2ijk} < min(T_{1ijk}, IT_i). \end{cases} \tag{2}$$

For example, if we conduct the ALT under temperatures $w_j =$ 35, 45, 55, 65, and inspection times $IT_i = 10, 20, 30$, with $K = 10$ items placed under every test condition, then the data observed will be of the form in Table I.

We also denote p_{0ij}, p_{1ij}, and p_{2ij} for the survival probability, failure probability due to cause 1, and failure probability due to cause 2, respectively, as follows.

$$\begin{aligned} p_{0ij} &= (1 - F_1(IT_i|w_j))\,(1 - F_2(IT_i|w_j)) \\ &= \exp\left(-(\lambda_{1j} + \lambda_{2j}) IT_i\right), \end{aligned} \tag{3}$$

$$p_{1ij} = \left(\frac{\lambda_{1j}}{\lambda_{1j} + \lambda_{2j}}\right)\left(1 - \exp\left(-(\lambda_{1j} + \lambda_{2j}) IT_i\right)\right), \tag{4}$$

$$p_{2ij} = \left(\frac{\lambda_{2j}}{\lambda_{1j} + \lambda_{2j}}\right)\left(1 - \exp\left(-(\lambda_{1j} + \lambda_{2j}) IT_i\right)\right). \tag{5}$$

Now, given the data $\boldsymbol{\delta} = \{\delta_{ijk}\}$, the numbers of $\delta_{ijk} = 0$, $\delta_{ijk} = 1$, and $\delta_{ijk} = 2$, given by S_{ij}, d_{1ij}, and d_{2ij}, respectively, collected at temperatures $\boldsymbol{w} = \{w_j, j = 1, \ldots, J\}$, and inspection times $\boldsymbol{IT} = \{IT_i, i = 1, \ldots, I\}$, the likelihood function of $\boldsymbol{\alpha} = \{\alpha_{10}, \alpha_{11}, \alpha_{20}, \alpha_{21}\}$ is given by

$$L(\boldsymbol{\alpha}|\boldsymbol{\delta}, \boldsymbol{IT}, \boldsymbol{w}) = \prod_{i=1}^{I} \prod_{j=1}^{J} p_{0ij}^{S_{ij}} p_{1ij}^{d_{1ij}} p_{2ij}^{d_{2ij}}. \tag{6}$$

III. Bayesian Estimation

A Bayesian estimation method is developed in this section for the estimation of the model parameters. Fan *et al.* [3] developed a Bayesian approach for estimating the parameters of one-shot devices. For developing results along their lines, several modifications need to be done to their algorithm.

Let $\pi(\boldsymbol{\alpha})$ be the joint prior. Then, the joint posterior density of $\boldsymbol{\alpha}$, given $\boldsymbol{\delta}$, $\boldsymbol{IT}$, and $\boldsymbol{w}$, is

$$\pi(\boldsymbol{\alpha}|\boldsymbol{\delta}, \boldsymbol{IT}, \boldsymbol{w}) = \frac{L(\boldsymbol{\alpha}|\boldsymbol{\delta}, \boldsymbol{IT}, \boldsymbol{w})\pi(\boldsymbol{\alpha})}{\int L(\boldsymbol{\alpha}|\boldsymbol{\delta}, \boldsymbol{IT}, \boldsymbol{w})\pi(\boldsymbol{\alpha})d\boldsymbol{\alpha}}. \tag{7}$$

The denominator in (7) is usually not in a closed-form. Following the method of [3], the Bayesian point estimate of $\hat{\boldsymbol{\alpha}} = \sum_{n=1}^{N} \boldsymbol{\alpha}^{(n)}/N$, where $\boldsymbol{\alpha}^{(n)}$ is the nth value out of N samples from the posterior distribution. We may be interested in estimates of some parameters of interest at room temperature (typical, normal condition), such as the failure rate, λ_r, the survival probability at IT_i, p_{0i}, the s-mean lifetimes for cause r, $E(T_r)$, the s-expected life time, $E(T)$, and the probability of failure due to cause 1 given failure, P.d1 $= P(T_1 < T_2|min(T_1, T_2) < IT)$. They can be estimated as follows.

$$\begin{aligned}
\hat{\lambda}_r &= N^{-1} \sum_{n=1}^{N} \alpha_{r0}^{(n)} e^{\alpha_{r1}^{(n)} w_j} = N^{-1} \sum_{n=1}^{N} \lambda_r^{(n)} \\
\hat{p}_{0i} &= N^{-1} \sum_{n=1}^{N} \exp\left(-\left(\lambda_1^{(n)} + \lambda_2^{(n)}\right) IT_i\right) \\
\hat{\text{P.d1}} &= N^{-1} \sum_{n=1}^{N} \lambda_1^{(n)} / \left(\lambda_1^{(n)} + \lambda_2^{(n)}\right) \\
\hat{E}(T_r) &= N^{-1} \sum_{n=1}^{N} \left(\lambda_r^{(n)}\right)^{-1} \\
\hat{E}(T) &= N^{-1} \sum_{n=1}^{N} \left(\lambda_1^{(n)} + \lambda_2^{(n)}\right)^{-1}
\end{aligned}$$

The mean lifetime is a useful quantity for finding the $q_0 \times 100\%$ quantile t_0 of the devices with exponential lifetimes, and the relationship is simply

$$P(T > t_0) = \exp(-\lambda t_0) \Longrightarrow t_0 = -\ln(q_0)/\lambda.$$

Consequently, the Bayesian estimate of the quantile will be $-\ln(q_0) N^{-1} \sum_{n=1}^{N} \left(\lambda^{(n)}\right)^{-1}$, which is a constant multiple of the Bayesian estimate of the mean lifetime. So, the MSE of the estimate of the quantile will simply be the multiplication of the MSE of the mean lifetime by $(\ln(q_0))^2$.

Several prior distributions are considered in this study, as described below.

A. Exponential Prior Distribution

Because the parameters $(\alpha_{10}, \alpha_{11}, \alpha_{20}, \alpha_{21})$ are all positive, a simple prior distribution for them is an exponential one of the form

$$\pi_1(\boldsymbol{\alpha}) = \prod_{r=1}^{R} \theta_{r0}^{-1} e^{-\alpha_{r0}/\theta_{r0}} \theta_{r1}^{-1} e^{-\alpha_{r1}/\theta_{r1}}, \tag{8}$$

where $\alpha_{rm}, \theta_{rm} > 0$ for $r = 1, 2$, $m = 0, 1$.

1) Least-Square Estimates for Hyperparameters: The θ_{rm} are the unknown hyperparameters, and $\mathrm{E}(\alpha_{rm}) = \theta_{rm}$. Fan *et al.* [3] assumed that the reliability of the items under w_j and IT_i are around $\hat{p}_{0ij}, \hat{p}_{1ij}, \hat{p}_{2ij}$, for $i = 1, \cdots, I$, and $j = 1, \cdots, J$, so $\hat{p}_{0ij}, \hat{p}_{1ij}, \hat{p}_{2ij}$ can be empirically estimated as $S_{ij}/K_{ij}, d_{1ij}/K_{ij}, d_{2ij}/K_{ij}$. If one of these estimates is zero, then it will be hard to determine the initial value. The zero-frequency problem has long been discussed in the literature, and Lee and Cohen [9] suggested using

$$(\tilde{p}_{0ij}, \tilde{p}_{1ij}, \tilde{p}_{2ij}) = \left(\frac{S_{ij}+1}{K_{ij}+3}, \frac{d_{1ij}+1}{K_{ij}+3}, \frac{d_{2ij}+1}{K_{ij}+3}\right). \tag{9}$$

Recall that we can estimate $\lambda_{1ij} + \lambda_{2ij} = -\ln(p_{0ij})/IT_i$ by (3). Then, by (4) and (5), we can rewrite

$$\begin{aligned}
\ln(\alpha_{r0}) + \alpha_{r1} w_j = \ln(p_{rij}) &- \ln(1 - p_{0ij}) \\
&+ \ln\left(-\ln(p_{0ij})\right) - \ln(IT_i)
\end{aligned} \tag{10}$$

for $l = 1, 2$. By replacing p_{rij} by the estimates in (9), we can obtain the least-squares estimate of $\boldsymbol{\alpha}$ by minimizing

$$S(\boldsymbol{\alpha}) = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{r=1}^{2} \left(\hat{y}_{rij} - \ln(\alpha_{r0}) - \alpha_{r1} w_j - \ln(IT_i)\right)^2, \tag{11}$$

where

$$\hat{y}_{rij} = \ln(\tilde{p}_{rij}) - \ln(1 - \tilde{p}_{0ij}) + \ln\left(-\ln(\tilde{p}_{0ij})\right). \tag{12}$$

By performing the necessary algebra, we derive the least-squares estimates as shown in (13)–(15) at the bottom of the next page.

However, the least-square estimates are not guaranteed to be positive, which will violate the assumption that both α_{11} and α_{21} are positive. To have a least-square estimate with non-negative constraints, Liew [10] suggested using the Inequality Constraints Least-Square (ICLS) method: if $\hat{\alpha}_{r1}^{LSE}$ is negative, then

$$\begin{aligned}
\hat{\alpha}_{r1}^{\text{ICLS}} &= 0, \\
\ln\left(\hat{\alpha}_{r0}^{\text{ICLS}}\right) &= \ln\left(\hat{\alpha}_{r0}^{\text{LSE}}\right) + \hat{\alpha}_{r1}^{\text{LSE}} \times \frac{\sum_{i=1}^{I}\sum_{j=1}^{J} K_{ij}^2 w_j}{\sum_{i=1}^{I}\sum_{j=1}^{J} K_{ij}^2};
\end{aligned}$$

otherwise,

$$\begin{aligned}
\hat{\alpha}_{r1}^{\text{ICLS}} &= \hat{\alpha}_{r1}^{LSE}, \\
\hat{\alpha}_{r0}^{\text{ICLS}} &= \hat{\alpha}_{r0}^{LSE}.
\end{aligned} \tag{16}$$

If $\hat{\alpha}_{r1}^{\mathrm{ICLS}} = 0$, we will add a very small number to it, and make it positive: $\hat{\alpha}_{r1}^{\mathrm{ICLS}} = 10^{-14}$, say. By presuming that the ICLS estimates are close to the prior means, we can use them as hyperparameter values for α_{r0} and α_{r1}, i.e., $\theta_{rm} = \hat{\alpha}_{rm}^{\mathrm{ICLS}}$ for $r = 1, 2$ and $m = 0, 1$. Upon combining with (7), the posterior density becomes

$$\begin{aligned}\pi_1(\boldsymbol{\alpha}|\boldsymbol{\delta}, \boldsymbol{IT}, \boldsymbol{w}) &\propto L(\boldsymbol{\alpha}|\boldsymbol{\delta}, \boldsymbol{IT}, \boldsymbol{w})\pi_1\left(\boldsymbol{\alpha}|\boldsymbol{\theta} = \boldsymbol{\alpha}^{\mathrm{ICLS}}\right)\\ &\propto \prod_{i=1}^{I}\prod_{j=1}^{J} p_{0ij}^{S_{ij}} p_{1ij}^{d_{1ij}} p_{2ij}^{d_{2ij}}\\ &\quad\times \prod_{r=1}^{R} \exp\left(-\left(\frac{\alpha_{r0}}{\hat{\alpha}_{r0}^{\mathrm{ICLS}}} + \frac{\alpha_{r1}}{\hat{\alpha}_{r1}^{\mathrm{ICLS}}}\right)\right). \end{aligned} \tag{17}$$

B. Normal Prior

Let ϵ_{rij} be the error such that

$$\tilde{p}_{rij} = p_{rij} + \epsilon_{rij}, \tag{18}$$

and let us now assume that the errors ϵ_{rij} are i.i.d. $N(0, \sigma^2)$ variables. Then, the conditional likelihood function of $\boldsymbol{\alpha}$, given σ^2, is

$$\begin{aligned}&L(\boldsymbol{\alpha}|IT_i, w_j, \tilde{p}_{rij}, \sigma^2)\\ &\quad\propto \prod_{r=1}^{2} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}\left(p_{rij} - \tilde{p}_{rij}\right)^2\right\},\end{aligned}$$

where p_{rij}, and $\tilde{p}_{rij}$ are as specified in (4), (5), and (9), respectively. We will now adopt the likelihood function as the prior distribution of $\boldsymbol{\alpha}$:

$$\begin{aligned}&\pi_2(\boldsymbol{\alpha}|\boldsymbol{IT}, \boldsymbol{w}, \sigma^2)\\ &\quad\propto \prod_{i=1}^{I}\prod_{j=1}^{J}\prod_{r=1}^{2} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(p_{rij} - \tilde{p}_{rij})^2\right\}.\end{aligned} \tag{19}$$

As σ^2 is unknown, we adopt the non-informative prior

$$\pi(\sigma^2) \propto \frac{1}{\sigma^2}, \quad \sigma^2 > 0,$$

which yields the joint prior density of $\boldsymbol{\alpha}$ as

$$\begin{aligned}&\pi_2(\boldsymbol{\alpha}|\boldsymbol{IT}, \boldsymbol{w})\\ &\quad\propto \int_0^\infty \pi_2(\boldsymbol{\alpha}|\boldsymbol{IT}, \boldsymbol{w}, \sigma^2)\pi(\sigma^2)d\sigma^2\\ &\quad\propto \int_0^\infty (\sigma^2)^{-\frac{2IJ+2}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{r=1}^{2}(p_{rij} - \tilde{p}_{rij})^2\right\}d\sigma^2\\ &\quad\propto \left\{\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{r=1}^{2}(p_{rij} - \tilde{p}_{rij})^2\right\}^{-IJ}.\end{aligned} \tag{20}$$

Then, by (7), the joint posterior density of $\boldsymbol{\alpha}$ becomes

$$\begin{aligned}\pi_2(\boldsymbol{\alpha}|\boldsymbol{\delta}, \boldsymbol{IT}, \boldsymbol{w}) &\propto \prod_{i=1}^{I}\prod_{j=1}^{J} p_{0ij}^{S_{ij}} p_{1ij}^{d_{1ij}} p_{2ij}^{d_{2ij}}\\ &\quad\times \left\{\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{r=1}^{2}(p_{rij} - \tilde{p}_{rij})^2\right\}^{-IJ}.\end{aligned} \tag{21}$$

C. Dirichlet Prior

The natural extension of the beta prior discussed in [3] to the competing risks scenario is the Dirichlet prior. The prior density corresponding to p_{rij} is

$$f_{ij}(p_{0ij}, p_{1ij}, p_{2ij}) = \frac{p_{0ij}^{\beta_{0ij}-1} p_{1ij}^{\beta_{1ij}-1} p_{2ij}^{\beta_{2ij}-1}}{B(\boldsymbol{\beta}_{ij})},$$

where

$$\begin{aligned}p_{0ij} + p_{1ij} + p_{2ij} = 1,\ p_{0ij}, p_1, p_{2ij} > 0,\ \text{and } B(\boldsymbol{\beta}_{ij})\\ = \frac{\Gamma(\beta_{0ij})\Gamma(\beta_{1ij})\Gamma(\beta_{2ij})}{\Gamma(\beta_{0ij} + \beta_{1ij} + \beta_{2ij})}.\end{aligned}$$

The hyperparameters $\boldsymbol{\beta}_{ij}$ are then chosen so as to match

$$\mathrm{E}(p_{rij}) = \frac{\beta_{rij}}{\beta_{0ij} + \beta_{1ij} + \beta_{2ij}} = \tilde{p}_{rij}, \quad r = 1, 2. \tag{22}$$

$$\hat{\alpha}_{r1}^{\mathrm{LSE}} = \left\{\begin{array}{c}\left(\sum_{i=1}^{I}\sum_{j=1}^{J} K_{ij}\right)\sum_{i=1}^{I}\sum_{j=1}^{J} K_{ij}w_j\left(\hat{y}_{rij} - \ln(IT_i)\right) -\\ \left(\sum_{i=1}^{I}\sum_{j=1}^{J} w_jK_{ij}\right)\sum_{i=1}^{I}\sum_{j=1}^{J} K_{ij}\left(\hat{y}_{rij} - \ln(IT_i)\right)\end{array}\right\} \tag{13}$$

$$\times\left\{\left(\sum_{i=1}^{I}\sum_{j=1}^{J} w_j^2K_{ij}\right)\left(\sum_{i=1}^{I}\sum_{j=1}^{J} K_{ij}\right) - \left(\sum_{i=1}^{I}\sum_{j=1}^{J} w_jK_{ij}\right)^2\right\}^{-1}, \tag{14}$$

$$\hat{\alpha}_{r0}^{\mathrm{LSE}} = \exp\left\{\begin{array}{c}\left\{\begin{array}{c}\left(\sum_{i=1}^{I}\sum_{j=1}^{J} w_j^2K_{ij}\right)\sum_{i=1}^{I}\sum_{j=1}^{J} K_{ij}(\hat{y}_{rij} - \ln(IT_i)) -\\ \left(\sum_{i=1}^{I}\sum_{j=1}^{J} w_jK_{ij}\right)\sum_{i=1}^{I}\sum_{j=1}^{J} K_{ij}w_j(\hat{y}_{rij} - \ln(IT_i))\end{array}\right\}\\ \times\left\{\left(\sum_{i=1}^{I}\sum_{j=1}^{J} w_j^2K_{ij}\right)\left(\sum_{i=1}^{I}\sum_{j=1}^{J} K_{ij}\right) - \left(\sum_{i=1}^{I}\sum_{j=1}^{J} w_jK_{ij}\right)^2\right\}^{-1}\end{array}\right\} \tag{15}$$

Clearly, one more equation is needed for the determination of the hyperparameter $\boldsymbol{\beta}_{ij}$. For this equation, we may focus on the variance of p_{0ij}, which corresponds to the accuracy of the survival probability of one-shot devices. With the prior belief that $\mathrm{Var}(p_{0ij}) = c^2$, the last equation to be used for determining the hyperparameter $\boldsymbol{\beta}_{ij}$ is given by

$$\mathrm{Var}(p_{0ij}) = \frac{\beta_{0ij}(\beta_{1ij} + \beta_{2ij})}{\left(\sum_{r=0}^{2} \beta_{rij}\right)^2 \left(\sum_{r=0}^{2} \beta_{rij} + 1\right)} = c^2. \quad (23)$$

Using (22) and (23), we then obtain the hyperparameters as

$$\beta_{1ij} = \tilde{p}_{1ij} \sum_{r=0}^{2} \beta_{rij}, \beta_{2ij} = \tilde{p}_{2ij} \sum_{r=0}^{2} \beta_{rij},$$
$$\beta_{0ij} = \sum_{r=0}^{2} \beta_{rij} - \beta_{1ij} - \beta_{2ij}, \quad (24)$$

where

$$\sum_{r=0}^{2} \beta_{rij} = \left(\frac{\tilde{p}_{0ij}(1-\tilde{p}_{0ij})}{c^2} - 1\right),$$

which yields the posterior distribution to be

$$\pi_3(\boldsymbol{\alpha}|\boldsymbol{\delta}, \boldsymbol{IT}, \boldsymbol{w}) \propto \prod_{i=1}^{I} \prod_{j=1}^{J} p_{0ij}^{S_{ij}+\beta_{0ij}-1} p_{1ij}^{d_{1ij}+\beta_{1ij}-1} p_{2ij}^{d_{2ij}+\beta_{2ij}-1}. \quad (25)$$

Here, p_{0ij}, p_{1ij}, and p_{2ij} are as specified in (3), (4), and (5), respectively; and $\tilde{p}_{0ij}$, $\tilde{p}_{1ij}$, and $\tilde{p}_{2ij}$ are as specified in (9).

D. Prior Belief on p_{rij}

In the study of Fan *et al.* [3], they supposed that the prior belief of p_{ij}, denoted by $\hat{p}_{ij}$, is very reliable with regard to the true unknown parameter p_{ij}. So, they generated p_{0ij} from a beta distribution with a specific choice of parameters. Now, by incorporating competing risks into the one-shot device testing, we suppose that the prior belief of p_{rij}, denoted by $\hat{p}_{rij}$, is also very reliable in the sense that the variance of the prior belief on the survival probability, $\mathrm{Var}(\hat{p}_{0ij}) = c^2$, is small, with c^2 being a small constant. We also assume that $E(\hat{p}_{rij}) = p_{rij}$. Then, with the choice of parameter being similar to the one in (24), we have

$$f(\hat{p}_{0ij}, \hat{p}_{1ij}, \hat{p}_{2ij}) \propto \hat{p}_{0ij}^{\beta^*_{0ij}-1} \hat{p}_{1ij}^{\beta^*_{1ij}-1} \hat{p}_{2ij}^{\beta^*_{2ij}-1}, \quad (26)$$

where $\hat{p}_{0ij} + \hat{p}_{1ij} + \hat{p}_{2ij} = 1, \hat{p}_{0ij}, \hat{p}_{1ij}, \hat{p}_{2ij} > 0$. The parameters β^*_{rij} are chosen to be

$$\beta^*_{1ij} = \hat{p}_{1ij} \sum_{r=0}^{2} \beta^*_{rij}, \quad \beta^*_{2ij} = \hat{p}_{2ij} \sum_{r=0}^{2} \beta^*_{rij},$$
$$\beta^*_{0ij} = \sum_{r=0}^{2} \beta_{rij} - \beta^*_{1ij} - \beta^*_{2ij},$$

where

$$\sum_{r=0}^{2} \beta^*_{rij} = \left(\frac{\hat{p}_{0ij}(1-\hat{p}_{0ij})}{c^2} - 1\right). \quad (27)$$

TABLE II
PARAMETER VALUES USED IN THE SIMULATION STUDY FOR DEVICES WITH HIGH RELIABILITY

Parameters	Symbols	Values
Risk 1	α_{10}, α_{11}	(0.001,0.05)
Risk 2	α_{20}, α_{21}	(0.0001,0.08)
Temperature(°C)	w_1, w_2, w_3, w_4	(35,45,55,65)
Inspection Time (days)	IT_1, IT_2, IT_3	(10,20,30)
Sample size	K_1, K_2, K_3	(10,50,100)
Prior belief variance	c^2	0.001
Scale parameters for MH	(σ_0, σ_1)	(0.02, 0.03)

The prior belief on the parameter can be used to replace $\tilde{p}_{rij}$ in (12), (21), and (24); and the corresponding posterior distribution will then result. Note that $\sum_{r=0}^{2} \beta^*_{rij}$ have to be larger than zero. From (27), this means that $c^2 < p_{0ij}(1 - p_{0ij})$.

IV. SIMULATION STUDY

In this section, we will compare the performance of the Bayesian estimation with three prior distributions and two prior beliefs, $\tilde{p}_{rij}$ and $\hat{p}_{rij}$, with that of the MLEs obtained by the EM algorithm [8]. We consider an accelerated life test with $I = 3$ inspection times, and $J = 4$ levels of temperatures. We repeat this experiment with different sample sizes of K devices allocated to each condition. The parameter settings used in this simulation study are as given in Table II.

Observe that risk 2 has a larger coefficient (0.08) than risk 1 (0.05). This outcome means, for example, that risk 2, the leakage of organic fuel, is more sensitive to the temperature as compared to risk 1, the burnout of resistance wire, in an electro-explosive device under a one-shot device test. This simulation setting is imitating the higher chances of having cracks on the case with increasing temperature while the resistance wire is not being sensitive to a change in temperature. When risk 1 has a higher intercept (0.001) than risk 2 (0.0001), the setting mimics the fact that most of the common failures are due to a disconnection of resistance wires before ignition.

The disconnection may be due to shocks on daily-use basis or manufacturing problems, and leads to a higher intercept of risk 1. The intercept of risks are small, and so the products are indeed of high reliability.

In this simulation study, 1000 sets of data were simulated under the specified settings. For the EM algorithm, the iteration was terminated when the sum of squares of the difference in parameter estimates was less than 10^{-10}. For the Bayesian estimate, we used the Metropolis-Hastings algorithm [11] to simulate the posterior distributions. We generated a vector of four log-normal random variables with mean parameters equal to the logarithms of the previous estimates, and the scale parameters as $(\sigma_0, \sigma_1) = (0.02, 0.03)$ for $(\alpha_{r0}, \alpha_{r1})$, $r = 1, 2$, which gave an acceptance rate of about 0.25; this approach is considered to be optimal in practice as stated in [12], and [13]. We simulated a sequence of 10,000 random variables from the algorithm, and the first 1000 data points were discarded to account for burn-in. We then chose one sample for

TABLE III
BIAS, AND MSE OF THE ESTIMATES OF THE PARAMETERS FOR DEVICES WITH HIGH RELIABILITY UNDER DIFFERENT ESTIMATION METHODS

	Bias			MSE		
α_{10}=0.001	$K=10$	$K=50$	$K=100$	$K=10$	$K=50$	$K=100$
EM	**1.704e-03**	3.056e-04	1.632e-04	**7.087e-06**	4.656e-07	1.779e-07
Exp($\tilde{p}$)	**1.64e-03**	**-1.007e-04**	-2.16e-04	9.558e-06	**2.34e-07**	1.298e-07
Norm($\tilde{p}$)	1.923e-03	**3.39e-05**	**-1.151e-04**	1.058e-05	2.62e-07	**1.155e-07**
Dir($\tilde{p}$)	2.117e-03	**6.467e-05**	**-1.316e-04**	**9.429e-06**	**2.232e-07**	**1.012e-07**
Exp($\hat{p}$)	**7.381e-04**	-1.022e-04	-1.812e-04	**2.87e-06**	**1.878e-07**	1.201e-07
Norm($\hat{p}$)	3.757e-03	1.696e-04	**-7.962e-05**	2.478e-05	4.367e-07	**1.185e-07**
Dir($\hat{p}$)	4.36e-03	5.442e-03	5.137e-03	2.363e-05	3.082e-05	2.725e-05
α_{11}=0.05	$K=10$	$K=50$	$K=100$	$K=10$	$K=50$	$K=100$
EM	-1.201e-02	-2.621e-03	**-1.643e-03**	**3.346e-04**	**8.141e-05**	**4.007e-05**
Exp($\tilde{p}$)	**-8.47e-03**	4.979e-03	5.868e-03	4.783e-04	1.153e-04	7.776e-05
Norm($\tilde{p}$)	-1.045e-02	**1.856e-03**	**3.252e-03**	4.086e-04	**8.523e-05**	**5.255e-05**
Dir($\tilde{p}$)	-1.354e-02	**1.25e-03**	**3.735e-03**	3.593e-04	**6.647e-05**	**5.041e-05**
Exp($\hat{p}$)	**-3.102e-03**	4.733e-03	5.086e-03	**2.817e-04**	1.039e-04	6.821e-05
Norm($\hat{p}$)	-1.288e-02	**2.003e-03**	3.849e-03	3.618e-04	9.056e-05	5.741e-05
Dir($\hat{p}$)	**-7.48e-03**	-1.683e-02	-1.964e-02	**7.921e-05**	2.951e-04	3.952e-04
α_{20}=0.0001	$K=10$	$K=50$	$K=100$	$K=10$	$K=50$	$K=100$
EM	1.444e-03	**1.639e-04**	**6.97e-05**	3.178e-06	**5.172e-08**	**1.309e-08**
Exp($\tilde{p}$)	1.261e-03	**1.653e-04**	**1.18e-04**	3.086e-06	**5.727e-08**	**2.66e-08**
Norm($\tilde{p}$)	1.539e-03	2.799e-04	1.968e-04	4.655e-06	1.34e-07	5.924e-08
Dir($\tilde{p}$)	1.842e-03	3.334e-04	2.115e-04	5.939e-06	1.71e-07	6.383e-08
Exp($\hat{p}$)	**5.762e-04**	**1.867e-04**	**1.484e-04**	**6.071e-07**	**6.701e-08**	**3.723e-08**
Norm($\hat{p}$)	**1.096e-03**	2.228e-04	1.566e-04	**2.58e-06**	1.046e-07	4.135e-08
Dir($\hat{p}$)	**5.446e-04**	7.254e-04	7.603e-04	**3.327e-07**	5.743e-07	6.236e-07
α_{21}=0.08	$K=10$	$K=50$	$K=100$	$K=10$	$K=50$	$K=100$
EM	-3.982e-02	**-1.276e-02**	**-6.323e-03**	1.749e-03	**2.765e-04**	**1.281e-04**
Exp($\tilde{p}$)	-4.004e-02	**-1.228e-02**	**-1.048e-02**	2.033e-03	**2.905e-04**	**1.899e-04**
Norm($\tilde{p}$)	-3.768e-02	-1.798e-02	-1.558e-02	1.778e-03	4.518e-04	3.157e-04
Dir($\tilde{p}$)	-4.083e-02	-2.073e-02	-1.682e-02	1.9e-03	5.379e-04	3.459e-04
Exp($\hat{p}$)	**-2.883e-02**	**-1.363e-02**	**-1.278e-02**	**1.133e-03**	**3.188e-04**	**2.405e-04**
Norm($\hat{p}$)	**-2.968e-02**	-1.408e-02	-1.283e-02	**1.279e-03**	3.576e-04	2.456e-04
Dir($\hat{p}$)	**-1.042e-02**	-2.055e-02	-2.487e-02	**1.425e-04**	4.479e-04	6.409e-04

every 100 random variables simulated to avoid correlation between the iterated samples, and thus a sample of 90 observations was finally obtained. To generate $(\hat{p}_{0ij}, \hat{p}_{1ij}, \hat{p}_{2ij})$, we set $\mathrm{Var}(\hat{p}_{0ij})$ to be $\min(c^2, p_{0ij}(1-p_{0ij})/12)$ to avoid negative β values. For each method, we set the initial guess of parameters as the ICLS estimates, $\hat{\alpha}_{10}^{\mathrm{ICLS}}, \hat{\alpha}_{11}^{\mathrm{ICLS}}, \hat{\alpha}_{20}^{\mathrm{ICLS}}, \hat{\alpha}_{21}^{\mathrm{ICLS}}$, which were quite often close to the true parameters. To evaluate the performance of the estimators, we will compare their bias, $N^{-1}\sum_{i=1}^{N}\theta^{(i)}-\theta$, and the MSE, $N^{-1}\sum_{i=1}^{N}\left(\theta^{(i)}-\theta\right)^2$, where $\theta^{(i)}$ is the estimate of a quantity of interest from the ith sample out of $N=1000$ simulations, while θ is the true value of that quantity.

Table III shows the bias, and MSEs of the estimates of the model parameters for different estimation methods. Table IV shows the bias, and MSEs of the estimates of some probabilities of interest for different estimation methods. Table V shows the bias, and MSEs of the estimates of the mean lifetimes. For a better readability, the tables in this section are specially formatted. The values in bold indicate the three smallest absolute values of bias and MSEs in each simulation setting. Also, the values within borders indicate which method is the best under a particular simulation setting. From these results, we see that the EM algorithm is generally better than Bayesian methods with prior information $\tilde{p}$, which is solely

TABLE IV
BIAS, AND MSE OF THE ESTIMATES OF SOME PROBABILITIES OF INTEREST FOR DEVICES WITH HIGH RELIABILITY UNDER DIFFERENT ESTIMATION METHODS

	Bias			MSE		
p_{01}=0.9586	$K = 10$	$K = 50$	$K = 100$	$K = 10$	$K = 50$	$K = 100$
EM	-4.907e-02	-9.502e-03	-4.753e-03	2.988e-03	1.995e-04	7.401e-05
Exp($\tilde{p}$)	**-3.068e-02**	**-6.128e-04**	**1.776e-03**	**1.774e-03**	**1.008e-04**	**5.234e-05**
Norm($\tilde{p}$)	**-4.42e-02**	**-6.539e-03**	**-2.585e-03**	**2.908e-03**	**1.556e-04**	6.022e-05
Dir($\tilde{p}$)	-5.895e-02	-9.499e-03	-3.164e-03	4.238e-03	1.924e-04	**5.726e-05**
Exp($\hat{p}$)	**-1.56e-02**	**-1.305e-03**	**2.476e-04**	**5.974e-04**	**9.269e-05**	**4.973e-05**
Norm($\hat{p}$)	-7.415e-02	-8.094e-03	-2.615e-03	6.801e-03	2.108e-04	6.707e-05
Dir($\hat{p}$)	-1.265e-01	-1.224e-01	-1.073e-01	1.626e-02	1.511e-02	1.16e-02
p_{02}=0.9189	$K = 10$	$K = 50$	$K = 100$	$K = 10$	$K = 50$	$K = 100$
EM	-9.109e-02	-1.802e-02	-9.037e-03	1.018e-02	7.165e-04	2.684e-04
Exp($\tilde{p}$)	**-5.679e-02**	**-1.025e-03**	**3.483e-03**	**6.009e-03**	**3.671e-04**	**1.929e-04**
Norm($\tilde{p}$)	**-8.159e-02**	**-1.234e-02**	**-4.875e-03**	**9.752e-03**	**5.592e-04**	2.19e-04
Dir($\tilde{p}$)	-1.087e-01	-1.8e-02	-5.995e-03	1.424e-02	6.91e-04	**2.081e-04**
Exp($\hat{p}$)	**-2.911e-02**	**-2.359e-03**	**5.535e-04**	**2.077e-03**	**3.375e-04**	**1.823e-04**
Norm($\hat{p}$)	-1.347e-01	-1.523e-02	-4.912e-03	2.218e-02	7.517e-04	2.434e-04
Dir($\hat{p}$)	-2.261e-01	-2.194e-01	-1.939e-01	5.18e-02	4.85e-02	3.787e-02
p_{03}=0.8808	$K = 10$	$K = 50$	$K = 100$	$K = 10$	$K = 50$	$K = 100$
EM	-1.269e-01	-2.563e-02	-1.289e-02	1.956e-02	1.448e-03	5.477e-04
Exp($\tilde{p}$)	**-7.89e-02**	**-1.26e-03**	**5.122e-03**	**1.148e-02**	**7.53e-04**	**4.002e-04**
Norm($\tilde{p}$)	**-1.131e-01**	**-1.747e-02**	**-6.894e-03**	**1.846e-02**	**1.131e-03**	4.479e-04
Dir($\tilde{p}$)	-1.505e-01	-2.558e-02	-8.517e-03	2.7e-02	1.396e-03	**4.255e-04**
Exp($\hat{p}$)	**-4.074e-02**	**-3.188e-03**	**9.09e-04**	**4.069e-03**	**6.917e-04**	**3.762e-04**
Norm($\hat{p}$)	-1.84e-01	-2.15e-02	-6.917e-03	4.086e-02	1.509e-03	4.971e-04
Dir($\hat{p}$)	-3.037e-01	-2.955e-01	-2.633e-01	9.325e-02	8.788e-02	6.975e-02
P.d1=0.8253	$K = 10$	$K = 50$	$K = 100$	$K = 10$	$K = 50$	$K = 100$
EM	-2.192e-01	**-7.153e-02**	**-3.341e-02**	6.046e-02	**1.089e-02**	**4.501e-03**
Exp($\tilde{p}$)	-1.846e-01	-1.095e-01	-9.886e-02	5.212e-02	1.824e-02	1.314e-02
Norm($\tilde{p}$)	-2.037e-01	-1.384e-01	-1.248e-01	5.947e-02	2.583e-02	1.94e-02
Dir($\tilde{p}$)	-2.158e-01	-1.488e-01	-1.31e-01	6.112e-02	2.824e-02	2.051e-02
Exp($\hat{p}$)	**-1.486e-01**	-1.16e-01	-1.081e-01	**3.414e-02**	1.976e-02	1.52e-02
Norm($\hat{p}$)	**-7.008e-02**	**-9.386e-02**	**-9.63e-02**	**1.583e-02**	**1.515e-02**	**1.259e-02**
Dir($\hat{p}$)	**-1.466e-02**	**-1.876e-02**	**-2.599e-02**	**9.506e-04**	**1.01e-03**	**1.281e-03**

based on the data. However, the EM algorithm becomes less efficient than Bayesian methods with prior information $\hat{p}$, based on past knowledge.

Also, it is of interest to compare the estimation methods at different reliability, and for this purpose we simulated data sets for different $\boldsymbol{\alpha}$ values. We set $(\alpha_{10}, \alpha_{11}, \alpha_{20}, \alpha_{21}) = (0.004, 0.05, 0.0004, 0.08)$, which represent devices with moderate reliability. The prior belief variance for devices with moderate reliability, c_M^2, was set to be 0.001, and the scale parameters for the proposed distributions in the Metropolis-Hastings sampling were set to be $(\sigma_0, \sigma_1) \times \tau_k^M = (0.02, 0.03) \times \tau_k^M$ for $(\alpha_{r0}, \alpha_{r1})$ $r = 1, 2$, where $(\tau_1^M, \tau_2^M, \tau_3^M) = (3.5, 2, 1.5)$ for $K = 10$, $K = 50$, and $K = 100$, respectively. We also considered the setting $(\alpha_{10}, \alpha_{11}, \alpha_{20}, \alpha_{21}) = (0.008, 0.05, 0.0008, 0.08)$, representing devices with low reliability. The prior belief variance for devices with low reliability, c_L^2, was set to be 0.001 as well, and the scale parameters for the proposed distributions in the Metropolis-Hastings sampling were set to be $(\sigma_0, \sigma_1) \times \tau_k^L = (0.02, 0.03) \times \tau_k^L$ for $(\alpha_{r0}, \alpha_{r1})$ $r = 1, 2$, where $(\tau_1^L, \tau_2^L, \tau_3^L) = (4.6, 1.8, 1.3)$ for $K = 10$, $K = 50$, and $K = 100$, respectively. All other settings were kept the same as before. The choices of the parameters used in the simulation study for devices with moderate, and low reliability are summarized in Tables VI, and VII, respectively.

TABLE V
BIAS, AND MSE OF THE ESTIMATES OF MEAN LIFETIMES FOR DEVICES WITH HIGH RELIABILITY UNDER DIFFERENT ESTIMATION METHODS

	Bias			MSE		
E(T)=236.4	$K=10$	$K=50$	$K=100$	$K=10$	$K=50$	$K=100$
EM	-1.231e+02	-3.628e+01	-1.947e+01	1.615e+04	**3.331e+03**	**1.653e+03**
Exp($\tilde{p}$)	**-6.776e+01**	**1.814e+01**	2.343e+01	**1.1e+04**	4.878e+03	2.938e+03
Norm($\tilde{p}$)	**-1.032e+02**	**-1.802e+01**	**-5.069e+00**	**1.354e+04**	**3.315e+03**	**1.708e+03**
Dir($\tilde{p}$)	-1.332e+02	-3.502e+01	**-9.925e+00**	1.873e+04	**3.201e+03**	**1.476e+03**
Exp($\hat{p}$)	**-3.196e+01**	**1.267e+01**	1.32e+01	**6.699e+03**	4.419e+03	2.264e+03
Norm($\hat{p}$)	-1.389e+02	-2.036e+01	**-2.823e+00**	2.108e+04	3.794e+03	1.933e+03
Dir($\hat{p}$)	-1.813e+02	-1.799e+02	-1.736e+02	3.29e+04	3.238e+04	3.017e+04
E(T_1)=286.5	$K=10$	$K=50$	$K=100$	$K=10$	$K=50$	$K=100$
EM	3.104e+02	6.655e+02	6.72e+02	3.769e+05	7.006e+05	5.698e+05
Exp($\tilde{p}$)	**1.165e+01**	7.992e+01	7.616e+01	3.604e+04	1.971e+04	1.157e+04
Norm($\tilde{p}$)	**-4.941e+01**	**3.966e+01**	**4.792e+01**	**1.953e+04**	**1.139e+04**	**7.04e+03**
Dir($\tilde{p}$)	-1.073e+02	**1.703e+01**	**4.33e+01**	**1.686e+04**	**6.574e+03**	**5.918e+03**
Exp($\hat{p}$)	**4.567e+01**	7.569e+01	6.662e+01	3.437e+04	1.936e+04	9.819e+03
Norm($\hat{p}$)	-1.464e+02	**1.738e+01**	**3.836e+01**	**2.934e+04**	**9.495e+03**	**6.236e+03**
Dir($\hat{p}$)	-2.183e+02	-2.162e+02	-2.078e+02	4.768e+04	4.68e+04	4.321e+04
E(T_2)=1353	$K=10$	$K=50$	$K=100$	$K=10$	$K=50$	$K=100$
EM	-1.342e+03	-1.349e+03	-1.35e+03	1.802e+06	1.82e+06	1.822e+06
Exp($\tilde{p}$)	**-7.837e+02**	**-3.51e+02**	**-3.355e+02**	**7.253e+05**	**2.818e+05**	**2.014e+05**
Norm($\tilde{p}$)	-9.306e+02	-5.874e+02	-5.351e+02	9.296e+05	4.291e+05	3.366e+05
Dir($\tilde{p}$)	-1.059e+03	-6.853e+02	-5.774e+02	1.139e+06	5.205e+05	3.712e+05
Exp($\hat{p}$)	**-5.96e+02**	**-3.92e+02**	**-4.062e+02**	**5.134e+05**	**2.971e+05**	**2.394e+05**
Norm($\hat{p}$)	**-8.475e+02**	**-4.383e+02**	**-4.271e+02**	**8.184e+05**	**3.318e+05**	**2.552e+05**
Dir($\hat{p}$)	-1.052e+03	-1.051e+03	-1.029e+03	1.108e+06	1.106e+06	1.06e+06

TABLE VI
PARAMETER VALUES USED IN THE SIMULATION STUDY FOR DEVICES WITH MODERATE RELIABILITY

Parameters	Symbols	Values
Risk 1	α_{10}, α_{11}	(0.004,0.05)
Risk 2	α_{20}, α_{11}	(0.0004,0.08)
Temperature(°C)	w_1, w_2, w_3, w_4	(35,45,55,65)
Inspection Time (days)	IT_1, IT_2, IT_3	(10,20,30)
Sample size	K_1, K_2, K_3	(10,50,100)
Prior belief variance	c_M^2	0.001
Scale parameter weights	$(\tau_1^M, \tau_2^M, \tau_3^M)$	(3.5, 2 ,1.5)

TABLE VII
PARAMETER VALUES USED IN THE SIMULATION STUDY FOR DEVICES WITH LOW RELIABILITY

Parameters	Symbols	Values
Risk 1	α_{10}, α_{11}	(0.008,0.05)
Risk 2	α_{20}, α_{11}	(0.0008,0.08)
Temperature(°C)	w_1, w_2, w_3, w_4	(35,45,55,65)
Inspection Time (days)	IT_1, IT_2, IT_3	(10,20,30)
Sample size	K_1, K_2, K_3	(10,50,100)
Prior belief variance	c_L^2	0.001
Scale parameter weights	$(\tau_1^L, \tau_2^L, \tau_3^L)$	(4.6,1.8,1.3)

The corresponding results obtained for the estimates of the parameters are presented in Tables VIII and IX, the estimates of some probabilities of interest are presented Tables X and XI, and those of the estimates of the mean lifetimes are given in Tables XII and XIII.

From all these results, the method with the best performance in terms of the least bias, and the MSE are summarized in Tables XIV, and XV, respectively. From Table XIV, we observe that Dir($\hat{p}$) is generally good for estimating parameters α_{r0}, α_{r1}, $r = 1, 2$, when the devices' reliability is high. However, when the devices' reliability is low or moderate, the EM algorithm works generally better, and this result agrees with the finding in [4]. When we estimate the survival probabilities under normal operating conditions, Bayesian estimates with various priors are good for devices with high and moderate reliability, but EM works better for devices with low reliability. However, in

TABLE VIII
BIAS, AND MSE OF THE ESTIMATES OF THE PARAMETERS FOR DEVICES WITH MODERATE RELIABILITY UNDER DIFFERENT ESTIMATION METHODS

	Bias			MSE		
α_{10}=0.004	$K=10$	$K=50$	$K=100$	$K=10$	$K=50$	$K=100$
EM	**1.502e-03**	**2.873e-04**	**1.242e-04**	2.367e-05	2.179e-06	**8.762e-07**
Exp($\tilde{p}$)	2.495e-03	-8.77e-04	-1.058e-03	5.968e-05	2.077e-06	1.679e-06
Norm($\tilde{p}$)	2.814e-03	-4.472e-04	-7.468e-04	4.807e-05	1.719e-06	1.26e-06
Dir($\tilde{p}$)	2.65e-03	-5.547e-04	-9.071e-04	2.495e-05	**1.589e-06**	1.364e-06
Exp($\hat{p}$)	-1.562e-03	-1.302e-03	-1.241e-03	**5.713e-06**	2.433e-06	1.958e-06
Norm($\hat{p}$)	**8.122e-04**	**-5.55e-05**	**-4.261e-04**	**6.836e-06**	**9.813e-07**	**7.076e-07**
Dir($\hat{p}$)	**3.617e-04**	**-5.607e-05**	**-3.213e-04**	**3.136e-06**	**4.527e-07**	**5.088e-07**
α_{11}=0.05	$K=10$	$K=50$	$K=100$	$K=10$	$K=50$	$K=100$
EM	**-6.024e-04**	**-3.337e-04**	**-9.176e-05**	2.456e-04	**4.291e-05**	**1.909e-05**
Exp($\tilde{p}$)	**1.103e-03**	6.563e-03	6.764e-03	3.193e-04	9.162e-05	6.915e-05
Norm($\tilde{p}$)	-3.878e-03	3.323e-03	4.512e-03	2.567e-04	5.539e-05	4.46e-05
Dir($\tilde{p}$)	-9.855e-03	3.771e-03	5.723e-03	**2.134e-04**	5.432e-05	5.332e-05
Exp($\hat{p}$)	1.54e-02	9.068e-03	7.897e-03	3.954e-04	1.203e-04	8.247e-05
Norm($\hat{p}$)	-1.68e-03	**1.192e-03**	**2.872e-03**	**5.202e-05**	**2.749e-05**	**2.513e-05**
Dir($\hat{p}$)	**-1.149e-03**	**4.588e-04**	**1.818e-03**	**2.471e-05**	**1.309e-05**	**1.442e-05**
α_{20}=0.0004	$K=10$	$K=50$	$K=100$	$K=10$	$K=50$	$K=100$
EM	3.808e-04	**5.883e-05**	**3.505e-05**	7.709e-07	**5.813e-08**	**2.564e-08**
Exp($\tilde{p}$)	1.21e-03	4.475e-04	4.379e-04	4.646e-06	3.565e-07	2.744e-07
Norm($\tilde{p}$)	2.538e-03	8.284e-04	6.783e-04	1.805e-05	9.915e-07	5.937e-07
Dir($\tilde{p}$)	2.607e-03	8.28e-04	6.366e-04	1.249e-05	1.01e-06	5.356e-07
Exp($\hat{p}$)	**1.872e-04**	3.665e-04	4.191e-04	**1.72e-07**	2.253e-07	2.443e-07
Norm($\hat{p}$)	**1.844e-04**	**1.936e-04**	**2.687e-04**	**1.279e-07**	**9.545e-08**	**1.222e-07**
Dir($\hat{p}$)	**1.141e-04**	**1.665e-04**	**2.334e-04**	**3.63e-08**	**5.034e-08**	**7.916e-08**
α_{21}=0.08	$K=10$	$K=50$	$K=100$	$K=10$	$K=50$	$K=100$
EM	-5.149e-03	**-4.093e-04**	**-3.831e-04**	3.37e-04	**8.212e-05**	**4.253e-05**
Exp($\tilde{p}$)	-1.454e-02	-1.068e-02	-1.174e-02	5.537e-04	1.91e-04	1.778e-04
Norm($\tilde{p}$)	-2.528e-02	-1.75e-02	-1.618e-02	9.815e-04	3.798e-04	3.022e-04
Dir($\tilde{p}$)	-3.422e-02	-1.973e-02	-1.701e-02	1.345e-03	4.656e-04	3.32e-04
Exp($\hat{p}$)	**5.342e-04**	-9.259e-03	-1.138e-02	**1.942e-04**	1.458e-04	1.653e-04
Norm($\hat{p}$)	**-3.88e-03**	**-5.109e-03**	**-7.862e-03**	**1.07e-04**	**8.447e-05**	**1.023e-04**
Dir($\hat{p}$)	**-3.688e-03**	**-5.726e-03**	**-7.872e-03**	**4.726e-05**	**5.935e-05**	**8.314e-05**

Table XV, Dir($\hat{p}$) is seen to be the best method in terms of the minimum MSE. We also observe that EM is generally better than other methods for large samples with moderate and low reliability. This result may be true because we are likely to observe failures in the case of large samples with moderate and low reliability. The EM algorithm performs generally well with more observed failures. If the mean lifetimes are the quantities we are interested in, then the Bayesian estimates are much better than the EM estimate. Dir($\hat{p}$) generally estimates the mean lifetime with the least bias. But, Dir($\hat{p}$) only performs well in this case for devices with high reliability, and Norm($\hat{p}$) provides better estimates for devices with moderate and low reliability.

In some cases, the prior belief $\hat{p}$ is not always available. The methods based only on observed data that provide the least bias, and MSE in this case are listed in Tables XVI, and XVII, respectively. From Table XVI, we see that no particular method is best for the estimation in the case of devices with high reliability, but EM turns out to be the best in general for devices with moderate and low reliability. From Table XVII, EM is seen to perform well in estimating the parameters and the probability of failure due to cause 1, P.d1. Exp($\tilde{p}$) is good for estimating the survival probabilities for devices with high reliability, while Dir($\tilde{p}$) is better for devices with moderate and low reliability. If the mean lifetimes are the quantities of interest, then the Bayesian esti-

TABLE IX
BIAS, AND MSE OF THE ESTIMATES OF THE PARAMETERS FOR DEVICES WITH LOW RELIABILITY UNDER DIFFERENT ESTIMATION METHODS

	Bias			MSE		
α_{10}=0.008	$K=10$	$K=50$	$K=100$	$K=10$	$K=50$	$K=100$
EM	**1.145e-03**	**2.29e-04**	**8.212e-05**	**7.553e-05**	8.202e-06	**3.821e-06**
Exp($\tilde{p}$)	1.216e-02	-1.31e-03	-1.776e-03	6.438e-04	**7.508e-06**	5.902e-06
Norm($\tilde{p}$)	7.623e-03	**-7.946e-04**	**-1.381e-03**	3.786e-04	**7.506e-06**	**4.912e-06**
Dir($\tilde{p}$)	8.171e-03	**-6.264e-04**	**-1.487e-03**	1.801e-04	**6.317e-06**	**4.794e-06**
Exp($\hat{p}$)	-4.772e-03	-3.559e-03	-2.997e-03	**4.207e-05**	1.47e-05	1.05e-05
Norm($\hat{p}$)	**-1.57e-03**	-2.049e-03	-2.038e-03	8.649e-05	8.059e-06	6.322e-06
Dir($\hat{p}$)	**-3.28e-03**	-3.69e-03	-3.563e-03	**2.222e-05**	1.431e-05	1.323e-05
α_{11}=0.05	$K=10$	$K=50$	$K=100$	$K=10$	$K=50$	$K=100$
EM	6.195e-03	**8.521e-04**	**4.251e-04**	4.722e-04	**5.75e-05**	**2.568e-05**
Exp($\tilde{p}$)	-6.132e-03	5.587e-03	5.941e-03	5.308e-04	8.911e-05	6.526e-05
Norm($\tilde{p}$)	**-3.147e-03**	3.616e-03	**4.438e-03**	**4.534e-04**	7.563e-05	**4.873e-05**
Dir($\tilde{p}$)	-1.88e-02	**2.002e-03**	4.747e-03	4.623e-04	**5.086e-05**	4.907e-05
Exp($\hat{p}$)	2.853e-02	1.396e-02	1.049e-02	1.136e-03	2.45e-04	1.37e-04
Norm($\hat{p}$)	**6.9e-05**	5.575e-03	5.701e-03	**9.547e-05**	7.791e-05	5.88e-05
Dir($\hat{p}$)	**-1.385e-03**	**1.41e-03**	**2.39e-03**	**4.367e-05**	**1.759e-05**	**1.804e-05**
α_{20}=0.0008	$K=10$	$K=50$	$K=100$	$K=10$	$K=50$	$K=100$
EM	3.315e-04	**5.232e-05**	**3.605e-05**	**2.765e-06**	**1.753e-07**	**8.46e-08**
Exp($\tilde{p}$)	3.64e-03	1.167e-03	1.02e-03	4.3e-05	2.17e-06	1.373e-06
Norm($\tilde{p}$)	5.931e-03	1.692e-03	1.356e-03	1.05e-04	4.172e-06	2.31e-06
Dir($\tilde{p}$)	5.901e-03	1.528e-03	1.118e-03	5.975e-05	3.589e-06	1.646e-06
Exp($\hat{p}$)	**-1.07e-04**	**4.123e-04**	**5.997e-04**	**2.174e-07**	**4.023e-07**	**5.346e-07**
Norm($\hat{p}$)	**2.7e-04**	7.029e-04	8.093e-04	7.259e-06	9.623e-07	9.053e-07
Dir($\hat{p}$)	**-2.611e-04**	**-1.266e-04**	**9.933e-06**	**9.664e-08**	**5.007e-08**	**3.611e-08**
α_{21}=0.08	$K=10$	$K=50$	$K=100$	$K=10$	$K=50$	$K=100$
EM	**7.227e-03**	**1.078e-03**	**2.712e-04**	7.094e-04	**9.085e-05**	**4.675e-05**
Exp($\tilde{p}$)	-2.07e-02	-1.399e-02	-1.395e-02	8.57e-04	2.772e-04	2.355e-04
Norm($\tilde{p}$)	-2.621e-02	-1.837e-02	-1.706e-02	1.199e-03	4.223e-04	3.338e-04
Dir($\tilde{p}$)	-4.295e-02	-2.042e-02	-1.687e-02	2.017e-03	5.057e-04	3.274e-04
Exp($\hat{p}$)	1.315e-02	**-4.517e-03**	**-8.683e-03**	**4.482e-04**	**8.79e-05**	**1.12e-04**
Norm($\hat{p}$)	**-4.76e-03**	-9.869e-03	-1.197e-02	**1.685e-04**	1.719e-04	1.816e-04
Dir($\hat{p}$)	**-4.034e-03**	**-6.581e-03**	**-8.703e-03**	**5.231e-05**	**7.163e-05**	**9.671e-05**

mates are much better than the EM estimate. We also note that Dir($\tilde{p}$) generally estimates $\mathrm{E}(T_1)$ with the least bias and MSE, Exp($\tilde{p}$) generally estimates $\mathrm{E}(T_2)$ with the least bias and MSE, and Norm($\tilde{p}$) generally estimates $\mathrm{E}(T)$ with the least bias, but Dir($\tilde{p}$) generally estimates $\mathrm{E}(T)$ with the least MSE. Thus, no particular prior distribution turns out to be best overall.

V. SENSITIVITY ANALYSIS ON PRIOR ACCURACY

The prior information with a different variance c^2 may affect the resultant estimation. This section is devoted to examine the sensitivity of the estimation for varying c^2. It is common in practice to have good prior information in the case of most one-shot device testing analyses. For this reason, we set the values of c so that $c^2 = 0.05$, 0.001, and 0.0005, in our simulation study, to reflect different levels of accuracy with respect to prior information. For different c^2, the bias, and the MSE of the estimate of α_{21} with an exponential prior distribution using prior information $\hat{p}$ are presented in Table XVIII. The values in bold represent the best settings in terms of the least absolute values of bias or MSEs.

From Table XVIII, we observe that, when c^2 decreases, the bias, and MSE also decrease. It is more obvious when the reliability of the product is moderate or low, and the sample size is small. The reason for this result is that the prior information has

TABLE X
BIAS, AND MSE OF THE ESTIMATES OF SOME PROBABILITIES OF INTEREST FOR DEVICES WITH MODERATE RELIABILITY UNDER DIFFERENT ESTIMATION METHODS

	Bias			MSE		
p_{01}=0.8444	$K=10$	$K=50$	$K=100$	$K=10$	$K=50$	$K=100$
EM	**-1.692e-02**	**-3.448e-03**	-1.91e-03	3.257e-03	5.129e-04	2.448e-04
Exp($\tilde{p}$)	-2.462e-02	7.007e-03	7.737e-03	5.017e-03	5.834e-04	3.23e-04
Norm($\tilde{p}$)	-4.865e-02	-8.277e-03	-2.892e-03	6.81e-03	6.259e-04	3.08e-04
Dir($\tilde{p}$)	-5.511e-02	-6.29e-03	**5.339e-04**	5.24e-03	**5.121e-04**	**2.399e-04**
Exp($\hat{p}$)	3.332e-02	1.552e-02	1.121e-02	**2.045e-03**	5.971e-04	3.374e-04
Norm($\hat{p}$)	**-1.073e-02**	**-2.178e-03**	**3.506e-04**	**6.788e-04**	**2.618e-04**	**1.701e-04**
Dir($\hat{p}$)	**-5.082e-03**	**-1.745e-03**	**-2.86e-05**	**2.473e-04**	**1.247e-04**	**1.024e-04**
p_{02}=0.713	$K=10$	$K=50$	$K=100$	$K=10$	$K=50$	$K=100$
EM	**-2.531e-02**	**-5.311e-03**	-2.98e-03	8.393e-03	1.433e-03	6.909e-04
Exp($\tilde{p}$)	-3.47e-02	1.276e-02	1.356e-02	1.181e-02	1.692e-03	9.428e-04
Norm($\tilde{p}$)	-7.365e-02	-1.306e-02	-4.417e-03	1.567e-02	1.711e-03	8.623e-04
Dir($\tilde{p}$)	-8.765e-02	-1.002e-02	**1.213e-03**	1.292e-02	**1.415e-03**	**6.818e-04**
Exp($\hat{p}$)	5.914e-02	2.71e-02	1.942e-02	**6.284e-03**	1.779e-03	9.944e-04
Norm($\hat{p}$)	**-1.697e-02**	**-3.207e-03**	**8.98e-04**	**1.722e-03**	**7.333e-04**	**4.846e-04**
Dir($\hat{p}$)	**-8.215e-03**	**-2.732e-03**	**1.276e-04**	**6.543e-04**	**3.524e-04**	**2.889e-04**
p_{03}=0.602	$K=10$	$K=50$	$K=100$	$K=10$	$K=50$	$K=100$
EM	**-2.821e-02**	**-6.086e-03**	-3.466e-03	1.234e-02	2.256e-03	1.098e-03
Exp($\tilde{p}$)	-3.628e-02	1.733e-02	1.781e-02	1.614e-02	2.769e-03	1.551e-03
Norm($\tilde{p}$)	-8.4e-02	-1.54e-02	-5.011e-03	2.078e-02	2.641e-03	1.361e-03
Dir($\tilde{p}$)	-1.048e-01	-1.193e-02	**1.931e-03**	1.809e-02	**2.205e-03**	**1.091e-03**
Exp($\hat{p}$)	7.869e-02	3.548e-02	2.524e-02	**1.09e-02**	2.986e-03	1.65e-03
Norm($\hat{p}$)	**-2.014e-02**	**-3.474e-03**	**1.524e-03**	**2.503e-03**	**1.158e-03**	**7.777e-04**
Dir($\hat{p}$)	**-9.957e-03**	**-3.188e-03**	**3.831e-04**	**9.84e-04**	**5.609e-04**	**4.589e-04**
P.d1=0.8253	$K=10$	$K=50$	$K=100$	$K=10$	$K=50$	$K=100$
EM	**-1.835e-01**	**-1.425e-01**	**-1.597e-01**	4.655e+02	8.279e+01	**4.07e+01**
Exp($\tilde{p}$)	2.61e+00	5.766e+00	4.736e+00	6.355e+02	1.564e+02	8.14e+01
Norm($\tilde{p}$)	-7.863e+00	-1.28e+00	**-1.092e-02**	4.109e+02	8.777e+01	5.082e+01
Dir($\tilde{p}$)	-1.435e+01	-1.155e+00	1.061e+00	**3.275e+02**	**7.457e+01**	4.416e+01
Exp($\hat{p}$)	2.826e+01	9.551e+00	6.25e+00	1.489e+03	1.966e+02	9.128e+01
Norm($\hat{p}$)	**-2.129e+00**	**3.012e-01**	9.733e-01	**6.704e+01**	**4.39e+01**	**3.221e+01**
Dir($\hat{p}$)	**-1.286e+00**	**-1.633e-01**	**4.479e-01**	**2.85e+01**	**2.084e+01**	**1.707e+01**

a smaller variance, and so the estimation will depend on the accurate prior information more instead of the imprecise information from the sample. Besides, c^2 also represents the closeness between the estimate and the prior information. We can see that the bias and MSE do not decrease with sample size when the prior information is of small variance, $c^2 = 0.001$ and 0.0005. This observation is so because the estimation will put more weight on the prior information. An increase of sample size may not necessarily result in a reduction in the bias and MSE of the estimates because the samples may significantly different from the prior information. We observe similar results for other parameters as well with different priors, under different c^2 and K.

VI. MASKED CAUSES OF FAILURE

Sometimes, the cause of the failure cannot be identified precisely. In this case, the cause is said to be masked, and let us denote the masked cause by the indicator $\Delta_{ijk} = -1$. We then modify (2) as follows.

$$\Delta_{ijk} = \begin{cases} -1 & \text{for } \{\min(T_{1ijk}, T_{2ijk}) \leq IT_i\} \cap \{\text{masked}\}, \\ 0 & \text{for } \{\min(T_{1ijk}, T_{2ijk}) \geq IT_i\}, \\ 1 & \text{for } \{T_{1ijk} < \min(IT_i, T_{2ijk})\} \cap \{\text{not masked}\}, \\ 2 & \text{for } \{T_{2ijk} < \min(IT_i, T_{1ijk})\} \cap \{\text{not masked}\}. \end{cases} \tag{28}$$

TABLE XI
BIAS, AND MSE OF THE ESTIMATES OF SOME PROBABILITIES OF INTEREST FOR DEVICES WITH LOW RELIABILITY UNDER DIFFERENT ESTIMATION METHODS

	Bias			MSE		
p_{01}=0.713	$K=10$	$K=50$	$K=100$	$K=10$	$K=50$	$K=100$
EM	**4.557e-03**	**-2.623e-04**	**2.704e-04**	**8.595e-03**	**1.415e-03**	**7.142e-04**
Exp($\tilde{p}$)	-7.447e-02	**1.673e-03**	6.399e-03	2.093e-02	**1.431e-03**	**8.121e-04**
Norm($\tilde{p}$)	**-6.555e-02**	-1.324e-02	**-4.945e-03**	1.873e-02	1.834e-03	8.256e-04
Dir($\tilde{p}$)	**-7.173e-02**	**-7.551e-03**	**2.132e-03**	**1.04e-02**	**1.385e-03**	**7.077e-04**
Exp($\hat{p}$)	9.662e-02	4.55e-02	3.069e-02	**1.151e-02**	2.943e-03	1.509e-03
Norm($\hat{p}$)	8.954e-02	2.934e-02	1.851e-02	1.23e-02	2.14e-03	1.035e-03
Dir($\hat{p}$)	1.209e-01	1.118e-01	9.917e-02	1.507e-02	1.265e-02	9.975e-03
p_{02}=0.5083	$K=10$	$K=50$	$K=100$	$K=10$	$K=50$	$K=100$
EM	**1.509e-02**	**1.041e-03**	**1.1e-03**	**1.664e-02**	**2.848e-03**	**1.447e-03**
Exp($\tilde{p}$)	**-8.038e-02**	**4.807e-03**	1.042e-02	2.82e-02	**2.885e-03**	1.69e-03
Norm($\tilde{p}$)	**-7.023e-02**	-1.618e-02	**-5.785e-03**	**2.511e-02**	3.473e-03	**1.626e-03**
Dir($\tilde{p}$)	-9.147e-02	**-9.075e-03**	**3.974e-03**	**1.637e-02**	**2.692e-03**	**1.444e-03**
Exp($\hat{p}$)	1.509e-01	6.848e-02	4.567e-02	2.808e-02	6.673e-03	3.332e-03
Norm($\hat{p}$)	1.413e-01	4.489e-02	2.791e-02	2.708e-02	4.763e-03	2.247e-03
Dir($\hat{p}$)	1.875e-01	1.722e-01	1.515e-01	3.63e-02	3.007e-02	2.332e-02
p_{03}=0.3624	$K=10$	$K=50$	$K=100$	$K=10$	$K=50$	$K=100$
EM	**2.494e-02**	**2.614e-03**	**1.938e-03**	**1.934e-02**	**3.256e-03**	**1.656e-03**
Exp($\tilde{p}$)	**-6.507e-02**	**7.7e-03**	1.255e-02	2.383e-02	**3.33e-03**	1.992e-03
Norm($\tilde{p}$)	**-5.646e-02**	-1.457e-02	**-4.861e-03**	**2.141e-02**	3.759e-03	**1.816e-03**
Dir($\tilde{p}$)	-8.815e-02	**-7.951e-03**	**5.249e-03**	**1.498e-02**	**2.975e-03**	**1.668e-03**
Exp($\hat{p}$)	1.771e-01	7.736e-02	5.1e-02	3.889e-02	8.538e-03	4.145e-03
Norm($\hat{p}$)	1.663e-01	5.144e-02	3.153e-02	3.524e-02	6.009e-03	2.753e-03
Dir($\hat{p}$)	2.188e-01	1.993e-01	1.738e-01	4.942e-02	4.035e-02	3.076e-02
P.d1=0.8253	$K=10$	$K=50$	$K=100$	$K=10$	$K=50$	$K=100$
EM	**-1.111e-02**	**-1.756e-03**	**-1.607e-03**	**8.243e-03**	**1.521e-03**	**7.792e-04**
Exp($\tilde{p}$)	-8.66e-02	-9.62e-02	-9.601e-02	1.827e-02	1.146e-02	1.039e-02
Norm($\tilde{p}$)	-1.5e-01	-1.166e-01	-1.094e-01	3.236e-02	1.572e-02	1.322e-02
Dir($\tilde{p}$)	-1.404e-01	-9.228e-02	**-8.763e-02**	2.823e-02	1.111e-02	**8.883e-03**
Exp($\hat{p}$)	-8.138e-02	-8.868e-02	-9.139e-02	1.326e-02	9.501e-03	9.396e-03
Norm($\hat{p}$)	**-3.127e-02**	**-8.275e-02**	-9.042e-02	**2.771e-03**	**8.037e-03**	9.017e-03
Dir($\hat{p}$)	**-1.339e-02**	**-3.526e-02**	**-4.928e-02**	**9.768e-04**	**1.924e-03**	**2.939e-03**

If we assume that the occurrence of masked causes is s-independent of the underlying unobserved causes responsible for the failure, we then have the following expressions.

$$\begin{aligned}
Pr(\Delta_{ijk}=-1) &= Pr\left(\{\min(T_{1ijk},T_{2ijk}) \le IT_i\}\right.\\
&\qquad \left.\cap\{\text{masked}\}\right)\\
&= qPr\left(\{\min(T_{1ijk},T_{2ijk}) \le IT_i\}\right)\\
&= q(1-p_{0ij}),\\
Pr(\Delta_{ijk}=0) &= p_{0ij},\\
Pr(\Delta_{ijk}=1) &= Pr\left(\{T_{1ijk} < \min(IT_i,T_{2ijk})\}\right.\\
&\qquad \left.\cap\{\text{not masked}\}\right)\\
&= (1-q)Pr\left(\{T_{1ijk} < \min(IT_i,T_{2ijk})\}\right)\\
&= (1-q)p_{1ij},\\
Pr(\Delta_{ijk}=2) &= Pr\left(\{T_{2ijk} < \min(IT_i,T_{1ijk})\}\right.\\
&\qquad \left.\cap\{\text{not masked}\}\right)\\
&= (1-q)Pr\left(\{T_{2ijk} < \min(IT_i,T_{1ijk})\}\right)\\
&= (1-q)p_{2ij},
\end{aligned}$$

where $q = Pr(\{\text{masked}\})$, the probability of masked event; and p_{0ij}, p_{1ij}, and p_{2ij} are as defined in (3), (4), and (5), respectively. Suppose the number of survived items is S_{ij}, the number of failures due to cause r is d_{rij}, $r = 1, 2$, and the number of

TABLE XII
BIAS, AND MSE OF THE ESTIMATES OF MEAN LIFETIMES OF DEVICES WITH MODERATE RELIABILITY UNDER DIFFERENT ESTIMATION METHODS

	Bias			MSE		
E(T)=59.11	$K=10$	$K=50$	$K=100$	$K=10$	$K=50$	$K=100$
EM	**-1.835e-01**	**-1.425e-01**	**-1.597e-01**	4.655e+02	8.279e+01	**4.07e+01**
Exp($\tilde{p}$)	2.61e+00	5.766e+00	4.736e+00	6.355e+02	1.564e+02	8.14e+01
Norm($\tilde{p}$)	-7.863e+00	-1.28e+00	**-1.092e-02**	4.109e+02	8.777e+01	5.082e+01
Dir($\tilde{p}$)	-1.435e+01	-1.155e+00	1.061e+00	**3.275e+02**	**7.457e+01**	4.416e+01
Exp($\hat{p}$)	2.826e+01	9.551e+00	6.25e+00	1.489e+03	1.966e+02	9.128e+01
Norm($\hat{p}$)	**-2.129e+00**	**3.012e-01**	9.733e-01	**6.704e+01**	**4.39e+01**	**3.221e+01**
Dir($\hat{p}$)	**-1.286e+00**	**-1.633e-01**	**4.479e-01**	**2.85e+01**	**2.084e+01**	**1.707e+01**
E(T_1)=71.63	$K=10$	$K=50$	$K=100$	$K=10$	$K=50$	$K=100$
EM	2.591e+02	1.854e+02	1.804e+02	1.732e+05	4.201e+04	3.58e+04
Exp($\tilde{p}$)	2.01e+01	1.789e+01	1.654e+01	2.55e+03	6.18e+02	4.16e+02
Norm($\tilde{p}$)	8.022e+00	1.135e+01	**1.216e+01**	1.212e+03	3.595e+02	2.816e+02
Dir($\tilde{p}$)	**-2.501e+00**	**1.032e+01**	1.247e+01	**4.821e+02**	**2.989e+02**	**2.643e+02**
Exp($\hat{p}$)	5.29e+01	2.369e+01	1.911e+01	5.024e+03	8.313e+02	4.949e+02
Norm($\hat{p}$)	**-9.646e-01**	**3.653e+00**	**6.41e+00**	**1.251e+02**	**1.02e+02**	**1.092e+02**
Dir($\hat{p}$)	**-1.867e-01**	**2.537e+00**	**4.875e+00**	**5.45e+01**	**4.615e+01**	**5.865e+01**
E(T_2)=338.3	$K=10$	$K=50$	$K=100$	$K=10$	$K=50$	$K=100$
EM	-3.342e+02	-3.354e+02	-3.355e+02	1.117e+05	1.125e+05	1.126e+05
Exp($\tilde{p}$)	**-4.625e+01**	-7.933e+01	-9.506e+01	**3.37e+04**	1.07e+04	1.105e+04
Norm($\tilde{p}$)	-1.538e+02	-1.341e+02	-1.297e+02	3.513e+04	2.045e+04	1.822e+04
Dir($\tilde{p}$)	-1.912e+02	-1.27e+02	-1.18e+02	3.956e+04	1.883e+04	1.558e+04
Exp($\hat{p}$)	7.5e+01	**-7.126e+01**	**-9.325e+01**	5.995e+04	**8.787e+03**	**1.049e+04**
Norm($\hat{p}$)	**-1.624e+01**	**-3.944e+01**	**-6.581e+01**	**6.819e+03**	**5.241e+03**	**6.503e+03**
Dir($\hat{p}$)	**-2e+01**	**-4.124e+01**	**-6.241e+01**	**2.873e+03**	**3.486e+03**	**5.045e+03**

masked causes is m_{ij} so that $K_{ij} = S_{ij} + d_{1ij} + d_{2ij} + m_{ij}$. In this case, the likelihood function presented in (6) can be modified as

$$L_{\text{masked}}(\boldsymbol{\alpha}|\boldsymbol{\delta}, \boldsymbol{IT}, \boldsymbol{w}) = \prod_{i=1}^{I}\prod_{j=1}^{J} p_{0ij}^{S_{ij}}(1-p_{0ij})^{m_{ij}}(1-q)^{K_{ij}-m_{ij}}q^{m_{ij}}\prod_{r=1}^{R} p_{rij}^{d_{rij}}, \tag{29}$$

where $R = 2$ in our study. If the probability of masking is not of interest, we may treat q to be an unknown constant; and the posterior distribution, given the prior distribution $\pi(\alpha)$, will be

$$\begin{aligned}\pi_{\text{masked}}(\boldsymbol{\alpha}|\boldsymbol{\delta}, \boldsymbol{IT}, \boldsymbol{w}) &= \frac{L_{\text{masked}}(\boldsymbol{\alpha}|\boldsymbol{\delta}, \boldsymbol{IT}, \boldsymbol{w})\pi(\boldsymbol{\alpha})}{\int L_{\text{masked}}(\boldsymbol{\alpha}|\boldsymbol{\delta}, \boldsymbol{IT}, \boldsymbol{w})\pi(\boldsymbol{\alpha})d\boldsymbol{\alpha}} \\ &\propto \prod_{i=1}^{I}\prod_{j=1}^{J} p_{0ij}^{S_{ij}}(1-p_{0ij})^{m_{ij}}\prod_{r=1}^{R} p_{rij}^{d_{rij}}\pi(\boldsymbol{\alpha}),\end{aligned} \tag{30}$$

where $R = 2$ in our study.

The resultant posterior distribution in (30) in the masked cause case is similar to the original one presented in (7). So, it is easy to derive the posterior distributions for different prior distributions, namely, the exponential, normal, and Dirichlet distributions; and the results will be quite similar to those in (17), (21), and (25), respectively. If we have precise prior knowledge, $\hat{p}_{0ij}, \hat{p}_{1ij}, \hat{p}_{2ij}$, we can use them to fix the hyperparameters in the same way as described before. If such information is not available, then we may modify $\tilde{p}_{0ij}, \tilde{p}_{1ij}, \tilde{p}_{2ij}$ presented in Table XIX. In the above, we have made use of the facts that

$$\begin{aligned}&Pr\left(T_{1ijk} < \min(IT_i, T_{2ijk})\right) \\ &\quad = \left(\frac{\lambda_{1j}}{\lambda_{1j}+\lambda_{2j}}\right)\left(1-\exp\left(-(\lambda_{1j}+\lambda_{2j})IT_i\right)\right), \\ &Pr\left(T_{2ijk} < \min(IT_i, T_{1ijk})\right) \\ &\quad = \left(\frac{\lambda_{2j}}{\lambda_{1j}+\lambda_{2j}}\right)\left(1-\exp\left(-(\lambda_{1j}+\lambda_{2j})IT_i\right)\right),\end{aligned}$$

and

$$\begin{aligned}Pr(T_{1ij} < T_{2ij}) &= Pr\left(\min(T_{1ij}, T_{2ij}) < IT_i\right) \\ &\quad \times Pr\left(T_{1ij} < T_{2ij} \middle| \min(T_{1ij}, T_{2ij}) < IT_i\right) \\ &= (1-p_{0ij}) \times \frac{\lambda_{1ij}}{\lambda_{1ij}+\lambda_{2ij}}, \\ Pr(T_{2ij} < T_{1ij}) &= (1-p_{0ij}) \times \frac{\lambda_{2ij}}{\lambda_{1ij}+\lambda_{2ij}}.\end{aligned}$$

TABLE XIII
BIAS, AND MSE OF THE ESTIMATES OF MEAN LIFETIMES OF DEVICES WITH LOW RELIABILITY UNDER DIFFERENT ESTIMATION METHODS

	Bias			MSE		
E(T)=29.56	$K=10$	$K=50$	$K=100$	$K=10$	$K=50$	$K=100$
EM	**5.039e+00**	**5.778e-01**	**3.38e-01**	2.845e+02	**2.33e+01**	**1.121e+01**
Exp($\tilde{p}$)	**-2.172e+00**	1.277e+00	1.378e+00	**1.682e+02**	2.638e+01	1.528e+01
Norm($\tilde{p}$)	**-1.849e+00**	**-5.872e-01**	**-9.11e-02**	**1.506e+02**	**2.466e+01**	**1.187e+01**
Dir($\tilde{p}$)	-5.838e+00	**-2.515e-01**	**6.686e-01**	**6.952e+01**	**1.947e+01**	**1.176e+01**
Exp($\hat{p}$)	2.377e+01	7.725e+00	4.789e+00	8.103e+02	8.835e+01	3.697e+01
Norm($\hat{p}$)	2.008e+01	5.3e+00	3.036e+00	5.016e+02	6.078e+01	2.352e+01
Dir($\hat{p}$)	2.645e+01	2.284e+01	1.886e+01	7.33e+02	5.38e+02	3.673e+02
E(T_1)=35.81	$K=10$	$K=50$	$K=100$	$K=10$	$K=50$	$K=100$
EM	2.359e+02	9.883e+01	9.257e+01	3.194e+05	1.236e+04	9.559e+03
Exp($\tilde{p}$)	**3.998e+00**	**6.912e+00**	**6.826e+00**	**5.282e+02**	**1.079e+02**	**7.817e+01**
Norm($\tilde{p}$)	**6.825e+00**	**5.363e+00**	**5.523e+00**	**4.727e+02**	**8.799e+01**	**5.937e+01**
Dir($\tilde{p}$)	**-2.694e-01**	**4.435e+00**	**5.306e+00**	**1.193e+02**	**6.513e+01**	**5.351e+01**
Exp($\hat{p}$)	3.936e+01	1.522e+01	1.121e+01	2.373e+03	2.993e+02	1.59e+02
Norm($\hat{p}$)	2.687e+01	1.129e+01	8.686e+00	8.837e+02	1.876e+02	1.052e+02
Dir($\hat{p}$)	3.342e+01	3.068e+01	2.669e+01	1.181e+03	9.762e+02	7.366e+02
E(T_2)=169.2	$K=10$	$K=50$	$K=100$	$K=10$	$K=50$	$K=100$
EM	-1.659e+02	-1.664e+02	-1.664e+02	2.755e+04	2.769e+04	2.769e+04
Exp($\tilde{p}$)	**-3.608e+01**	**-4.94e+01**	-5.176e+01	**8.277e+03**	**3.24e+03**	3.077e+03
Norm($\tilde{p}$)	**-7.161e+01**	-6.612e+01	-6.315e+01	**8.225e+03**	4.941e+03	4.316e+03
Dir($\tilde{p}$)	-8.624e+01	-5.413e+01	-5.105e+01	**8.537e+03**	3.752e+03	**3.034e+03**
Exp($\hat{p}$)	**8.302e+01**	**-2.148e+01**	**-3.692e+01**	2.402e+04	**1.398e+03**	**1.793e+03**
Norm($\hat{p}$)	9.028e+01	**-2.847e+01**	**-4.347e+01**	1.423e+04	**1.855e+03**	**2.301e+03**
Dir($\hat{p}$)	1.388e+02	8.653e+01	**5.075e+01**	2.161e+04	8.714e+03	3.273e+03

TABLE XIV
THE METHOD OF ESTIMATION WITH THE LEAST BIAS

Reliability	High			Moderate			Low		
Size	$K=10$	$K=50$	$K=100$	$K=10$	$K=50$	$K=100$	$K=10$	$K=50$	$K=100$
α_{10}	Exp($\hat{p}$)	Norm($\tilde{p}$)	Norm($\hat{p}$)	Dir($\hat{p}$)	Norm($\hat{p}$)	EM	EM	EM	EM
α_{11}	Exp($\hat{p}$)	Dir($\tilde{p}$)	EM	EM	EM	EM	Norm($\hat{p}$)	EM	EM
α_{20}	Dir($\hat{p}$)	EM	EM	Dir($\hat{p}$)	EM	EM	Exp($\hat{p}$)	EM	Dir($\hat{p}$)
α_{21}	Dir($\hat{p}$)	Exp($\tilde{p}$)	EM	Exp($\hat{p}$)	EM	EM	Dir($\hat{p}$)	EM	EM
p_{01}	Exp($\hat{p}$)	Exp($\tilde{p}$)	Exp($\hat{p}$)	Dir($\hat{p}$)	Dir($\hat{p}$)	Dir($\hat{p}$)	EM	EM	EM
p_{02}	Exp($\hat{p}$)	Exp($\tilde{p}$)	Exp($\hat{p}$)	Dir($\hat{p}$)	Dir($\hat{p}$)	Dir($\hat{p}$)	EM	EM	EM
p_{03}	Exp($\hat{p}$)	Exp($\tilde{p}$)	Exp($\hat{p}$)	Dir($\hat{p}$)	Dir($\hat{p}$)	Dir($\hat{p}$)	EM	EM	EM
P.d1	Dir($\hat{p}$)	Dir($\hat{p}$)	Dir($\hat{p}$)	Dir($\hat{p}$)	EM	EM	EM	EM	EM
E(T)	Exp($\hat{p}$)	Exp($\hat{p}$)	Norm($\hat{p}$)	EM	EM	Norm($\tilde{p}$)	Norm($\tilde{p}$)	Dir($\tilde{p}$)	Norm($\tilde{p}$)
E(T_1)	Exp($\hat{p}$)	Dir($\tilde{p}$)	Norm($\hat{p}$)	Dir($\hat{p}$)	Dir($\hat{p}$)	Dir($\hat{p}$)	Dir($\tilde{p}$)	Dir($\tilde{p}$)	Dir($\tilde{p}$)
E(T_2)	Exp($\hat{p}$)	Exp($\tilde{p}$)	Exp($\tilde{p}$)	Norm($\hat{p}$)	Norm($\hat{p}$)	Dir($\hat{p}$)	Exp($\tilde{p}$)	Exp($\hat{p}$)	Exp($\tilde{p}$)

Then, the Metropolis-Hastings algorithm can be applied to the general random variable from the posterior distribution, and the Bayesian estimation and inference can be developed in a similar manner.

VII. AN EXAMPLE FROM A TUMORIGENICITY EXPERIMENT

As an illustration, we apply the model and the methods developed in the preceding sections to a dataset described in [14]

TABLE XV
THE METHOD OF ESTIMATION WITH THE LEAST MSE

Reliability	High			Moderate			Low		
Size	$K=10$	$K=50$	$K=100$	$K=10$	$K=50$	$K=100$	$K=10$	$K=50$	$K=100$
α_{10}	Exp($\hat{p}$)	Exp($\hat{p}$)	Dir($\tilde{p}$)	Dir($\hat{p}$)	Dir($\hat{p}$)	Dir($\hat{p}$)	Dir($\hat{p}$)	Dir($\tilde{p}$)	EM
α_{11}	Dir($\hat{p}$)	Dir($\tilde{p}$)	EM	Dir($\hat{p}$)	Dir($\hat{p}$)	Dir($\hat{p}$)	Dir($\hat{p}$)	Dir($\hat{p}$)	Dir($\hat{p}$)
α_{20}	Dir($\hat{p}$)	EM	EM	Dir($\hat{p}$)	Dir($\hat{p}$)	EM	Dir($\hat{p}$)	Dir($\hat{p}$)	Dir($\hat{p}$)
α_{21}	Dir($\hat{p}$)	EM	EM	Dir($\hat{p}$)	Dir($\hat{p}$)	EM	Dir($\hat{p}$)	Dir($\hat{p}$)	EM
p_{01}	Exp($\hat{p}$)	Exp($\hat{p}$)	Exp($\hat{p}$)	Dir($\hat{p}$)	Dir($\hat{p}$)	Dir($\hat{p}$)	EM	Dir($\tilde{p}$)	Dir($\tilde{p}$)
p_{02}	Exp($\hat{p}$)	Exp($\hat{p}$)	Exp($\hat{p}$)	Dir($\hat{p}$)	Dir($\hat{p}$)	Dir($\hat{p}$)	Dir($\tilde{p}$)	Dir($\tilde{p}$)	Dir($\tilde{p}$)
p_{03}	Exp($\hat{p}$)	Exp($\hat{p}$)	Exp($\hat{p}$)	Dir($\hat{p}$)	Dir($\hat{p}$)	Dir($\hat{p}$)	Dir($\tilde{p}$)	Dir($\tilde{p}$)	EM
P.d1	Dir($\hat{p}$)	Dir($\hat{p}$)	Dir($\hat{p}$)	Dir($\hat{p}$)	Dir($\hat{p}$)	EM	Dir($\hat{p}$)	EM	EM
E(T)	Exp($\hat{p}$)	Dir($\tilde{p}$)	Dir($\tilde{p}$)	Dir($\hat{p}$)	Dir($\hat{p}$)	Dir($\hat{p}$)	Dir($\tilde{p}$)	Dir($\tilde{p}$)	EM
E(T_1)	Dir($\tilde{p}$)	Dir($\tilde{p}$)	Dir($\tilde{p}$)	Dir($\hat{p}$)	Dir($\hat{p}$)	Dir($\hat{p}$)	Dir($\tilde{p}$)	Dir($\tilde{p}$)	Dir($\tilde{p}$)
E(T_2)	Exp($\hat{p}$)	Exp($\tilde{p}$)	Exp($\tilde{p}$)	Dir($\hat{p}$)	Dir($\hat{p}$)	Dir($\hat{p}$)	Norm($\tilde{p}$)	Exp($\hat{p}$)	Exp($\hat{p}$)

TABLE XVI
THE METHOD OF ESTIMATION WITH THE LEAST BIAS AMONG THE METHODS BASED ONLY ON OBSERVED DATA

Reliability	High			Moderate			Low		
Size	$K=10$	$K=50$	$K=100$	$K=10$	$K=50$	$K=100$	$K=10$	$K=50$	$K=100$
α_{10}	Exp($\tilde{p}$)	Norm($\tilde{p}$)	Norm($\tilde{p}$)	EM	EM	EM	EM	EM	EM
α_{11}	Exp($\tilde{p}$)	Dir($\tilde{p}$)	EM	EM	EM	EM	Norm($\tilde{p}$)	EM	EM
α_{20}	Exp($\tilde{p}$)	EM	EM	EM	EM	EM	EM	EM	EM
α_{21}	Norm($\tilde{p}$)	Exp($\tilde{p}$)	EM	EM	EM	EM	EM	EM	EM
p_{01}	Exp($\tilde{p}$)	Exp($\tilde{p}$)	Exp($\tilde{p}$)	EM	EM	Dir($\tilde{p}$)	EM	EM	EM
p_{02}	Exp($\tilde{p}$)	Exp($\tilde{p}$)	Exp($\tilde{p}$)	EM	EM	Dir($\tilde{p}$)	EM	EM	EM
p_{03}	Exp($\tilde{p}$)	Exp($\tilde{p}$)	Exp($\tilde{p}$)	EM	EM	Dir($\tilde{p}$)	EM	EM	EM
P.d1	Exp($\tilde{p}$)	EM	EM	EM	EM	EM	EM	EM	EM
E(T)	Exp($\tilde{p}$)	Norm($\tilde{p}$)	Norm($\tilde{p}$)	EM	EM	Norm($\tilde{p}$)	Norm($\tilde{p}$)	Dir($\tilde{p}$)	Norm($\tilde{p}$)
E(T_1)	Exp($\tilde{p}$)	Dir($\tilde{p}$)	Dir($\tilde{p}$)	Dir($\tilde{p}$)	Dir($\tilde{p}$)	Norm($\tilde{p}$)	Dir($\tilde{p}$)	Dir($\tilde{p}$)	Dir($\tilde{p}$)
E(T_2)	Exp($\tilde{p}$)	Exp($\tilde{p}$)	Exp($\tilde{p}$)	Exp($\tilde{p}$)	Exp($\tilde{p}$)	Exp($\tilde{p}$)	Exp($\tilde{p}$)	Exp($\tilde{p}$)	Dir($\tilde{p}$)

(Table I). The table shows the experimental results conducted by the National Center for Toxicological Research in 1974. 3355 out of 24000 female mice are randomized to a control group ($w_1 = 0$), or groups that will be injected with a high dose (150 parts per million) of a known carcinogen, called 2-AAF ($w_2 = 1$). The inspection times used on the mice were 12, 18, and 33 months; and the outcome on mice were death without tumour (DNT), death with tumour (DWT), sacrificed without tumour (SNT), and sacrificed with tumour (SWT). In this analysis, we combine SNT and SWT as the sacrificed group ($r = 0$); and denote the cause of DNT as natural death ($r = 1$), and the cause of DWT as death due to cancer ($r = 2$). The data modified in this fashion are presented in Table XX, and the estimates of model parameters obtained by the proposed estimation methods are presented in Table XXI.

From Table XXI, we note that the EM estimate of α_{11} is negative. This result may be due to the fact that the true value of α_{11} may be quite close to zero. Note that the Bayesian estimates of α_{11} are all very small, which means that the drug will not increase the hazard rate of the natural death outcome. When looking at the estimates of mean lifetimes, the Bayesian estimate shows a reduction when the carcinogenic drug is administered, but the EM estimate does not show this behavior. Thus, in this case, we observe that the Bayesian approach gives a more meaningful result in the context of the laboratory experiment.

VIII. CONCLUDING REMARKS

In this work, a Bayesian approach has been developed for the evaluation of one-shot devices with competing causes of failure. Three different prior distributions have been considered, and their corresponding posterior distributions have been derived. The performance of the Bayesian approach has been compared with the maximum likelihood estimation based on the EM algorithm through Monte Carlo simulations. If mean lifetimes are of interest, the Bayesian approach turns out to be the best. For some quantities of interest, if we have a prior belief $\hat{p}$, with small variance c^2, then a Dirichlet prior using

TABLE XVII
THE METHOD OF ESTIMATION WITH THE LEAST MSE AMONG THE METHODS BASED ONLY ON OBSERVED DATA

Reliability	High			Moderate			Low		
Size	$K = 10$	$K = 50$	$K = 100$	$K = 10$	$K = 50$	$K = 100$	$K = 10$	$K = 50$	$K = 100$
α_{10}	EM	Dir($\tilde{p}$)	Dir($\tilde{p}$)	EM	Dir($\tilde{p}$)	EM	EM	Dir($\tilde{p}$)	EM
α_{11}	EM	Dir($\tilde{p}$)	EM	Dir($\tilde{p}$)	EM	EM	Norm($\tilde{p}$)	Dir($\tilde{p}$)	EM
α_{20}	Exp($\tilde{p}$)	EM	EM	EM	EM	EM	EM	EM	EM
α_{21}	EM	EM	EM	EM	EM	EM	EM	EM	EM
p_{01}	Exp($\tilde{p}$)	Exp($\tilde{p}$)	Exp($\tilde{p}$)	EM	Dir($\tilde{p}$)	Dir($\tilde{p}$)	EM	Dir($\tilde{p}$)	Dir($\tilde{p}$)
p_{02}	Exp($\tilde{p}$)	Exp($\tilde{p}$)	Exp($\tilde{p}$)	EM	Dir($\tilde{p}$)	Dir($\tilde{p}$)	Dir($\tilde{p}$)	Dir($\tilde{p}$)	Dir($\tilde{p}$)
p_{03}	Exp($\tilde{p}$)	Exp($\tilde{p}$)	Exp($\tilde{p}$)	EM	Dir($\tilde{p}$)	Dir($\tilde{p}$)	Dir($\tilde{p}$)	Dir($\tilde{p}$)	EM
P.d1	Exp($\tilde{p}$)	EM	EM	EM	EM	EM	EM	EM	EM
E(T)	Exp($\tilde{p}$)	Dir($\tilde{p}$)	Dir($\tilde{p}$)	Dir($\tilde{p}$)	Dir($\tilde{p}$)	EM	Dir($\tilde{p}$)	Dir($\tilde{p}$)	EM
E(T_1)	Dir($\tilde{p}$)	Dir($\tilde{p}$)	Dir($\tilde{p}$)	Dir($\tilde{p}$)	Dir($\tilde{p}$)	Dir($\tilde{p}$)	Dir($\tilde{p}$)	Dir($\tilde{p}$)	Dir($\tilde{p}$)
E(T_2)	Exp($\tilde{p}$)	Exp($\tilde{p}$)	Exp($\tilde{p}$)	Exp($\tilde{p}$)	Exp($\tilde{p}$)	Exp($\tilde{p}$)	Norm($\tilde{p}$)	Exp($\tilde{p}$)	Dir($\tilde{p}$)

TABLE XVIII
COMPARISON OF DIFFERENT PRIOR VARIANCES c^2 IN THE ESTIMATE OF α_{21} UNDER Exp($\hat{p}$)

$\alpha_{21} = 0.08$		Bias			MSE		
Reliability	c^2	$K = 10$	$K = 50$	$K = 100$	$K = 10$	$K = 50$	$K = 100$
high	0.05	**-1.431e-02**	**-8.234e-03**	**-8.822e-03**	**7.721e-04**	**2.529e-04**	**2.057e-04**
	0.001	-2.883e-02	-1.363e-02	-1.278e-02	1.133e-03	3.188e-04	2.405e-04
	0.0005	-2.818e-02	-1.406e-02	-1.3e-02	1.081e-03	3.309e-04	2.438e-04
moderate	0.05	1.293e-02	**-1.521e-03**	**-5.274e-03**	6.357e-04	1.96e-04	**1.496e-04**
	0.001	5.342e-04	-9.259e-03	-1.138e-02	1.942e-04	**1.458e-04**	1.653e-04
	0.0005	**1.861e-04**	-9.667e-03	-1.157e-02	**1.893e-04**	1.561e-04	1.684e-04
low	0.05	2.822e-02	5.362e-03	**-1.085e-04**	1.415e-03	2.901e-04	1.881e-04
	0.001	1.315e-02	**-4.517e-03**	-8.683e-03	4.482e-04	**8.79e-05**	**1.12e-04**
	0.0005	**1.286e-02**	-5.092e-03	-9.096e-03	**4.241e-04**	8.851e-05	1.179e-04

TABLE XIX
MODIFICATION WITH ZERO CORRECTION ON $\tilde{p}_{0ij}, \tilde{p}_{1ij}, \tilde{p}_{2ij}$

$\tilde{p}s$	Without Zero Correction	With Zero Correction
$\tilde{p}_{0ij}$	S_{ij}/K_{ij}	$(S_{ij}+1)/(K_{ij}+3)$
$\tilde{p}_{1ij}$	$(1-\tilde{p}_{0ij})d_{1ij}/(d_{1ij}+d_{2ij})$	$(1-\tilde{p}_{0ij})(d_{1ij}+1)/(d_{1ij}+d_{2ij}+2)$
$\tilde{p}_{2ij}$	$(1-\tilde{p}_{0ij})d_{2ij}/(d_{1ij}+d_{2ij})$	$(1-\tilde{p}_{0ij})(d_{2ij}+1)/(d_{1ij}+d_{2ij}+2)$

$\hat{p}$ is recommended for use. If, however, the devices are with moderate or low reliability, the EM method turns out to be a good method of estimation.

For further study, we can extend the model to the case of Weibull or gamma lifetimes. One important aspect of the present work is that the probability of failure due to cause 1, P.d1, does not depend on inspection time IT because the hazard rate is constant over time for the exponential distribution. However, when we consider Weibull or gamma lifetime distributions, P.d1 will not be a constant of IT anymore, and thus may provide a more practical model, even though it will render the ensuing analysis a lot more complicated. We are currently looking into this problem, and hope to report the findings in a future paper.

TABLE XX
THE NUMBER OF MICE SACRIFICED ($r = 0$) AND DIED (WITHOUT TUMOUR $r = 1$, WITH TUMOUR $r = 2$) FROM THE ED01 EXPERIMENT DATA

		$\delta_{ijk} = 0$	$\delta_{ijk} = 1$	$\delta_{ijk} = 2$
$IT_1 = 12$	$w_1 = 0$	115	22	8
	$w_2 = 1$	110	49	16
$IT_2 = 18$	$w_1 = 0$	780	42	8
	$w_2 = 1$	540	54	26
$IT_3 = 33$	$w_1 = 0$	675	200	85
	$w_2 = 1$	510	64	51

APPENDIX

A. EM Algorithm

To compare the performance of the Bayesian estimates, we developed the EM algorithm to compute the MLEs. The missing

TABLE XXI
BIAS, AND MSE OF THE ESTIMATES OF VARIOUS QUANTITIES OF INTEREST UNDER DIFFERENT ESTIMATION METHODS

Parameter	EM	Exp($\tilde{p}$)	Norm($\tilde{p}$)	Dir($\tilde{p}$)	Parameter	EM	Exp($\tilde{p}$)	Norm($\tilde{p}$)	Dir($\tilde{p}$)
α_{10}	6.169e-03	5.979e-03	5.817e-03	6.235e-03	α_{20}	2.347e-03	2.467e-03	2.52e-03	2.777e-03
α_{11}	-1.279e-01	5.035e-03	1.572e-02	9.042e-02	α_{21}	2.532e-01	1.147e-01	1.367e-01	5.631e-02
$p_{01\|w=0}$	9.029e-01	9.036e-01	9.048e-01	8.975e-01	$p_{01\|w=1}$	9.036e-01	9.001e-01	8.998e-01	8.895e-01
$p_{02\|w=0}$	8.579e-01	8.59e-01	8.607e-01	8.503e-01	$p_{02\|w=1}$	8.589e-01	8.54e-01	8.535e-01	8.39e-01
$p_{03\|w=0}$	7.55e-01	7.568e-01	7.595e-01	7.428e-01	$p_{03\|w=1}$	7.566e-01	7.487e-01	7.48e-01	7.248e-01
P.d1$\|w=0$	7.244e-01	7.083e-01	6.981e-01	6.92e-01	P.d1$\|w=1$	6.423e-01	6.853e-01	6.719e-01	6.995e-01
E($T\|w=0$)	1.174e+02	1.187e+02	1.202e+02	1.111e+02	E($T\|w=1$)	1.183e+02	1.142e+02	1.138e+02	1.026e+02
E($T_1\|w=0$)	1.621e+02	1.676e+02	1.721e+02	1.607e+02	E($T_1\|w=1$)	1.842e+02	1.667e+02	1.694e+02	1.467e+02
E($T_2\|w=0$)	4.26e+02	4.093e+02	3.997e+02	3.63e+02	E($T_2\|w=1$)	3.307e+02	3.644e+02	3.493e+02	3.425e+02

data in the one-shot device with competing risk is T_{rijk}. The complete log-likelihood would be

$$l(\boldsymbol{\alpha}) = \sum_{r=1}^{R}\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K_{ij}} \left\{ \ln(\lambda_{rj}) - \lambda_{rj} T_{rijk}^{(\Delta_{ijk})} \right\}, \qquad (31)$$

where $T_{rijk}^{(\Delta_{ijk})}$ is the lifetime given Δ_{ijk}. In the E-step, $T_{rijk}^{(\Delta_{ijk})}$ in (31) is replaced by the expectation $E(T_{rijk}^{(\Delta_{ijk})}|\boldsymbol{\alpha}^{(t)}) = E(T_{rijk}|\Delta_{ijk}, \boldsymbol{\alpha}^{(t)})$, where $\boldsymbol{\alpha}^{(t)}$ is the current estimate of $\boldsymbol{\alpha}$ at the tth steps. The expectation can be computed either by simulation, or by tedious algebraic derivation. In the M-step, we maximize the expected complete log-likelihood function by the Newton-Raphson method: we derive the gradient of the function $\partial l(\boldsymbol{\alpha}|\boldsymbol{\alpha}^{(t)})/\partial\boldsymbol{\alpha} = I^{(t)}(\boldsymbol{\alpha})$, and the Hessian matrix of the function $\partial^2 l(\boldsymbol{\alpha}|\boldsymbol{\alpha}^{(t)})/\partial\boldsymbol{\alpha}/\partial\boldsymbol{\alpha}' = J^{(t)}(\boldsymbol{\alpha})$; and then solve $I^{(t)}(\boldsymbol{\alpha}) = \mathbf{0}$ by the updating equation

$$\boldsymbol{\alpha}_{(m+1)} = \boldsymbol{\alpha}_{(m)} - \left(J^{(t)}\left(\boldsymbol{\alpha}_{(m)}\right)\right)^{-1} I^{(t)}\left(\boldsymbol{\alpha}_{(m)}\right),$$

where $\boldsymbol{\alpha}_{(m)}$ is the approximation of the root at the mth iteration. When the iteration converges, the expected complete log-likelihood is maximized, and we update the current estimate, $\boldsymbol{\alpha}^{(t+1)} = \boldsymbol{\alpha}_{(\infty)}$. We repeat the E-step and M-step iteratively until $\boldsymbol{\alpha}^{(t+1)}$ converges. The converged value will be the MLE of $\boldsymbol{\alpha}$. For the initial values $\boldsymbol{\alpha}^{(0)}$, ICLS is a possible candidate. Through the E- and M-steps, we can obtain the MLEs efficiently.

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