Assignment 9 Problem 4

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April 1, 2016

4. The Koch Snowflake

(a/c) Find the perimeter of the nth snowflake (call it L_n)

Perimeter of:

$C_1 = 3(1) = 3$

$C_2 = 12(\frac{1}{3}) = 4$

$C_3 = 48(\frac{1}{9}) = \frac{16}{3}$

$C_4 = 192(\frac{1}{27}) = \frac{64}{9}$

$L_n = 3(\frac{4}{3})^n$

And to show that the $\lim_{n\to\infty} L_n = \infty$

We simply need to cite the definition of a converging/diverging geometric series.

A geometric series is diverging if the multiplier, r, from Σar^k is $|r| \geq 1$, which in this case it is since $\frac{4}{3} > 1$.

Thus the series must be diverging.

(b/d) Find the area of the nth snowflake (call it A_n)

Recall that the formula for the area of an equilateral triangle is: $A = \frac{s^2\sqrt{3}}{4}$

Let us notice that area is accrued in the form of partial sums, in that each subsequent area is the sum of the area of the previous iteration of the fractal with the areas of the new triangles.

$A_n = \frac{s^2\sqrt{3}}{4} + 3\frac{(\frac{s}{3})^2\sqrt{3}}{4} + 12\frac{(\frac{s}{9})^2\sqrt{3}}{4} + 48\frac{(\frac{s}{27})^2\sqrt{3}}{4} + ...$

To observe the form of the series, let us do some factoring and re-configuration:

$A_n = \frac{s^2\sqrt{3}}{4}\left(1 + 3(\frac{1}{3})^2 + 12(\frac{1}{9})^2 + 48(\frac{1}{27})^2 + ...\right)$

So a few things to note:

Starting from the third term, the outer coefficient increases by multiples of 4, while starting from the second the denominator of the inner fraction increases by multiples of 3.

Thus to make the iteration more deliberate, I re-write the series as:

$A_n = \frac{s^2\sqrt{3}}{4}\left(1 + 3 * 4^0(\frac{1}{9^1}) + 3 * 4^1(\frac{1}{9^2}) + 3 * 4^2(\frac{1}{9^3}) + ...\right)$

Because we see that the powers of 4 and 9 do not match up, I will now manipulate the series by multiplying and dividing by 4 to "align" the indices of the series.

$A_n = \frac{1}{4}(\frac{s^2\sqrt{3}}{4})\left(4 + 3(\frac{4}{9})^1 + 3(\frac{4}{9})^2 + 3(\frac{4}{9})^3 + ...\right)$

The general formula for A_n thus becomes:

$A_n = \frac{s^2\sqrt{3}}{16}\left(4 + 3\Sigma_{i=1}^{n-1}(\frac{4}{9})^i\right)$ for all $n > 1$, and where $A_1 = \frac{s^2\sqrt{3}}{4}$

In this case since we assume $s = 1$ we get:

$A_n = \frac{\sqrt{3}}{16}\left(4 + 3\Sigma_{i=1}^{n-1}(\frac{4}{9})^i\right)$ for all $n > 1$, and where $A_1 = \frac{\sqrt{3}}{4}$

To show that the $\lim_{n\to\infty} A_n = (8/5)A_1$ we simply need to further simplify the series notation we had earlier:

$A_n = \frac{1}{4}(\frac{s^2\sqrt{3}}{4})\left(4 + 3(\frac{4}{9})^1 + 3(\frac{4}{9})^2 + 3(\frac{4}{9})^3 + ...\right)$

Let us just observe this portion of the series, which happens to be geometric:

$3(\frac{4}{9}) + 3(\frac{4}{9})^2 + 3(\frac{4}{9})^3 + ... = \frac{12}{9}(1 + \frac{4}{9} + (\frac{4}{9})^2 + ...)$

The geometric series sum is:
$\frac{\frac{12}{9}}{1-\frac{4}{9}} = \frac{12}{5}$
Thus $\lim_{n\to\infty} A_n = \frac{s^2\sqrt{3}}{16}(4 + \frac{12}{5}) = \frac{s^2\sqrt{3}}{16}(\frac{32}{5})$
Which when simplified, knowing $A_1 = \frac{s^2\sqrt{3}}{4}$ equals:
$\lim_{n\to\infty} A_n = \frac{8}{5}A_1$