Assignment 9

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Exercise 1

given:

$$z = (-\sqrt{3} + i)$$

we must find: z^{100}

we can use the formula: $z^n = r^n[\cos(n \cdot \theta) + i \sin(n \cdot \theta)]$ where r is the length of the complex vector and θ is the angle from the x axis; to find the solution for any z^n given n, instead of having to find it recursively. thus:

$$r = \sqrt{\sqrt{3}^2 + 1^2} = 2$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{(3)}}{1}\right) = \frac{\pi}{3}$$

thus:

$$z^n = 2^n[\cos(n \cdot \frac{\pi}{3}) + i \sin(n \cdot \frac{\pi}{3})]$$

therefore:

$$z^{100} = 2^{100}[\cos \frac{100\pi}{3} + i \sin \frac{100\pi}{3}]$$

$$z^{100} = -2^{100}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

Exercise 2

given:

$$z = (3 + 4i)$$

2.1 i

find z^5 using brute force recursion.

$$\begin{aligned}
z^1 &= (3 + 4i) \\
&= (3 + 4i)(3 + 4i) = (-7 + 24i) \\
&= (3 + 4i)(-7 + 24i) = (-177 + 44i) \\
&= (3 + 4i)(-177 + 44i) = (-527 - 336i) \\
z^5 &= (3 + 4i)(-527 - 336i) = (-237 - 3116i)
\end{aligned}$$

2.2 ii

find z^5 using the polar coordinates method.

$$\begin{aligned}
\theta &= \tan^{-1}\left(\frac{3}{4}\right) \\
r &= \sqrt{3^2 + 4^2} = 5
\end{aligned}$$

$$\begin{aligned}
z^n &= r^n[\cos(n \cdot \theta) + i\sin(n \cdot \theta)] \\
z^5 &= 5^5[\cos(5 \cdot \tan^{-1}\left(\frac{3}{4}\right)) + i\sin(5 \cdot \tan^{-1}\left(\frac{3}{4}\right))] \\
z^5 &= (-3116 - 237i)
\end{aligned}$$

side note: i have no idea why this one is backwards...

Exercise 3

3.1 a

given the equations:

$$x_{n+1} = -4y_n$$

$$y_{n+1} = x_n$$

we must find x & y for $\{n|3 \leq n \geq 9\}$ thus:

n	x_n	y_n
1	-4	0
2	0	-4
3	16	0
4	0	16
5	-64	0
6	0	-64
7	256	0
8	0	256
9	-1024	0

3.2 b

in order to find the equation for x_n in this system one must split it into two: an Even and Odd equation.

from looking at the above able one can infer these 2 equations

even:

$$x_n = 0$$

odd:

$$x_n = (-4)^{\frac{n+1}{2}}$$

3.3 c

if one wanted to find *one* equation to represent x_n however, one would have to use complex numbers.

in order to do this i first want to find the roots of our original equation for x_n

$$x_{n+1} = -4y_n$$
$$y_n = x_{n-1}$$
$$x_{n+1} = -4x_{n-1}$$
$$x_n = \lambda^n$$
$$\lambda^{n+1} = -4\lambda^{n-1}$$

now, if we input $n = 1$ and solve for λ:

$$\begin{aligned} \lambda^2 &= -4\lambda^0 \\ \lambda^2 &= -4 \\ \lambda &= \sqrt{-4} \\ \lambda &= \pm 2i \end{aligned}$$

thus λ and its conjugate $\bar{\lambda}$ are as follows:

$$\lambda = +2i$$

$$\bar{\lambda} = -2i$$

now, using the formula $x_n = a\lambda^n + b\bar{\lambda}^n$ we can find the coefficients a and b that, when satisfying 2 conditions of n, will satisfy all conditions of n via. recursion.

$n = 1$:

$$-4 = a \cdot (2i)^1 + b \cdot (-2i)^1 \mapsto -4 = (a \cdot 2i) + (b \cdot -2i) \tag{3.1}$$

$n = 3$

$$16 = a \cdot (2i)^3 + b \cdot (-2i)^3 \mapsto 16 = (a \cdot -8i) + (b \cdot 8i) \tag{3.2}$$

(3.1) simplifies to:

$$a = b + 2i$$

(3.2) simplifies to:

$$a = b + 2i$$

inputting (3.1) as a in (3.2) results in:

$$b + 2i = b + 2i \mapsto b = b$$

which means b can be any arbitrary number.
because of this, a is simply:

$$a = b + 2i$$

and the final formula is:

$$x_n = (b + 2i)\lambda^n + b\bar{\lambda}^n$$

thus:

$$\begin{aligned} x_1 &= (36 + 2i)(2i)^1 + 36(-2i)^1 \\ \mapsto -4 & \\ x_3 &= (7 + 2i)(2i)^3 + 7(-2i)^3 \\ \mapsto 16 & \\ x_9 &= (100 + 2i)(2i)^9 + 100(-2i)^9 \\ \mapsto -1024 & \end{aligned}$$

therefore the equation holds!

Exercise 4

4.1 a

$$b_x = 0.2b_x + 0.4b_x$$
$$b_y = 0.6b_x + 0.2b_y + 1$$

thus:

$$0.8b_x - 0.4b_y = 0 \tag{4.1}$$
$$-0.6b_x + 0.8b_y = 1 \tag{4.2}$$

2·(4.1):

$$1.6b_x - 0.8b_y = 0 \tag{4.3}$$

(4.2)+(4.3):

$$b_x = 1$$

thus:

$$0.8(1) = 0.4b_y \mapsto b_y = 2$$

4.2 b

$$L_x = 0.2(L_x + 1) + 0.4(L_y + 1)$$
$$L_y = 0.6(L_x + 1) + 0.2(L_y + 1)$$

thus:

$$0.8L_x - 0.4L_y = 0.6 \tag{4.4}$$
$$-0.6L_x + 0.8L_y = 0.8 \tag{4.5}$$

2·(4.4):

$$1.6L_x - 0.8L_y = 1.2 \tag{4.6}$$

(4.5)+(4.6):

$$L_x = 2$$

thus:

$$0.8(2) - 0.6 = 0.4L_y \mapsto L_y = 2.5$$

4.3 c

$$V_x = 0.2(V_x + 1) + 0.6(V_y + 1)$$
$$V_y = 0.4(V_x + 1) + 0.2(V_y + 1)$$

thus:

$$0.8V_x - 0.6V_y = 0.2 \tag{4.7}$$
$$-0.4V_x + 0.8V_y = 0.4 \tag{4.8}$$

2·(4.8):

$$-0.8V_x + 1.6V_y = 0.8 \tag{4.9}$$

(4.7)+(4.9):

$$V_y = 1$$

thus:

$$0.8V_x - 0.6 = 0.2 \mapsto V_x = 1$$

4.4 d

starting from x, there are 3 moves that can be taken:

$$\begin{aligned} X &\mapsto X \;:\; probability = 0.2 \\ X &\mapsto Y \;:\; probability = 0.4 \\ X &\mapsto ☠ \;:\; probability = 0.4 \end{aligned}$$

therefore the equation for the probability of an agent staring on X, moving to Y (P_y) is:

$$P_y = 0.4 + 0.4(0.2) + 0.4(0.2)^2 + 0.4(0.2)^3 + ... + 0.4(0.2)^n$$

such that $n \geq 0$.
we can simplify P_y to:

$$= \sum_{n=0}^{\infty} 0.4(0.2)^n$$

$$= \frac{0.4}{1 - 0.2}$$

$$P_y = 0.5$$

4.5 e

the average wait time (WT) on Y can be found by the formula:

$$WT_{avg} = \frac{WT_{ex}}{P_y}$$

where $WT_{avg},\ WT_{ex}$ are the average and expected wait times at Y respectively.
WT_{ex} can be found by a similar formula to P_y:

$$\begin{aligned} WT_{ex} &= 1(0.4) + 2(0.4)(0.2) + 3(0.4)(0.2)^2 + 4(0.4)(0.2)^3 + ... + n(0.4)(0.2)^{n-1} \\ &= 0.4(1(0.2)^0 + 2(0.2)^1 + 3(0.2)^2 + ... + n(0.2)^{n-1}) \\ &= \frac{0.4}{0.2}(1(0.2) + 2(0.2)^2 + 3(0.2)^3 + ... + n(0.2)^n) \\ &= \left(\frac{0.4}{0.2}\right)\left(\frac{0.2}{(1 - 0.2)^2}\right) \\ WT_{ex} &= 0.625 \end{aligned}$$

therfore:

$$WT_{avg} = \frac{WT_{ex}}{P_y} = \frac{0.625}{0.5} = 1.25$$