Numerical Solutions of the Geodesic Equations

MATH 552 - Final Project

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Introduction

In this work we study geodesics defined as curves that in some sense minimize the distance between two points on a surface. We derive the geodesic equation by means of differential geometry and solve it numerically. We find a curve in two dimensional space that corresponds to straight lines on a surface. Our numerical scheme consists in reducing the considered initial value to a system of first order ordinary differntial equations and solve it using forward Euler's method. There is no assessment of performance nor of stability in this work.

Definitions

Let U be a simply connected domain and $p : U \to \mathbb{R}^3; p(u^1, u^2)$ be the parametrization of a regular surface. Let $I \subset \mathbb{R}$ and $c : I \to U; c(c^1(t), c^2(t))$ be a regular curve. Let $\alpha = p \circ c$.

Vector Field

Let $V : I \to \mathbb{R}^3$. We call V a **vector field** if $V(t) \in T_{\alpha(t)}p$ for all $t \in I$.

Covariant Derivative

The **covariant derivative** of a vector field V along α is the projection of $\frac{dV}{dt}(t)$ on $T_{\alpha(t)}p$. We denoted it as $\frac{\nabla V}{dt}$.

Geodesic

We call α a **geodesic** on the image of p if and only if $\frac{\nabla \alpha'}{dt} = 0$.

Geodesic Equation Derivation

We addopt the notation $\frac{\partial p}{\partial u^1} := p_1$ and $\frac{\partial p}{\partial u^2} := p_2$. Let $V = V^1(t)p_1(c) + V^2(t)p_2(c)$ be a vector field. Let's compute $\frac{\nabla V}{dt}$. Write

$$\begin{aligned} V &= V^1(t)p_1(c) + V^2(t)p_2(c) \\ \implies \frac{dV}{dt} &= \frac{dV^1}{dt}p_1 + V^1\left(p_{11}\frac{dc^1}{dt} + p_{12}\frac{dc^2}{dt}\right) + \frac{dV^2}{dt}p_2 + V^2\left(p_{21}\frac{dc^1}{dt} + p_{22}\frac{dc^2}{dt}\right) \end{aligned} \tag{$\star$}$$

Now, recall the gaus equations

$$\begin{aligned} p_{11} &= \Gamma^1_{11}p_1 + \Gamma^2_{11}p_2 + h_{11}\nu \\ p_{12} &= \Gamma^1_{12}p_1 + \Gamma^2_{12}p_2 + h_{12}\nu \\ p_{21} &= \Gamma^1_{21}p_1 + \Gamma^2_{21}p_2 + h_{21}\nu \\ p_{22} &= \Gamma^1_{22}p_1 + \Gamma^2_{22}p_2 + h_{22}\nu \end{aligned}$$

Recall that the metric which is the positive definite matrix

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

where, with respect to our parametrization, $g_{11} = \langle p_1, p_1 \rangle, g_{12} = g_{21} = \langle p_1, p_2 \rangle, g_{22} = \langle p_2, p_2 \rangle$ Here, $\Gamma^1_{11}, \Gamma^2_{11}, \ldots, \Gamma^2_{22}$ are the **Christoffel's symbols**. The Christoffel's symbols can be derived from the metric alone,

$$\begin{aligned} \Gamma^1_{ij} &= \frac{1}{2}g^{1\alpha}\left(\frac{\partial g_{\alpha i}}{\partial u^j} + \frac{\partial g_{\alpha j}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^\alpha}\right) \\ \Gamma^2_{ij} &= \frac{1}{2}g^{2\alpha}\left(\frac{\partial g_{\alpha i}}{\partial u^j} + \frac{\partial g_{\alpha j}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^\alpha}\right) \end{aligned}$$

The $h_{11}, h_{12}, \ldots, h_{22}$ are the coefficients of the second fundamental form given by $h_{ij} = -\langle \nu_j, p_i \rangle$. So we can rewrite $(\star)$ as

$$\begin{aligned}
\frac{dV}{dt} &= \frac{dV^1}{dt}p_1 + V^1\left[\frac{dc^1}{dt}\left(\Gamma^1_{11}p_1 + \Gamma^2_{11}p_2 + h_{11}\nu\right) + \frac{dc^2}{dt}\left(\Gamma^1_{12}p_1 + \Gamma^2_{12}p_2 + h_{12}\nu\right)\right] \\
&+ \frac{dV^2}{dt}p_2 + V^2\left[\frac{dc^1}{dt}\left(\Gamma^1_{21}p_1 + \Gamma^2_{21}p_2 + h_{21}\nu\right) + \frac{dc^2}{dt}\left(\Gamma^1_{22}p_1 + \Gamma^2_{22}p_2 + h_{22}\nu\right)\right] \\
&= \sum_{i=1}^{2}\left[\frac{dV^i}{dt}p_i + V^i\sum_{k=1}^{2}\sum_{j=1}^{2}\left(\frac{dc^j}{dt}\Gamma^k_{ij}p_k + \frac{dc^j}{dt}h_{ij}\nu\right)\right] \\
&= \sum_{i=1}^{2}\left[\frac{dV^i}{dt}p_iV^i\sum_{i,j=1}^{2}\left(\frac{dc^j}{dt}\Gamma^k_{ij}p_k\right) + V^i\sum_{k,j=1}^{2}\left(\frac{dc^j}{dt}h_{ij}\nu\right)\right] \\
&= \sum_{i=1}^{2}\left\{\left[\frac{dV^i}{dt}V^i\sum_{i,j=1}^{2}\frac{dc^j}{dt}\Gamma^k_{ij}\right]p_i + V^i\sum_{k,j=1}^{2}\left(\frac{dc^j}{dt}h_{ij}\nu\right)\right\}
\end{aligned}$$

So, we get

$$\frac{\nabla V}{dt} = \sum_{i=1}^{2}\left[\frac{dV^i}{dt}V^i\sum_{i,j=1}^{2}\frac{dc^j}{dt}\Gamma^k_{ij}\right]p_1 \tag{$\star\star$}$$

Now, if we take

$$\begin{aligned}
V = \frac{d}{dt}\alpha &= \frac{d}{dt}X \circ c \\
&= \frac{dc^1}{dt}X_1 + \frac{dc^2}{dt}X_2
\end{aligned}$$

then we get that $V^1 = \frac{dc^1}{dt}$ and $V^2 = \frac{dc^2}{dt}$. Substituting these into $(\star\star)$ and setting $\frac{\nabla V}{dt} = 0$ we get precisely the geodesic equation as encountered in most literature.

$$\frac{d^2c^k}{dt^2} + \sum_{i,j=1}^{2}\frac{dc^i}{dt}\frac{dc^j}{dt}\Gamma^k_{ij} = 0, \quad k = 1, 2.$$

Numerical Solution

To solve the geodesic equation numerically we have to pick our initial conditions. The explicit solution requires an initial coordinate position c_0 and initial coordinate velocity $\frac{dc_0}{dt}$. Our goal is to numerically solve the following initial value problem.

$$\begin{cases}
\frac{d^2c^1}{dt^2} + \sum_{i,j=1}^{2}\frac{dc^i}{dt}\frac{dc^j}{dt}\Gamma^1_{ij} & = 0 \\
\frac{d^2c^2}{dt^2} + \sum_{i,j=1}^{2}\frac{dc^i}{dt}\frac{dc^j}{dt}\Gamma^2_{ij} & = 0 \\
c^1(0) = c^1_0, \left.\frac{dc^1}{dt}\right|_0 & = \frac{dc^1_0}{dt} \\
c^2(0) = c^1_0, \left.\frac{dc^2}{dt}\right|_0 & = \frac{dc^2_0}{dt}.
\end{cases}$$

One rewrites the two nonlinear 2nd order ODEs as a system of four first-order ODEs.

$$\frac{d}{dt}\begin{pmatrix} c^1 \\ \frac{dc^1}{dt} \\ c^2 \\ \frac{dc^2}{dt} \end{pmatrix} = \begin{pmatrix} \frac{dc^1}{dt} \\ -\sum\limits_{i,j=1}^{2} \frac{dc^i}{dt}\frac{dc^j}{dt}\Gamma^1_{ij} \\ \frac{dc^2}{dt} \\ -\sum\limits_{i,j=1}^{2} \frac{dc^i}{dt}\frac{dc^j}{dt}\Gamma^2_{ij}. \end{pmatrix}$$

The chosen scheme to numerically solve the above equations is the forward Euler scheme. Let $c^1_m \approx c^1(\Delta t)$ and $c^2_m \approx c^2(\Delta t)$. We replace the continous derivatives by a first order forward difference approximation to get the following update formulas.

$$\frac{c^k_{m+1} - c^k_m}{\Delta t} = \frac{dc^k_m}{dt} \implies c^k_{m+1} = \frac{dc^k_m}{dt}\Delta t + c^k_m$$

and
$$\frac{\frac{dc^k_{m+1}}{dt} - \frac{dc^k_m}{dt}}{\Delta t} = -\sum_{i,j=1}^{2} \frac{dc^i_m}{dt}\frac{dc^j_m}{dt}\Gamma^k_{ij} \implies \frac{dc^k_{m+1}}{dt} = \frac{dc^k_m}{dt} - \sum_{i,j=1}^{2} \frac{dc^i_m}{dt}\frac{dc^j_m}{dt}\Gamma^1_{ij}\Delta t, \quad k = 1, 2.$$

Results

Unit Sphere

We consider the regular parametrizaton of a unit sphere.

$$p(u^1, u^2) = \left(\cos u^2 \sin u^1, \sin u^2 \sin u^1, \cos u^1\right), (u^1, u^2) \in [0, \pi] \times [0, 2\pi).$$

The partial derivatives are given.

$$p_1 = \left(\cos u^1 \cos u^2, \sin u^2 \cos u^1, -\sin u^1\right), \quad p_2 = \left(-\sin u^2 \sin u^1, \cos u^2 \sin u^1, 0\right).$$

So our metric is

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 u^1. \end{pmatrix}$$

By using the formulas presented in previous section to compute the Christoffel's symbols one eventually gets.

$$\Gamma^1_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & -\sin u^1 \cos u^1 \end{pmatrix}, \quad \Gamma^2_{ij} = \begin{pmatrix} 0 & \cot u^1 \\ \cot u^1 & 0. \end{pmatrix}$$

In particular, the geodesic equation for the unit sphere is given by

$$\begin{aligned} \frac{d^2c^1}{dt^2} - \sin^2 u^1 \left(\frac{dc^2}{dt}\right)^2 &= 0 \\ \frac{d^2c^2}{dt^2} + 2\cot u^1 \frac{dc^1}{dt}\frac{dc^2}{dt} &= 0. \end{aligned}$$

We integrate the system by the discretization outlined in the previous section. Below is a plot of one geodesic circle (great circle) on the unit sphere.

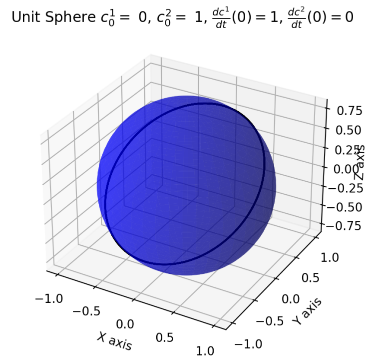

Figure 1: Great circle plotted in black. Here $\Delta t = 2^{-4}$ and solution was simulated until $t = 50$.

Torus

One can plot geodesics on general discretized surfaces using our approach. We ommit the paramerization alongside the metric and the Christoffel's symbols used for the torus. There are different kinds of geodesics that exist on the torus. Below is a plot of three of these geodesics, each belonging to a different family.

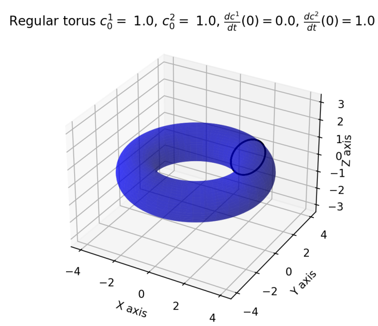

Figure 2: Meridian geodesic. $\Delta t = 2^{-8}$ and solution simulated until $t = 15$.

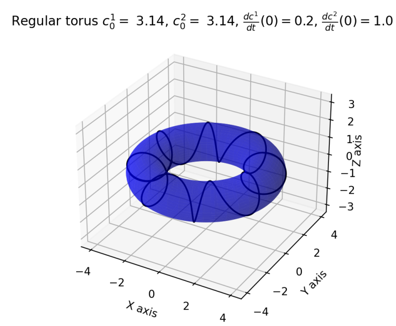

Figure 3: Unbounded geodesic that cross both inner and outer equators and can pass through all the points on the surface. $\Delta t = 2^{-8}$ and solution simulated until $t = 50$.

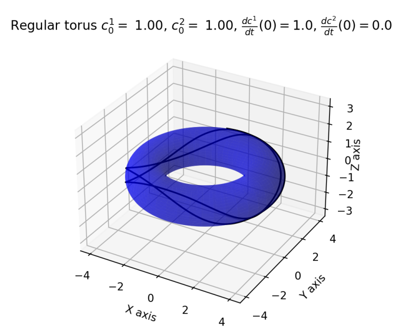

Figure 4: Bounded geodesics that spiral close to the inner equator but never touch it. $\Delta t = 2^{-8}$ and solution simulated until $t = 15$.

Hyperbolic Plane

One can even study abstract surfaces's geodesics where we define all concepts / quantities that depend solely on the first fundamental form. Consider, for instance the Poincaré's half plane model which is the upper half plane $H = \{(x, u) : y > 0; x, y \in \mathbb{R}\}$ together with the metric

$$(ds)^2 = \frac{(dx)^2 + (dy)^2}{y^2}.$$

This makes a model of two-dimensional hyperbolic geometry. Below is a plot of the geodesics of the hyperbolic plane. In hyperbolic geometry, Euclidean parallel postulate is false. Given a point and a line, there exist at least two lines parallel to the first passing through the point. In fact one can draw infinitely many lines that pass through the points and do not intersect the line.

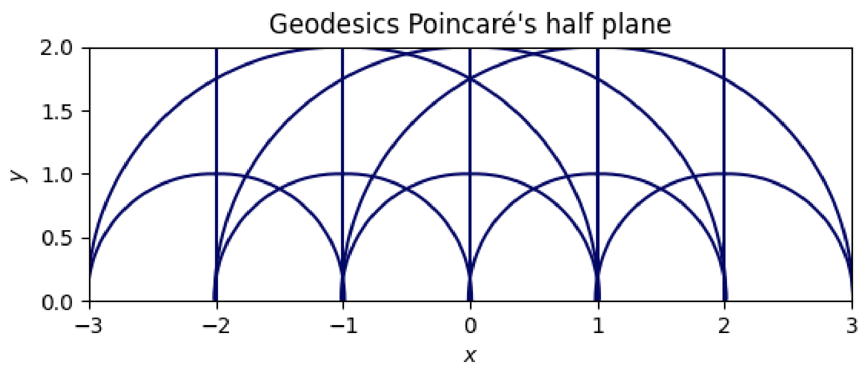

Figure 5: Geodesics plotted in blue. Each solution was simulated until $t = 10$ with $\Delta t = 2^{-4}$.

Conclusion and Further Analysis

We have presented of the geodesic inital value problem for different surfaces. For the sake of a complete analysis, precise analytical solutions of the problems would enable us to examine the perfomance of our scheme. One could explore the length minimizing property of the geodesics by means of calculus of variations since our solutions satisfy the Euler-Lagrange equations seeking to minimize the local distance between two points on a surface. Finally, we can apply this concept to serve as basis for many problems in reallity. For instance, one could want to calculate horizontal distances on Earth or understand how particles travel in a gravitational field. We learned that challenges associated with geodesics can be transformed into numerical analysis problems and approximated by simple schemes easily and accurately.

References

[1] Kobayashi, Shoshichi. Differential Geometry of Curves and Surfaces. Springer, 2019.