# Linear Data Analysis
# Orthogonal Projection

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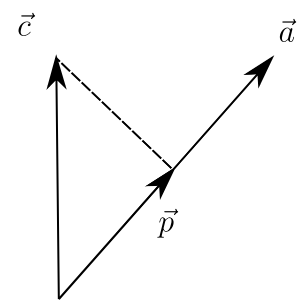
# a Concepts in Orthogonal Projection

We now ask the question, if we are given a vector space, what vector in a vector space is closest to the given vector. This is whats known as projecting.

We will first look at projecting a vector in 1 Dimensions.

Consider the vector $\vec{a}$ as a basis for a 1D space. Given a new vector $\vec{c}$ what multiple of $\vec{a}$ is nearest to $\vec{c}$




# b Projecting a Vector to a Vector

We will explore how we can project a vector in a 1-dimensional vector space.

Given: basis $\vec{a}$ and s new vector $\vec{c}$

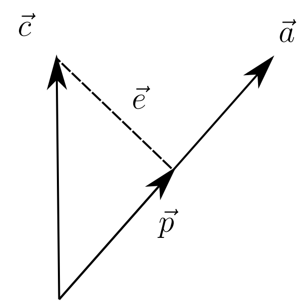
We want the vector in the vector space of $\vec{a}$ that is nearest to $\vec{c}$. We refer the nearest vector as:

$$\vec{p} =^{def} w\vec{a}.$$

To find $\vec{p}$ we must use the **error vector**. The error vector is defined as

$$\vec{e} =^{def} \vec{c} - \vec{p}.$$

Using the example from section a of this lesson, $\vec{e}$ is:




Thus we require the vector $\vec{a}$ perpendicular to vector $\vec{e}$

$$\vec{a} \perp \vec{e}.$$

IN linear algebra, this can be represented as:

$$\begin{aligned} \vec{a} \cdot \vec{e} &= 0 \\ \equiv \vec{a}^\top \vec{e} &= 0 \\ \equiv \vec{a} =^\top (\vec{c} - \vec{p}) &= 0 \\ \equiv \vec{a}^\top \vec{c} - \vec{a}^\top \vec{p} &= 0 \\ \equiv \vec{a}^\top w \vec{a} &= \vec{a}^\top \vec{c} \\ \equiv w(\vec{a}^\top \vec{a}) &= \vec{a}^\top \vec{c} \\ \equiv w &= \frac{\vec{a}^\top \vec{c}}{\vec{a}^\top \vec{a}}. \end{aligned}$$

Thus a projection vector is:

$$\vec{p} = w\vec{a} = \frac{\vec{a}^\top \vec{c}}{\vec{a}^\top \vec{a}} \vec{a}.$$

Which is the nearest vector to $\vec{c}$ in vector space $\vec{a}$

# c Projecting a Vector to a Vector Space

We will now explore how to project a given vector to a 2 dimensional vector space.
Given: vector space $\mathbb{V}$ with basis $\vec{a}_1, \vec{a}_2$ and new vector $\vec{c}$
What $\vec{p} \in \mathbb{V}$ is nearest to $\vec{c}$

We will thus define $\vec{p}$ for 2D space as:

$$\begin{aligned}\vec{p} &=^{def} w_1\vec{a}_1 + w_2\vec{a}_2 \\ &= \begin{bmatrix}\vec{a}_1 & \vec{a}_2\end{bmatrix}\begin{bmatrix}w_1 \\ w_2\end{bmatrix} \\ &= A\vec{w}\end{aligned}$$

.

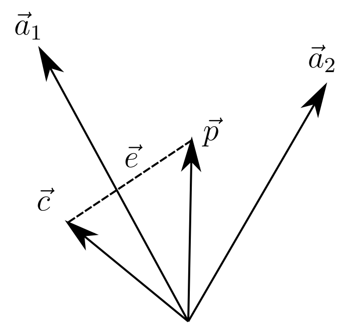
and the error vector is define as:

$$\vec{e} =^{def} \vec{c} - \vec{p}.$$

Where:

$$\vec{e}\perp\vec{a}_1 \wedge \vec{e}\perp\vec{a}_2.$$

This can be represented visually as: Note that this is a 2D representation of




3D space so one cannot see how it is perpendicular to both $a$'s.
Mathematically, this is represented as:

$$\begin{aligned}\vec{a}_1 \cdot \vec{e} &= 0 \\ \vec{a}_2 \cdot \vec{e} &= 0 \\ \vec{a}_1^\top \cdot \vec{e} &= 0 \\ \vec{a}_2^\top \cdot \vec{e} &= 0.\end{aligned}$$

We then gather up our observations into a matrix:

$$\begin{bmatrix}\vec{a}_1^\top \vec{a}_2^\top\end{bmatrix}\vec{e} = \begin{bmatrix}0 \\ 0\end{bmatrix} \implies A^\top\vec{e} = 0.$$

Observe that this means the vector e is in the null space of $A$ such that: $\vec{e} \in null(A)$. Thus we can also say that $\vec{e}$ in the orthogonal complement of $\mathbb{V}$

# d The Normal Equation

Consider $\mathbb{V}$ with basis $\vec{a}_1, \vec{a}_2$ and the new vector $\vec{c}$
What $\vec{p} \in \mathbb{V}$ is nearest to $\vec{c}$

$$\begin{aligned} A^\top \vec{e} &= \vec{0} \\ A^\top (\vec{c} - \vec{e}) &= \vec{0} \\ A^\top \vec{c} - A^\top \vec{p} &= \vec{0} \\ [A^\top A]\vec{w} &= A^\top \vec{c}. \end{aligned}$$

Observe that $A = \Sigma V^\top$ which means that A is symmetric: $A^\top A = V \Sigma V^\top$. Thus, the following matrix is positive definite

$$[A^\top A] \succ 0.$$

Whic implies that there is an explicit solution of $\vec{w}$ such that

$$\vec{w} = [A^\top A]^{-1} A^\top \vec{c}.$$

And thus $\vec{p}$ is:

$$\begin{aligned} p &= A\vec{w} \\ &= A[A^\top A]^{-1} A^\top \vec{c} \\ &= P\vec{c} \\ . \end{aligned}$$

Thus $\vec{p}$ is equal to the projection matrix $p$ times $\vec{c}$. The projection matrix $P$ is based only on $A$ thus we can use $P$ with any $\vec{c}$ to find the nearest vector to $\vec{c}$ in $A$
Observe that $\vec{p}$ is not the sum of $\vec{p}_1$ nearest to $\vec{a}_1$ and $\vec{p}_2$ to $\vec{a}_2$ You can use these as examples to practice

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \vec{a}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \vec{c} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

# e Overdetermined Linear Equations

We will now consider 2D projection as a linear equation.
For 2 vectors $\vec{a}_1, \vec{a}_2$ of size 3. Matrix $A$ is $3 \times 2$, vector $\vec{c}$ is size 3. The entries of $A$ where $A\vec{w} \approx \vec{c}$ are:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

While we have not solved this overdetermined system, we have *approximately* solved it.

# Learning Summary

Students should now be able to

- formulate a projection problem from $\vec{a}_j$ and $\vec{c}$
- solve for the weight vector $\vec{w}$
- solve for the projection vector $\vec{p}$
- solve for the error vector $\vec{e}$