# Assignment 10 Problem 4

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**4. Determine whether the following series converge or diverge. You may use any test that you find appropriate (ratio test, comparison test, alternating series test, limit comparison test, root test, integral test.)** (a) $\Sigma_{n=1}^{\infty} \frac{(2n)!}{n!(n+1)!}$

Using the Ratio Test:

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{\frac{(2(n+1))!}{(n+1)!(n+2)!}}{\frac{(2n)!}{n!(n+1)!}} = \frac{n!(n+1)!}{(n+1)!(n+2)!} \times \frac{(2n+2)!}{(2n)!}$

$= \frac{n!}{(n+2)!} \times \frac{(2n+2)!}{(2n)!}$

By the properties that factorials have:

$(n+2)! = n!(n+1)(n+2)$

$(2n+2)! = (2n)!(2n+1)(2n+2)$

Thus the original equation factors to:

$= \frac{n!}{n!(n+1)(n+2)} \times \frac{(2n)!(2n+1)(2n+2)}{(2n)!}$

$= \frac{1}{(n+1)(n+2)} \times \frac{(2n+1)(2n+2)}{1}$

$= \frac{(2n+1)(2n+2)}{(n+1)(n+2)}$

$= \frac{4n^2+6n+2}{n^2+3n+3}$

Because we have taken the limit, the fraction is in indeterminate form. Thus we apply L'Hospital's Rule (denoted $L^x()$)

$L'(\frac{4n^2+6n+2}{n^2+3n+3}) = \frac{8n+6}{2n+3}$

$L'' = \frac{8}{2} = 4 > 1$

Because the $r$ is greater than 1 the series diverges.

(b) $\Sigma_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+2n+1}$

This series is similar in form to $\frac{1}{n^2}$ so I will prove that it converges absolutely by using the Limit Comparison Test.

Let $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n^2+2n+1}$

$\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n^2+2n+1}} = \frac{n^2+2n+1}{n^2} = 1 + \frac{2n}{n^2} + \frac{1}{n^2} = 1$

Thus because the limit is between 0 and infinity, $a_n$ and $b_n$ must both converge or diverge.

Because we know that $a_n$ converges by the p-test for series, then $b_n$ must converge.

Therefore the original series converges absolutely.

(c) $\Sigma_{n=1}^{\infty}(1+\frac{1}{n})^{n^2}$

For a series $\Sigma a_n$ to converge the $\lim_{n\to\infty} a_n$ must equal 0.

If we observe the exponentiated portion of the series $1+\frac{1}{n}$ will always be strictly greater than 1 for $n \geq 1$.

Thus the quantity being exponentiated, let's call it $\gamma$, will always be strictly greater than 1 for $n \geq 1$ and will equal 1 at the limit.

Because $n \geq 1$ the exponent term $n^2$ will be a positive, increasing quantity.

$\lim_{n\to\infty} \gamma \to 1$

$\lim_{n\to\infty} \gamma^{n^2} \to 1$

Because the sequence $\lim a_{n\to\infty} \neq 0 \implies \Sigma a_n$ does not converge.

(d) $\Sigma_{n=1}^{\infty}(\frac{1}{2n-1} - \frac{1}{2n})$

$= \Sigma_{n=1}^{\infty}(\frac{2n}{2n(2n-1)} - \frac{2n-1}{(2n-1)(2n)})$

$= \Sigma_{n=1}^{\infty} \frac{-1}{4n^2-2n}$

For ease of convergence proofing purposes I multiply by the constant $-1$.

$= \Sigma_{n=1}^{\infty} \frac{1}{4n^2-2n}$

I will use the Limit Comparison Test to prove convergence:

Let $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{4n^2-2n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\frac{1}{n^2}}{\frac{1}{4n^2-2n}} = \frac{4n^2-2n}{n^2}$

Using L'Hospitals:

$L'(\frac{4n^2-2n}{n^2}) = \frac{8n-2}{2n}$

$L'' = \frac{8}{2} = 4$

Because this is a constant between 0 and infinity, then both $a_n$ and $b_n$ must converge or diverge.

Because $a_n$ converges by the p-test for series, $b_n$ must also converge, which means the original series converges.