Problem Three

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March 3, 2016

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Output

$y_t = e^{z_t} k_t^{\alpha}$

Suppose y_t and k_t changes around their steady states y and k

$\hat{y_t} = log(\frac{y_t}{y})$

$y_t = ye^{\hat{y_t}}$ and k_t follows the same form.

Then $ye^{\hat{y_t}} = e^{z_t}(ke^{\hat{k_t}})^{\alpha}$

In the steady-state $z_t = 0$, thus $e^{z_t} = 1$

$\therefore y = k^{\alpha}$

Thus dividing both sides by y/k^{α} equivalently, we get:

$= e^{\hat{y_t}} = e^{z_t}(e^{\hat{k_t}})^{\alpha}$

And taking logs we get:

$\hat{y_t} = z_t + \alpha\hat{k_t}$

Capital

$k_{t+1} = (1-\delta)k_t + se^{z_t}k_t^{\alpha}$

k_t and k_{t+1} changes around its steady state k.

Therefore:

$ke^{\widehat{k_{t+1}}} = (1-\delta)ke^{\hat{k_t}} + se^{z_t}(ke^{\hat{k_t}})^{\alpha}$

Dividing by k we get:

$e^{\widehat{k_{t+1}}} = (1-\delta)e^{\hat{k_t}} + \frac{sk^{\alpha}}{k}e^{z_t}(e^{\hat{k_t}})^{\alpha}$

Recall: $k_{t+1} = (1-\delta)k_t + sy_t$

In the steady-state:

$k = (1-\delta)k + sy$

$k - (1-\delta)k = sy$

$\delta k = sy$

Because $y = k^{\alpha}$

$\delta k = sk^{\alpha}$

$\delta = \frac{sk^{\alpha}}{k}$

Thus replacing $\frac{sk^{\alpha}}{k}$ in our last equation with δ we get:

$e^{\widehat{k_{t+1}}} = (1-\delta)e^{\hat{k_t}} + \delta e^{z_t}(e^{\hat{k_t}})^\alpha$

Recall from the Taylor Series expansion of e^x

$e^x = 1 + x + \frac{1}{2}x^2 + ...$

$\approx 1 + x$

Thus $e^{\widehat{k_{t+1}}} \approx 1 + \widehat{k_{t+1}} = (1-\delta)(1+\hat{k_t}) + \delta(1 + z_t + \alpha\hat{k_t})$

$= (1-\delta) + (1-\delta)\hat{k_t} + \delta + delta(z_t + \alpha\hat{k_t})$

$\widehat{k_{t+1}} = (1-\delta)\hat{k_t} + \delta z_t + \alpha\delta\hat{k_t}$

$\widehat{k_{t+1}} = \delta z_t + (1-\delta+\alpha\delta)\hat{k_t}$

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Expressing as ARMA processes

$\widehat{k_{t+1}} = (1-\delta-\alpha\delta)\hat{k_t} + \delta z_t$

Recall:

$z_t = \rho z_{t-1} + w_t$

$= \rho L z_t + w_t$

$(1-\rho L)z_t = w_t$

$z_t = \frac{1}{1-\rho L}w_t$

Plugging this back into our equation for $\widehat{k_{t+1}}$

$\widehat{k_{t+1}} = (1-\delta-\alpha\delta)\hat{k_t} + \delta(\frac{1}{1-\rho L}w_t)$

Then plugging in the lag operator, L

$\widehat{k_{t+1}} = (1-\delta-\alpha\delta)L\widehat{k_{t+1}} + \delta(\frac{1}{1-\rho L}w_t)$

$\widehat{k_{t+1}} - (1-\delta-\alpha\delta)L\widehat{k_{t+1}} = \delta(\frac{1}{1-\rho L}w_t)$

$[1-(1-\delta-\alpha\delta)L]\widehat{k_{t+1}} = \delta(\frac{1}{1-\rho L}w_t)$

$\widehat{k_{t+1}} = \delta(\frac{1}{[1-(1-\delta-\alpha\delta)L](1-\rho L)})w_t$

Lets denote $1-(1-\delta-\alpha\delta)L$ as ρ_1 and the previous ρ as ρ_2, which will make the equation of the form:

$\widehat{k_{t+1}} = \delta(\frac{1}{[1-\rho_1 L](1-\rho_2 L)})w_t$

$= (1-\rho_1 L)(1-\rho_2 L)\widehat{k_{t+1}} = \delta w_t$

$[1-(\rho_1+\rho_2)L + \rho_1\rho_2 L^2]\widehat{k_{t+1}} = \delta w_t$

$\widehat{k_{t+1}} - (\rho_1+\rho_2)\hat{k_t} + \rho_1\rho_2\widehat{k_{t-1}} = \delta w_t$

Let us denote $\rho_1+\rho_2$ as ϕ_1 and $\rho_1\rho_2$ as ϕ_2

Now $\widehat{k_{t+1}}$ is rewritten as an AR(2) process:

$\widehat{k_{t+1}} = \phi_1\hat{k_t} + \phi_2\widehat{k_{t-1}} + \delta w_t$

Now to represent output as well.

Recall: $\hat{y_t} = z_t + \alpha k_t = \frac{1}{1-\rho L}w_t + \frac{\alpha\delta}{(1-(1-\delta+\alpha\delta)L)(1-\rho L)}w_t$

$\hat{y_t} = \frac{1+\alpha\delta-(1-\delta+\alpha\delta)L}{(1-(1-\delta+\alpha\delta)L)(1-\rho L)}w_t$

$\hat{y_t} + \phi_1\widehat{y_{t-1}} + \phi_2\widehat{y_{t-2}} = [1+\alpha\delta-(1-\delta+\alpha\delta)L]w_t$

$= (1 + \alpha\delta)w_t - (1 - \delta + \alpha\delta)w_{t-1}$

$\boxed{\hat{y_t} = \phi_1 \widehat{y_{t-1}} + \phi_2 \widehat{y_{t-2}} + (1 + \alpha\delta)w_t - (1 - \delta + \alpha\delta)w_{t-1}}$

Thus output is represented as an ARMA(2,1) process.