Assignment 5 Problem Two

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2. In this problem, we explore whether for any continuous functions $f(x)$,

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{b\to\infty} \int_{-b}^{b} f(x)dx$$

(a) Find $\int_0^\infty \frac{2x}{x^2+1}dx$

$\int_0^\infty \frac{2x}{x^2+1}dx = \lim_{b\to\infty} \int_0^b \frac{2x}{x^2+1}dx$

Now we use u-substitution and let $u = x^2+1$ and thus $du = 2xdx$.

This changes the upper limit to be equal to b^2+1 and the lower limit equals 1.

Thus $\lim_{b\to\infty} \int_0^b \frac{2x}{x^2+1}dx = \lim_{b\to\infty} \int_1^{b^2+1} \frac{du}{u}$

$= ln|u|\Big|_1^{b^2+1} = ln|b^2+1| - ln|1| = ln|b^2+1|$ as b approaches ∞, which thus equals ∞.

(b) Using your solution to part (b), determine whether $\int_{-\infty}^{\infty} \frac{2x}{x^2+1}dx$ is convergent or divergent.

To solve an integral which has the bounds $-\infty$ and ∞, each of the separate components of the integral when split into the pieces $\int_{-\infty}^0$and $\int_0^\infty$must be separately convergent.

Because we have already checked $\int_0^\infty$ and found it to be divergent, then we can conclude that the two-sided $[-\infty, \infty]$ integral must also be divergent.

(c) Find $\lim_{b\to\infty} \int_{-b}^{b} \frac{2x}{x^2+1}dx$

We have already solved for the portion, $\lim_{b\to\infty} \int_0^b \frac{2x}{x^2+1}dx$, so now we just need to solve for the portion from the bound $-\infty$ to 0.

$\lim_{b\to-\infty} \int_b^0 \frac{2x}{x^2+1}dx$

Using the same u-substitution as we used in the previous problem, we get the answer:

$= ln|u|\Big|_{b^2+1}^{1} = ln|1| - ln|b^2+1| = -ln|b^2+1|$ as b approaches $-\infty$, which thus equals ∞.

Therefore the $\lim_{b\to\infty} \int_{-b}^{b} \frac{2x}{x^2+1}dx = \infty - \infty$, which is still divergent.

(d) State your conclusion. Is it true that for all continuous functions $f(x)$,

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{b\to\infty} \int_{-b}^{b} f(x)dx?$$

Yes, this statement is true for all continuous functions because of the definition of a limit.

If the limit, which is the approximation of a finite number, b, as it approaches an infinite value, ∞, exists then the improper integral is convergent, but if the limit does not exist as a finite number then the improper integral is divergent.