Linear Data Analysis
Single Value Decomposition

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a Intro to the Single Value Decomposition

Recall a diagonalizable matrix is $A \in \mathbb{R}^{m \times m}$ where

$$A = E\Lambda E^{-1}.$$

For a matrix of this type there are multiple situations we have not considered: For any matrix $A \in \mathbb{R}^{m \times n}$ what if:

- $A \in \mathbb{R}^{m \times m}$ and is full rank with no eigenvector basis?
- $A \in \mathbb{R}^{m \times m}$ and is rank deficient?
- $A \in \mathbb{R}^{m \times n}$ is full rank?
- $A \in \mathbb{R}^{m \times n}$ is rank deficient?

We will use a method to handle these matrices by first examining the following matrices

$$[A^\top A] \preceq 0 \quad [AA^\top] \preceq 0.$$

where both are diagonalizable

b The Left Transpose Product

The Left transpose product is $A^\top A$.
Consider the rectangular matrix $A \in \mathbb{R}^{m\times n}$.
Suppose that the columns of A are linearly independent. (meaning must be either square or full rank).
Let:

$$\begin{aligned}
B_v &= A^\top A \\
&= V\Lambda V^\top && \leftarrow \text{by the spectral theorum} \\
V &= [\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n] \\
\Lambda &= \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_n \end{bmatrix} \\
.
\end{aligned}$$

Now, suppose that the rank of A equals $rank(A) = r$ such that:

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_r & \\ & & & 0\ldots \end{bmatrix}.$$

where all values on the diagonal past λ_r are 0.

Example

Given the matrix rectangular matrix A

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix} \qquad A^\top A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$\lambda_1 = 4 \qquad \lambda_2 = 2.$$

Because A is full rank and tall thin, it's EigenVectors are an orthonormal basis such that:

$$\vec{v}_1 \propto \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{v}_2 \propto \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Which is expected for a symmetric, positive semidefinite matrix.

c The Right Transpose Product

The right transpose product is $AA^\top$.
Consider the rectangular matrix $A \in \mathbb{R}^{m\times n}$
Suppose that A has linearly independent **rows** (meaning A must be square or full rank).
Let:

$$\begin{aligned} B_u &= AA^\top \\ &= U\Lambda U^\top \\ U &= [\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m] \\ \Lambda &= \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_m \end{bmatrix} \end{aligned}$$

.

Same as Left Transpose, Suppose $rank(A) = r$ such that:

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_r & \\ & & & 0\ldots \end{bmatrix}.$$

Example

Given the rectangular matrix A:

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix} \qquad B_u = AA^\top$$

$$\lambda_1 = 4 \ \lambda_2 = 2 \ \lambda_3 = 0 \qquad B_u = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}.$$

Where:

$$\vec{v}_1 \propto \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \vec{v}_2 \propto \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \vec{v}_3 \propto \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Such that these EigenVectors create a orthonormal basis. Furthermore, the nullspace of B_u should be spanned by $\vec{v}_3$

d Structure of The Singular Value Decomposition

Here, we will state the Single Value Decomposition (SVD) theorum. A decomposition is a way of representing a matrix using the products of factors. Sometimes referred to as a decomposition or factorization.
A general $A \in \mathbb{R} \gtrdot \times \ltimes$ can be factorized as:

$$A = U\Sigma V^\top.$$

Where:

$$\begin{aligned} &U \in \mathbb{R}^{m\times m} && \text{real, Orthogonal Matrix, Orthonormal columns: basis for } \mathbb{R}^m \\ &V \in \mathbb{R}^{n\times n} && \text{real, Orthogonal Matrix, Orthonormal columns: basis for } \mathbb{R}^n \\ &\Sigma \in \mathbb{R}^{m\times n} && \text{`diagonal' matrix of singular values such that:} \\ & && \sigma \in \mathbb{R},\;\; rank(A) = r \rightarrow \text{if } j \leq r : \sigma > 0 \end{aligned}$$

Note, diagonal is in quotes as while Σ isn't necessarily square, it has values only on its diagonal Additionally, the values in Σ are ordered largest to smallest such that σ_1 is the largest value and σ_r is the smallest non zero value.
The Singular matrix $\Sigma \in \mathbb{R}^{m\times n}$ is structured as:

$$\Sigma = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_r & \\ & & & 0\ldots \end{bmatrix}.$$

where all values are 0 accept for the diagonal: where values below λ_r are 0

Example

Given:

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix} \quad A = U\Sigma V^\top.$$

Observe that U is the same as B_u and V is the same as B_v. Additionally, Σ must have the same shape as A, thus:

$$\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}.$$

Now, we can use the SVD to decompose A.
First, we will to the left transpose:

$$\begin{aligned} A^\top A &= [U\Sigma V^\top]^\top [U\Sigma V^\top] \\ &= V\Sigma^\top U^\top U\Sigma V^\top \\ &= V\Sigma^2 V^\top \end{aligned}$$
$$.$$

Thus $A^\top A$ is the spectral decomposition of a positive semidefinite symmetric matrix. Note Σ^2 is the EigenValue matrix and thus, $\sigma^2 \in \Sigma$ are the EigenValues of $A^\top A$
Next, we will do the right transpose:

$$\begin{aligned} AA^\top &= [U\Sigma V^\top][U\Sigma V^\top]^\top \\ &= U\Sigma V^\top V\Sigma^\top U^\top \\ &= U\Sigma^2 U^\top \end{aligned}$$
$$.$$

Note that $AA^\top$ and $A^\top A$ share most of their EigenValues

Learning Summary

students should now be able to:

- state dimensions of U and V from A
- write the structure of Σ from rank r of A
- compute EigenValues of $A^\top A$ and $AA^\top$