Non-homogeneous Poisson Process (NHPP)

Miao Cai

2018-12-02

1 Non-homogeneous Poisson Process - essential part for Stan

2 Real data example

3 Mathematical prove

3.1 Definitions

Intensity function The intensity function of a point process is:

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{P(N(t, t + \Delta t) \geq 1)}{\Delta t}$$

Nonhomogeneous Poisson Process The Nonhomogeneous Poisson Process (NHPP) is a Poisson process whose intensity function is non-constant.

When the intensity function of a NHPP has the form $\lambda(t) = (\beta/\theta)(t/\theta)^{\beta-1}$, where $\beta > 0$ and $\theta > 0$, the process is called **power law process** (PLP).

1. **Failure truncation**: When testing stops after a predetermined number of failures, the data are said to be failure truncated.
2. **Time truncation**: Data are said to be time truncated when testing stops at a predetermined time t.

Conditional probability

$$\begin{aligned}
P(A \cap B) &= P(A)P(B|A) \\
P(A \cap B \cap C) &= P(A)P(B|A)P(C|A \cap B)
\end{aligned}$$

3.2 Failure Truncated Case

3.2.1 The first event

The cumulative density function (cdf) of time to the first event is $F(t_1)$: $F_1(t_1) = P(T_1 \leq t_1) = 1 - S(t_1)$.

The survival function for the first event $S_1(t_1)$ is:

$$\begin{aligned}
S_1(t_1) &= P(T_1 > t_1) \\
&= P(N(0, t_1) = 0) \quad N \text{is the number of events} \\
&= e^{-\int_0^{t_1} \lambda_u du}(e^{-\int_0^{t_1} \lambda_u du})^0/0! \\
&= e^{-\int_0^{t_1} \lambda_u du}
\end{aligned}$$

The probability density function (pdf) of time to the first event can be calculated by taking the first order derivative of the cdf $F_1(t_1)$:

$$
\begin{aligned}
f_1(t_1) &= \frac{d}{dt_1} F_1(t_1) \\
&= \frac{d}{dt_1}[1 - S_1(t_1)] \\
&= -\frac{d}{dt_1} S_1(t_1) \\
&= -\frac{d}{dt_1} e^{-\int_0^{t_1} \lambda(u)du} \\
&= -(-\lambda_{t_1}) e^{-\int_0^{t_1} \lambda(u)du} \\
&= \lambda(t_1) e^{-\int_0^{t_1} \lambda(u)du}
\end{aligned}
$$

If this NHPP is a PLL, we plug in the intensity function $\lambda(t) = (\beta/\theta)(t/\theta)^{\beta-1}$, then we have:

$$
f_1(t_1) = \frac{\beta}{\theta}(\frac{t_1}{\theta})^{\beta-1} e^{-(\frac{t_1}{\theta})^\beta}, \quad t_1 > 0
$$

This pdf is identical with the pdf of Weibull distribution, so we have:

$$
T_1 \sim \text{Weibull}(\beta, \theta)
$$

3.2.2 The second event

The Survival function of the second event given the first event occurred at t_2 is:

$$
\begin{aligned}
S_2(t_2|t_1) &= P(T_2 > t_2 | T_1 = t) \\
&= P(N(t_1, t_2) = 0 | T_1 = t_1) \\
&= e^{-\int_{t_1}^{t_2} \lambda_u du} [\int_{t_1}^{t_2} \lambda_u du]^0 / 0! \\
&= e^{-\int_{t_1}^{t_2} \lambda_u du}
\end{aligned}
$$

The we can derive the pdf of t_2 conditioned on t_1

$$
\begin{aligned}
f(t_2|t_1) &= -\frac{d}{dt_2} S_2(t_2) \\
&= -\frac{d}{dt_2} e^{-\int_{t_1}^{t_2} \lambda(u)du} \\
&= \lambda(t_2) e^{-\int_{t_1}^{t_2} \lambda(u)du} \\
&= \frac{\beta}{\theta}(\frac{t_2}{\theta})^{\beta-1} e^{-[(\frac{t_2}{\theta})^\beta - (\frac{t_1}{\theta})^\beta]} \\
&= \frac{\frac{\beta}{\theta}(\frac{t_2}{\theta})^{\beta-1} e^{-(t_2/\theta)^\beta}}{e^{-(t_1/\theta)^\beta}}, \quad t_2 > t_1
\end{aligned}
$$

3.2.3 All events

$$
\begin{aligned}
f(t_1, t_2, \cdots, t_n) &= f(t_1) f(t_2|t_1) f(t_3|t_1, t_2) \cdots f(t_n|t_1, t_2, \ldots, t_{n-1}) \\
&= \lambda(t_1) e^{-\int_0^{t_1} \dot{\lambda}(u)du} \lambda(t_2) e^{-\int_{t_1}^{t_2} \dot{\lambda}(u)du} \cdots \lambda(t_n) e^{-\int_{t_{n-1}}^{t_n} \lambda(u)du} \\
&= \Big(\prod_{i=1}^{n} \lambda(t_i)\Big) e^{-\int_0^t \lambda(u)du} \\
&= \Big(\prod_{i=1}^{n} \frac{\beta}{\theta} (\frac{t_i}{\theta})^{\beta-1}\Big) e^{-(t_n/\theta)^\beta}, \quad t_1 < t_2 < \cdots < t_n
\end{aligned}
$$

3.3 Time Truncated Case

3.3.1 Conditional likelihood function

We assume that the truncated time is τ. The derivation of $f(t_1, t_2, \cdots, t_n|n)$ is messy in math, we directly give the conclusion here:

$$
f(t_1, t_2, \cdots, t_n|n) = n! \prod_{i=1}^{n} \frac{\lambda(t_i)}{\Lambda(\tau)}
$$

3.3.2 Joint likelihood function

Therefore, the joint likelihood function for $f(n, t_1, t_2, \cdots, t_n)$ is:

$$
\begin{aligned}
f(n, t_1, t_2, \cdots, t_n) &= f(n) f(t_1, t_2, \cdots, t_n|n) \\
&= \frac{e^{-\int_0^\tau \lambda(u)du} [\int_0^\tau \lambda(u)du]^n}{n!} n! \frac{\prod_{i=1}^n \lambda(t_i)}{[\Lambda(\tau)]^n} \\
&= \Big(\prod_{i=1}^{n} \lambda(t_i)\Big) e^{-\int_0^\tau \lambda(u)du} \\
&= \Big(\prod_{i=1}^{n} \frac{\beta}{\theta} (\frac{t_i}{\theta})^{\beta-1}\Big) e^{-(\tau/\theta)^\beta}, \\
n &= 0, 1, 2, \cdots, \quad 0 < t_1 < t_2 < \cdots < t_n
\end{aligned}
$$

3.3.3 Log likelihood function

The log likelihood function l is then:

$$
\begin{aligned}
l &= \log \Big(\prod_{i=1}^{n} \frac{\beta}{\theta} (\frac{t_i}{\theta})^{\beta-1}\Big) e^{-(\tau/\theta)^\beta} \\
&= \sum_{i=1}^{n} \log \Big(\frac{\beta}{\theta} (\frac{t_i}{\theta})^{\beta-1}\Big) - (\frac{\tau}{\theta})^\beta \\
&= n \log \beta - n\beta \log \theta + (\beta - 1) \sum_{i=1}^{n} \log t_i - (\frac{\tau}{\theta})^\beta
\end{aligned}
$$