Jump-point PLP (JPLP) simulation

Miao Cai miao.cai@slu.edu

2020-04-08

Contents

1 Power law process (PLP)

1.1 PLP intensity function

Power law process (PLP): When the intensity function of a NHPP is:

$$\lambda(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1} = \beta\theta^{-\beta}t^{\beta},$$

where $\beta > 0$ and $\theta > 0$, the process is called the power law process (PLP). The mean function $\Lambda(t)$ is the integral of the intensity function:

$$\Lambda(t) = \int_0^t \lambda(t)dt = \int_0^t \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1} = \left(\frac{t}{\theta}\right)^{\beta}$$

1.2 PLP simulation

```
# simulating PLP - time truncated case
sim_plp_tau = function(tau = 30,
                       beta = 1.5,
                       theta = 10){
  # initialization
  s = 0; t = 0
  while (max(t) <= tau) {
    u <- runif(1)
    s <- s - log(u)
    t_new <- theta*s^(1/beta)
    t <- c(t, t_new)
```

```r
  }
  t = t[c(-1, -length(t))]

  return(t)
}

# simulate multiple NHPPs - time truncated case
sim_mul_plp_tau = function(n_shift = 20,
                           shift_len_mean = 20, shift_len_sd = 5,
                           theta = 10, beta = 2, mean_n = 5){
  tau_vector = rnorm(n_shift, shift_len_mean, shift_len_sd)#difference1

  t_list = list()
  len_list = list()
  # end_time1 = list() # not needed for time truncated case


  for (i in 1:n_shift) {
    t_list[[i]] = sim_plp_tau(tau_vector[i], beta, theta)
    len_list[[i]] = length(t_list[[i]])
  }

  event_dat = data.frame(
    shift_id = rep(1:n_shift, unlist(len_list)),
    event_time = Reduce(c, t_list)
  )

  start_end_dat = data.frame(
    shift_id = 1:n_shift,
    start_time = rep(0, n_shift),
    end_time = tau_vector #difference2
  )

  return(list(event_dat = event_dat,
              start_end_dat = start_end_dat,
              shift_length = unlist(len_list)))
}


sim_hier_plp_tau = function(N, beta = 1.5, theta){
  t_list = list()
  len_list = list()
  tau_vector = rnorm(N, 10, 1.3)

  for (i in 1:N) {
    t_list[[i]] = sim_plp_tau(tau_vector[i], beta = beta, theta = theta[i])
    len_list[[i]] = length(t_list[[i]])
```

```
  }

  event_dat = data.frame(
    shift_id = rep(1:N, unlist(len_list)),
    event_time = Reduce(c, t_list)
  )

  start_end_dat = data.frame(
    shift_id = 1:N,
    start_time = rep(0, N),
    end_time = tau_vector #difference2
  )

  return(list(event_dat = event_dat,
              start_end_dat = start_end_dat,
              shift_length = unlist(len_list)))
}

plot_est = function(data, var = "beta", hline_var = 1.5){
  p = data %>%
    filter(term == var) %>%
    ggplot(aes(id, est_mean)) +
    geom_point() +
    geom_line(linetype = "dashed", color = "red")+
    geom_errorbar(aes(ymax = est_mean + 1.96*est_sd,
                      ymin = est_mean - 1.96*est_sd),
                  width = 1)+
    geom_segment(aes(x = 10, xend = 100,
                     y = hline_var, yend = hline_var),
                 color = "green")+
    scale_x_continuous(breaks = c(0, 10, 25, 50, 75, 100),
                       labels = c("0", "10", "25", "50", "75", "100")) +
    labs(x = "The number of drivers (random effects)",
         y = var) +
    theme_bw()
  return(p)
}
```

```
# plot events
plot_events = function(event_dat, start_end_dat, cross_size = 2){
  p = event_dat %>%
    ggplot(aes(x = event_time, y = shift_id)) +
    geom_point(alpha = 0.8, shape = 4, color = 'red', size = cross_size) +
    scale_y_continuous("shift ID",
                       labels = as.character(start_end_dat$shift_id),
                       breaks = start_end_dat$shift_id)+
    xlab('Time to event (minutes)') +
```

```
    geom_segment(data = start_end_dat,
                 aes(x = start_time, xend = end_time,
                     y = shift_id, yend = shift_id),
                 lineend = 'butt',
                 arrow = arrow(length = unit(0.2, "cm"))) +
    theme_classic()
  return(p)
}
```

2 Jump Power Law Process (JPLP)

2.1 JPLP intensity function

A Bayesian hierarchical JPLP has the following intensity function:

$$\lambda_{\text{JPLP}}(t|d,s,r,\beta,\gamma_{0,d},\gamma,\mathbf{X}_d,\mathbf{W}) = \begin{cases} \kappa^0\lambda(t|\beta,\gamma_{0,d},\gamma,\mathbf{X}_d,\mathbf{W}) & 0 \le t \le a_{d,s,1} \\ \kappa^1\lambda(t|\beta,\gamma_{0,d},\gamma,\mathbf{X}_d,\mathbf{W}) & a_{d,s,1} \le t \le a_{d,s,2} \\ \dots & \dots \\ \kappa^{R-1}\lambda(t|\beta,\gamma_{0,d},\gamma,\mathbf{X}_d,\mathbf{W}) & a_{d,s,R-1} \le t \le a_{d,s,R} \end{cases} \tag{1}$$
$$= \kappa^{r-1}\lambda(t|d,s,r,\kappa,\beta,\gamma_{0,d},\gamma,\mathbf{X}_d,\mathbf{W}) \quad a_{d,s,r-1} \le t \le a_{d,s,r},$$

where the introduced parameter κ is the percent of intensity function recovery once the driver takes a break. We assume that this κ is constant across drivers and shifts.

2.2 JPLP simulation

> Let $T_1, T_2, \dots$ be random variables representing the event times of a nonhomogeneous Poisson process with continuous expectation function $\Lambda(t)$, and let N_t represent the total number of events occurring before time t in the process. Then, conditional on the number of events $N_{t_0} = n$, the event times $T_1, T_2, \dots, T_n$ are distributed as order statistics from a sample with distribution function $F(t) = \Lambda(t)/\Lambda(t_0)$ for $t \in [0, t_0]$.

This is a generalization of the result for homogeneous Poisson processes. It naturally gives rise to the following algorithm for generating random variates from a nonhomogeneous Poisson process with expectation function $\Lambda(t)$ in a fixed interval $[0, t_0]$.

(1) Generate $n \sim \text{Poisson}(\Lambda(t_0))$.
(2) Independently generate n random variates $t'_1, t'_2, \dots, t'_n$ from the cdf $F(t) = \Lambda(t)/\Lambda(t_0)$.
(3) Order $t'_1, t'_2, \dots, t'_n$ to obtain $t_1 = t'_{(1)}, t_2 = t'_{(2)}, \dots, t_n = t'_{(n)}$.
(4) Deliver $t_1, t_2, \dots, t_n$.

2.2.1 Mean function $\Lambda(t)$

```
# Mean function Lambda for JPLP
L_plp = function(t, beta = 1.5, theta = 4) return((t/theta)^beta)
Lambda = function(t,
                  tau = 12,
                  kappa = 0.8,
```

```
                  t_trip = c(3.5, 6.2, 9),
                  beta = 1.5,
                  theta = 4)
{
  t_trip1 = c(0, t_trip)
  n_trip = length(t_trip1)
  comp = L_plp(t_trip, beta, theta)
  kappa_vec0 = rep(kappa, n_trip - 1)^(0:(n_trip - 2))
  kappa_vec1 = rep(kappa, n_trip - 1)^(1:(n_trip - 1))
  cum_comp0 = comp*kappa_vec0
  cum_comp1 = comp*kappa_vec1
  index_trip = max(cumsum(t > t_trip1)) - 1

  if(index_trip == 0){
    return((t/theta)^beta)
  }else{
    return(sum(cum_comp0[1:index_trip]) - sum(cum_comp1[1:index_trip]) +
             kappa^index_trip*(t/theta)^beta)
  }
}
```

2.2.2 Test the mean fucntion $\Lambda(t)$

```
# test Lambda
kappa = 0.8
t_trip = c(3.5, 6.2, 9)
beta = 1.5
theta = 4

Lambda(3.1)
```

```
## [1] 0.6822642
```

```
L_plp(3.1)
```

```
## [1] 0.6822642
```

```
Lambda(4.1)
```

```
## [1] 0.9938842
```

```
kappa^0*L_plp(t_trip[1]) +
  kappa^1*L_plp(4.1) - kappa^1*L_plp(t_trip[1])
```

```
## [1] 0.9938842
```

```
Lambda(8.9)
```

```
## [1] 2.596555
```

```
kappa^0*L_plp(t_trip[1]) +
  kappa^1*L_plp(t_trip[2]) - kappa^1*L_plp(t_trip[1]) +
```

```
  kappa^2*L_plp(8.9) - kappa^2*L_plp(t_trip[2])
```

```
## [1] 2.596555
```

```
Lambda(12)
```

```
## [1] 3.564885
```

```
kappa^0*L_plp(t_trip[1]) +
  kappa^1*L_plp(t_trip[2]) - kappa^1*L_plp(t_trip[1]) +
  kappa^2*L_plp(t_trip[3]) - kappa^2*L_plp(t_trip[2]) +
  kappa^3*L_plp(12) - kappa^3*L_plp(t_trip[3])
```

```
## [1] 3.564885
```

2.2.3 Plot the mean function of PLP and JPLP

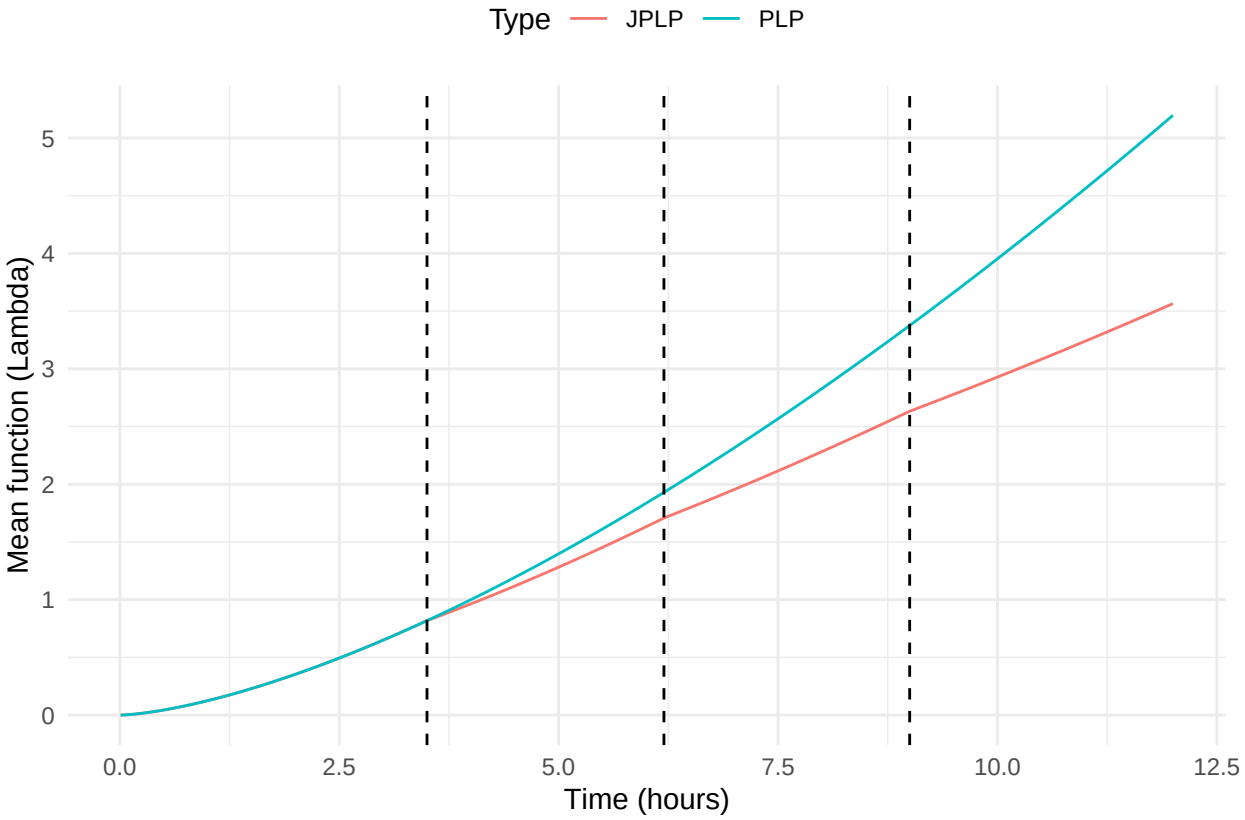
```
pacman::p_load(ggplot2, dplyr, tidyr)

t_trip = c(3.5, 6.2, 9)
beta = 1.5
theta = 4

x = seq(0.01, 12, 0.01)
y0 = (x/theta)^beta
y1 = rep(NA_real_, length(x))

for (i in 1:length(x)) {
  y1[i] = Lambda(t = x[i], kappa = 0.8)
}

data.frame(x, y0, y1) %>%
  tidyr::pivot_longer(cols = c('y0', 'y1'),
                      names_to = "Type",
                      values_to = 'y') %>%
  mutate(Type = case_when(Type == 'y0' ~ 'PLP',
                          Type == 'y1' ~ 'JPLP')) %>%
  ggplot(aes(x = x, y = y, group = Type, color = Type)) +
  geom_line() +
  geom_vline(xintercept = t_trip, linetype = "dashed") +
  labs(x = 'Time (hours)',
       y = 'Mean function (Lambda)') +
  theme_minimal() +
  theme(legend.position = "top")
```



2.2.4 Simulation JPLP events

```
inverse = function (f, lower = 0.0001, upper = 100) {
   function (y) uniroot((function (x) f(x) - y), lower = lower, upper = upper)[1]
}

inv_Lambda = inverse(Lambda, 0.0001, 100)
inv_Lambda(3.564885)$root
```

```
## [1] 12
```

```
# simulating PLP - time truncated case
sim_jplp = function(tau = 12,
                    kappa = 0.8,
                    t_trip = c(3.5, 6.2, 9),
                    beta = 1.5,
                    theta = 4)
{ s = 0; t = 0
  while (max(t) <= tau) {
    u <- runif(1)
    s <- s - log(u)
    t_new <- inv_Lambda(s)
    t_new <- t_new$root
    t <- c(t, t_new)
```

```
  }
  t = t[c(-1, -length(t))]

  return(t)
}

sim_jplp()
```

```
## [1] 2.010403 2.738316 5.817421 7.509946
```