Assignment 3 Problem 4

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Consider the following integral:
$\int_0^{\pi/2} \frac{sin^n(x)}{sin^n(x)+cos^n(x)} dx$ where n is any positive integer.

(a) *Use Wolfram Alpha or another computer algebra system to evaluate the above integral (for a general n). Write down the result.*
Standard computational time exceeded.

(b) *Using a computer algebra system, evaluate the above integral for* $n = 1, 4, 7$.
For $n = 1$:
$\int_0^{\pi/2} \frac{sin(x)}{sin(x)+cos(x)} dx = \frac{\pi}{4}$

For $n = 4$:
$\int_0^{\pi/2} \frac{sin^4(x)}{sin^4(x)+cos^4(x)} dx = \frac{\pi}{4}$

For $n = 7$:
$\int_0^{\pi/2} \frac{sin^7(x)}{sin^7(x)+cos^7(x)} dx = \frac{\pi}{4}$

(c) *Evaluate the integral by hand as follows.*

(i) Rewrite the integral using the substitution $u = \frac{\pi}{2} - x$.
If $u = \frac{\pi}{2} - x$ then $x = \frac{\pi}{2} - u$.
The limits then become $\frac{\pi}{2}$ and 0 respectively, but with $du = -dx$ as well, the two negative cancel out and we get:
$= \int_0^{\frac{\pi}{2}} \frac{sin^n(\frac{\pi}{2}-u)}{sin^n(\frac{\pi}{2}-u)+cos^n(\frac{\pi}{2}-u)} du$
Using the trigonometric identity for angle addition and subtraction:
$sin(x-y) = sinxcosy - cosxsiny$ and $cos(x-y) = cosxcosy + sinxsiny$ we get:
$\int_0^{\frac{\pi}{2}} \frac{(sin\frac{\pi}{2}cosu-cos\frac{\pi}{2}sinu)^2}{(sin\frac{\pi}{2}cosu-cos\frac{\pi}{2}sinu)^n+(cos\frac{\pi}{2}cosu+sin\frac{\pi}{2}sinu)^n} du$, which simplifies to:
$\int_0^{\frac{\pi}{2}} \frac{cos^n u}{cos^n u+sin^n u} du$

(ii/iii) Add the integral you obtain above to the original integral.
Because the u can be any arbitrary variable, I am going to re-label u as x.
Also, I am going to label the original integral as I.
Therefore:
$2I = \int_0^{\frac{\pi}{2}} \frac{cos^n(x)}{cos^n(x)+sin^n(x)}dx + \int_0^{\frac{\pi}{2}} \frac{sin^n(x)}{sin^n(x)+cos^n(x)}dx$
$2I = \int_0^{\frac{\pi}{2}} \frac{sin^n(x)+cos^n(x)}{sin^n(x)+cos^n(x)dx}$
$2I = \int_0^{\frac{\pi}{2}} 1dx$
$2I = \frac{\pi}{2}$
$I = \frac{\pi}{4}$
The original integral will always equal $\frac{\pi}{4}$ no matter what n is input.