Assignment 10 Problem 2

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2(a) Suppose that Σa_n and Σb_n are series with positive terms and Σb_n is convergent. Show that if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ then Σa_n is also convergent.
For Σb_n to converge then by definition $\lim_{n\to\infty} b_n = 0$.

However, if this is true then the only way $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ is if $\frac{a_n}{b_n}$ (without first applying L'Hospitals Rule) is in an indeterminate form since you cannot divide by 0 outright.

Thus as $n \to \infty$, a_n must approach 0 an order of magnitude more quickly than b_n, such that upon the use of L'Hospitals Rule the numerator because a constant whilst the denominator remains a function of n.

For example:

If $a_n = \frac{1}{n^3}$ and $b_n = \frac{1}{n^2}$ (and let $L^x()$ represent the use of L'Hospitals, where x is the x-th derivative)

Then $\frac{a_n}{b_n} = \frac{n^2}{n^3}$

$L'(\frac{a_n}{b_n}) = \frac{2n}{3n^2}$

$L''(\frac{a_n}{b_n}) = \frac{2}{6n}$

Thus evaluating this new fraction at the limit as $n \to \infty$, we get zero.

Thus if a_n approaches 0 an order of magnitude more quickly than b_n and b_n is convergent then a_n must also be convergent.

2(b) Use part (a) to show that the following series converges: $\Sigma_{n=1}^{\infty} \frac{ln(n)}{e^n\sqrt{n}}$

To simplify this problem let's use the Comparison Test.

Because $ln(n) < n$ for all $n \geq 1$ then we can say that:

$\frac{ln(n)}{e^n\sqrt{n}} < \frac{n}{e^n\sqrt{n}} < \frac{n}{e^n}$

So let's test whether or not $\Sigma_{n=1}^{\infty} \frac{n}{e^n}$ is convergent.

Suppose we let $a_n = \frac{n}{e^n}$ and $b_n = \frac{n^2}{e^n}$

First we must prove that b_n is convergent.

Let's use the Ratio Test.

$\left|\frac{b_{n+1}}{b_n}\right| = \frac{\frac{(n+1)^2}{e^{n+1}}}{\frac{n^2}{e^n}} = \frac{e^n}{e^{n+1}} \times \frac{(n+1)^2}{n^2}$

$= \frac{e^n}{e(e^n)} \times (\frac{n^2}{n^2} + \frac{2n}{n^2} + \frac{1}{n^2})$

$\lim_{n\to\infty} \frac{1}{e} \times (1 + 0 + 0) = \frac{1}{e} < 1$

Thus by the Ratio Test, Σb_n is convergent.

Thus because we have proven that Σb_n is convergent, we can now use part (a) to justify that Σa_n is convergent.

We now show that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\frac{n}{e^n}}{\frac{n^2}{e^n}} = \frac{e^n}{e^n} \times \frac{n}{n^2} = 1 \times \frac{1}{n} = 0$

Thus because $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, Σa_n is also convergent.

And finally because a_n is convergent, by the Comparison Test, $\Sigma_{n=1}^{\infty} \frac{ln(n)}{e^n\sqrt{n}}$ must also converge.