Linear Data Analysis
Orthonormal Vectors and the Singular Variance Decomposition

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February 9, 2022

a I Did Not Shoot the SVD

We shall explore some application of the SVD

- Left Singular Vectors: basis of data space
- right singular vectors: basis of weight space
- singular values are generalized EigenValues
- approximations of data space

An interesting application of this is splitting an audio into ts 2 distingct components of Music and Speaking, as demonstrated by Professor Ellis. (Song: I Shot The Sheriff)

b Examples of the SVD

Consider the matrix:

$$A_1 = [1 \;\; 5 \;\; ; \;\; 0 \;\; 1].$$

It has the SVD of:

$$[U_1, S_1, V_1] = svd\,(A_1)\,.$$

$$\begin{aligned} U_1 &= [0.98 \;\; -0.19 \;\; ; \;\; 0.19 \;\; 0.98] \\ S_1 &= [5.19 \;\; 0 \;\; ; \;\; 0 \;\; 0.19] \\ V_1 &= [0.19 \;\; -0.98 \;\; ; \;\; 0.98 \;\; 0.19] \\ &. \end{aligned}$$

Note that this is all in MatLab notation.

Consider the next example:

```
>> A2 = [1 -1 ; -1 1]
A2 =
 1 -1
-1  1
>> [U2,S2,V2] = svd(A2)
U2 =
-0.71  0.71
 0.71  0.71
S2 =
 2.00  0.00
 0.00  0.00
V2 =
-0.71 -0.71
 0.71 -0.71
```

Note that:

- the matrix A_2 has a rank of 1 as S_2 only has 1 non zero value on the diagonal

- the absence of a value in the second column of S_2 denotes that the second column of V_2 is the nullspace of A_2

- the presence of a value in the first column of S_2 denotes to use that the first column of U_2 is a basis vector for the column space of A_2.

These analysis are general and so if the above conditions apply, the corresponding inference holds (for SVD)

c Matrix Spaces and the SVD

Consider the following computation:

```
>> A3 = [1 -1 ; -1 1 ; 2 0]
A3 =
 1 -1
-1  1
 2  0
>> [U3,S3,V3] = svd(A3)
U3 =
-0.50  0.50 -0.71
 0.50 -0.50 -0.71
-0.71 -0.71 -0.00
S3 =
 2.61  0.00
 0.00  1.08
 0.00  0.00
V3 =
-0.92 -0.38
 0.38 -0.92
```

Note that:

- the third vector in U_3 is a basis for the complement of the column space of A_3

Consider the Rank deficient matrix and its computation:

```
>> A4 = [1 -1 2 ; -1 1 -2]
A4 =
 1 -1  2
-1  1 -2
>> [U4,S4,V4] = svd(A4)
U4 =
-0.71  0.71
 0.71  0.71
S4 =
 3.46  0.00  0.00
 0.00  0.00  0.00
V4 =
-0.41 -0.91 -0.00
 0.41 -0.18  0.89
-0.82  0.37  0.45
```

Note that:

- there is only one non zero value in S_4 which means that the rank of A_4 is 1
- there is a value in the first column of S_4 which means that the first column of U_4 is a basis vector for A_4

- there is no value in column 2 of S_4, which means column 2 of U_4 is a basis for the complement of the column space of A
- columns where a 0 value is on the diagonal of S indicates that the corresponding column in V is a basis vector for the nullspace of A. Additionally, if a column in S does not correspond to a column in A, then said column in S is a basis vector for the nullspace of A

d Nullspace and the SVD

Consider a rank deficient matrix: $A \in \mathbb{R}^{m \times m}$

$$\begin{aligned}
&A \in \mathbb{R}^{2\times 2}\\
&A = U\Sigma V^\top\\
&U = \begin{bmatrix}\vec{u}_1 & \vec{u}_2\end{bmatrix}\\
&\Sigma = \begin{bmatrix}\sigma_1 & 0\\ 0 & 0\end{bmatrix}\\
&V = \begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix}\\
&\vec{w} \in \mathbb{R}.
\end{aligned}$$

What is $A\vec{w}$? It is the following:

$$A\vec{w} = U\Sigma V^\top \vec{w}.$$

So then what is $V^\top \vec{w}$

$$V^\top \vec{w} = \begin{bmatrix}\vec{v}_1^\top \cdot \vec{w}\\ \vec{v}_2^\top \cdot \vec{w}\end{bmatrix}.$$

Note: '$\cdot$' is the dot product.
And continuing to derive the equation:

$$\begin{aligned}
\Sigma V^\top \vec{w} &= \begin{bmatrix}\sigma_1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\vec{v}_1^\top \cdot \vec{w}\\ \vec{v}_2^\top \cdot \vec{w}\end{bmatrix}\\
&= \begin{bmatrix}\sigma_1\vec{v}_1 \cdot \vec{w}\\ 0\end{bmatrix}
\end{aligned}$$

.

When is the first entry $\sigma_1\vec{v}_1 \cdot \vec{w}$ equal to 0?
If and only if $\vec{v}_1 \cdot \vec{w} = 0$, which is true when $\vec{w} \propto \vec{v}_2$ Thus, $A\vec{w} = 0$ if and only if $\vec{w} \propto \vec{v}_2$. Therefore the nullspace of A is $\vec{v}_2$

e Orthonormal Basis Vectors and the SVD

Consider the following computation: Note that:

```
>> A5 = [1 -1 ; -1 1 ; 2 -2.02]
A5 =
 1.00 -1.00
-1.00  1.00
 2.00 -2.02
>> [U5,S5,V5] = svd(A5)
U5 =
-0.41  0.58 -0.71
 0.41 -0.58 -0.71
-0.82 -0.58  0.00
S5 =
 3.48  0.00
 0.00  0.01
 0.00  0.00
V5 =
-0.70  0.71
 0.71  0.70
```

- a large ratio between non zero values in S_5 is indicative of that the matrix A_5 is *almost* rank deficient.

$$\frac{S_5(1,1)}{S_5(2,2)} = 427.1043$$

  which suggest that an error of 1% could have a whole number change of 4 on the result. If we ignore smaller values like 0.01 we can say that mathematically, this matrix is rank deficient

- corresponding vectors in U_5 to these smaller values in S_5 are ‘almost’ the null space of A_5

SVD Properties

for a general matrix $A \in \mathbb{R}^{m\times n}$

- $U \in \mathbb{R}^{m\times m}$ is an Orthonormal basis for the data space of A
- $V \in \mathbb{R}^{n\times n}$ is an Orthonormal basis for the weight space of A

- $\Sigma \in \mathbb{R}^{m \times n}$: singular value 'diagonal' matrix which is made up of the singular values (pseudo-Eigenvalues) of A. Same dimensions as A
- the last $(n-r)$ columns of V are a basis for the nullspace of A

Learning Summary

Students should now be able to

- determine the rank of A from SVD
- find column space from left singular vectors
- find the nullspace from right singular vectors
- find the orthogonal complements from U and V