Assignment 2 Problem 4

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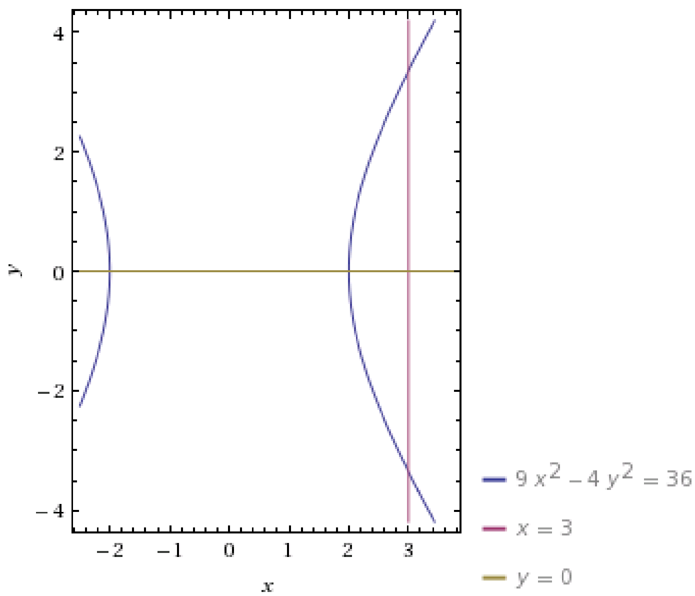
4. *Find the area of the region bounded by the hyperbola* $9x^2 - 4y^2 = 36$ *and the vertical line* $x = 3$.

Input:

$$\{9x^2 - 4y^2 = 36, x = 3, y = 0\}$$

Plot of solution set:



The area bounded by the vertical line $x = 3$ and $9x^2 - 4y^2 = 36$ is given by finding x when $y = 0$ and $x = 3$.
When $y = 0$ then $9x^2 - 4y^2 = 36$ becomes $9x^2 = 36$.
Thus $x = \sqrt(36/9) = 6/3 = 2$
Therefore we must solve the integral bounded by x = 2 and x = 3 with respect to x.

We find the integrand by isolating the y-variable.
$9x^2 - 4y^2 = 36$ becomes $y = \sqrt{\frac{9}{4}x^2 - \frac{36}{4}^2}$
Also, because we are seeking to find the area bounded by the hyperbola and the vertical line, we must account for the portions of the function above and below the x axis.
Thus, we multiply the integral with respect to the positive root of the hyperbola function by two since the function is symmetric across the x-axis.

$2\int_2^3 \sqrt{(\frac{3}{2}x)^2 - (3)^2}$
Because the integrand contains the form $\sqrt{u^2 - a^2}$ we use trigonometric substitution.
Thus, I set the "u" component, or $\frac{3}{2}x$, equal to $asec\theta = 3sec\theta$
Which means that $x = 2sec\theta$, and thus $dx = 2sec\theta tan\theta$.
We now see that the equation under the radicand becomes $(3sec\theta)^2 - 3^2$, which simplifies to $9tan^2\theta$.

The new bounds under the u-substitution are now:
Upper limit, $ulim = sec^-1(\frac{3}{2})$
Lower limit, $llim = sec^-1(\frac{2}{2}) = 0$
For simplicity, we will indicate $sec^-1(\frac{3}{2})$ by s until the end when we evaluate the definite integral.

$2\int_0^s \sqrt{9tan^2\theta}\ 2sec\theta tan\theta\ d\theta$.
$2\int_0^s 3tan\theta 2sec\theta tan\theta d\theta$
$12\int_0^s sec\theta tan^2\theta d\theta$
$= 12\int_0^s sec\theta - sec^3\theta d\theta$
$= 12\int_0^s sec\theta d\theta - 12\int_0^s sec^3\theta d\theta$

First we evaluate $\int_0^s sec\theta d\theta$
We multiply the whole integral by $\frac{tan\theta + sec\theta}{tan\theta + sec\theta}$ to get $\frac{sec^2\theta + sec\theta tan\theta}{tan\theta + sec\theta}$
Then we use u substitution to set $u = tan\theta + sec\theta$ and thus $du = sec^2\theta + sec\theta tan\theta d\theta$
We then must alter the bounds.
The lower bound $= tan(0) + sec(0) = 1$ and the upper bound $= tan(s) + sec(s)$, where $s = sec^{-1}(\frac{3}{2})$.

Therefore the upper bound $= tan(sec^{-1}(\frac{3}{2})) + sec(sec^{-1}(\frac{3}{2}))$
$tan(sec^-1(x)) = \sqrt{1-\frac{1}{x^2}}x$
Thus the upper bound $= \sqrt{1-\frac{1}{(\frac{3}{2})^2}}\frac{3}{2} + \frac{3}{2}$
This simplifies to $\frac{3}{2} + \frac{\sqrt{5}}{2}$, which we set equal to s'.
Therefore the integral is now $\int_1^{s'} u^{-1}du$, which equals $ln|u| + C$
$= ln|tan\theta + sec\theta|$ evaluated at lower bound 1 and upper bound s'.

Now we consider $\int_0^s sec^3\theta d\theta$
$= \int_0^s \sec^2\theta sec\theta d\theta$
Because this is a product, we use integration by parts.
I set $u = sec\theta$ and thus $du = sec\theta tan\theta d\theta$.
I set $dv = sec^2\theta d\theta$ and thus $v = tan\theta$
Therefore the upper bound $sec(sec^{-1}(\frac{3}{2})) = \frac{3}{2}$ and the lower bound $= sec(0) = 1$

Therefore $\int_1^{\frac{3}{2}} sec^3\theta d\theta = tan\theta sec\theta - \int_1^{\frac{3}{2}} tan^2\theta sec\theta d\theta$.
We use the Pythagorean trigonometric identity to substitute $sec^2\theta - 1$ for $tan^2\theta$.
Thus $tan\theta sec\theta - \int_1^{\frac{3}{2}} tan^2\theta sec\theta d\theta = tan\theta sec\theta - \int_1^{\frac{3}{2}}(sec^2\theta - 1)sec\theta d\theta$
$= tan\theta sec\theta - \int_1^{\frac{3}{2}} sec^3\theta d\theta + \int_1^{\frac{3}{2}} sec\theta d\theta$
By moving $\int_1^{\frac{3}{2}} sec^3\theta d\theta$ to the left hand-side, where there is a like term, and by expanding $\int_1^{\frac{3}{2}} sec\theta d\theta$ we get:
$= 2\int_1^{\frac{3}{2}} sec^3\theta d\theta = tan\theta sec\theta|_1^{\frac{3}{2}} + ln|tan\theta + sec\theta||_1^{\frac{3}{2}}$
$\int_0^s sec^3\theta d\theta = \int_1^{\frac{3}{2}} sec^3\theta d\theta = \frac{1}{2}tan\theta sec\theta|_1^{\frac{3}{2}} + \frac{1}{2}ln|tan\theta + sec\theta||_1^{\frac{3}{2}}$.

Now plugging everything back into the original formula: $= 12\int_0^s sec\theta d\theta - 12\int_0^s sec^3\theta d\theta$
We get $12(ln|tan\theta + sec\theta||_0^{s'}) - 12(\frac{1}{2}tan\theta sec\theta|_1^{\frac{3}{2}} + \frac{1}{2}ln|tan\theta + sec\theta||_1^{\frac{3}{2}})$
Recall $s' = \frac{3}{2} + \frac{\sqrt{5}}{2}$
Therefore this definite integral equals:
$= 12(ln|tan(\frac{3}{2}+\frac{\sqrt{5}}{2})+sec(\frac{3}{2}+\frac{\sqrt{5}}{2})|-ln|tan(0)+sec(0)|-12(\frac{1}{2}(tan(\frac{3}{2})sec(\frac{3}{2})-tan(1)sec(1))+\frac{1}{2}ln|tan(\frac{3}{2}) + sec(\frac{3}{2})| - \frac{1}{2}ln|tan(1)sec(1)|$

**I can't really evaluate any further. I checked my answer with the Wolfram calculation, and it seems that my process was the same up until the usage of some obscure "reduction" formula for the integral of $sec^3\theta$. Feedback on this would be appreciated!