Linear Data Analysis
Design Matrices and Standard Data

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a Variables and Observations

the main concepts for the this lecture are: what is data and how do we prepare it.
A variable is a computational object with a Type. For example, Mass is weighted in Kilograms, whereas the waist in ratio to the hip is measured in Real Numbers $\mathbb{R}$. A variable can hold any unit of any type of object.

b Variables as Vectors

We shall explore variables as column vectors. Consider this dataset

$$\begin{bmatrix} 33.15 \\ 135.28 \\ 73.3 \\ 0.82 \\ 52.8 \end{bmatrix}.$$

Assume that any vector like this is a **column** vector representing a variable like the measurements of height. Any row vector is thus an **observation** of many variables, like a many measurement over time.

$$\begin{aligned} \textit{column} = & \vec{x} & \implies \textbf{Variable} \\ \textit{row} = & \vec{x}^\top & \implies \textbf{Observation}. \end{aligned}$$

consider the observation of a persons height, weight, etc. . . :

$$\vec{a}^\top = \begin{bmatrix} 33.15 & 135.28 & \cdots \end{bmatrix}.$$

We can gather many observations into a data matrix $A \in \mathbb{R}^{m \times n}$

$$A = \begin{bmatrix} \vec{a}_1^\top \\ \vec{a}_2^\top \\ \vdots \\ \vec{a}_m^\top \end{bmatrix} = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}.$$

The conundrum is we cannot add observations.
To add, we need to covert an observation into a real number. For example, one could use a vector of weights (in the sense of a designed vector) in order to summarize in a standard fashion each observation. Thus, A summarized:

$$A\vec{w}.$$

c Zero Mean Data

a problem often found in data analytics is: How to relate observations to dependent variables. For example, if $\vec{a}_i^\top \in A$ has dependent variable c_i. This means that

$$A\vec{w} \approx \vec{c}.$$

Standardization One of the concepts this course will take from the field of Statistics is Data Standardization. There are 2 steps to standardizing data:

1. Zero Mean
2. Unit Variance

These are represented as, given the data: $\vec{a} \in \mathbb{R}^m$

$$\begin{aligned}
Mean \quad &= \frac{\sum_{i=1}^m a_i}{m} = \frac{\vec{1}^\top \vec{a}}{m} \quad &= \overline{a} \\
\vec{a} \in A \quad &\implies \overline{A} = \frac{\vec{1}^\top A}{m}
\end{aligned}$$

.

In Matlab the mean of a matrix is found by using `A-mean(A)`.
Consider the data:

$$\vec{a} = \begin{bmatrix} 15 \\ 17 \\ 31 \\ 19 \\ 3 \end{bmatrix} \implies \overline{a} = 17.$$

Thus we have found the mean of the data $\vec{a}$. To then turn it into **zero mean data** –as per professor Karl Pearsons's first step of standardization–we will do the following

$$\vec{a} - \vec{1}\overline{a} = \begin{bmatrix} -2 \\ 0 \\ 14 \\ 2 \\ -14 \end{bmatrix}.$$

We can verify this is a zero mean vector my calculating it's mean, which should result to be 0. Observe that $-2, 2$ cancel out as well as $-14, 14$

d Unit Variance Data

In Statistics there are 2 types of variance in data. Sample Variance and Population Variance. This course will focus in **Sample Variance**. The sample variance of $\vec{a} \in \mathbb{R}^m$ is:

$$\begin{aligned}\sigma^2 &= \frac{\sum_{i=1}^{m} (a_i - \overline{a})^2}{m-1} \\ &= \frac{||\vec{a} - \vec{1}\overline{a}||^2}{m-1}.\end{aligned}$$

Recall that Professor Karl's second step to standardization is Unit Variance. Thus the equation for standardizing the data $\vec{a}$ is:

$$\vec{z}(\vec{a}) = \frac{\vec{a} - \vec{1}\overline{a}}{\sigma}.$$

The Sample variance if $\vec{a}$ is:

$$\sigma^2 = \frac{400}{4} = 100.$$

Therefore the z vector of $\vec{a}$ i s:

$$\vec{z} = \begin{bmatrix} -0.2 \\ 0.0 \\ 1.4 \\ 0.2 \\ = 1.4 \end{bmatrix}.$$

And thus we have the standardzation of the original data vector

In Matlab this can be done by using `zscore(a)`

e Standardized Data for Regression

We will Now define what a **Design Matrix** is:
Given $A \in \mathbb{R}^{m \times n}$ and the dependent variable $\vec{c} \in \mathbb{R}^m$, We wouls like to show a relation such that

$$A\vec{w} \approx \vec{c}.$$

To do this we can form a design matrix where each column j is standardized:

$$\vec{a}_i.$$

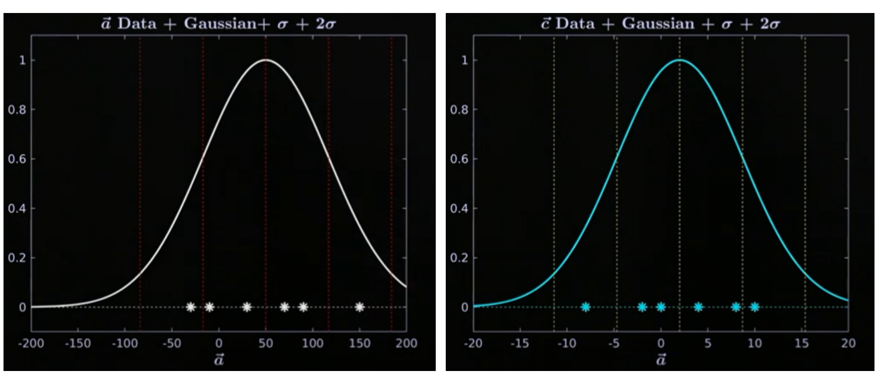
We can then score A as X, score $\vec{c}$ as $\vec{y}$ and relate $X\vec{u} \approx \vec{y}$.
Where $\vec{u}$ is a weight vector like $\vec{w}$

f Measuring Standard Deviations

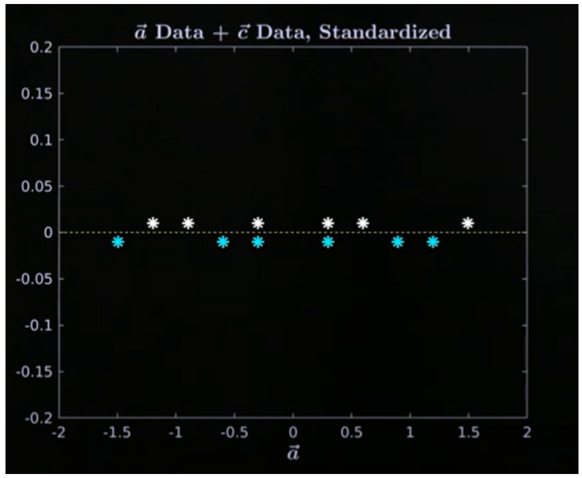
We will use the Gaussian Distribution to measure standard deviations. This distribution is also known as the bell curve.
The method of doing this is to divide the zero mean data of a dataset by its sample standard deviation, like so:

$$\frac{\vec{a} - \vec{1}\overline{a}}{\sigma}.$$

This measures for each a_i how many standard deviations it is from $\overline{a}$
Thus, if we standardized $\vec{a}$ and $\vec{c}$ like so:



Thus we can measure and plot each data point in $\vec{a}$ and $\vec{c}$ in terms of their respective standard deviations like so:



Learning Outcomes

Students should now be able to:

- Determine whether data needs transposition
- Create Zero Mean Data
- Create Unit Variance Data
- Prepare data for analysis