Assignment 3 Problem 2

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A rumor (that Calculus is a fun class) is spread in a school. The time t at which a fraction p of the school population has heard the rumor is given by: $t(p) = \int_a^p \frac{b}{x(1-x)} dx$, where a and b are constants, with $0 < a < 1$ and $b > 0$.

(a) Evaluate the integral to find an explicit formula for $t(p)$. Write your answer so that it only has one ln term.

Because the integral is in the form of a rational function, we use Partial Fraction Decomposition.

$\frac{b}{x(1-x)} = \frac{A}{x} + \frac{B}{1-x}$

Cross-multiplying and comparing terms to the numerator on the LHS we get:

$b = A(1-x) + B(x)$

When $x = 1$, $B = b$

When $x = 0$, $A = b$

Therefore, $\frac{b}{x(1-x)} = \frac{b}{x} + \frac{b}{1-x}$

And thus, $\int_a^p \frac{b}{x(1-x)} = \int_a^p \frac{b}{x} + \int_a^p \frac{b}{1-x}$

$= b\ln x|_a^p - b\ln(1-x)|_a^p$

$= b\ln p - b\ln(1-p) + b\ln(1-a) - b\ln a$

$= b\ln(\frac{p}{1-p}) + b\ln(\frac{1-a}{a})$

$= b\ln(\frac{p(1-a)}{a(1-p)})$

(b) It turns out that at time $t = 0$, one percent of the school population ($p = 0.01$) has heard the rumor. Solve for the constant a.

$t(p) = t(0.01) = 0 = b\ln(\frac{p(1-a)}{a(1-p)})$

$\ln(\frac{0.01 - 0.01a}{a - 0.01a}) = 0$

$\ln(0.01 - 0.01a) - \ln(a - 0.01a) = 0$

$\ln(0.01 - 0.01a) = \ln(a - 0.01a)$

$e^{\ln(0.01 - 0.01a)} = e^{\ln(a - 0.01a)}$

$0.01 - 0.01a = a - 0.01a$

$a = 0.01$

This makes sense because for an integral to equal 0 the lower and upper bounds of the

integral must equal the same number. Thus a must equal p for t to equal 0.

(c) In addition to the information in part (b) above, it turns out that at time $t = 1$, half the school population has heard the rumor. Solve for the constant b.

$1 = bln(\frac{0.5(1-0.01)}{0.01(1-0.5)})$

$\frac{1}{b} = ln(\frac{.495}{.005})$

$\frac{1}{b} = ln(99)$

$b = \frac{1}{ln99}$

(d) At what time has 90 percent of the school population heard the rumor?

$t = \frac{1}{ln99}(ln(\frac{.9(1-.01)}{.01(1-.9)}))$

$t = \frac{1}{ln99}(ln(\frac{.891}{.001}))$

$t = \frac{ln891}{ln99}$

$t \approx 1.4782$

This means that at time $t = 1.4782$, 90 percent of the school population has heard the rumor.