Assignment 3 Problem 3

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(a) $\int \frac{tan^{-1}x}{4x^2}$
$= \frac{1}{4} \int x^{-2} tan^{-1}x$
Using rule 95 from the Table of Integrals, which states:
$\int u^n tan^{-1}udu = \frac{1}{n+1}[u^{n+1}tan^{-1}u - \int \frac{u^{n+1}du}{1+u^2}]$
Therefore $\frac{1}{4} \int x^{-2}tan^{-1}x = \frac{1}{4} - 1[x^{-1}tan^{-1}x - \int \frac{1}{x(1+x^2)}$

Because $\int \frac{1}{x(1+x^2)}$ contains a rational function, I choose to use Partial Fraction Decomposition to evaluate the integral.
$\int \frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$
If we cross-multiply and consider the equality of the numerator of the integrand to the cross multiplied partial fractions we get:
$1 = A(x^2 + 1) + (Bx + C)(x)$
Let us consider $x = 0$.
If $x = 0$ then $A = 1$.
Now if we further expand the RHS to solve for the coefficients we get:
$1 = Ax^2 + A + Bx^2 + Cx$
Collecting terms we get:
$1 = (A + B)x^2 + Cx + A$
Therefore, $C = 0$ because the linear term, x, on the LHS has the coefficient of 0.
And, $B = -1$ since $A + B$ must equal 0 for the same reason.
Thus the completed PFD is $\int \frac{1}{x(1+x^2)} = \int \frac{1}{x} - \frac{x}{1+x^2}$
If we separate the integrals and use a simple u-substitution on the second fraction, setting $u = 1 + x^2$ and thus $du = 2xdx \rightarrow \frac{1}{2}du = xdx$ then:
$\int \frac{1}{x(1+x^2)} = ln|x| + \frac{1}{2}ln|1 + x^2| + C$
And therefore plugging that back into the original equation to evaluate the entire integral we get:
$\frac{1}{4}(ln|x| - \frac{1}{2}ln|1 + x^2| - \frac{tan^{-1}x}{x}) + C$

(b) $\int \frac{x}{x^4+2x^2+5}$
First we have to complete the square in the denominator.
$x^4 + 2x^2 + 5 = (x^2 + 1) + 4$
Thus the integral equals $\int \frac{x}{(x^2+1)^2+2^2}$
This fits the form of rule 25 in the Table of Integrals if we first use u-substitution to set $u = x^2 + 1$ thus making $du = 2xdx \rightarrow \frac{1}{2}du = xdx$.
Therefore the integral equals $\frac{1}{2}\int \frac{du}{u^2+2^2} = \frac{1}{2}ln(u + \sqrt{a^2 + u^2}) + C$
$= \frac{1}{2}ln(x^2 + 1 + \sqrt{4 + (x^2 + 1)^2}) + C$

(c) $\int \frac{(x^2-1)^{3/2}}{x}dx$
First to make the substitution more apparent, we rewrite the integral as $\int \frac{\sqrt{x^2-1}^3}{x}dx$.
Because there is a square root, we know that we must make a trigonometric substitution.
The best substitution for the form $\sqrt{u^2 - a^2}$ is $x = sec\theta$, which makes $dx = sec\theta tan\theta d\theta$.
Therefore the integral simplifies to $\int \frac{tan^3\theta}{sec\theta} sec\theta tan\theta d\theta$, which further simplifies to $\int tan^4\theta d\theta$.
Using rule number 75 on the Table of Integrals, we get:
$= \frac{1}{3}tan^3\theta - \int tan^2\theta d\theta$
And then again using rule 65 on the Table of Integrals, we get:
$= \frac{1}{3}tan^3\theta - [tan\theta - \theta + C]$
Substitution x back into the equation we get:
$= \frac{1}{3}tan^3 sec^{-1}x - [tansec^{-1}x - sec^{-1}x + C]$
We use the formula $tan(sec^{-1}x) = \sqrt{1 - \frac{1}{x^2}}x$ and thus the final form is:
$= \frac{1}{3}(x^2 - 1)^{3/2} - \sqrt{x^2 - 1} + sec^{-1}x + C$