On Bayesian quantile regression and outliers

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Abstract: In this work we discuss the progress of Bayesian quantile regression models since their first proposal and we discuss the importance of all parameters involved in the inference process. Using a representation of the asymmetric Laplace distribution as a mixture of a normal and an exponential distribution, we discuss the relevance of the presence of a scale parameter to control for the variance in the model. Besides that we consider the posterior distribution of the latent variable present in the mixture representation to showcase outlying observations given the Bayesian quantile regression fits, where we compare the posterior distribution for each latent variable with the others. We illustrate these results with simulation studies and also with data about Gini indexes in Brazilian states from years with census information.

Keywords and phrases: Bayesian quantile regression, Asymmetric Laplace, Kullback-Leibler divergence, Outliers, Gini index.

1. Introduction

Quantile regression can no longer be considered an outsider in the regression analysis framework, as it has been widely studied in the literature and can be found in most statistical software these days. This technique was introduced by Koenker and Bassett (1978) as a minimization problem, where the conditional quantiles of the response variable is the answer. It was even first coined as "regression quantiles", instead of quantile regression, by the authors. In fact, the second term is the name of the book by Koenker (2005) which brings several examples of application, while also presenting key asymptotic results that, for instance, allow the construction of confidence intervals and hypothesis tests using a few different inferential procedures.

First, this frequentist procedure was not attached to any probability distribution, as parameter estimation was made possible through linear programming algorithms, while inferential methods, such as hypothesis tests and confidence intervals could rely on asymptotic results or bootstrap, for instance. Koenker and Machado (1999) connected the asymmetric Laplace distribution to these models, where they defined a likelihood ratio test using the assumption of this distribution.

Yu and Moyeed (2001) introduced Bayesian quantile regression models, assuming in the likelihood the asymmetric Laplace distribution, but fixing its scale parameter equal to one. In this first proposal, they used an improper prior distribution for the regression parameters, but the authors showed that they still obtained a proper posterior. Later, Kozumi and Kobayashi (2011) adopted a location-scale mixture of the asymmetric Laplace distribution to build a more flexible Markov Chain Monte Carlo (MCMC) scheme to draw samples from the posterior distribution. Khare and Hobert (2012) proved that this new sampling algorithm converges at a geometric rate.

Recently, Sriram, Ramamoorthi and Ghosh (2013) demonstrated posterior consistency for quantile estimates using the assumption of the asymmetric Laplace distribution, as a misspecified model. In fact, when building these models for the same dataset one considers that for each quantile of interest a different likelihood should be properly combined with the prior, to produce a posterior distribution. This makes the misspecified model assumption very reasonable. Using a similar idea, Yang, Wang and He (2015) argue that fixing the σ parameter, one needs to make a small modification in the posterior covariance matrix of the regression parameters, in order to get reliable confidence intervals. Although, we agree with the misspecified model result, we discuss here in this paper that one should not fix σ , but instead should learn from its posterior distribution.

In the nonparametrics and semiparametric literature, there are also proposals for Bayesian quantile regression models. For instance, using Dirichlet processes, Kottas and Gelfand (2001) suggest a model for median regression, while Kottas and Krnajajić (2009) and Taddy and Kottas (2010) study models for all quantiles. Non-crossing quantiles planes, which is a concern when dealing with quantile regression, are proposed by Reich, Fuentes and Dunson (2011) and Tokdar and Kadane (2011), considering Bernstein polynomial bases and functions of Gaussian processes, respectively. In an interesting way, these proposals are able to produce quantile estimates, without relying on the asymmetric Laplace distribution.

Concerning outlying observations, in the frequentist literature, Santos and Elian (2015) proposed influence measures to identify observations that might affect the model fit. They considered the likelihood displacement function to determine whether one observation would be deemed influential or not. In the process, the model is fit again for every observation, in order to obtain the parameter estimates without each point. This could become computationally challenging for data with high dimensions. Instead, we propose in this paper, in the light of the Bayesian model, to compare the posterior distribution of the latent variable v_i for each observation, in order to find those most distant points from the others.

The paper is organized as follows. In Section 2, we give a brief review of Bayesian quantile regression, discussing some parameters, which usually do not receive enough attention in the literature. In Section 3, we propose the use of the posterior distribution of the latent variable v_i as a measure of distance between the observations, suggesting a possible manner to identify outliers in the sample. Moreover, in Section 4, we present two simulation studies to check how these

proposed methods vary in different scenarios, with zero, one or two outliers. We illustrate our proposal with an application in Section 5, where we argue about the presence of more than one outlier in data about the Gini indexes in Brazilian states. We finish with our final remarks in Section 6.

2. Bayesian quantile regression

In quantile regression models, the interest lies, for example, considering just linear terms, in the following model

$$Q_y(\tau|x) = x'\beta(\tau),$$

which states basically that the τ th conditional quantile of Y given X is assumed to follow a linear model with coefficients $\beta(\tau)$. A first model to produce such estimates goes back to Koenker and Bassett (1978), where the authors proposed, given a sample of n pairs (y_i, X_i) , to minimize the following weighted absolute sum

$$\sum_{i=1}^{n} \rho_{\tau}(y_i - x_i'\beta), \tag{2.1}$$

where $\rho_{\tau}(u) = u(\tau - \mathbb{I}(u < 0))$ and I(.) is the indicator function, through linear programming algorithms.

In the Bayesian paradigm, Yu and Moyeed (2001) used the asymmetric Laplace distribution in the likelihood, with density

$$f(y|\mu, \sigma, \tau) = \frac{\tau(1-\tau)}{\sigma} \exp\left\{-\rho_{\tau} \left(\frac{y-\mu}{\sigma}\right)\right\},$$

due to the fact that its location parameter, $\mu \in \mathbb{R}$, is the τ th quantile of the distribution. In fact, the maximum likelihood estimator when we replace μ for $x'\beta$, matches the estimator obtained by the minimization in (2.1), for the frequentist model.

Still about the asymmetric Laplace distribution, its mean and variance can be written as

$$E(Y) = \mu + \frac{\sigma(1 - 2\tau)}{\tau(1 - \tau)}, \quad \text{Var}(Y) = \sigma^2 T(\tau),$$

where $\sigma > 0$ is the scale parameter and $T(\tau) = (1 - 2\tau + 2\tau^2)/((1 - \tau)^2\tau^2)$. The function $T(\tau)$, from which depends the variance of Y is presented in Figure 1(a). One can see that this function is U-shaped, so for fixed σ the variance is greater for smaller or larger quantiles. In their first proposal, Yu and Moyeed (2001) assumed $\sigma = 1$, automatically increasing the variability for lower and greater quantiles, and followed their inference drawing posterior samples for $\beta(\tau)$.

By giving σ a prior distribution, for example, the inverse gamma distribution, one can carry on the inference in a more complete way, because the posterior distribution for σ takes into account the data variation and the variation due

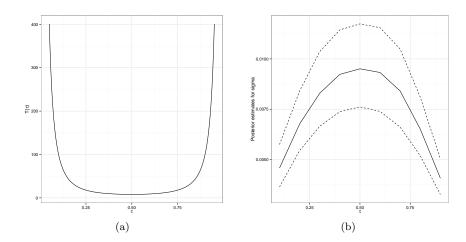


FIG 1. (a) $T(\tau)$ function which is part of the variance in an asymmetric Laplace distribution. (b) Posterior estimates for σ in the model analyzed in the application section.

to the asymmetric Laplace in the likelihood. For instance, in Figure 1(b), we have the mean posterior estimates for different quantiles, $\tau = 0.1, 0.2, \dots, 0.9$, in the application studied in Section 5. It is easy to see that the estimates for σ adapt according to the quantile and the function $T(\tau)$, and that by fixing σ , which was done by Yu and Moyeed (2001) and suggested by Yang, Wang and He (2015), one loses such result.

Yet about Bayesian quantile regression, in the modeling framework, Kozumi and Kobayashi (2011) proposed a location-scale mixture representation of the asymmetric Laplace distribution, combining a normal distribution conditional on an exponential distribution with mean σ , as follows

$$Y|v \sim N(\mu + \theta v, \psi^2 \sigma v),$$

where $\theta = (1 - 2\tau)/(\tau(1 - \tau))$, $\psi^2 = 2/(\tau(1 - \tau))$. The marginal distribution of Y is the asymmetric Laplace with parameters μ , σ and τ . Now, if we substitute $\mu = x'\beta(\tau)$ and give a normal prior distribution for $\beta(\tau)$ we have that the full conditional posterior distribution for the quantile regression parameters is also normal, making it easier to draw samples from the posterior. In a similar way, the full conditional posterior distribution for σ is inverse gamma, if we assume an inverse gamma distribution in the prior.

Moreover, the latent variable v_i , which by construction have an exponential prior distribution also needs to be updated in the MCMC algorithm. The full conditional posterior distribution for each v_i is proportional to

$$v_i^{\nu-1} \exp\left\{-\frac{1}{2}(\delta_i^2 v_i^{-1} + \zeta^2 v_i)\right\},$$
 (2.2)

that is the kernel of a generalized inverse Gaussian distribution. Because each

 v_i has its own posterior distribution, that depends on the residual value for each observation, this information can be used to compare all observations, even to identify possible outliers.

All the details of the posterior distributions of all parameters can be found in Kozumi and Kobayashi (2011).

3. Outliers observations given the quantile regression fits

Due to the location-scale mixture representation of the asymmetric Laplace, a latent variable v_i is added in the modeling scheme for each observation. Before updating with data, every v_i is assumed to have an exponential distribution with mean σ , that with the likelihood produces a posterior distributed according to a generalized inverse Gaussian as in (2.2) with parameters,

$$\nu = \frac{1}{2}, \quad \delta_i^2 = \frac{(y_i - x_i'\beta(\tau))^2}{\psi^2 \sigma}, \quad \zeta^2 = \frac{2}{\sigma} + \frac{\theta^2}{\psi^2 \sigma}.$$
 (3.1)

From the parameters in the posterior distribution of v_i , just δ_i^2 varies for each observation. And its value is the weighted squared residual of the quantile fit. One can see that for larger values of δ_i^2 , while the other parameters are kept fixed, the posterior distribution of the latent variable v_i has a greater expected value. Therefore, more extreme observations present a posterior distribution for its latent variable more distant from zero.

From empirical evidence, we see that points that have a completely different pattern than the one proposed by the model, have their latent variable distributed in a region far from the other observations. Given that difference, we propose to use that information to label these data points as possible outliers, i.e., observations that show an extreme pattern that can not be explained by the quantile regression model. These points often cause bias in the parameter estimates, so it could be discussed even if its presence is indeed necessary.

We propose to measure this distance between one observation from the others, by comparing the posterior distribution of its latent variable in two different ways. First, we propose to measure the mean probability of the posterior conditional latent variable of being greater than the other respective latent variables. Second, we use the Kullback-Leibler divergence to assess the difference between the conditional posterior distributions of latent variables based on the MCMC samples.

3.1. Mean probability posterior

If we define the variable O_i , which takes value equal to 1 when the *i*th observation is an outlier, and 0 otherwise, then we propose to calculate the probability of an observation being an outlier as

$$P(O_i = 1) = \frac{1}{n-1} \sum_{j \neq i} P(v_i > v_j | \text{data}).$$
 (3.2)

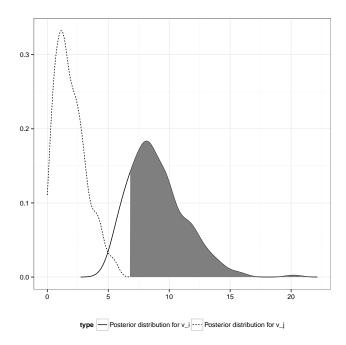


FIG 2. Example of how part of the probability in (3.2) is calculated, where the area under the dashed line in gray is the probability.

An example of this calculus is presented in Figure 2, where in the probability proposed in (3.2), we average over all observations.

We believe that for points, which are not outliers, this probability should be small, possibly close to zero. Given the natural ordering of the residuals, due to the fact of the posterior parameters depending solely on them as in (3.1), it is expected that some observations present greater values for this probability in comparison to others. What we think that should be deemed as an outlier, ought to be those observations with a higher $P(O_i = 1)$, and possibly one that is particularly distant from the others.

The probability in (3.2) can be approximated given the MCMC draws, as follows

$$P(O_i = 1) = \frac{1}{M} \sum_{l=1}^{M} \mathbb{I}(v_i^{(l)} > \max_{k \in 1:M} v_j^{(k)}),$$

where M is the size of the chain of v_i after the burn-in period and $v_i^{(l)}$ is the lth draw of this chain.

An important note about this proposal of calculating the probability of an observation being an outlier is that this result depends on the quantile, therefore a point can be considered an extreme observation for one quantile, but not the

others. This brings more information about the data variation, as it is more flexible in determining these possible outliers.

3.2. Kullback-Leibler divergence

As a second proposal to address these differences between the posterior distributions from the distinct latent variables in the model, we suggest the use of the Kullback-Leibler divergence proposed by Kullback and Leibler (1951), as a more precise method of measuring the distance between those latent variables in the Bayesian quantile regression framework, in its posterior information. This divergence is defined as

$$K(f_i, f_j) = \int \log \left(\frac{f_i(x)}{f_j(x)}\right) f_i(x) dx, \tag{3.3}$$

where in our problem f_i could be the posterior conditional distribution of v_i and f_j the posterior conditional distribution of v_j . Similar to the probability proposal in the previous subsection, we should average this divergence for one observation based on the distance from all others, i.e.,

$$KL(f_i) = \frac{1}{n-1} \sum_{j \neq i} K(f_i, f_j)$$

This proposal should be seen as a ratification to the previous probability, using a more precise measure of distance between the posterior latent variables. We expect that when an observation presents a higher value for this divergence, it should also present a high probability value of being an outlier. On one hand, there is the probability value in the range (0, 1), which should give some insight of whether one observation should be regarded as too extreme. On the other hand, there is the Kullback-Leibler, a positive valued measure, that could always be analyzed relatively among the observations, i.e., instead of using its absolute, one could compare how many times this value is greater than the others. This approach could be helpful to identify observations that, for instance, show a not so high probability value, but still are distributed, in its relative posterior conditional distribution, far from the others.

Here, based on the MCMC draws from the posterior of each latent variable, we estimate the densities in (3.3) using a normal kernel and we compute the integral using the trapezoidal rule.

4. Simulation studies

In this section, we propose two simulation studies in order to understand how these measures defined in the previous section vary according to the presence or not of an outlying observation in the case with multivariate explanatory variables. In the first study, we study the distribution of the probability of being an outlier in the absence of such observation. Following, we discuss the results of the case when there are more than one outlier, showing results both for the probability as for the Kullback-Leibler divergence measure.

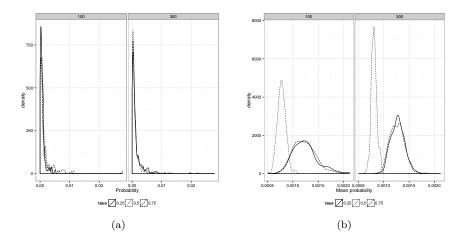


Fig 3. (a) Distribution of the probabilities for all observations in a randomly selected replication in the simulation study. (b) Distribution of the mean probability for each replication.

4.1. Simulation 1

In this first simulation, we try to deal with the scenario where there are no outliers, in order to learn the distribution for the probability of being an outlier in these situations. We do not present summaries for the Kullback-Leibler divergence, as this quantity is not limited and its distribution is dependent on other parameters, such as the quantile regression parameter, σ and the quantile of interest.

We consider the following linear model

$$Y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \beta_3 x_{3i} + \epsilon_i, \quad i = 1, \dots, n,$$

where we set $\beta_0 = 0$, $\beta_1 = 1$, $\beta_2 = -1$, $\beta_3 = 2$, $\epsilon_i \sim N(0, 4)$, and we draw the three explanatory variables from an uniform distribution between 0 and 10. We use two samples sizes in this study, n = 100, 300. Each sample was replicated 250 times to produce the summaries that we discuss next. And three different quantiles were estimated, $\tau = 0.25, 0.50, 0.75$.

The results for this study can be seen in Figure 3. In the left part of the figure, we show the probabilities for one of the replications, which was randomly selected, and where we can see that the probability varies between 0 and 0.03, approximately. As expected, with the absence of extreme observations, one should not expect greater values for this probability, as the posterior distribution of all latent variables should be relatively close, given that residuals should be rather small as well.

If we compare the summaries of all these probabilities in each replication, we get the distribution shown in Figure 3(b). Between the different quantiles, the probabilities in the conditional median presented smaller mean values in

comparison with the 0.25th and 0.75th quantiles. For the different sample sizes, the probabilities decrease slightly as we increase the sample size.

Given the results in this simulation study, we suggest as a rule of thumb to consider outlying observations the ones with non-negligible probability values, possibly greater than 0.10 at least.

4.2. Simulation 2

For this second simulation study, we add until two outliers and record both measures to study the presence of outlying observations, while we replicate these scenarios 250 times as well. We are interested in checking the influence of one outlier on the other, when both are present in the model. We verify this by analyzing the results with just one of these observations separately and then with both of these in the model. We use the same setup as in the previous simulation study, but only considering the sample size equal to 100.

The two outlier observations have the following values for the response variables and their respective explanatory variables

$$y^* = 30, \quad x_1^* = \bar{x}_1, \quad x_2^* = 20, \quad x_3^* = \bar{x}_3$$

 $y^* = 0, \quad x_1^* = 20, \quad x_2^* = \bar{x}_2, \quad x_3^* = \bar{x}_3,$

where \bar{x}_i represents the mean for the *i*th explanatory variable without any possible outlier. We argue that y^* should be considered an outlier because both the response variable value and x_2^* are definitely a lot greater than expected, specially given the fact that the coefficient for x_2 is negative and all other observations for the predictor are drawn from an uniform distribution from 0 to 10. Moreover, for similar reasons y^* also should be defined as an outlier, as x_1^* is outside the range (0,10) and it produces a response variable smaller than expected.

In the following summary results, we use the setup presented in Table 1, where \times represents the presence of the extreme observation in the scenario.

 ${\it Table~1} \\ Setup~for~the~different~scenarios~in~Simulation~study~2.$

	Outlier		
	*	*	
Scenario 1			
Scenario 2		×	
Scenario 3	×	×	
Scenario 4	×		

The summaries for the probabilities in all scenarios are presented in Table 2. It is easy to see when each outlier is added separately in the model then their respective probability is always high, greater than 0.40 on average. For most scenarios, y^* always presents a greater probability value in comparison with y^* . For both outliers, the probability decreases in the presence of the other, but still

Table 2
Summary results for the probabilities in each scenario.

		Outlier *			Outlier *				
au	Scenario	Mean	Median	2.5%	97.5%	Mean	Median	2.5%	97.5%
	2					0.505	0.488	0.228	0.783
0.1	3	0.981	0.984	0.957	0.994	0.452	0.436	0.197	0.721
	4	1.000	1.000	0.998	1.000				
	2					0.433	0.431	0.265	0.631
0.5	3	0.656	0.657	0.506	0.809	0.273	0.266	0.159	0.419
	4	0.780	0.781	0.636	0.914				
	2					0.987	0.992	0.948	1.000
0.9	3	0.810	0.823	0.596	0.935	0.765	0.778	0.543	0.911
	4	0.841	0.852	0.656	0.961				

Table 3
Summary results for the mean relative Kullback-Leibler divergence in each scenario.

		Outlier *				Outlier *			
au	Scenario	Mean	Median	2.5%	97.5%	Mean	Median	2.5%	97.5%
	2					11.056	10.436	3.259	20.847
0.1	3	13.988	14.880	5.968	21.908	9.614	9.248	3.084	17.532
	4	9.125	9.380	3.321	16.125				
	2					26.343	26.871	10.636	38.251
0.5	3	28.581	29.704	15.871	38.061	17.153	17.365	8.042	25.147
	4	35.511	36.723	15.151	48.379				
	2					10.616	10.926	4.120	18.575
0.9	3	14.882	15.448	7.055	23.127	14.403	14.913	7.104	22.025
	4	17.217	18.383	7.424	28.111				

show values far from zero. Overall, these probabilities are smaller for quantile 0.5.

In a interesting way, when we look for the Kullback-Leibler divergences, we have an opposite outcome, as we see the greater disparities in the models for the conditional median. In Table 3, we show the mean relative Kullback-Leibler divergence for both outliers, i.e., the mean ratio between the divergence between the outliers and a randomly selected observation in the sample. We used the comparison with just one observation due to the computation burden to calculate for all observations, but also because we believe that between all observations, which are not extreme, the difference would be small. In general, we see that these ratios are always greater than 9, on average, approximately. In other words, we can say that, these outliers show a Kullback-Leibler divergence at least 9 times the divergence from a non-outlier observation.

Another interesting aspect of these measures is how they give different conclusions in respect to these two outliers, y^* and y^* . For instance, in the 0.1th quantile, in the models only with one outlier, the probability is greater for y^* , while the Kullback-Leibler divergence presents higher values for the y^* . On the other hand, in the 0.9th quantile the Kullback-Leibler divergence is greater for

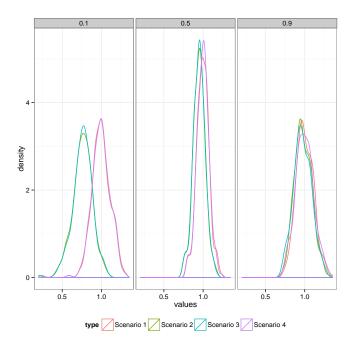


Fig 4. Distribution of $\hat{\beta}_1(\tau)$ for $\tau = \{0.1, 0.5, 0.9\}$.

 y^* , even though y^* presents higher values of being an outlier.

Moreover, we present the distribution of the estimates for $\beta_1(\tau)$ and $\beta_2(\tau)$ in Figure 4 and Figure 5, respectively. For $\beta_1(\tau)$, we can see that its estimates are only influenced by the presence of y^* , in Scenarios 2 and 3. And even then just in the lower quantiles, for instance, the 0.10th quantile.

On the other hand, for $\beta_2(\tau)$, we have that the presence of y^* adds a bias in its estimates for greater quantiles, only when this outlier is present in Scenario 4, but also when both outliers are present in Scenario 3.

For $\beta_3(\tau)$, we found that neither outlier presented a challenge in its estimates, as for all scenarios the distribution of $\hat{\beta}_3(\tau)$ was not affected by those observations.

5. Application

In the interest of using Bayesian quantile regression models to analyze possible outlying observations, we consider data about Gini indexes in Brazilian states in the years 1991, 2000 and 2010, when censuses were conducted countrywide. This data comprises the information about 26 states and the Federal District, where the Brazilian capital is located, completing 81 observations.

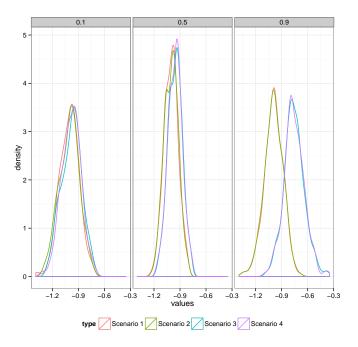


Fig 5. Distribution of $\hat{\beta}_2(\tau)$ for $\tau = \{0.1, 0.5, 0.9\}.$

If we consider data about the whole country, Brazil is usually regarded as a highly unequal country, when compared to European countries, for instance. Using the Gini index, which gives an indicator of the income inequality, one can see that, at least, there was an advance between 1991 and 2010, when this measure decreased for several states, as depicted in Figure 6, in spite of the increase for some states at first in 2000.

The following model was proposed to study the conditional quantiles of the Gini index,

$$Q_{Y_i}(\tau|x_i) = \beta_0(\tau) + \beta_1(\tau) \text{EDUC}_i + \beta_2(\tau) \text{INCPC}_i + \beta_3(\tau) \text{Y} 2000_i + \beta_4(\tau) \text{Y} 2010_i$$
(5.1)

where EDUC is the average years of education and INCPC is the income per capita of each state, and two indicator variables were used to control for the difference between the three years, using 1991 as reference. We decided not to transform the response variable, the Gini index, which is a number between 0 and 1, as suggested by Santos and Bolfarine (2015), because even at the most extreme quantiles, the conditional estimates were far from the boundaries 0 and 1

The posterior estimates were considered using a chain of size 3000, discarding the first 1000 as burn-in. We used a normal distribution N(0, 100I) for $\beta(\tau)$, where I stands for the identity matrix. For σ , we adopted IG(3/2, 0.1/2). The

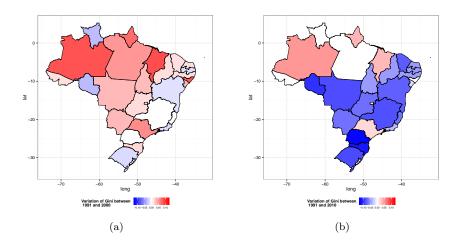


FIG 6. Variation for the Gini index in the 26 Brazilian states and the Federal District, in comparison with 1991 (a) 2000, (b) 2010.

posterior mean and its respective credible interval for σ in the different quantiles can be seen in Figure 1(b), where we can clearly see that the shape of posterior estimates, along with its credible intervals, have the inverse form of the function $T(\tau)$, presented in Section 2. Given these results, we defend the importance of using a prior distribution for σ , instead of fixing its value, arguing that the posterior distribution naturally adapts to the different sources of variation in the modeling process. The posterior mean and 95% credible intervals for $\beta(\tau)$ is presented in Figure 7.

For years of education, the estimates for $\beta_1(\tau)$ are negative for all quantiles, but with greater absolute values for τ 's closer to 1. For income per capita, the estimates for its respective parameter are also negative, but not significant for greater quantiles, $\tau \geq 0.6$. Both variables for years presented similar estimates with values decreasing along the quantiles, despite having a different evolution as shown in Figure 6. Controlling for other variables, we estimate that the Gini indexes in the year 2000 and 2010 in comparison with 1991 are greater, with this difference being smaller for greater quantiles.

If we calculate the probability proposed in Section 3 for all observations we get Figure 8. And the Kullback-Leibler divergences are presented in Figure 9. Here we focus the attention on three quantiles, even though we analyzed the others quantiles, as only in these quantiles there were observations which are separated from the others in these plots. In the 0.1th quantile, these observations are #27, #54 and #81, which are the three observations from Federal District, in the three years that the data was collected. For quantile 0.9, the observation #76 is the one most distant from the others, and it is about the state of Santa Catarina in the year 2010. Comparing figures 8 and 9, we have the same pattern of observations which are detached from the others.

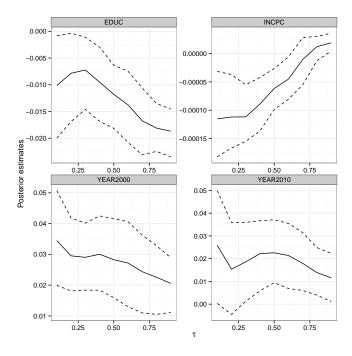


FIG 7. Posterior estimates for the quantile regression parameters proposed in the model (5.1).

In the first case, those three observations from Federal District have high values of income per capita, with R\$917, R\$1,204, and R\$1,717 in the years 1991, 2000 and 2010, respectively. Ordering this variable in the sample we find that these values are number 8, 2, 1 in this list, respectively. Also for years of education, these points present high values considering the data. Meanwhile, the effect of income per capita in the lower quantiles is estimated to be negative as shown in Figure 7, as it is the effect of years of education. On the other hand, their Gini indexes are among the highest in the dataset. Therefore, it is suiting that these observations are marked as outliers in the lower quantiles of the conditional distribution, given these unexpected results, as for all three it was likely that they present small values for the Gini index.

Moreover, observation #76 from the state of Santa Catarina, measured in 2010, has the lowest Gini in the sample, of 0.49. It is important to note that this observation presented a greater probability of being an outlier just in the higher quantiles. It can be argued that this observation should be considered an outlier since it presented the lower value of Gini in the sample despite happening in the year 2010, while the estimated coefficients for this dummy variable are positive for all quantiles, even though not significant for some quantiles. Besides that, this observation presents a big difference to next state in the sample, as the second lowest value is 0.53. Such a difference between two points is not seen

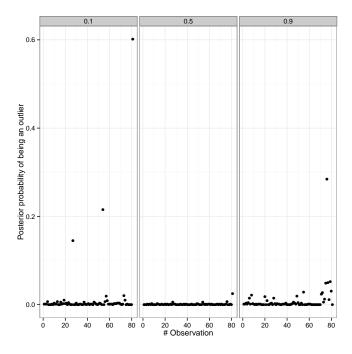


Fig. 8. Probabilities of being an outlier for $\tau = \{0.1, 0.5, 0.9\}$, considering the model in (5.1)

in the entire sample, making it even more fitting this observation as an outlier. These observations from two different states could be considered outliers in different parts of the conditional distribution of the Gini index, and this was only possible examining their latent variables in each quantile of interest, as we propose here in this work.

6. Final discussion

Quantile regression models have become a great tool in the regression analysis framework given its flexibility in studying the conditional quantiles of the response variable. The Bayesian version of this model, taking into account the misspecified model assumption, is well established now with the asymmetric Laplace distribution and its mixture representation, which readily provides a setup to identify possible outlying observations in the regression analysis, while also controlling for the variance in the data with the σ parameter. We showed how the posterior inference for σ varies with τ , and how it could be vague when its value is fixed from the beginning. We also showed how the posterior distribution for each latent variable v_i provides evidence regarding observations that are too far apart from the others, which could be seen as outliers. We demonstrated these results with simulated examples to illustrate how this approach

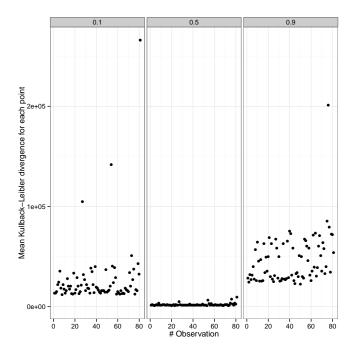


Fig 9. $KL(f_i)$ for $\tau = \{0.1, 0.5, 0.9\}$, considering the model in (5.1)

works, showing how when there are more than one outlier, they can affect the estimates differently for distinct quantiles.

In a real dataset, about Gini indexes in Brazilian states, we were able to find extreme observations from two different states that affected the quantile regression fits in different parts of the conditional distribution, one being in the lower quantiles and the other in the greater quantiles. This was only possible using our approach that gives attention to each quantile separately. It is important to note that in our method we are not checking whether this observation influences the regression models or not, as some diagnostics measures are concerned with, but we are more interested in identifying these most distant observations from the others, based on the posterior posterior distribution from their latent variable v_i , even though we did observe in the simulation studies that the outliers increased the bias in the quantile regression estimates. As a future study, case-deletion diagnostics for this type of model could be proposed, in addition to our approach.

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