Assignment 12 Problem Four

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4.

(a) $f(x) = \frac{x}{1-x-x^2}$

Using the quadratic formula we get:

$x_{1,2} = \frac{1}{2}(1 \pm \sqrt{5})$

Let $a = x_1$ (the positive term)

Then the denominator factors into:

$f(x) = \frac{x}{(x+a)(x-a)}$

First find $\frac{1}{x+a} = \frac{1}{a}(\frac{1}{1-(-x)}) = \frac{1}{a}\Sigma_{n=0}^{\infty}(-x)^n$

Then find $\frac{1}{x-a} = \frac{-1}{a}(\frac{1}{1-x}) = \frac{-1}{a}\Sigma_{n=0}^{\infty}x^n$

$\frac{x}{(x+a)(x-a)} = x(\frac{1}{a}\Sigma_{n=0}^{\infty}(-x)^n)(\frac{-1}{a}\Sigma_{n=0}^{\infty}x^n) = -\frac{1}{a^2}\Sigma_{n=0}^{\infty}(-1)^n x^{2n+1}$, where $a = \frac{1}{2}(1+\sqrt{5})$

(b) $f(x) = \frac{x}{1-x-x^2} = \Sigma_{n=0}^{\infty}c_n x^n$

$x = \Sigma_{n=0}^{\infty}c_n x^n(1-x-x^2)$

$= \Sigma_{n=0}c_n x^n - \Sigma_{n=0}c_n x^{n+1} - \Sigma_{n=0}c_n x^{n+2}$

Then we re-index so that the x_n terms align

$= \Sigma_{n=0}c_n x^n - \Sigma_{n=1}c_{n-1}x^n - \Sigma_{n=2}c_{n-2}x^n$

The $n = 0$ partial sum is C_0

The $n = 1$ partial sum is $C_0 + C_1 x - C_0 x$

When $n \geq 2$ the partial sum is $C_0 + C_1 x - C_0 x + \Sigma_{n=2}^{\infty}(C_n - C_{n-1} - C_{n-2})x^n$

*Don't know where to go from here

(c) $f(x) = \frac{x}{1-x-x^2}$

Because we have already found the roots let β_1 equal the positive root $\frac{1}{2}(1+\sqrt{5})$ and β_2 equal the negative root $\frac{1}{2}(1-\sqrt{5})$

$\frac{x}{1-x-x^2} = \frac{\alpha_1}{1-\beta_1 x} + \frac{\alpha_2}{1-\beta_2 x}$, where α_1 and α_2 are unknowns.

Using partial fraction decomposition we get:

$x = (1-\beta_2 x)\alpha_1 + (1-\beta_1 x)\alpha_2$

When $x = \frac{1}{\beta_1}$

$\frac{1}{\beta_1} = (1 - \frac{\beta_2}{\beta_1})\alpha_1$

$\alpha_1 = \frac{1}{\beta_1 - \beta_2} = \frac{1}{\sqrt{5}}$

When $x = \frac{1}{\beta_2}$

$\frac{1}{\beta_2} = (1 - \frac{\beta_1}{\beta_2})\alpha_2$

$\alpha_2 = \frac{1}{\beta_2 - \beta_2} = -\frac{1}{\sqrt{5}}$

$\frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}}(\frac{1}{1-\beta_1 x} - \frac{1}{1-\beta_2 x})$

Note that the two terms inside the parentheses are in the form of an infinite geometric series.

$\frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}}((1 + \beta_1 x + \beta_1^2 x^2 + ...) - (1 + \beta_2 x + \beta_2^2 x^2 ...))$

$= \frac{1}{\sqrt{5}} = ((\beta_1 - \beta_2)x + (\beta_1^2 - \beta_2^2)x^2 + ...))$

Thus the closed form expression for $\{c_n\}$ is $c_n = \frac{1}{\sqrt{5}}(\beta_1^n - \beta_2^n) = \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n)$