2.

a)

$$X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}$$

$$X^T X = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} = \begin{bmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{bmatrix}$$

$$\frac{1}{n} X^T X = \begin{bmatrix} 1 & \bar{x}_n \\ \bar{x}_n & \frac{1}{n}\sum_{i=1}^{n} x_i^2 \end{bmatrix}$$

b)

Inverse Formula for 2x2: $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$a = 1$

$b = \bar{x}_n$

$c = \bar{x}_n$

$d = \frac{1}{n}\sum_{i=1}^{n} x_i^2$

$$= \frac{1}{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}_n^2} \begin{bmatrix} \frac{1}{n}\sum_{i=1}^{n} x_i^2 & -\bar{x}_n \\ -\bar{x}_n & 1 \end{bmatrix}$$

c) $\hat{\beta} = (X^T X)^{-1} (X^T Y)$ $\qquad \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$

$$= \left(\frac{1}{n} X^T X\right)^{-1} \left(\frac{1}{n} X^T Y\right)$$

$$\frac{1}{n} X^T Y = \frac{1}{n} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \overset{\bar{y}_n}{\overset{\shortparallel}{\frac{1}{n} \sum y_i}} \\ \frac{1}{n} \sum x_i y_i \end{bmatrix}$$

$$\left(\frac{1}{n} X^T X\right)^{-1} \left(\frac{1}{n} X^T Y\right) = \frac{1}{\frac{1}{n} \sum x_i^2 - \bar{x}_n^2} \begin{bmatrix} \frac{1}{n} \sum x_i^2 & -\bar{x}_n \\ -\bar{x}_n & 1 \end{bmatrix} \begin{bmatrix} \bar{y}_n \\ \frac{1}{n} \sum x_i y_i \end{bmatrix}$$

$$= \frac{1}{\cdots} \begin{bmatrix} \left(\frac{1}{n} \sum x_i^2\right) \bar{y}_n - (\bar{x}_n)\left(\frac{1}{n} \sum x_i y_i\right) \\ \frac{1}{n} \sum x_i y_i - \bar{x}_n \bar{y}_n \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\left(\frac{1}{n} \sum x_i^2\right) \bar{y}_n - (\bar{x}_n)\left(\frac{1}{n} \sum x_i y_i\right)}{\frac{1}{n} \sum x_i^2 - \bar{x}_n^2} \\ \dfrac{\frac{1}{n} \sum x_i y_i - \bar{x}_n \bar{y}_n}{\frac{1}{n} \sum x_i^2 - \bar{x}_n^2} \end{bmatrix}$$

Let's examine just this first row.

$$\hat{\beta}_0 \left[\frac{\left(\frac{1}{n} \sum x_i^2\right) \bar{y}_n - (\bar{x}_n)\left(\frac{1}{n} \sum x_i y_i\right)}{\frac{1}{n} \sum x_i^2 - \bar{x}_n^2} \right.$$

we know $\hat{\beta}_1 = \dfrac{\frac{1}{n} \sum x_i y_i - \bar{x}_n \bar{y}_n}{\frac{1}{n} \sum x_i^2 - \bar{x}_n^2}$

$$\bar{y}_n - \hat{\beta}_1 \bar{x}_n = \bar{y}_n - \frac{\bar{x}_n \left(\frac{1}{n} \sum x_i y_i - \bar{x}_n \bar{y}_n\right)}{\frac{1}{n} \sum x_i^2 - \bar{x}_n^2} = \bar{y}_n - \frac{\left(\frac{1}{n} \sum x_i y_i\right) \bar{x}_n - \bar{x}_n^2 \bar{y}_n}{\frac{1}{n} \sum x_i^2 - \bar{x}_n^2}$$

$$= \frac{\bar{y}_n \left(\frac{1}{n} \sum x_i^2 - \bar{x}_n^2\right) - \left(\frac{1}{n} \sum x_i y_i\right) \bar{x}_n + \bar{x}_n^2 \bar{y}_n}{\frac{1}{n} \sum x_i^2 - \bar{x}_n^2}$$

$$= \frac{\left(\frac{1}{n} \sum x_i^2\right) \bar{y}_n - \cancel{\bar{x}_n^2 \bar{y}_n} - \left(\frac{1}{n} \sum x_i y_i\right) \bar{x}_n + \cancel{\bar{x}_n^2 \bar{y}_n}}{\frac{1}{n} \sum x_i^2 - \bar{x}_n^2}$$

$$= \frac{\left(\frac{1}{n} \sum x_i^2\right) \bar{y}_n - \left(\frac{1}{n} \sum x_i y_i\right) \bar{x}_n}{\frac{1}{n} \sum x_i^2 - \bar{x}_n^2} \qquad \square$$