Assignment 11 Problem One

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1. Find the radius of convergence and interval of convergence of the following power series.
(a) $\Sigma_{n=1}^{\infty} \sqrt[n]{n}(2x+5)^n$

Use the Ratio Test:

$\left|\frac{(n+1)^{\frac{1}{n+1}}(2x+5)^{n+1}}{n^{\frac{1}{n}}(2x+5)^n}\right| = \frac{(n+1)^{\frac{1}{n+1}}}{n^{\frac{1}{n}}} \times \left|\frac{(2x+5)^{n+1}}{(2x+5)^n}\right|$

To discover what the first component converges to lets assume:

$\lim_{n\to\infty} \frac{(n+1)^{\frac{1}{n+1}}}{n^{\frac{1}{n}}} = L$

$(n+1)^{\frac{1}{n+1}} = L(n^{\frac{1}{n}})$

$((n+1)^{\frac{1}{n+1}})^{n+1} = (L(n^{\frac{1}{n}}))^{n+1}$

$n+1 = L^{n+1}(n^{\frac{1}{n}})^{n+1} = L^{n+1}n$

$\frac{n+1}{n} = L^{n+1}$

$\lim_{n\to\infty} \frac{n}{n} + \frac{1}{n} = L^{n+1}$

$\lim_{n\to\infty} 1 = L^{n+1} \implies L = 1$

Thus the Ratio Test produces $|2x+5| < 1 \implies |x + \frac{5}{2}| < \frac{1}{2}$

$-\frac{1}{2} < x + \frac{5}{2} < \frac{1}{2}$

$-3 < x < -2$

Thus the radius of convergence is $\frac{1}{2}$.

We must check the endpoints -3 and -2 to see if the series is convergent on the ends.

When $x = -3$

The series equals $\Sigma_{n=1}^{\infty} \sqrt[n]{n}(-1)^n = \Sigma(n)^{\frac{1}{n}}(-1)^n$

Note that $n^{\frac{1}{n}}$ can be re-written as $e^{\frac{log(n)}{n}} = (e^{log(n)})^{\frac{1}{n}} = n^{\frac{1}{n}}$

Thus we can apply L'Hospital's to the exponential.

$\lim_{n\to\infty} \frac{log(n)}{n}$

$L' = \frac{\frac{1}{n}}{1} = \frac{1}{n} = 0$

Thus as $\lim_{n\to\infty} n^{\frac{1}{n}} = e^{\frac{log(n)}{n}} = e^0 = 1$

Thus the original series just becomes the Grandeis Series at the limit, which we know to be divergent (oscillating between -1 and 1).

When $x = -2$

The series equals $\Sigma_{n=1}^{\infty} \sqrt[n]{n}$, which as proven earlier is also divergent since the limit of $\sqrt[n]{n}$ approaches 1.

Thus the interval of convergence is $(-3, -2)$.

(b) $\Sigma_{n=1}^{\infty} \frac{(n!)^2}{2^n(2n)!}x^n$

Use the Ratio Test:

$\left|\frac{\frac{((n+1)!)^2}{2^{n+1}(2(n+1))!}x^{n+1}}{\frac{(n!)^2}{2^n(2n)!}x^n}\right| = |\frac{x^{n+1}}{x^n}| \times \frac{((n+1)!)^2}{(n!)^2} \times \frac{2^n}{2^{n+1}} \times \frac{(2n)!}{(2n+2)!}$

$= x \times \frac{((n+1)n!)^2}{(n!)^2} \times \frac{1}{2} \times \frac{(2n)!}{(2n+2)(2n+1)(2n)!}$

$= x \times (n+1)^2 \times \frac{1}{2} \times \frac{1}{(2n+2)(2n+1)}$

$= \frac{1}{2}x \times \frac{n^2+2n+1}{4n^2+6n+2}$

We know that the polynomial on the right converges to $\frac{1}{4}$ as we take the limit (split the polynomial into each numerator term over the denominator, divide by the greatest multiple of n in the numerator and all the terms should approach zero with the exception of the $\frac{n^2}{4n^2...}$ term which equals $\frac{1}{4}$).

Thus $|\frac{1}{8}x| < 1 \implies |x| < 8$

Thus the radius of convergence is equal to 8.

Now we must check the endpoints of -8 and 8.

When $x = -8$

The series equals $\Sigma_{n=1}^{\infty} \frac{(n!)^2}{2^n(2n)!}(-8)^n = \Sigma(-1)^n \times \frac{8^n}{2^n} \times \frac{(n!)^2}{(2n)!}$

To check what $\frac{(n!)^2}{(2n)!}$ converges to we employ the Ratio Test again:

$\left|\frac{\frac{((n+1)!)^2}{(2n+2)!}}{\frac{(n!)^2}{(2n)!}}\right| = \frac{((n+1)!)^2}{(n!)^2} \times \frac{(2n)!}{(2n+2)!} = \frac{((n+1)n!)^2}{(n!)^2} \times \frac{(2n)!}{(2n+2)(2n+1)2n!}$

$= (n+1)^2 \times \frac{1}{(2n+2)(2n+1)}$

$= \frac{1}{2} \times \frac{1}{2n+1} \times (n+1)$, which converges to $\frac{1}{4}$ as you take the limit.

Thus the expression we were evaluating before as you take the limit is:

$\Sigma(-1)^n \times 4^n \times \frac{1}{4}$, which is divergent.

When $x = 8$

The series equals $\Sigma_{n=1}^{\infty} \frac{(n!)^2}{2^n(2n)!}(8^n)$, which is also divergent (because if we've proven that the series is not conditionally convergent then it cannot be absolutely convergent).

Thus the interval of convergence is $(-8, 8)$.