1-concentration_of_measure

August 5, 2019

[1]:
```
import matplotlib.pyplot as plot
import scipy.stats as stats
import numpy
import math

light = "#DCBCBC"
light_highlight = "#C79999"
mid = "#B97C7C"
mid_highlight = "#A25050"
dark = "#8F2727"
dark_highlight = "#7C0000"
green = "#00FF00"
```

To facilitate the computation of Monte Carlo estimators let's define a *Welford accumulator* that **computes empirical means and variances of a sample in a single pass**.

[2]:
```
def welford_summary(x):
  summary = [0.0, 0.0]
  for n in range(len(x)):
    delta = x[n] - summary[0]
    summary[0] += delta / (n + 1)
    summary[1] += delta * (x[n] - summary[0])
  summary[1] /= (len(x) - 1)
  return summary
```

We can then use the Welford accumulator output to **compute the Monte Carlo estimator of a function and an estimate of its Monte Carlo Standard Error**.

[3]:
```
def compute_mc_stats(x):
  summary = welford_summary(x)
  return [summary[0], math.sqrt(summary[1] / len(x))]

# To generate our samples we'll use numpy's pseudo random number
# generator which needs to be seeded to achieve reproducible
# results
numpy.random.seed(seed=8675309)
```

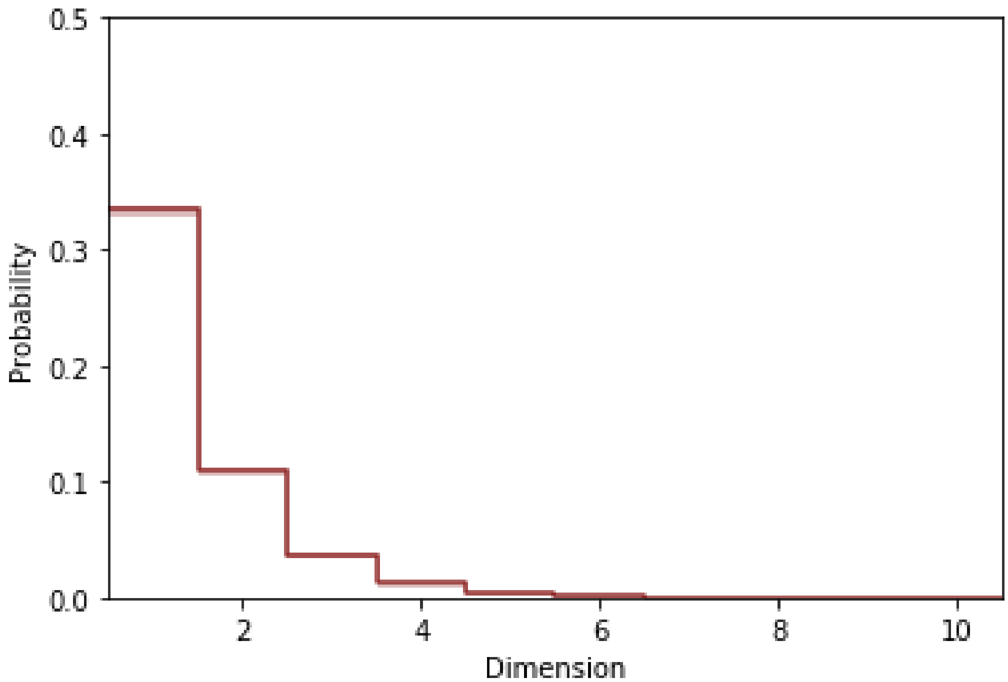
[4]:
```
# To ensure accurate results let's generate pretty large samples
N = 100000
```

```
# To see how results scale with dimension we'll consider
# behavior one thorugh ten dimensions
Ds = [ n + 1 for n in range(10) ]

idxs = [ idx for idx in range(Ds[-1]) for r in range(2) ]
plot_Ds = [ D + delta for D in Ds for delta in [-0.5, 0.5]]

# Quantile probabilities that we'll use to quantify distributions
quant_probs = [10, 20, 30, 40, 50, 60, 70, 80, 90]
```

0.0.1 What is the volume of central rectangular box that spans [-1, +1] in each dimension relative to the volume of a box spanning [-3, +3] in each dimension?

[5]:
```
prob_means = [0] * len(Ds)
prob_ses = [0] * len(Ds)

for D in Ds:
  # Is the sampled point in the central interval?
  is_central_samples = [0] * N

  for n in range(N):
    # We start by assuming that the point will be
    # in the central interval
    is_central = 1

    # Sample a new point one dimension at a time
    for d in range(D):
      x_d = stats.uniform.rvs(-3, 3, size=1)

      # If the component of the point in the current
      # dimension is not contained within the central
      # interval then set the flag to false
      if -1 < x_d and x_d < 1:
        is_central = is_central & 1
      else:
        is_central = is_central & 0

    is_central_samples[n] = is_central

  # Estimate the relative volume as a probability
  s = compute_mc_stats(is_central_samples)
  prob_means[D - 1] = s[0]
  prob_ses[D - 1] = s[1]

# Plot probabilities verses dimension
plot.fill_between(plot_Ds,
```

```
                    [ prob_means[idx] - 2 * prob_ses[idx] for idx in idxs ],
                    [ prob_means[idx] + 2 * prob_ses[idx] for idx in idxs ],
                    facecolor=light, color=light)
plot.plot(plot_Ds, [ prob_means[idx] for idx in idxs], color=dark)

plot.gca().set_xlim([plot_Ds[0], plot_Ds[-1]])
plot.gca().set_xlabel("Dimension")
plot.gca().set_ylim([0, 0.5])
plot.gca().set_ylabel("Probability")

plot.show()
```



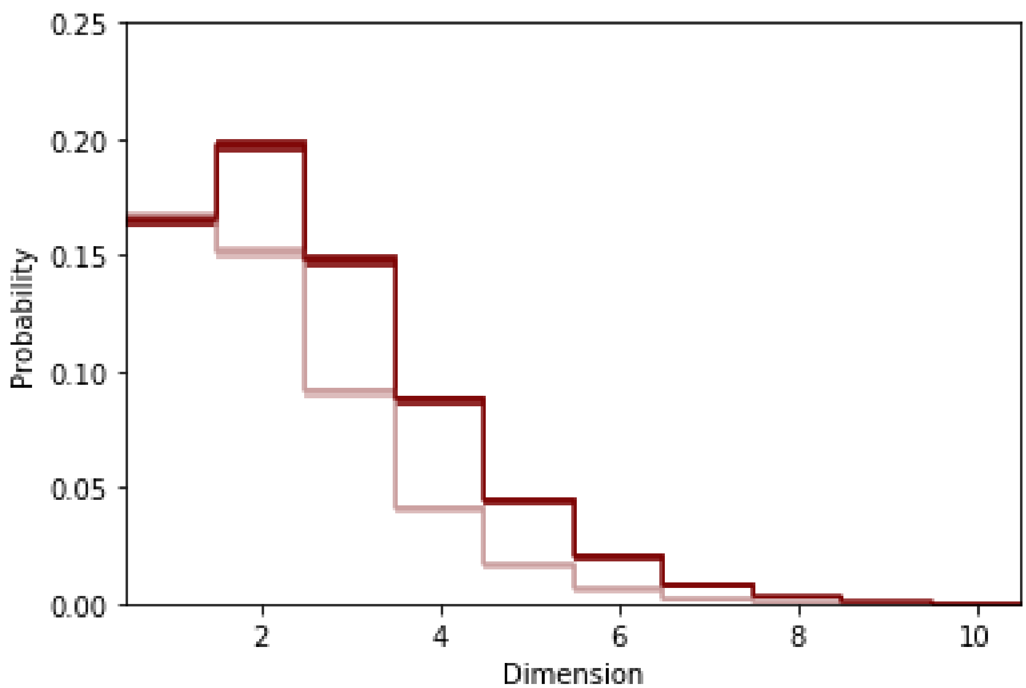
0.0.2 How much volume is in the neighborhood immediately outside a sphere, between a radius of 2 and 2.5, relative to the volume that lies in a neighborhood immediately inside that sphere, between a radius of 1.5 and 2?

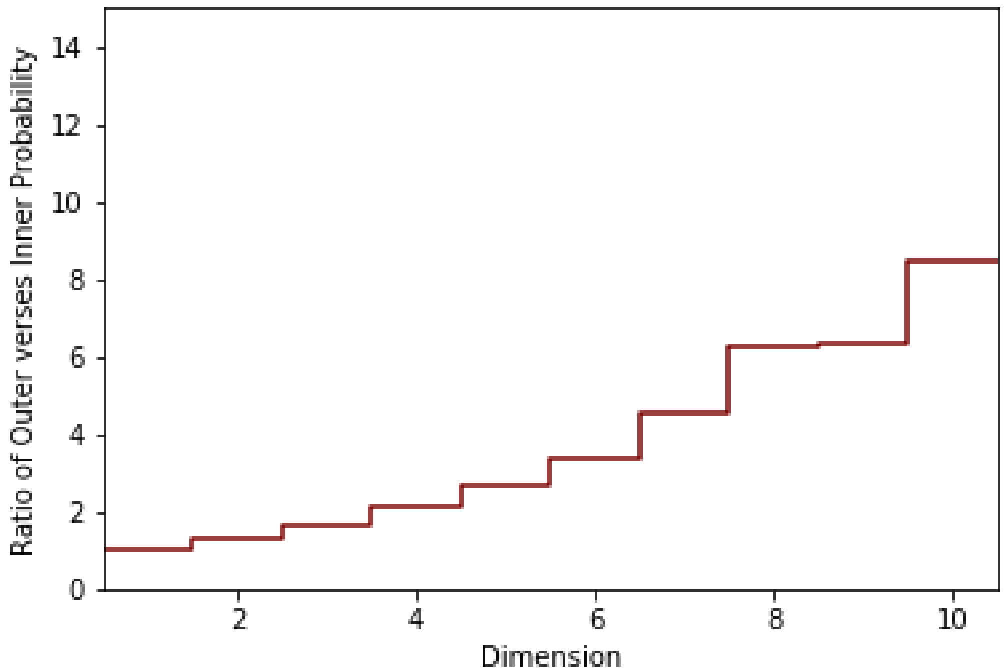
[6]:

```
prob_inner_means = [0] * len(Ds)
prob_inner_ses = [0] * len(Ds)

prob_outer_means = [0] * len(Ds)
prob_outer_ses = [0] * len(Ds)

R = 2
delta = 0.5
```

```python
for D in Ds:
  # Does the sampled point fall in the inside neighborhood?
  is_inner_samples = [0] * N
  # Does the sampled point fall in the outside neighborhood?
  is_outer_samples = [0] * N

  for n in range(N):
    # Sample a new point
    x = stats.uniform.rvs(-3, 3, size=D)

    # Compute distance from origin
    r = math.sqrt(sum([ x_d**2 for x_d in x]))

    # Check if point falls in the inside neighborhood
    if R - delta < r and r < R:
      is_inner_samples[n] = 1

    # Check if point falls in the outside neighborhood
    if R < r and r < R + delta:
      is_outer_samples[n] = 1;

  # Estimate the relative volumes as probabilies
  s1 = compute_mc_stats(is_inner_samples)
  prob_inner_means[D - 1] = s1[0]
  prob_inner_ses[D - 1] = s1[1]

  s2 = compute_mc_stats(is_outer_samples)
  prob_outer_means[D - 1] = s2[0]
  prob_outer_ses[D - 1] = s2[1]

# Plot probabilities verses dimension
plot.fill_between(plot_Ds,
                  [ prob_inner_means[idx] - 2 * prob_inner_ses[idx] for idx in idxs ],
                  [ prob_inner_means[idx] + 2 * prob_inner_ses[idx] for idx in idxs ],
                  facecolor=light, color=light)
plot.plot(plot_Ds, [ prob_inner_means[idx] for idx in idxs], color=light_highlight)

plot.fill_between(plot_Ds,
                  [ prob_outer_means[idx] - 2 * prob_outer_ses[idx] for idx in idxs ],
                  [ prob_outer_means[idx] + 2 * prob_outer_ses[idx] for idx in idxs ],
                  facecolor=dark, color=dark)
```

```
plot.plot(plot_Ds, [ prob_outer_means[idx] for idx in idxs],␣
↪color=dark_highlight)

plot.gca().set_xlim([plot_Ds[0], plot_Ds[-1]])
plot.gca().set_xlabel("Dimension")
plot.gca().set_ylim([0, 0.25])
plot.gca().set_ylabel("Probability")

plot.show()

# Plot ratio of probabilities verses dimension
plot.plot(plot_Ds,
          [ prob_outer_means[idx] / prob_inner_means[idx] for idx in idxs],
          color=dark_highlight)

plot.gca().set_xlim([plot_Ds[0], plot_Ds[-1]])
plot.gca().set_xlabel("Dimension")
plot.gca().set_ylim([0, 15])
plot.gca().set_ylabel("Ratio of Outer verses Inner Probability")

plot.show()
```

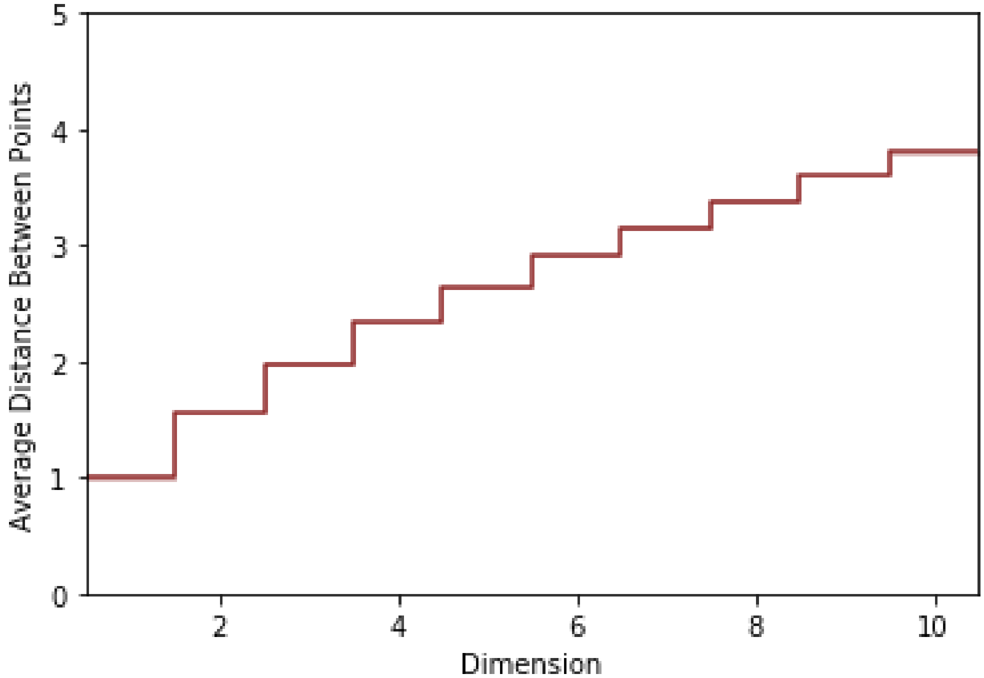


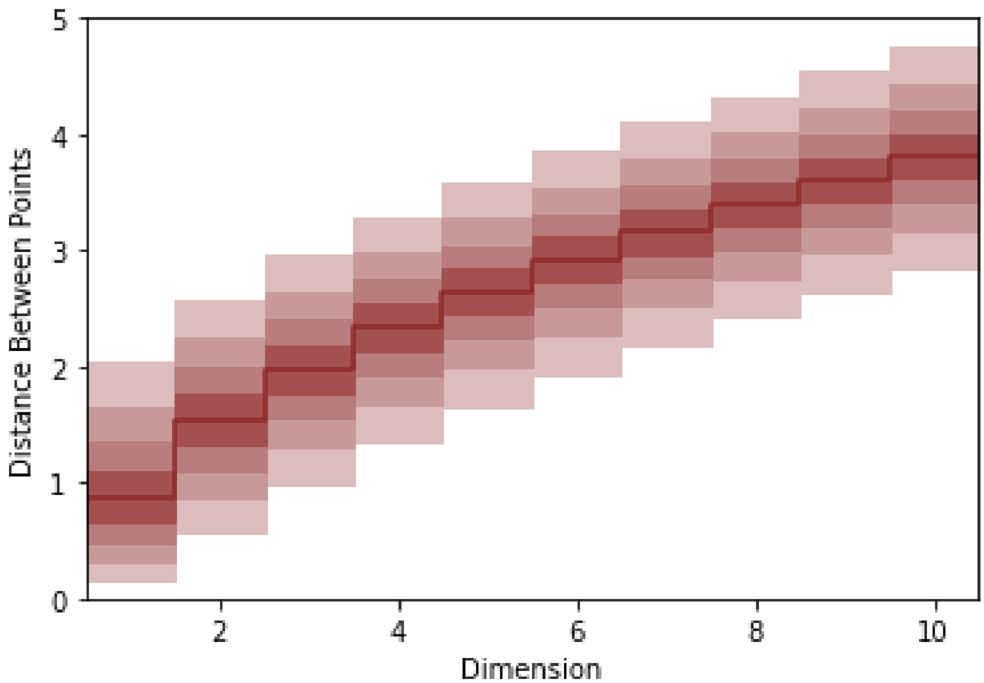


0.0.3 How does the distance between two sampled points behave as the dimensionality of the box increases?

[7]:
```
delta_means = [0] * len(Ds)
delta_ses = [0] * len(Ds)
delta_quantiles = [ [0] * 9 ] * len(Ds)

for D in Ds:
  # Distances between two sampled points
  delta_samples = [0] * N

  for n in range(N):
    # Sample two points
    x1 = stats.uniform.rvs(-3, 3, size=D)
    x2 = stats.uniform.rvs(-3, 3, size=D)

    # Compute distance between them
    delta_samples[n] = math.sqrt(sum([ (x1[d] - x2[d])**2 for d in range(D)]))

  # Estimate average distance
  s = compute_mc_stats(delta_samples)
  delta_means[D - 1] = s[0]
  delta_ses[D - 1] = s[1]
```

```
  # Estimate distance quantiles
  delta_quantiles[D - 1] = numpy.percentile(delta_samples, quant_probs)

# Plot average distance between points verses dimension
plot.fill_between(plot_Ds,
                  [ delta_means[idx] - 2 * delta_ses[idx] for idx in idxs ],
                  [ delta_means[idx] + 2 * delta_ses[idx] for idx in idxs ],
                  facecolor=light, color=light)
plot.plot(plot_Ds, [ delta_means[idx] for idx in idxs], color=dark)

plot.gca().set_xlim([plot_Ds[0], plot_Ds[-1]])
plot.gca().set_xlabel("Dimension")
plot.gca().set_ylim([0, 5])
plot.gca().set_ylabel("Average Distance Between Points")

plot.show()

# Plot distance quantiles verses dimension
plot.fill_between(plot_Ds,
                  [ delta_quantiles[idx][0] for idx in idxs ],
                  [ delta_quantiles[idx][8] for idx in idxs ],
                  facecolor=light, color=light)
plot.fill_between(plot_Ds,
                  [ delta_quantiles[idx][1] for idx in idxs ],
                  [ delta_quantiles[idx][7] for idx in idxs ],
                  facecolor=light_highlight, color=light_highlight)
plot.fill_between(plot_Ds,
                  [ delta_quantiles[idx][2] for idx in idxs ],
                  [ delta_quantiles[idx][6] for idx in idxs ],
                  facecolor=mid, color=mid)
plot.fill_between(plot_Ds,
                  [ delta_quantiles[idx][3] for idx in idxs ],
                  [ delta_quantiles[idx][5] for idx in idxs ],
                  facecolor=mid_highlight, color=mid_highlight)
plot.plot(plot_Ds, [ delta_quantiles[idx][4] for idx in idxs ], color=dark)

plot.gca().set_xlim([plot_Ds[0], plot_Ds[-1]])
plot.gca().set_xlabel("Dimension")
plot.gca().set_ylim([0, 5])
plot.gca().set_ylabel("Distance Between Points")

plot.show()
```

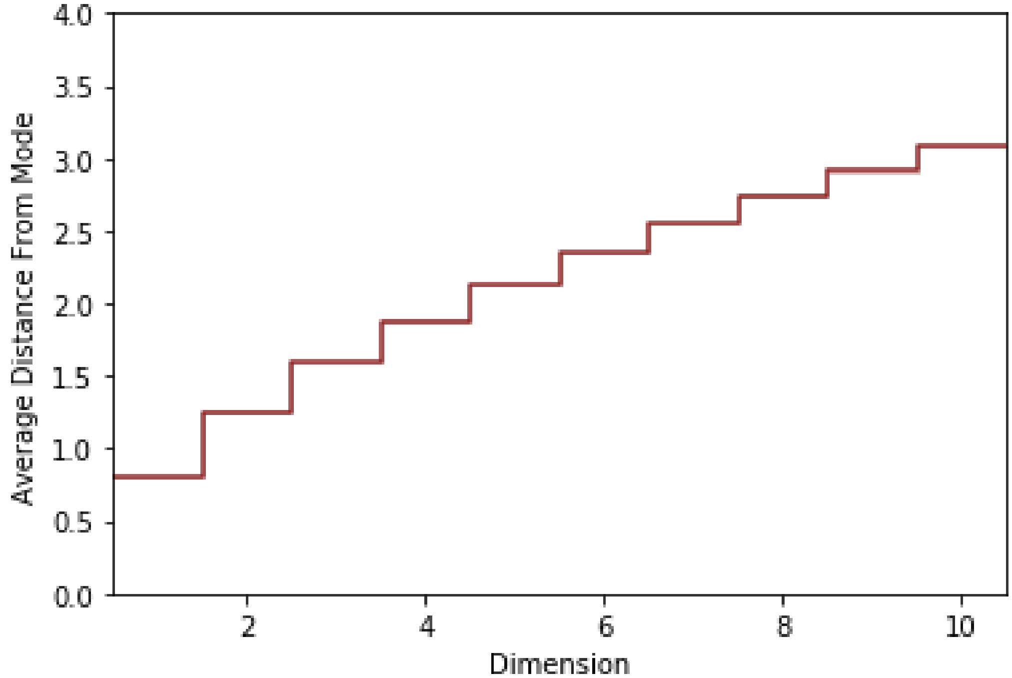

Average Distance Between Points
Dimension
0
1
2
3
4
5
2
4
6
8
10

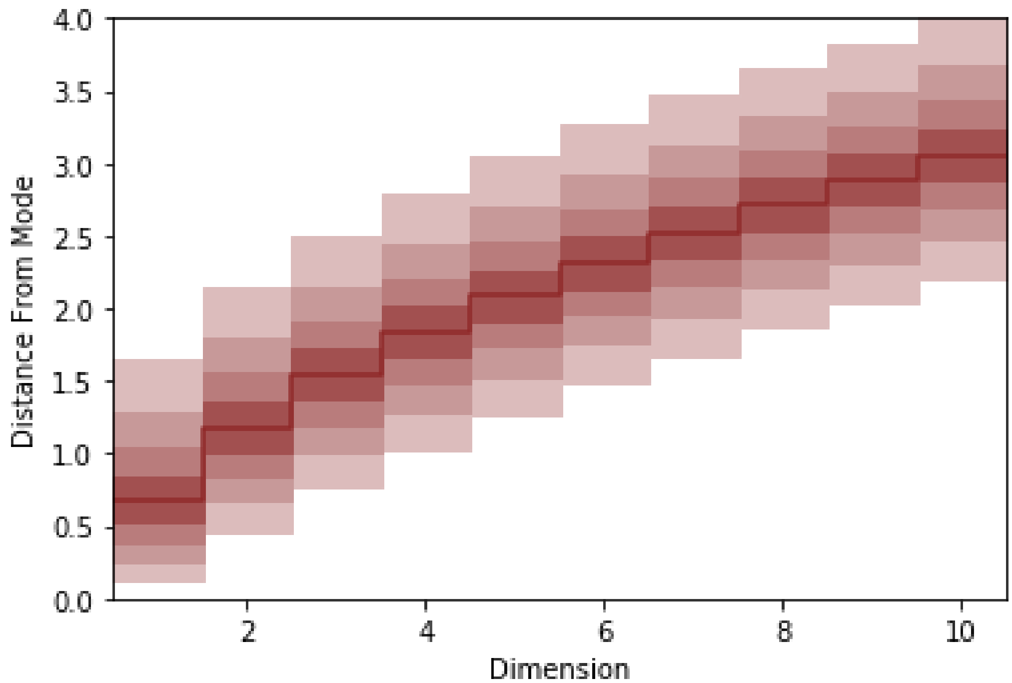

Distance Between Points
Dimension
0
1
2
3
4
5
2
4
6
8
10

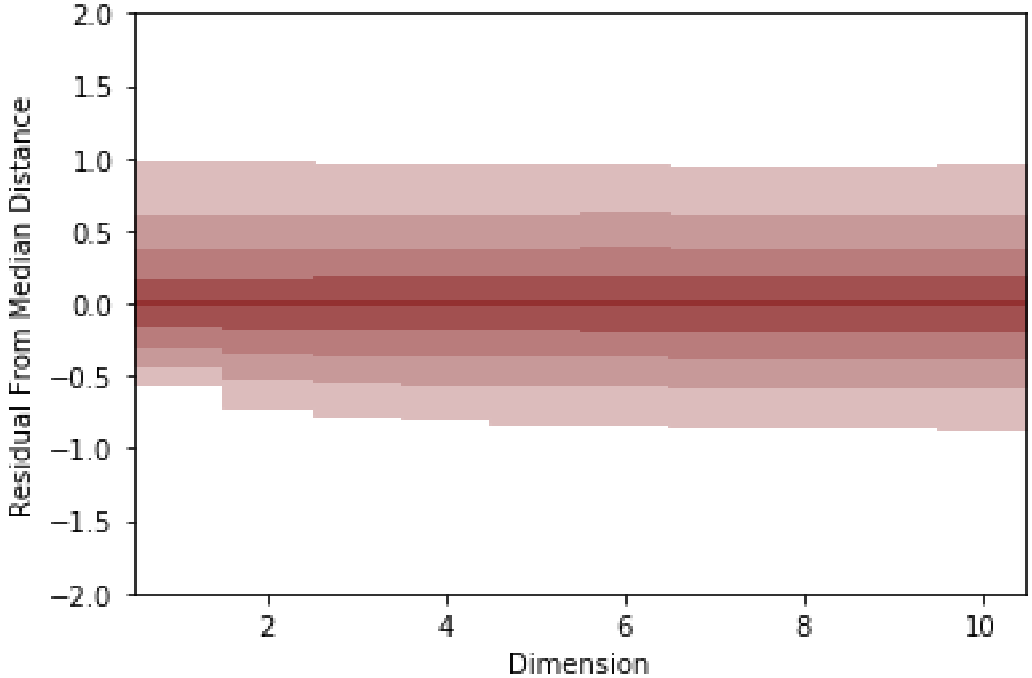
0.0.4 How does the distance from a Gaussian sample and the Gaussian mode behave as the dimensionality increases?

[8]:
```
r_means = [0] * len(Ds)
r_ses = [0] * len(Ds)
r_quantiles = [ [0] * 9 ] * len(Ds)

for D in Ds:
  # Distance from Gaussian samples to mode at zero
  r_samples = [0] * N

  for n in range(N):
    # Sample point
    x = stats.norm.rvs(0, 1, size=D)

    # Compute distance from point to mode at zero
    r_samples[n] = math.sqrt(sum([ x_d**2 for x_d in x]))

  # Estimate average distance
  s = compute_mc_stats(r_samples)
  r_means[D - 1] = s[0]
  r_ses[D - 1] = s[1]

  # Estimate distance quantiles
  r_quantiles[D - 1] = numpy.percentile(r_samples, quant_probs)

# Plot average distance from mode verses dimension
plot.fill_between(plot_Ds,
                  [ r_means[idx] - 2 * r_ses[idx] for idx in idxs ],
                  [ r_means[idx] + 2 * r_ses[idx] for idx in idxs ],
                  facecolor=light, color=light)
plot.plot(plot_Ds, [ r_means[idx] for idx in idxs], color=dark)

plot.gca().set_xlim([plot_Ds[0], plot_Ds[-1]])
plot.gca().set_xlabel("Dimension")
plot.gca().set_ylim([0, 4])
plot.gca().set_ylabel("Average Distance From Mode")

plot.show()

# Plot distance quantiles verses dimension
plot.fill_between(plot_Ds,
                  [ r_quantiles[idx][0] for idx in idxs ],
                  [ r_quantiles[idx][8] for idx in idxs ],
                  facecolor=light, color=light)
plot.fill_between(plot_Ds,
                  [ r_quantiles[idx][1] for idx in idxs ],
```

```
                 [ r_quantiles[idx][7] for idx in idxs ],
                 facecolor=light_highlight, color=light_highlight)
plot.fill_between(plot_Ds,
                 [ r_quantiles[idx][2] for idx in idxs ],
                 [ r_quantiles[idx][6] for idx in idxs ],
                 facecolor=mid, color=mid)
plot.fill_between(plot_Ds,
                 [ r_quantiles[idx][3] for idx in idxs ],
                 [ r_quantiles[idx][5] for idx in idxs ],
                 facecolor=mid_highlight, color=mid_highlight)
plot.plot(plot_Ds, [ r_quantiles[idx][4] for idx in idxs ], color=dark)

plot.gca().set_xlim([plot_Ds[0], plot_Ds[-1]])
plot.gca().set_xlabel("Dimension")
plot.gca().set_ylim([0, 4])
plot.gca().set_ylabel("Distance From Mode")

plot.show()

# Plot residual quantiles verses dimension
plot.fill_between(plot_Ds,
                 [ r_quantiles[idx][0] - r_quantiles[idx][4] for idx in idxs ],
                 [ r_quantiles[idx][8] - r_quantiles[idx][4] for idx in idxs ],
                 facecolor=light, color=light)
plot.fill_between(plot_Ds,
                 [ r_quantiles[idx][1] - r_quantiles[idx][4] for idx in idxs ],
                 [ r_quantiles[idx][7] - r_quantiles[idx][4] for idx in idxs ],
                 facecolor=light_highlight, color=light_highlight)
plot.fill_between(plot_Ds,
                 [ r_quantiles[idx][2] - r_quantiles[idx][4] for idx in idxs ],
                 [ r_quantiles[idx][6] - r_quantiles[idx][4] for idx in idxs ],
                 facecolor=mid, color=mid)
plot.fill_between(plot_Ds,
                 [ r_quantiles[idx][3] - r_quantiles[idx][4] for idx in idxs ],
                 [ r_quantiles[idx][5] - r_quantiles[idx][4] for idx in idxs ],
                 facecolor=mid_highlight, color=mid_highlight)
plot.plot(plot_Ds, [ r_quantiles[idx][4] - r_quantiles[idx][4] for idx in idxs␣
 ↪],
          color=dark)

plot.gca().set_xlim([plot_Ds[0], plot_Ds[-1]])
plot.gca().set_xlabel("Dimension")
plot.gca().set_ylim([-2, 2])
plot.gca().set_ylabel("Residual From Median Distance")

plot.show()
```


4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
Average Distance From Mode
2
4
6
8
10
Dimension


4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
Distance From Mode
2
4
6
8
10
Dimension



0.0.5 What is the probability of a Gaussian sample falling into a spherical neighborhood around the mode at zero?

[9]:
```
prob_means = [0] * len(Ds)
prob_ses = [0] * len(Ds)

R = 1

for D in Ds:
  # Does the sample fall into the spherical neighborhood?
  is_central_samples = [0] * N

  for n in range(N):
    # Sample a new point
    x = stats.norm.rvs(0, 1, size=D)

    # Compute radial distance from mode
    r = math.sqrt(sum([ x_d**2 for x_d in x]))

    # Check if sample is contained within spherical neighborhood
    if r < R:
      is_central_samples[n] = 1

  # Estimate probability of falling into spherical neighborhood
```

```
  s = compute_mc_stats(is_central_samples)
  prob_means[D - 1] = s[0]
  prob_ses[D - 1] = s[1]

# Plot inclusion probability verses dimension
plot.fill_between(plot_Ds,
                  [ prob_means[idx] - 2 * prob_ses[idx] for idx in idxs ],
                  [ prob_means[idx] + 2 * prob_ses[idx] for idx in idxs ],
                  facecolor=light, color=light)
plot.plot(plot_Ds, [ prob_means[idx] for idx in idxs], color=dark)

plot.gca().set_xlim([plot_Ds[0], plot_Ds[-1]])
plot.gca().set_xlabel("Dimension")
plot.gca().set_ylim([0, 0.7])
plot.gca().set_ylabel("Inclusion Probability")

plot.show()

# Done early?  Can you derive this probability analytically?
# Hint: convert to spherical coordinates and marginalize out
# the hyperspherical angles
```