CISC 203 Problem Set 1

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1. (a) let $P = (A \cup B)^c$, $Q = A^c \cap B^c$
let x be an arbitrary element of P such that $x \in P \to x \in (A \cup B)^c$

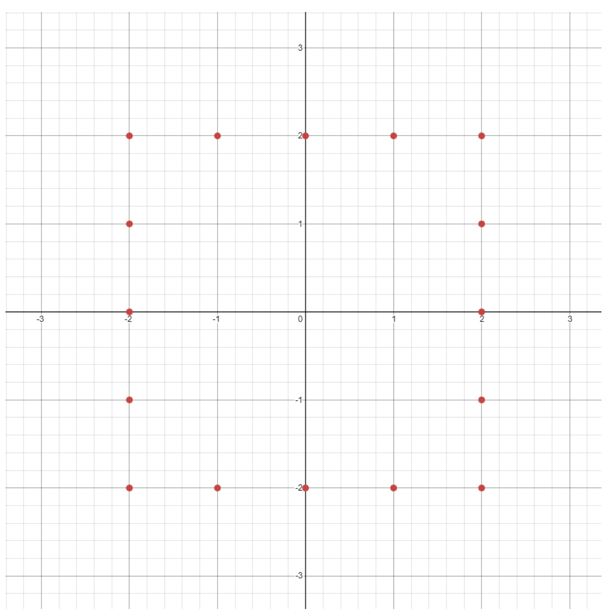
$$
\begin{aligned}
&\Rightarrow x \notin (A \cup B) && (1)\\
&\Rightarrow x \notin A \ \wedge \ x \notin B && (2)\\
&\Rightarrow x \in A^c \ \wedge \ x \in B^c && (3)\\
&\Rightarrow x \in (A^c \cap B^c) && (4)\\
&\Rightarrow x \in Q && (5)
\end{aligned}
$$

therefore, $P \subset Q$
and thus:

$$(A \cup B)^c = A^c \cap B^c$$

(b) given $A = \{-2, -1, 0, 1, 2\}$, $B = \{-2, 2\}$

$$
\begin{aligned}
C = A \times B =& \{(a, b) \in \mathbb{R} : |a| \leq 2, |b| = 2\}\\
=& \{(-2, -2), (-2, 2), (-1, -2), (-1, 2), (0, -2), (0, 2), (1, -2), (1, 2), (2, -2), (2, 2)\}\\
D = B \times A =& \{(b, a) \in \mathbb{R} : |a| \leq 2, |b| = 2\}\\
=& \{(-2, -2), (-2, -1), (-2, 0), (-2, 1), (-2, 2), (2, -2), (2, -1), (2, 0), (2, 1), (2, 2)\}\\
C \cup D =& \{(-2, -2), (-2, -1), (-2, 0), (-2, 1), (-2, 2), (-1, -2), (-1, 2), (0, -2), (0, 2), (1, -2),\\
& (1, 2), (2, -2), (2, -1), (2, 0), (2, 1), (2, 2)\}
\end{aligned}
$$



(c) given $A_n = \{x \in \mathbb{R} : -\frac{1}{n} \leq x \leq \frac{1}{n}\}$

i. to find $\bigcup_{n=1}^{\infty}$ we can start by calculating the first few values of A_n.

$$-\frac{1}{2} \leq x \leq \frac{1}{2}$$
$$-\frac{1}{3} \leq x \leq \frac{1}{3}$$
$$-\frac{1}{4} \leq x \leq \frac{1}{4}$$
$$...$$
$$-\frac{1}{100} \leq x \leq \frac{1}{100}$$

we can see that as n approaches ∞, the set A_n becomes infinitesimally small. because of this fact eventhough we have a union of infinite sets, they all exist within a finite range and so, all numbers contained in the sets A_n where n goes from 1 to ∞ are all contained within and thus equal the interval $[-1, 1]$, or in other words,

$$\bigcup_{n=1}^{\infty} A_n = [-1, 1]$$

ii. to find $\bigcap_{n=1}^{\infty}$ we can use the knowledge we gained from i. to help us. we know that as n approaches infinity the range of x approaches 0.in other words:

$$\lim_{x \to \infty} -\frac{1}{n} \leq x \leq \frac{1}{n}$$
$$-\frac{1}{\infty} \leq x \leq \frac{1}{\infty}$$
$$0 \leq x \leq 0$$

thus:

$$\bigcap_{n=1}^{\infty} A_n = \{0\}$$

2. given $a \ R \ b \Leftrightarrow 3|(a^2 - b^2)$

(a) prove its symmetric!

$$\begin{aligned} a \ R \ b &\Leftrightarrow a^2 - b^2 = 3m \\ &\Rightarrow b^2 - a^2 = -3m \qquad n = -m \\ &\Rightarrow b^2 - a^2 = 3n \\ &\Rightarrow b \ R \ a \end{aligned}$$

(b) prove its reflexive!

$$\begin{aligned} a \ R \ a &\Leftrightarrow a^2 - a^2 = 3m \qquad m = 0 \\ &\Rightarrow a \ R \ a \end{aligned}$$

(c) prove its transitive!

$$\begin{aligned} a \ R \ b &\Leftrightarrow a^2 - b^2 = 3m \\ b \ R \ c &\Leftrightarrow b^2 - c^2 = 3m \\ &\Rightarrow a^2 - b^2 + b^2 - c^2 = 3m + 3n \\ &\Rightarrow a \ R \ c \end{aligned}$$

(d) finally, we shall find all the distinct equivalence classes.

$$0^2 = 0(mod3)$$
$$1^2 = 1(mod3)$$
$$2^2 = 1(mod3)$$
$$3^2 = 0(mod3)$$
$$4^2 = 1(mod3)$$
$$5^2 = 1(mod3)$$
$$6^2 = 0(mod3)$$

thus we can see that the 2 distinct equivalence classes are:

$$[0] = \{0, 3, 6, 9, ...\}$$

$$[1] = \{1, 2, 4, 5, ...\}$$

3. for this question (3), lowercase letters denote a set while uppercase letters denote the cardinality of their corresponding set.

(a) let a=DOTA2, b=CS:GO, c=TF2

i. $A + B + C - A \cap B - A \cap C - B \cap C + A \cap B \cap C$
$92 + 104 + 100 - 50 - 42 - 46 + 32 = 190$

ii. $A - (A \cap B + A \cap C) + A \cap B \cap C$
$92 - (50 + 42) + 32 = 32$ people who only play DOTA2

$B - (B \cap A + B \cap C) + A \cap B \cap C$
$104 - (50 + 46) + 32 = 40$ people who only pay CS:GO

$C - (C \cap A + C \cap B) + A \cap B \cap C$
$100 - (42 + 46) + 32 = 44$ people who only play TF2
thus:

$$32 + 40 + 44 = 116$$

iii. total students - students who play at least one game
$240 - 190 = 50$

iv. $A - (A \cap B + A \cap C) + A \cap B \cap C$
$92 - (50 + 42) + 32 = 32$

v. $A + B - (A \cap C + B \cap C) + A \cap B \cap C$
$92 + 104 - (42 + 46) + 32 = 140$

(b) given $a = \{1, 2, ..., 9999\}$, let p denote the set of numbers divisible by 2, q divisible by 4, r divisible by 5, and s divisible by 7. let $f(x) = \lfloor \frac{9999}{x} \rfloor$
we know that we must use the inclusion-exclusion principle to find $P \cup Q \cup R \cup S$ which means we must fill out the equation:

$$\begin{aligned} P \cup Q \cup R \cup S =& P + Q + R + S \\ & - P \cap Q - P \cap R - P \cap S - Q \cap R - Q \cap S - R \cap S \\ & + P \cap Q \cap R + P \cap Q \cap S + P \cap R \cap S + Q \cap R \cap S \\ & - P \cap Q \cap R \cap S \end{aligned}$$

assuming that if we have the set x where x is the set of numbers divisible by 10. The cardinality of this set (X) would be $f(10) = \lfloor \frac{9999}{10} \rfloor$. furthermore, if we were to take the union of x and y (if y was the set of numbers divisible by 8) then the cardinality of this union would be $f(10*8) = \lfloor \frac{9999}{10*8} \rfloor$.
using this we can then calculate each term in the above equation.

they are:

$$
\begin{aligned}
P \to f(2) = \lfloor \frac{9999}{2} \rfloor & &= 4999 \\
Q \to f(3) & &= 3333 \\
R \to f(5) & &= 1999 \\
S \to f(7) & &= 1428 \\
P \cap Q \to f(2*3) = \lfloor \frac{9999}{6} \rfloor & &= 1666 \\
P \cap R \to f(2*5) & &= 999 \\
P \cap S \to f(2*7) & &= 714 \\
Q \cap R \to f(3*5) & &= 666 \\
Q \cap S \to f(3*7) & &= 476 \\
R \cap S \to f(5*7) & &= 285 \\
P \cap Q \cap R \to f(2*3*5) = \lfloor \frac{9999}{30} \rfloor & &= 333 \\
P \cap Q \cap S \to f(2*3*7) & &= 238 \\
P \cap R \cap S \to f(2*5*7) & &= 142 \\
Q \cap R \cap S \to f(3*5*7) & &= 95 \\
P \cap Q \cap R \cap S \to f(2*3*5*7) = \lfloor \frac{9999}{210} \rfloor & &= 47
\end{aligned}
$$

and using the inclusion-exclusion equation above, we can calculate the cardinality of $p \cup q \cup r \cup s$ which is thus: 7714
and the cardinality of the compliment of this union is thus: $9999 - 7714 = 2285$

4. VIA RAIL needs to organize 3 types of cars behind 3 different locomotives for service between Kingston and Montreal. each locomotive should have n cars behind it. Using this model, show LHS = RHS.

   (a) LHS:
   this side consists of 1 term. there are $\binom{3n}{3}$ ways to order the cars behind the locomotive. that is to say, we 3 locomotives with a target length of n and we have 3 different kinds of cars to choose from to fill out each trainset

   (b) RHS:
   this side contains 3 terms which each represent a different scenario:

      i. n^3 - this first case models if we were to put 1 car per locomotive. each locomotive would have n possible ways to place a car and there are 3 locomotives thus: $n \times n \times n = n^3$

      ii. $3\binom{n}{3}$ - in this second scenario, 1 locomotive gets all three cars and the other 2 get none. the locomotive will have n length and chooses 3 spots to put the cars in. thus: $\binom{n}{3}$. there are 3 ways to choose which backpack gets the three books, hence: $3\binom{n}{3}$

      iii. $6\binom{n}{1}\binom{n}{2}$ - in this final case, one locomotive gets one car, and another locomotive gets the other 2, leaving the final locomotive without any cars to choose. Thus: $\binom{n}{1}\binom{n}{2}$. there are 6 ways to choose the 2 locomotives that receive the cars, thus: $6\binom{n}{1}\binom{n}{2}$

   thus, these scenarios represent the ways to organize 3 cars behind 3 locomotives with target length of n $\therefore$ RHS = LHS

5. we must calculate the total amount of permutations and then subtract from it those permutations that contain 2 adjacent pairs.

   we can find all the permutations using a simple formula. there is a string of n lights and each time a light gets turned on it has 4 choices of colour. this repeats every time so if $n = 5$ then the number of different combinations of colours $= 5 * 5 * 5 * 5 * 5$. thus we can derive a general form of 4^n

   similarly, we can find all the permutations with adjacent pairs using the same kind of reasoning. There is a string of n lights and the first light to come on has 4 choices of colour. the next light however only has 3 choices as to prevent any adjacent pairs. for every subsequent light there are only 3 choices to prevent adjacent repetition. thus: $4 * 3^{n-1}$

   Therefore, the formula for finding the number of n length sequences which have at least one adjacent pair of colours is:

$$4^n - (4 \times 3^{n-1})$$