Notebook - Competitive Programming

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Contents

1 Data structures

1.1 Matrix

```
template <typename T>
struct Matrix {
  vector<vector<T>> d;

  Matrix() : Matrix(0) {}
  Matrix(int n) : Matrix(n, n) {}
  Matrix(int n, int m) : Matrix(vector<vector<T>>(n, vector<T>(m))) {}
  Matrix(const vector<vector<T>> &v) : d(v) {}

  constexpr int n() const { return (int)d.size(); }
  constexpr int m() const { return n() ? (int)d[0].size() : 0; }

  void rotate() { *this = rotated(); }

  Matrix<T> rotated() const {
    Matrix<T> res(m(), n());
    for (int i = 0; i < m(); i++) {
      for (int j = 0; j < n(); j++) {
        res[i][j] = d[n() - j - 1][i];
      }
    }
    return res;
  }

  Matrix<T> pow(int power) const {
    assert(n() == m());

    auto res = Matrix<T>::identity(n());
    auto b = *this;
    while (power) {
      if (power & 1) res *= b;
      b *= b;
      power >>= 1;
    }
    return res;
  }

  Matrix<T> submatrix(int start_i, int start_j, int rows = INT_MAX,
                      int cols = INT_MAX) const {
    rows = min(rows, n() - start_i);
    cols = min(cols, m() - start_j);
    if (rows <= 0 or cols <= 0) return {};

    Matrix<T> res(rows, cols);
    for (int i = 0; i < rows; i++)
      for (int j = 0; j < cols; j++) res[i][j] = d[i + start_i][j + start_j];
    return res;
  }

  Matrix<T> translated(int x, int y) const {
    Matrix<T> res(n(), m());
    for (int i = 0; i < n(); i++) {
      for (int j = 0; j < m(); j++) {
        if (i + x < 0 or i + x >= n() or j + y < 0 or j + y >= m()) continue;
        res[i + x][j + y] = d[i][j];
      }
    }
    return res;
  }

  static Matrix<T> identity(int n) {
    Matrix<T> res(n);
    for (int i = 0; i < n; i++) res[i][i] = 1;
    return res;
  }

  vector<T> &operator[](int i) { return d[i]; }
  const vector<T> &operator[](int i) const { return d[i]; }
  Matrix<T> &operator+=(T value) {
    for (auto &row : d) {
      for (auto &x : row) x += value;
    }
    return *this;
  }
  Matrix<T> operator+(T value) const {
    auto res = *this;
    for (auto &row : res) {
      for (auto &x : row) x = x + value;
    }
    return res;
  }
  Matrix<T> &operator-=(T value) {
    for (auto &row : d) {
      for (auto &x : row) x -= value;
    }
    return *this;
  }
  Matrix<T> operator-(T value) const {
    auto res = *this;
    for (auto &row : res) {
      for (auto &x : row) x = x - value;
    }
    return res;
  }
  Matrix<T> &operator*=(T value) {
    for (auto &row : d) {
      for (auto &x : row) x *= value;
    }
    return *this;
  }
  Matrix<T> operator*(T value) const {
    auto res = *this;
    for (auto &row : res) {
      for (auto &x : row) x = x * value;
    }
    return res;
  }
  Matrix<T> &operator/=(T value) {
    for (auto &row : d) {
      for (auto &x : row) x /= value;
    }
    return *this;
```

```cpp
}
Matrix<T> operator/(T value) const {
  auto res = *this;
  for (auto &row : res) {
    for (auto &x : row) x = x / value;
  }
  return res;
}
Matrix<T> &operator+=(const Matrix<T> &o) {
  assert(n() == o.n() and m() == o.m());
  for (int i = 0; i < n(); i++) {
    for (int j = 0; j < m(); j++) {
      d[i][j] += o[i][j];
    }
  }
  return *this;
}
Matrix<T> operator+(const Matrix<T> &o) const {
  assert(n() == o.n() and m() == o.m());
  auto res = *this;
  for (int i = 0; i < n(); i++) {
    for (int j = 0; j < m(); j++) {
      res[i][j] = res[i][j] + o[i][j];
    }
  }
  return res;
}
Matrix<T> &operator-=(const Matrix<T> &o) {
  assert(n() == o.n() and m() == o.m());
  for (int i = 0; i < n(); i++) {
    for (int j = 0; j < m(); j++) {
      d[i][j] -= o[i][j];
    }
  }
  return *this;
}
Matrix<T> operator-(const Matrix<T> &o) const {
  assert(n() == o.n() and m() == o.m());
  auto res = *this;
  for (int i = 0; i < n(); i++) {
    for (int j = 0; j < m(); j++) {
      res[i][j] = res[i][j] - o[i][j];
    }
  }
  return res;
}
Matrix<T> &operator*=(const Matrix<T> &o) {
  *this = *this * o;
  return *this;
}
Matrix<T> operator*(const Matrix<T> &o) const {
  assert(m() == o.n());
  Matrix<T> res(n(), o.m());
  for (int i = 0; i < res.n(); i++) {
    for (int j = 0; j < res.m(); j++) {
      auto &x = res[i][j];
      for (int k = 0; k < m(); k++) {
        x += (d[i][k] * o[k][j]);
      }
    }
  }
  return res;
}

friend istream &operator>>(istream &is, Matrix<T> &mat) {
  for (auto &row : mat)
    for (auto &x : row) is >> x;
  return is;
}
friend ostream &operator<<(ostream &os, const Matrix<T> &mat) {
  bool frow = 1;
  for (auto &row : mat) {
    if (not frow) os << '\n';
    bool first = 1;
    for (auto &x : row) {
      if (not first) os << ' ';
      os << x;
      first = 0;
    }

    frow = 0;
  }
  return os;
}

auto begin() { return d.begin(); }
auto end() { return d.end(); }
auto rbegin() { return d.rbegin(); }
auto rend() { return d.rend(); }

auto begin() const { return d.begin(); }
auto end() const { return d.end(); }
auto rbegin() const { return d.rbegin(); }
auto rend() const { return d.rend(); }
};
```

1.2 Merge Sort Tree

Like a segment tree but each node st_i stores a sorted subarray

- $inrange(l, r, a, b)$: counts the number of elements $x \in [l, r]$ such that $a \le x \le b$.

Memory: $O(N \log N)$
Build: $O(N \log N)$
inrange: $O(\log^2 N)$

```cpp
template <class T>
struct MergeSortTree {
  int n;
  vector<vector<T>> st;
  MergeSortTree(vector<T>& xs) : n(len(xs)), st(n << 1) {
    for (int i = 0; i < n; i++) st[i + n] = vector<T>({xs[i]});

    for (int i = n - 1; i > 0; i--) {
      st[i].resize(len(st[i << 1]) + len(st[i << 1 | 1]));
      merge(all(st[i << 1]), all(st[i << 1 | 1]), st[i].begin());
    }
  }
  int count(int i, T a, T b) {
    return upper_bound(all(st[i]), b) - lower_bound(all(st[i]), a);
  }

  int inrange(int l, int r, T a, T b) {
    int ans = 0;

    for (l += n, r += n + 1; l < r; l >>= 1, r >>= 1) {
      if (l & 1) ans += count(l++, a, b);
      if (r & 1) ans += count(--r, a, b);
    }

    return ans;
  }
};
```

1.3 Minimal Excluded With Updates (MEX-U)

In the problem you need to change individual numbers in the array, and compute the new MEX of the array after each such update.
Pre-compute: $O(N \log N)$
Update: $O(\log N)$
Query: $O(1)$

```
class Mex {
 private:
  map<ll, ll> frequency;
  set<ll> missing_numbers;
  vl A;

 public:
  Mex(vl const& A) : A(A) {
    for (ll i = 0; i <= A.size(); i++) missing_numbers.insert(i);

    for (ll x : A) {
      ++frequency[x];
      missing_numbers.erase(x);
    }
  }

  ll mex() { return *missing_numbers.begin(); }

  void update(ll idx, ll new_value) {
    if (--frequency[A[idx]] == 0) missing_numbers.insert(A[idx]);
    A[idx] = new_value;
    ++frequency[new_value];
    missing_numbers.erase(new_value);
  }
};
```

1.4 Minimal Excluded (MEX)

Given an array A of size N. You have to find the minimal non-negative element that is not present in the array. That number is commonly called the MEX (minimal excluded).
Time: $O(N)$

```
ll mex(vl const& A) {
  static bool used[MAX + 1ll] = {0};

  for (ll x : A) {
    if (x <= MAX) used[x] = true;
  }

  ll result = 0;
  while (used[result]) ++result;

  for (ll x : A) {
    if (x <= MAX) used[x] = false;
  }

  return result;
}
```

1.5 Range Min Query (RMQ)

Build: $O(N)$
Query: $O(1)$

```
// @brunomaletta
template <typename T>
struct rmq {
  vector<T> v;
  int n;
  static const int b = 30;
  vector<int> mask, t;

  int op(int x, int y) { return v[x] <= v[y] ? x : y; }
  int msb(int x) { return __builtin_clz(1) - __builtin_clz(x); }
  int small(int r, int sz = b) { return r - msb(mask[r] & ((1 << sz) - 1)); }
  rmq() {}
  rmq(const vector<T>& v_) : v(v_), n(v.size()), mask(n), t(n) {
    for (int i = 0, at = 0; i < n; mask[i++] = at |= 1) {
      at = (at << 1) & ((1 << b) - 1);
      while (at and op(i - msb(at & -at), i) == i) at ^= at & -at;
    }
    for (int i = 0; i < n / b; i++) t[i] = small(b * i + b - 1);
    for (int j = 1; (1 << j) <= n / b; j++)
      for (int i = 0; i + (1 << j) <= n / b; i++)
        t[n / b * j + i] =
          op(t[n / b * (j - 1) + i], t[n / b * (j - 1) + i + (1 << (j - 1))]);
  }
  int index_query(int l, int r) {
    if (r - l + 1 <= b) return small(r, r - l + 1);
    int x = l / b + 1, y = r / b - 1;
    if (x > y) return op(small(l + b - 1), small(r));
    int j = msb(y - x + 1);
    int ans = op(small(l + b - 1),
                 op(t[n / b * j + x], t[n / b * j + y - (1 << j) + 1]));
    return op(ans, small(r));
  }
  T query(int l, int r) { return v[index_query(l, r)]; }
};
```

1.6 Segment Tree (Parameterized OP)

Query: $O(\log N)$
Update: $O(\log N)$

```
template <typename T, auto op>
class SegTree {
 private:
  T e;
  ll N;
  vector<T> seg;

 public:
  SegTree(ll N, T e) : e(e), N(N), seg(N + N, e) {}

  void assign(ll i, T v) {
    i += N;
    seg[i] = v;
    for (i >>= 1; i; i >>= 1) seg[i] = op(seg[2 * i], seg[2 * i + 1]);
  }

  T query(ll l, ll r) {
    T la = e, ra = e;
    l += N;
    r += N;

    while (l <= r) {
      if (l & 1) la = op(la, seg[l++]);
      if (~r & 1) ra = op(seg[r--], ra);
      l >>= 1;
      r >>= 1;
    }

    return op(la, ra);
  }
};
```

1.7 Segment Tree 2D

Query: $O(\log N \cdot \log M)$
Update: $O(\log N \cdot \log M)$

```
template <typename T, auto op>
class SegTree {
 private:
  T e;
  ll n, m;
  vector<vector<T>> seg;

 public:
  SegTree(ll n, ll m, T e)
    : e(e), n(n), m(m), seg(2 * n, vector<T>(2 * m, e)) {}

  void assign(ll x, ll y, T v) {
    ll ny = y += m;
    for (x += n; x; x >>= 1, y = ny) {
      if (x >= n)
        seg[x][y] = v;
      else
        seg[x][y] = op(seg[2 * x][y], seg[2 * x + 1][y]);

      while (y >>= 1) seg[x][y] = op(seg[x][2 * y], seg[x][2 * y + 1]);
    }
  }

  T query(ll lx, ll rx, ll ly, ll ry) {
    ll ans = e, nx = rx + n, my = ry + m;

    for (lx += n, ly += m; lx <= ly; ++lx >>= 1, --ly >>= 1)
      for (rx = nx, ry = my; rx <= ry; ++rx >>= 1, --ry >>= 1) {
        if (lx & 1 and rx & 1) ans = op(ans, seg[lx][rx]);
        if (lx & 1 and !(ry & 1)) ans = op(ans, seg[lx][ry]);
        if (!(ly & 1) and rx & 1) ans = op(ans, seg[ly][rx]);
        if (!(ly & 1) and !(ry & 1)) ans = op(ans, seg[ly][ry]);
      }

    return ans;
  }
};
```

1.8 Segment Tree Lazy

Query (Range Sum): $O(\log N)$
Update (Sum Value): $O(\log N)$

```
template <typename T>
class SegTreeLazy {
 private:
  int N;
  vector<T> seg, lzy;

  void down(int k, int l, int r) {
    seg[k] += (r - l + 1) * lzy[k];
    if (l < r) {
      lzy[k << 1] += lzy[k];
      lzy[k << 1 | 1] += lzy[k];
    }
    lzy[k] = 0;
  }

  void update(int i, int j, int k, int l, int r, T v) {
    if (lzy[k]) down(k, l, r);
    if (i > r or j < l) return;
    if (i <= l and j >= r) {
      seg[k] += (r - l + 1) * v;
      if (l < r) {
        lzy[k << 1] += v;
        lzy[k << 1 | 1] += v;
      }
      return;
    }

    update(i, j, k << 1, l, (l + r) / 2, v);
    update(i, j, k << 1 | 1, (l + r) / 2 + 1, r, v);
    seg[k] = seg[k << 1] + seg[k << 1 | 1];
  }
  T query(int i, int j, int k, int l, int r) {
    if (lzy[k]) down(k, l, r);
    if (i > r or j < l) return 0;
    if (i <= l and j >= r) return seg[k];

    T lft = query(i, j, k << 1, l, (l + r) / 2);
    T rgt = query(i, j, k << 1 | 1, (l + r) / 2 + 1, r);
    return lft + rgt;
  }

 public:
  SegTreeLazy(int N) : N(N), seg(N << 2, 0), lzy(N << 2, 0) {}

  void update(int i, int j, T v) { update(i, j, 1, 0, N - 1, v); }

  T query(int i, int j) { return query(i, j, 1, 0, N - 1); }
};
```

1.9 Segtreelazy Generic

```
using SegT = ll;

struct QueryT {
  SegT mx, mn;
  QueryT() : mx(numeric_limits<SegT>::min()), mn(numeric_limits<SegT>::max())
    {}
  QueryT(SegT _v) : mx(_v), mn(_v) {}
};

inline QueryT combine(QueryT ln, QueryT rn, ii lr1, ii lr2) {
  ln.mx = max(ln.mx, rn.mx);
  ln.mn = min(ln.mn, rn.mn);
  return ln;
}

using LazyT = SegT;

inline QueryT applyLazyInQuery(QueryT q, LazyT l, ii lr) {
  if (q.mx == QueryT().mx) q.mx = SegT();
  if (q.mn == QueryT().mn) q.mn = SegT();
  q.mx += l, q.mn += l;
  return q;
}

inline LazyT applyLazyInLazy(LazyT a, LazyT b) { return a + b; }

using UpdateT = SegT;

inline QueryT applyUpdateInQuery(QueryT q, UpdateT u, ii lr) {
  if (q.mx == QueryT().mx) q.mx = SegT();
  if (q.mn == QueryT().mn) q.mn = SegT();
  q.mx += u, q.mn += u;
  return q;
}

inline LazyT applyUpdateInLazy(LazyT l, UpdateT u, ii lr) { return l + u; }
```

```
template <typename Qt = QueryT, typename Lt = LazyT, typename Ut = UpdateT,
          auto C = combine, auto ALQ = applyLazyInQuery,
          auto ALL = applyLazyInLazy, auto AUQ = applyUpdateInQuery,
          auto AUL = applyUpdateInLazy>
struct LazySegmentTree {
  int n, h;
  vector<Qt> ts;
  vector<Lt> ds;
  vector<ii> lrs;

  LazySegmentTree(int _n)
    : n(_n),
      h(sizeof(int) * 8 - __builtin_clz(n)),
      ts(n << 1),
      ds(n),
      lrs(n << 1) {
    for (int i = 0; i < n; i++) lrs[i + n] = {i, i};
    for (int i = n - 1; i > 0; i--) {
      lrs[i] = {lrs[i << 1].first, lrs[i << 1 | 1].second};
    }
  }

  LazySegmentTree(const vector<Qt> &xs) : LazySegmentTree(xs.size()) {
    copy(all(xs), ts.begin() + n);
    for (int i = 0; i < n; i++) lrs[i + n] = {i, i};
    for (int i = n - 1; i > 0; i--) {
      ts[i] = C(ts[i << 1], ts[i << 1 | 1], lrs[i << 1], lrs[i << 1 | 1]);
    }
  }

  void set(int p, Qt v) {
    ts[p + n] = v;
    build(p + n);
  }

  void upd(int l, int r, Ut v) {
    l += n, r += n + 1;
    int l0 = l, r0 = r;
    for (; l < r; l >>= 1, r >>= 1) {
      if (l & 1) apply(l++, v);
      if (r & 1) apply(--r, v);
    }
    build(l0), build(r0 - 1);
  }

  Qt qry(int l, int r) {
    l += n, r += n + 1;
    push(l), push(r - 1);
    Qt resl = Qt(), resr = Qt();
    ii lr1 = {l, l}, lr2 = {r, r};
    for (; l < r; l >>= 1, r >>= 1) {
      if (l & 1) resl = C(resl, ts[l], lr1, lrs[l]), l++;
      if (r & 1) r--, resr = C(ts[r], resr, lrs[r], lr2);
    }
    return C(resl, resr, lr1, lr2);
  }

  void build(int p) {
    while (p > 1) {
      p >>= 1;
      ts[p] = ALQ(C(ts[p << 1], ts[p << 1 | 1], lrs[p << 1], lrs[p << 1 | 1]),
                  ds[p], lrs[p]);
    }
  }

  void push(int p) {
    for (int s = h; s > 0; s--) {
      int i = p >> s;
      if (ds[i] != Lt()) {
        apply(i << 1, ds[i]), apply(i << 1 | 1, ds[i]);
        ds[i] = Lt();
      }
    }
  }

  inline void apply(int p, Ut v) {
    ts[p] = AUQ(ts[p], v, lrs[p]);
    if (p < n) ds[p] = AUL(ds[p], v, lrs[p]);
  }
};
```

1.10 Union Find Disjoint Set (UFDS)

Uncomment the lines to recover which element belong to each set.

Time: $\approx O(1)$ for everything.

```
class UFDS {
 public:
  vi ps, size;
  // vector<unordered_set<int>> sts;

  UFDS(int N) : size(N + 1, 1), ps(N + 1), sts(N) {
    iota(ps.begin(), ps.end(), 0);
    // for (int i = 0; i < N; i++) sts[i].insert(i);
  }

  int find_set(int x) { return x == ps[x] ? x : (ps[x] = find_set(ps[x])); }

  bool same_set(int x, int y) { return find_set(x) == find_set(y); }

  void union_set(int x, int y) {
    if (same_set(x, y)) return;

    int px = find_set(x);
    int py = find_set(y);

    if (size[px] < size[py]) swap(px, py);

    ps[py] = px;
    size[px] += size[py];
    // sts[px].merge(sts[py]);
  }
};
```

1.11 Wavelet Tree

Build: $O(N \cdot \log \sigma)$.
Queries: $O(\log \sigma)$.
σ = alphabet length

```
typedef vector<int>::iterator iter;

class WaveletTree {
 public:
  int L, H;
  WaveletTree *l, *r;
  vector<int> frq;

  WaveletTree(iter fr, iter to, int x, int y) {
    L = x, H = y;
    if (fr >= to) return;

    int M = L + ((H - L) >> 1);
    auto F = [M](int x) { return x <= M; };

    frq.reserve(to - fr + 1);
    frq.push_back(0);
    for (auto it = fr; it != to; it++) frq.push_back(frq.back() + F(*it));

    if (H == L) return;
    auto pv = stable_partition(fr, to, F);
    l = new WaveletTree(fr, pv, L, M);
    r = new WaveletTree(pv, to, M + 1, H);
  }

  // Find the k-th smallest element in positions [i,j)
  int quantile(int l, int r, int k) {
    if (l > r) return 0;
    if (L == H) return L;
    int inLeft = frq[r] - frq[l - 1];
    int lb = frq[l - 1], rb = frq[r];
    if (k <= inLeft) return this->l->quantile(lb + 1, rb, k);
    return this->r->quantile(l - lb, r - rb, k - inLeft);
  }

  // Count occurrences of number c until position i -> [0, i].
  int rank(int c, int i) { return until(c, min(i + 1, (int)frq.size() - 1)); }

  int until(int c, int i) {
    if (c > H or c < L) return 0;
    if (L == H) return i;

    int M = L + ((H - L) >> 1);
    int r = frq[i];
    if (c <= M)
      return this->l->until(c, r);
    else
      return this->r->until(c, i - r);
  }

  // Count number of occurrences of numbers in the range [a, b]
  int range(int i, int j, int a, int b) const {
    if (b < a or j < i) return 0;
    return range(i, j + 1, L, H, a, b);
  }

  int range(int i, int j, int a, int b, int L, int U) const {
    if (b < L or U < a) return 0;
    if (L <= a and b <= U) return j - i;
    int M = a + ((b - a) >> 1);
    int ri = frq[i], rj = frq[j];
    return this->l->range(ri, rj, a, M, L, U) +
           this->r->range(i - ri, j - rj, M + 1, b, L, U);
  }

  // Number of elements greater than or equal to k in [l, r];
  // Can count distinct in a range with aux vector of next pos
  int greater(int l, int r, int k) { return _greater(l + 1, r + 1, k); }

  int _greater(int l, int r, int k) {
    if (l > r or k > H) return 0;
    if (L >= k) return r - l + 1;

    int ri = frq[l - 1], rj = frq[r];
    return this->l->_greater(ri + 1, rj, k) +
           this->r->_greater(l - ri, r - rj, k);
  }
};
```

2 Dynamic programming

2.1 Kadane

```
int kadane(const vi& xs) {
  vi s(xs.size());
  s[0] = xs[0];

  for (size_t i = 1; i < xs.size(); ++i) s[i] = max(xs[i], s[i - 1] + xs[i]);

  return *max_element(all(s));
}
```

2.2 Longest Increasing Subsequence (LIS)

Time: $O(N \cdot \log N)$.

```
int lis(vi const& a) {
  int n = a.size();
  const int INF = 1e9;
  vi d(n + 1, INF);
  d[0] = -INF;

  for (int i = 0; i < n; i++) {
    int l = upper_bound(d.begin(), d.end(), a[i]) - d.begin();
    if (d[l - 1] < a[i] && a[i] < d[l]) d[l] = a[i];
  }

  int ans = 0;
  for (int l = 0; l <= n; l++) {
    if (d[l] < INF) ans = l;
  }

  return ans;
}
```

3 Extras

3.1 cin/cout __int128_t

Allows standard reading and writing with cin/cout for 128-bit integers using *__int128_t* type.

```
ostream& operator<<(ostream& dest, __int128_t value) {
  ostream::sentry s(dest);
  if (s) {
    __uint128_t tmp = value < 0 ? -value : value;
    char buffer[128];
    char* d = end(buffer);
    do {
      --d;
      *d = "0123456789"[tmp % 10];
      tmp /= 10;
    } while (tmp != 0);
    if (value < 0) {
      --d;
      *d = '-';
    }
    int len = end(buffer) - d;
    if (dest.rdbuf()->sputn(d, len) != len) dest.setstate(ios_base::badbit);
  }
  return dest;
}

istream& operator>>(istream& is, __int128_t& value) {
  string s;
  is >> s;

  __int128_t res = 0;
  size_t i = 0;

  bool neg = false;
  if (s[i] == '-') neg = 1, i++;
  for (; i < s.size(); ++i) (res *= 10) += (s[i] - '0');

  value = neg ? -res : res;
  return is;
}
```

4 Geometry

4.1 Circle

```
// çãDefinio da classe Point e da çãfuno equals()

template <typename T>
```

```
struct Circle {
  Point<T> C;
  T r;

  enum { IN, ON, OUT } PointPosition;

  PointPosition position(const Point& P) const {
    auto d = dist(P, C);

    return equals(d, r) ? ON : (d < r ? IN : OUT);
  }

  static std::optional<Circle> from_2_points_and_r(const Point<T>& P,
                                                   const Point<T>& Q, T r) {
    double d2 = (P.x - Q.x) * (P.x - Q.x) + (P.y - Q.y) * (P.y - Q.y);
    double det = r * r / d2 - 0.25;

    if (det < 0.0) return {};

    double h = sqrt(det);

    auto x = (P.x + Q.x) * 0.5 + (P.y - Q.y) * h;
    auto y = (P.y + Q.y) * 0.5 + (Q.x - P.x) * h;

    return Circle<T>{Point<T>(x, y), r};
  }

  static std::experimental::optional<Circle> from_3_points(const Point<T>& P,
                                                           const Point<T>& Q,
                                                           const Point<T>& R)

    {
    auto a = 2 * (Q.x - P.x);
    auto b = 2 * (Q.y - P.y);
    auto c = 2 * (R.x - P.x);
    auto d = 2 * (R.y - P.y);

    auto det = a * d - b * c;

    // Pontos colineares
    if (equals(det, 0)) return {};

    auto k1 = (Q.x * Q.x + Q.y * Q.y) - (P.x * P.x + P.y * P.y);
    auto k2 = (R.x * R.x + R.y * R.y) - (P.x * P.x + P.y * P.y);

    // çãSoluo do sistema por Regra de Cramer
    auto cx = (k1 * d - k2 * b) / det;
    auto cy = (a * k2 - c * k1) / det;

    Point<T> C{cx, cy};
    auto r = distance(P, C);

    return Circle<T>(C, r);
  }

  // çãInterseo entre o ícrculo c e a reta que passa por P e Q
  template <typename T>
  std::vector<Point<T>> intersection(const Circle<T>& c, const Point<T>& P,
                                     const Point<T>& Q) {
    auto a = pow(Q.x - P.x, 2.0) + pow(Q.y - P.y, 2.0);
    auto b = 2 * ((Q.x - P.x) * (P.x - c.C.x) + (Q.y - P.y) * (P.y - c.C.y));
    auto d = pow(c.C.x, 2.0) + pow(c.C.y, 2.0) + pow(P.x, 2.0) + pow(P.y, 2.0)
     +
             2 * (c.C.x * P.x + c.C.y * P.y);
    auto D = b * b - 4 * a * d;

    if (D < 0)
      return {};
    else if (equals(D, 0)) {
      auto u = -b / (2 * a);
      auto x = P.x + u * (Q.x - P.x);
      auto y = P.y + u * (Q.y - P.y);
      return {Point{x, y}};
    }

    auto u = (-b + sqrt(D)) / (2 * a);

    auto x = P.x + u * (Q.x - P.x);
    auto y = P.y + u * (Q.y - P.y);

    auto P1 = Point{x, y};

    u = (-b - sqrt(D)) / (2 * a);

    x = P.x + u * (Q.x - P.x);
    y = P.y + u * (Q.y - P.y);

    auto P2 = Point{x, y};

    return {P1, P2};
  }
};
```

4.2 Convex Hull Trick

Add lines of the form y = ax + b to a set and query the maximun value of y at a given x. add(a, b): add line y = ax + b query(x): find the maximum value of y at x
Time: $O(\log n)$ amortized for add(a, b) and $O(\log n)$ for query(x).

```
template <typename T = ll>
struct ConvexHullTrick {
  static constexpr T inf = numeric_limits<T>::max();

  struct Line {
    T a, b;
    mutable T x_inter;
    T eval(T x) const { return a * x + b; }
    bool operator<(const Line& rhs) const { return a < rhs.a; }
    bool operator<(T x) const { return x_inter < x; }
  };
  multiset<Line, less<>> ln;

  T query(T x) const {
    auto it = ln.lower_bound(x);
    if (it == ln.end()) return inf;
    return it->eval(x);
  }
```

```cpp
  void add(T a, T b) {
    auto it = ln.insert({a, b, 0});
    while (overlap(it)) ln.erase(next(it)), update(it);
    if (it != ln.begin() and !overlap(prev(it))) it = prev(it), update(it);
    while (it != ln.begin() and overlap(prev(it)))
      it = prev(it), ln.erase(next(it)), update(it);
  }

 private:
  void update(auto it) const {
    if (next(it) == ln.end())
      it->x_inter = inf;
    else if (it->a == next(it)->a)
      (it->x_inter = it->b >= next(it)->b ? inf : -inf);
    else {
      auto h = (it->b - next(it)->b);
      auto l = (next(it)->a - it->a);
      it->x_inter = h / l - ((h ^ l) < 0 && h % l);
    }
  }

  bool overlap(auto it) const {
    update(it);
    if (next(it) == ln.end()) return false;
    if (it->a == next(it)->a) return it->b >= next(it)->b;
    return it->x_inter >= next(it)->x_inter;
  }
};
```

4.3 Convex Hull

Given a set of points find the smallest convex polygon that contains all the given points.
Time: $O(N \cdot \log N)$
By default it removes the collinear points, set the boolean to true if you don't want that

```cpp
struct pt {
  double x, y;
};

int orientation(pt a, pt b, pt c) {
  double v = a.x * (b.y - c.y) + b.x * (c.y - a.y) + c.x * (a.y - b.y);
  if (v < 0) return -1;  // clockwise
  if (v > 0) return +1;  // counter-clockwise
  return 0;
}

bool cw(pt a, pt b, pt c, bool include_collinear) {
  int o = orientation(a, b, c);
  return o < 0 || (include_collinear && o == 0);
}
bool collinear(pt a, pt b, pt c) { return orientation(a, b, c) == 0; }

void convex_hull(vector<pt>& a, bool include_collinear = false) {
  pt p0 = *min_element(a.begin(), a.end(), [](pt a, pt b) {
    return make_pair(a.y, a.x) < make_pair(b.y, b.x);
  });
  sort(a.begin(), a.end(), [&p0](const pt& a, const pt& b) {
    int o = orientation(p0, a, b);
    if (o == 0)
      return (p0.x - a.x) * (p0.x - a.x) + (p0.y - a.y) * (p0.y - a.y) <
             (p0.x - b.x) * (p0.x - b.x) + (p0.y - b.y) * (p0.y - b.y);
    return o < 0;
  });
  if (include_collinear) {
    int i = (int)a.size() - 1;
    while (i >= 0 && collinear(p0, a[i], a.back())) i--;
    reverse(a.begin() + i + 1, a.end());
  }

  vector<pt> st;
  for (int i = 0; i < (int)a.size(); i++) {
    while (st.size() > 1 &&
           !cw(st[st.size() - 2], st.back(), a[i], include_collinear))
      st.pop_back();
    st.push_back(a[i]);
  }

  a = st;
}
```

4.4 Convex Hull Trick

Add lines $ax + b$ and query maximum value at x. If you want to get minimum value, set `inf = numeric_limits<T>::max()`. In case of overflow, try to compress x values.
Time: $O(\log(\text{HI} - \text{LO}))$ for query, $O(\log(\text{HI} - \text{LO}))$ for add, $O(\log^2(\text{HI} - \text{LO}))$ for add segment.

```cpp
template <typename T = ll, T LO = T(-1e9), T HI = T(1e9)>
struct LiChaoTree {
  // get max value at x by default
  // to get min value, set inf = numeric_limits<T>::max()
  static constexpr T inf = numeric_limits<T>::min();
  static constexpr bool compare(T a, T b) {
    if constexpr (inf == numeric_limits<T>::max()) {
      return a < b;
    } else {
      return a > b;
    }
  }
  static constexpr T best(T a, T b) { return (compare(a, b) ? a : b); }
  struct Line {
    T a, b;
    array<int, 2> ch;
    Line(T a_ = 0, T b_ = inf) : a(a_), b(b_), ch({-1, -1}) {}
    constexpr T eval(T x) const { return a * x + b; }
    constexpr bool is_leaf() const { return ch[0] == -1 and ch[1] == -1; }
  };
  vector<Line> ln;
  LiChaoTree() { ln.emplace_back(); }

  T query(T x, int v = 0, T l = LO, T r = HI) {
    auto m = l + (r - l) / 2, val = ln[v].eval(x);
    if (ln[v].is_leaf()) return val;
    if (x <= m)
      return best(val, query(x, ch(v, 0), l, m));
    else
      return best(val, query(x, ch(v, 1), m + 1, r));
  }

  void add(T a, T b) { add({a, b}, 0, LO, HI); }
  void add(Line s, int v, T l, T r) {
    auto m = l + (r - l) / 2;
    bool L = compare(s.eval(l), ln[v].eval(l));
    bool M = compare(s.eval(m), ln[v].eval(m));
    bool R = compare(s.eval(r), ln[v].eval(r));
    if (M) swap(ln[v], s), swap(ln[v].ch, s.ch);
    if (s.b == inf) return;
    if (L != M)
      add(s, ch(v, 0), l, m);
    else if (R != M)
      add(s, ch(v, 1), m + 1, r);
  }

  void add_segment(T a, T b, T l, T r) { add_segment({a, b}, l, r, 0, LO, HI);
     }
  void add_segment(Line s, T l, T r, int v, T L, T R) {
    if (l <= L and R <= r) return add(s, v, L, R);
    auto m = L + (R - L) / 2;
    if (l <= m) add_segment(s, l, r, ch(v, 0), L, m);
    if (r > m) add_segment(s, l, r, ch(v, 1), m + 1, R);
  }

 private:
  int ch(int v, bool b) {
    if (ln[v].ch[b] == -1) {
      ln[v].ch[b] = (int)ln.size();
      ln.emplace_back();
    }
    return ln[v].ch[b];
  }
};
```

4.5 Point in Polygon

Given the vertices of a polygon, we want to determine if a point lies inside the polygon.
Time: $O(num\_vertices)$
Note: The points must be sorted in increasing order of x-coordinates.

```
const double EPS = 1e-9;
template <typename T>
bool point_in_polygon(Point<T> point, vector<Point<T>> polygon) {
  int num_vertices = polygon.size();
  T x = point.x, y = point.y;
  bool inside = false;
  Point<T> p1 = polygon[0], p2;  // p1 is the first vertex
  for (int i = 1; i <= num_vertices; i++) {
    p2 = polygon[i % num_vertices];  // next vertex

    if (abs((p2.y - p1.y) * (x - p1.x) - (p2.x - p1.x) * (y - p1.y)) < EPS &&
        (x - p1.x) * (x - p2.x) <= 0 && (y - p1.y) * (y - p2.y) <= 0) {
      return true;  // point is on the boundary
    }

    if (y > min(p1.y, p2.y)) {
      if (y <= max(p1.y, p2.y)) {
        if (p1.x == p2.x) {
          if (x <= p1.x) {
            inside = !inside;
          }
        } else if (x <= max(p1.x, p2.x) &&
                   x <= (y - p1.y) * (p2.x - p1.x) / (p2.y - p1.y) + p1.x) {
          inside = !inside;
        }
      }
    }
    p1 = p2;
  }
  return inside;
}
```

4.6 Point To Segment

```
typedef pair<double, double> pdb;

double pt2segment(pdb A, pdb B, pdb E) {
  pdb AB = {B.fst - A.fst, B.snd - A.snd};
  pdb BE = {E.fst - B.fst, E.snd - B.snd};
  pdb AE = {E.fst - A.fst, E.snd - A.snd};

  double AB_BE = AB.fst * BE.fst + AB.snd * BE.snd;
  double AB_AE = AB.fst * AE.fst + AB.snd * AE.snd;

  double ans;
  if (AB_BE > 0) {
    double y = E.snd - B.snd;
    double x = E.fst - B.fst;
    ans = hypot(x, y);
  } else if (AB_AE < 0) {
    double y = E.snd - A.snd;
    double x = E.fst - A.fst;
    ans = hypot(x, y);
  } else {
    auto [x1, y1] = AB;
    auto [x2, y2] = AE;
    double mod = hypot(x1, y1);
    ans = abs(x1 * y2 - y1 * x2) / mod;
  }

  return ans;
}
```

4.7 Point Vector

```
template <typename T>
struct Point {
  T x, y;

  Point(T x = 0, T y = 0) : x(x), y(y) {}

  inline Point operator+(const Point &p) const {
    return Point(x + p.x, y + p.y);
}
inline Point operator-(const Point &p) const {
  return Point(x - p.x, y - p.y);
}
inline Point operator+(const T &k) const { return Point(x + k, y + k); }
inline Point operator-(const T &k) const { return Point(x - k, y - k); }
inline Point operator*(const T &k) const { return Point(x * k, y * k); }
inline Point operator/(const T &k) const { return Point(x / k, y / k); }

inline Point &operator+=(const Point &p) {
  x += p.x, y += p.y;
  return *this;
}
inline Point &operator-=(const Point &p) {
  x -= p.x, y -= p.y;
  return *this;
}
inline Point &operator+=(const T &k) {
  x += k, y += k;
  return *this;
}
inline Point &operator-=(const T &k) {
  x -= k, y -= k;
  return *this;
}
inline Point &operator*=(const T &k) {
  x *= k, y *= k;
  return *this;
}
inline Point &operator/=(const T &k) {
  x /= k, y /= k;
  return *this;
}

inline bool operator==(const Point &p) const {
  return eq(x, p.x) and eq(y, p.y);
}
inline bool operator<(const Point &p) const {
  return eq(x, p.x) ? y < p.y : x < p.x;
}
inline bool operator>(const Point &p) const {
  return eq(x, p.x) ? y > p.y : x > p.x;
}
inline bool operator<=(const Point &p) const {
  return *this == p or *this < p;
}
inline bool operator>=(const Point &p) const {
  return *this == p or *this > p;
}

friend ostream &operator<<(ostream &os, const Point &p) {
  return os << p.x << ' ' << p.y;
}
friend istream &operator>>(istream &is, Point &p) { return is >> p.x >> p.y;
   }

template <typename U>
void rotate(U rad) {
    tie(x, y) =
      make_pair(x * cos(rad) - y * sin(rad), x * sin(rad) + y * cos(rad));
  }
  template <typename U>
  Point<U> rotated(U rad) const {
    return Point<U>(x * cos(rad) - y * sin(rad), x * sin(rad) + y * cos(rad));
  }
  inline T dot(const Point &p) const { return x * p.x + y * p.y; }
  inline T cross(const Point &p) const { return x * p.y - y * p.x; }
  inline T cross(const Point &a, const Point &b) const {
    return (a - *this).cross(b - *this);
  }
  inline T dist2() const { return x * x + y * y; }
  inline double dist() const { return hypot(x, y); }
  inline double angle() const { return atan2(y, x); }
  inline double norm() const { return sqrt(dot(*this)); }
  inline Point rot90() const { return Point(-y, x); }
  inline Point to(const Point &p) const { return p - *this; }
};

template <typename T>
struct Vector {
  T x = 0, y = 0;

  Vector(const Point<T> &A, const Point<T> &B) : x(B.x - A.x), y(B.y - A.y) {}

  T length() const { return hypot(x, y); }
};

template <typename T>
struct Line {
  T a, b, c;

  Line(T av, T bv, T cv) : a(av), b(bv), c(cv) {}

  Line(const Point<T> &P, const Point<T> &Q)
    : a(P.y - Q.y), b(Q.x - P.x), c(P.x * Q.y - Q.x * P.y) {}
};
```

4.8 Polygon

```
template <typename T>
class Polygon {
 private:
  vector<Point<T>> vs;
  int n;

 public:
  // O âparmetro deve conter os n évrtices do ípolgono
  Polygon(const vector<Point<T>>& ps) : vs(ps), n(vs.size()) {
    vs.push_back(vs.front());
  }

 private:
  T D(const Point<T>& P, const Point<T>& Q, const Point<T>& R) const {
    return (P.x * Q.y + P.y * R.x + Q.x * R.y) -
           (R.x * Q.y + R.y * P.x + Q.x * P.y);
  }
```

```
public:
 bool convex() const {
   // Um ípolgono deve ter, no minimo, 3 évrtices
   if (n < 3) return false;

   int P = 0, N = 0, Z = 0;

   for (int i = 0; i < n; ++i) {
     auto d = D(vs[i], vs[(i + 1) % n], vs[(i + 2) % n]);
     d ? (d > 0 ? ++P : ++N) : ++Z;
   }

   return P == n or N == n;
 }

private:
 double distance(const Point<T>& P, const Point<T>& Q) {
   return hypot(P.x - Q.x, P.y - Q.y);
 }

public:
 double perimeter() const {
   auto p = 0.0;

   for (int i = 0; i < n; ++i) p += distance(vs[i], vs[i + 1]);

   return p;
 }

 double area() const {
   auto a = 0.0;

   for (int i = 0; i < n; ++i) {
     a += vs[i].x * vs[i + 1].y;
     a -= vs[i + 1].x * vs[i].y;
   }

   return 0.5 * fabs(a);
 }

private:
 // Ângulo APB, em radianos
 double angle(const Point<T>& P, const Point<T>& A, const Point<T>& B) {
   auto ux = P.x - A.x;
   auto uy = P.y - A.y;

   auto vx = P.x - B.x;
   auto vy = P.y - B.y;

   auto num = ux * vx + uy * vy;
   auto den = hypot(ux, uy) * hypot(vx, vy);

   // Caso especial: se den == 0, algum dos vetores é degenerado: os
   // dois pontos ãso iguais. Neste caso, o ângulo ãno áest definido

   return acos(num / den);
 }
```

```
 bool equals(double x, double y) {
   static const double EPS{1e-6};
   return fabs(x - y) < EPS;
 }

public:
 bool contains(const Point<T>& P) const {
   if (n < 3) return false;

   auto sum = 0.0;

   for (int i = 0; i < n - 1; ++i) {
     auto d = D(P, vs[i], vs[i + 1]);
     auto a = angle(P, vs[i], vs[i + 1]);
     sum += d > 0 ? a : (d < 0 ? -a : 0);
   }

   static const double PI = acos(-1.0);
   return equals(fabs(sum), 2 * PI);
 }

private:
 // çãInterseo entre a reta AB e o segmento de reta PQ
 Point<T> intersection(const Point<T>& P, const Point<T>& Q, const Point<T>&
   A,
                       const Point<T>& B) {
   auto a = B.y - A.y;
   auto b = A.x - B.x;
   auto c = B.x * A.y - A.x * B.y;
   auto u = fabs(a * P.x + b * P.y + c);
   auto v = fabs(a * Q.x + b * Q.y + c);

   // éMdia ponderada pelas âdistncias de P e Q éat a reta AB
   return {(P.x * v + Q.x * u) / (u + v), (P.y * v + Q.y * u) / (u + v)};
 }

public:
 // Corta o ípolgono com a reta r que passa por A e B
 Polygon cut_polygon(const Point<T>& A, const Point<T>& B) const {
   vector<Point<T>> points;
   const double EPS{1e-6};

   for (int i = 0; i < n; ++i) {
     auto d1 = D(A, B, vs[i]);
     auto d2 = D(A, B, vs[i + 1]);

     // éVrtice à esquerda da reta
     if (d1 > -EPS) points.push_back(vs[i]);

     // A aresta cruza a reta
     if (d1 * d2 < -EPS)
       points.push_back(intersection(vs[i], vs[i + 1], A, B));
   }

   return Polygon(points);
 }
```

```
  double circumradius() const {
    auto s = distance(vs[0], vs[1]);
    const double PI{acos(-1.0)};

    return (s / 2.0) * (1.0 / sin(PI / n));
  }

  double apothem() const {
    auto s = distance(vs[0], vs[1]);
    const double PI{acos(-1.0)};

    return (s / 2.0) * (1.0 / tan(PI / n));
  }
};
```

4.9 Polynominoes

Geometric figure made by equal squares, connected between themselves in a way that at least one side of each square coincide with a side of another square.
Watch out: the number of polynominoes increases fastly (size 12 has 63.600 figures)

```
// We consider the rotations
// as distinct (0, 10, 10+9, 10+9+8...)
vi pos = {0, 10, 19, 27, 34, 40, 45, 49, 52, 54, 55};

const int MAXP = 10;

struct Poly {
  ii v[MAXP];
  int64_t id;
  int n;

  Poly() {
    n = 1;
    v[0] = {0, 0};
    normalize();
  }

  Poly(vii &vp) {
    n = vp.size();
    for (int i = 0; i < n; i++) v[i] = vp[i];
    normalize();
  }

  ii &operator[](int i) { return v[i]; }

  bool add(int a, int b) {
    for (int i = 0; i < n; i++) {
      auto [f, s] = v[i];
      if (f == a and s == b) return false;
    }

    v[n++] = ii{a, b};
    normalize();
    return true;
  }

  void normalize() {
    int mx = 100, my = 100;
    for (int i = 0; i < n; i++) {
      auto [f, s] = v[i];
      mx = min(mx, f), my = min(my, s);
    }

    id = 0;
    for (int i = 0; i < n; i++) {
      auto &[f, s] = v[i];
      f -= mx, s -= my;
      id |= (1LL << (pos[f] + s));
    }
  }

  bool operator<(Poly oth) { return id < oth.id; }
};

vector<Poly> poly[MAXP + 1];

void buildPoly(int mxN) {
  for (int i = 0; i <= mxN; i++) poly[i].clear();

  Poly init;
  queue<Poly> q;
  unordered_set<int64_t> used;
  q.push(init);
  used.insert(init.id);
  while (not q.empty()) {
    Poly u = q.front();
    q.pop();
    poly[u.n].emplace_back(u);

    if (u.n == mxN) continue;

    for (int i = 0; i < u.n; i++) {
      for (auto [dx, dy] : dir4) {
        Poly to = u;
        auto [f, s] = to[i];
        bool ok = to.add(f + dx, s + dy);

        if (ok and not used.count(to.id)) {
          q.push(to);
          used.insert(to.id);
        }
      }
    }
  }
}
```

4.10 Sweep Line

```
struct Segment {
  double a, b, c;
  Point A, B;
  size_t idx;

  Segment(const Point& P, const Point& Q, size_t i)
    : a(P.y - Q.y),
      b(Q.x - P.x),
      c(P.x * Q.y - Q.x * P.y),
      A(P),
      B(Q),
      idx(i) {}

  bool operator<(const Segment& s) const {
    return (-a * sweep_x - c) * s.b < (-s.a * sweep_x - s.c) * b;
  }

  optional<Point> intersection(const Segment& s) const {
    auto det = a * s.b - b * s.a;

    if (not equals(det, 0.0))  // Concorrentes
    {
      auto x = (-c * s.b + s.c * b) / det;
      auto y = (-s.c * a + c * s.a) / det;

      if (min(A.x, B.x) <= x and x <= max(A.x, B.x) and
          min(s.A.x, s.B.x) <= x and x <= max(s.A.x, s.B.x)) {
        return Point{x, y};
      }
    }

    return {};
  }

  static double sweep_x;
};

double Segment::sweep_x;

struct Event {
  enum Type { OPEN, INTERSECTION, CLOSE };

  Point P;
  Type type;
  size_t i;

  bool operator<(const Event& e) const {
    if (P != e.P) return e.P < P;

    if (type != e.type) return type > e.type;

    return i > e.i;
  }
};

void add_neighbor_intersections(const Segment& s, const set<Segment>& sl,
                                set<Point>& ans,
                                priority_queue<Event>& events) {
  // TODO: garantir que a busca identifique unicamente o elemento s,
  // éatravs do ajuste fino da ávarivel Segment::sweep_x
  auto it = sl.find(s);

  if (it != sl.begin()) {
    auto L = *prev(it);
    auto P = s.intersection(L);
```

```
    if (P and ans.count(P.value()) == 0) {
      events.push(Event{P.value(), Event::INTERSECTION, s.idx});
      ans.insert(P.value());
    }
  }

  if (next(it) != sl.end()) {
    auto U = *next(it);
    auto P = s.intersection(U);

    if (P and ans.count(P.value()) == 0) {
      events.push(Event{P.value(), Event::INTERSECTION, s.idx});
      ans.insert(P.value());
    }
  }
}

set<Point> bentley_ottman(vector<Segment>& segments) {
  set<Point> ans;
  priority_queue<Event> events;

  for (size_t i = 0; i < segments.size(); ++i) {
    events.push(Event{segments[i].A, Event::OPEN, i});
    events.push(Event{segments[i].B, Event::CLOSE, i});
  }

  set<Segment> sl;

  while (not events.empty()) {
    auto e = events.top();
    events.pop();

    Segment::sweep_x = e.P.x;

    switch (e.type) {
      case Event::OPEN: {
        auto s = segments[e.i];
        sl.insert(s);

        add_neighbor_intersections(s, sl, ans, events);
      } break;

      case Event::CLOSE: {
        auto s = segments[e.i];
        auto it = sl.find(s);  // TODO: aqui étambm

        if (it != sl.begin() and it != sl.end()) {
          auto L = *prev(it);
          auto U = *next(it);
          auto P = L.intersection(U);

          if (P and ans.count(P.value()) == 0)
            events.push(Event{P.value(), Event::INTERSECTION, L.idx});
        }

        sl.erase(it);
      } break;
      default:
        auto r = segments[e.i];
        auto p = sl.equal_range(r);

        vector<Segment> range(p.first, p.second);

        // Remove os segmentos que se interceptam
        sl.erase(p.first, p.second);

        // Reinsere os segmentos
        Segment::sweep_x += 0.1;

        sl.insert(range.begin(), range.end());

        // Procura çõintersees com os novos vizinhos
        for (const auto& s : range)
          add_neighbor_intersections(s, sl, ans, events);
    }
  }

  return ans;
}
```

4.11 Triangulo

```
template <typename T>
struct Triangle {
  Point<T> A, B, C;

  // çãDefinio do émtodo area()

  // circulo inscrito no triangulo
  double circumradius() const {
    auto a = dist(B, C);
    auto b = dist(A, C);
    auto c = dist(A, B);

    return (a * b * c) / (4 * area());
  }

  Point<T> circumcenter() const {
    auto D = 2 * (A.x * (B.y - C.y) + B.x * (C.y - A.y) + C.x * (A.y - B.y));

    auto A2 = A.x * A.x + A.y * A.y;
    auto B2 = B.x * B.x + B.y * B.y;
    auto C2 = C.x * C.x + C.y * C.y;

    auto x = (A2 * (B.y - C.y) + B2 * (C.y - A.y) + C2 * (A.y - B.y)) / D;
    auto y = (A2 * (C.x - B.x) + B2 * (A.x - C.x) + C2 * (B.x - A.x)) / D;

    return {x, y};
  }

  // ortocentro do triangulo
  Point<T> orthocenter() const {
    Line<T> r(A, B), s(A, C);
    Line<T> u{r.b, -r.a, -(C.x * r.b - C.y * r.a)};
    Line<T> v{s.b, -s.a, -(B.x * s.b - B.y * s.a)};

    auto det = u.a * v.b - u.b * v.a;
    auto x = (-u.c * v.b + v.c * u.b) / det;
    auto y = (-v.c * u.a + u.c * v.a) / det;

    return {x, y};
  }
};
```

5 Graphs

5.1 Articulation Points

```
int dfs_num[MAX], dfs_low[MAX];
vi adj[MAX];

int dfs_articulation_points(int u, int p, int& next, set<int>& points) {
  int children = 0;
  dfs_low[u] = dfs_num[u] = next++;

  for (auto v : adj[u])
    if (not dfs_num[v]) {
      ++children;

      dfs_articulation_points(v, u, next, points);

      if (dfs_low[v] >= dfs_num[u]) points.insert(u);

      dfs_low[u] = min(dfs_low[u], dfs_low[v]);
    } else if (v != p)
      dfs_low[u] = min(dfs_low[u], dfs_num[v]);

  return children;
}

set<int> articulation_points(int N) {
  memset(dfs_num, 0, (N + 1) * sizeof(int));
  memset(dfs_low, 0, (N + 1) * sizeof(int));

  set<int> points;

  for (int u = 1, next = 1; u <= N; ++u)
    if (not dfs_num[u]) {
      auto children = dfs_articulation_points(u, u, next, points);

      if (children == 1) points.erase(u);
    }

  return points;
}
```

5.2 Bellman Ford

Time: $O(V \cdot E)$. Returns the shortest path from s to all other nodes.

```
using edge = tuple<int, int, int>;
```

```
pair<vi, vi> bellman_ford(int s, int N, const vector<edge>& edges) {
  vi dist(N + 1, oo), pred(N + 1, oo);

  dist[s] = 0;
  pred[s] = s;

  for (int i = 1; i <= N - 1; i++)
    for (auto [u, v, w] : edges)
      if (dist[u] < oo and dist[v] > dist[u] + w) {
        dist[v] = dist[u] + w;
        pred[v] = u;
      }

  return {dist, pred};
}
```

5.3 BFS 0/1

Time: $O(V + E)$.

```
vii adj[MAX];

vi bfs_01(int s, int N) {
  vi dist(N + 1, oo);
  dist[s] = 0;

  deque<int> q;
  q.emplace_back(s);

  while (not q.empty()) {
    auto u = q.front();
    q.pop_front();

    for (auto [v, w] : adj[u])
      if (dist[v] > dist[u] + w) {
        dist[v] = dist[u] + w;
        w == 0 ? q.emplace_front(v) : q.emplace_back(v);
      }
  }

  return dist;
}
```

5.4 Binary Lifting

Time: $O(N \cdot \log_2 K)$

```
const int MAXN = 2e5, MAXLOG2 = 60;
int bl[MAXN][MAXLOG2 + 1];

int jump(int u, ll k) {
  for (int i = 0; i <= MAXLOG2; i++)
    if (k & (1LL << i)) u = bl[u][i];

  return u;
}
```

```
void build(int N) {
  for (int i = 1; i <= MAXLOG2; i++)
    for (int j = 0; j < N; j++) bl[j][i] = bl[bl[j][i - 1]][i - 1];
}
```

5.5 Bridges

```
int dfs_num[MAX], dfs_low[MAX];
vi adj[MAX];

void dfs_bridge(int u, int p, int& next, vii& bridges) {
  dfs_low[u] = dfs_num[u] = next++;

  for (auto v : adj[u])
    if (not dfs_num[v]) {
      dfs_bridge(v, u, next, bridges);

      if (dfs_low[v] > dfs_num[u]) bridges.emplace_back(u, v);

      dfs_low[u] = min(dfs_low[u], dfs_low[v]);
    } else if (v != p)
      dfs_low[u] = min(dfs_low[u], dfs_num[v]);
}

vii bridges(int N) {
  memset(dfs_num, 0, (N + 1) * sizeof(int));
  memset(dfs_low, 0, (N + 1) * sizeof(int));

  vii bridges;

  for (int u = 1, next = 1; u <= N; ++u)
    if (not dfs_num[u]) dfs_bridge(u, u, next, bridges);

  return bridges;
}
```

5.6 Negative Cycle Bellman Ford

Time: $O(V \cdot E)$. Detects whether there is a negative cycle in the graph using Bellman Ford.

```
using edge = tuple<int, int, int>;

bool has_negative_cycle(int s, int N, const vector<edge>& edges) {
  vi dist(N + 1, oo);
  dist[s] = 0;

  for (int i = 1; i <= N - 1; i++)
    for (auto [u, v, w] : edges)
      if (dist[u] < oo and dist[v] > dist[u] + w) dist[v] = dist[u] + w;

  for (auto [u, v, w] : edges)
    if (dist[u] < oo and dist[v] > dist[u] + w) return true;

  return false;
}
```

5.7 Negative Cycle Floyd Warshall

Time: $O(n^3)$. Detects whether there is a negative cycle in the graph using Floyd Warshall.

```
int dist[MAX][MAX];
vii adj[MAX];

bool has_negative_cycle(int N) {
  for (int u = 1; u <= N; ++u)
    for (int v = 1; v <= N; ++v) dist[u][v] = u == v ? 0 : oo;

  for (int u = 1; u <= N; ++u)
    for (auto [v, w] : adj[u]) dist[u][v] = w;

  for (int k = 1; k <= N; ++k)
    for (int u = 1; u <= N; ++u)
      for (int v = 1; v <= N; ++v)
        if (dist[u][k] < oo and dist[k][v] < oo)
          dist[u][v] = min(dist[u][v], dist[u][k] + dist[k][v]);

  for (int i = 1; i <= N; ++i)
    if (dist[i][i] < 0) return true;

  return false;
}
```

5.8 Dijkstra

```
pair<vl, vl> Graph::dijkstra(ll src) {
  vl pd(this->N, LLONG_MAX), ds(this->N, LLONG_MAX);
  pd[src] = src;
  ds[src] = 0;

  set<pll> st;
  st.emplace(0, src);

  while (!st.empty()) {
    ll u = st.begin()->snd;
    ll wu = st.begin()->fst;
    st.erase(st.begin());

    if (wu != ds[u]) continue;
    for (auto& [v, w] : adj[u]) {
      if (ds[v] > ds[u] + w) {
        ds[v] = ds[u] + w;
        pd[v] = u;
        st.emplace(ds[v], v);
      }
    }
  }

  return {ds, pd};
}
```

5.9 Floyd Warshall

```
vii adj[MAX];

pair<vector<vi>, vector<vi>> floyd_warshall(int N) {
  vector<vi> dist(N + 1, vi(N + 1, oo));
  vector<vi> pred(N + 1, vi(N + 1, oo));

  for (int u = 1; u <= N; ++u) {
    dist[u][u] = 0;
    pred[u][u] = u;
  }

  for (int u = 1; u <= N; ++u)
    for (auto [v, w] : adj[u]) {
      dist[u][v] = w;
      pred[u][v] = u;
    }

  for (int k = 1; k <= N; ++k) {
    for (int u = 1; u <= N; ++u) {
      for (int v = 1; v <= N; ++v) {
        if (dist[u][k] < oo and dist[k][v] < oo and
            dist[u][v] > dist[u][k] + dist[k][v]) {
          dist[u][v] = dist[u][k] + dist[k][v];
          pred[u][v] = pred[k][v];
        }
      }
    }
  }

  return {dist, pred};
}
```

5.10 Graph

```
class Graph {
 private:
  ll N;
  bool undirected;
  vector<vll> adj;

 public:
  Graph(ll N, bool is_undirected = true) {
    this->N = N;
    adj.resize(N);
    undirected = is_undirected;
  }

  void add(ll u, ll v, ll w) {
    adj[u].emplace_back(v, w);
    if (undirected) adj[v].emplace_back(u, w);
  }
};
```

5.11 TopSort - Kahn

Works only on Directed Acyclic Graphs (DAGs). For each edge (u,v), u comes before v in the ordering. If the task A is a prerequisite for task B, then A comes before B in the ordering. Time: $O(E \cdot log(v))$

```
unordered_set<int> in[MAX], out[MAX];
```

```
vi topological_sort(int N) {
  vi o;
  queue<int> q;

  for (int u = 1; u <= N; ++u)
    if (in[u].empty()) q.push(u);

  while (not q.empty()) {
    auto u = q.front();
    q.pop();

    o.emplace_back(u);

    for (auto v : out[u]) {
      in[v].erase(u);

      if (in[v].empty()) q.push(v);
    }
  }

  return (int)o.size() == N ? o : vi{};
}
```

5.12 Kosaraju

Time: $O(V + E)$. Returns a vector of vectors indicating the directed strongly connected nodes.

```
vi adj[MAX], rev[MAX];
bitset<MAX> visited;

void dfs(int u, vi& order) {
  if (visited[u]) return;

  visited[u] = true;

  for (auto v : adj[u]) dfs(v, order);

  order.emplace_back(u);
}

vi dfs_order(int N) {
  vi order;

  for (int u = 1; u <= N; ++u) dfs(u, order);

  return order;
}

void dfs_cc(int u, vi& cc) {
  if (visited[u]) return;

  visited[u] = true;
  cc.emplace_back(u);

  for (auto v : rev[u]) dfs_cc(v, cc);
}

vector<vi> kosaraju(int N) {
  auto order = dfs_order(N);
  reverse(order.begin(), order.end());

  for (int u = 1; u <= N; ++u)
    for (auto v : adj[u]) rev[v].emplace_back(u);

  vector<vi> cs;
  visited.reset();

  for (auto u : order) {
    if (visited[u]) continue;

    cs.emplace_back(vi());
    dfs_cc(u, cs.back());
  }

  return cs;
}
```

5.13 Kruskal

Time: $O(e \cdot log(v))$

```
using edge = tuple<int, int, int>;

int kruskal(int N, vector<edge>& es) {
  sort(es.begin(), es.end());

  int cost = 0;
  UnionFind ufds(N);

  for (auto [w, u, v] : es) {
    if (not ufds.same_set(u, v)) {
      cost += w;
      ufds.union_set(u, v);
    }
  }

  return cost;
}
```

5.14 Minimax

A MST minimizes the maximum weight between the edges in any spanning tree. Time: $O(e \cdot log(v))$

```
vii adj[MAX];

int minimax(int u, int N) {
  set<int> C;
  C.insert(u);

  priority_queue<ii, vii, greater<ii>> pq;

  for (auto [v, w] : adj[u]) pq.push(ii(w, v));

  int minmax = -oo;
  while ((int)C.size() < N) {
    int v, w;

    do {
      w = pq.top().first, v = pq.top().second;
      pq.pop();
    } while (C.count(v));

    minmax = max(minmax, w);
    C.insert(v);

    for (auto [s, p] : adj[v]) pq.push(ii(p, s));
  }

  return minmax;
}
```

5.15 MSF

Minimum Spanning Forest - a forest of trees of length k that connects all vertices in a graph with minimum total weight. Time: $O(e \cdot log(v))$

```
using edge = tuple<int, int, int>;

int msf(int k, int N, vector<edge>& es) {
  sort(es.begin(), es.end());

  int cost = 0, cc = N;
  UnionFind ufds(N);

  for (auto [w, u, v] : es) {
    if (not ufds.same_set(u, v)) {
      cost += w;
      ufds.union_set(u, v);

      if (--cc == k) return cost;
    }
  }

  return cost;
}
```

5.16 Minimum Spanning Graph (MSG)

Given some obligatory edges es, find a minimum spanning graph that contains them. Time: $O(e \cdot \log(v))$

```
using edge = tuple<int, int, int>;

const int MAX{100010};

vector<ii> adj[MAX];

int msg(int N, const vector<edge>& es) {
  set<int> C;
  priority_queue<ii, vii, greater<ii>> pq;
  int cost = 0;

  for (auto [u, v, w] : es) {
    cost += w;

    C.insert(u);
    C.insert(v);

    for (auto [r, s] : adj[u]) pq.push(ii(s, r));

    for (auto [r, s] : adj[v]) pq.push(ii(s, r));
  }

  while ((int)C.size() < N) {
    int v, w;

    do {
      w = pq.top().first, v = pq.top().second;
      pq.pop();
    } while (C.count(v));

    cost += w;
    C.insert(v);

    for (auto [s, p] : adj[v]) pq.push(ii(p, s));
  }

  return cost;
}
```

5.17 Prim

A node u is chosen to start a connected component. Time: $O(e \cdot log(v))$

```
const int MAX{100010};

vector<ii> adj[MAX];

int prim(int u, int N) {
  set<int> C;
  C.insert(u);

  priority_queue<ii, vector<ii>, greater<ii>> pq;

  for (auto [v, w] : adj[u]) pq.push(ii(w, v));

  int mst = 0;

  while ((int)C.size() < N) {
    int v, w;

    do {
      w = pq.top().first, v = pq.top().second;
      pq.pop();
    } while (C.count(v));

    mst += w;
    C.insert(v);

    for (auto [s, p] : adj[v]) pq.push(ii(p, s));
  }

  return mst;
}
```

5.18 Retrieve Path 2d

```
vll Graph::retrieve_path_2d(ll src, ll trg, const vector<vl>& pred) {
  vll p;

  do {
    p.emplace_back(pred[src][trg], trg);
    trg = pred[src][trg];
  } while (trg != src);

  reverse(all(p));

  return p;
}
```

5.19 Retrieve Path

```
vll Graph::retrieve_path(ll src, ll trg, const vl& pred) {
  vll p;

  do {
    p.emplace_back(pred[trg], trg);
    trg = pred[trg];
  } while (trg != src);

  reverse(all(p));

  return p;
}
```

5.20 Second Best MST

Time: $O(v \cdot e)$

```
using edge = tuple<int, int, int>;

pair<int, vi> kruskal(int N, vector<edge>& es, int blocked = -1) {
  vi mst;
  UnionFind ufds(N);
  int cost = 0;

  for (int i = 0; i < (int)es.size(); ++i) {
    auto [w, u, v] = es[i];

    if (i != blocked and not ufds.same_set(u, v)) {
      cost += w;
      ufds.union_set(u, v);
      mst.emplace_back(i);
    }
  }

  return {(int)mst.size() == N - 1 ? cost : oo, mst};
}

int second_best(int N, vector<edge>& es) {
  sort(es.begin(), es.end());

  auto [_, mst] = kruskal(N, es);
  int best = oo;

  for (auto blocked : mst) {
    auto [cost, __] = kruskal(N, es, blocked);
    best = min(best, cost);
  }

  return best;
}
```

5.21 TopSort - Tarjan

Works only on Directed Acyclic Graphs (DAGs). For each edge (u,v), u comes before v in the ordering. If the task A is a prerequisite for task B, then A comes before B in the ordering. Time: $O(V + E)$

```
enum State { NOT_FOUND, FOUND, PROCESSED };

vi adj[MAX];

bool dfs(int u, vi& o, vi& state) {
  if (state[u] == PROCESSED) return true;

  if (state[u] == FOUND) return false;

  state[u] = FOUND;

  for (auto v : adj[u])
    if (not dfs(v, o, state)) return false;

  state[u] = PROCESSED;
  o.emplace_back(u);

  return true;
}

vi topological_sort(int N) {
  vi o, state(N + 1, NOT_FOUND);

  for (int u = 1; u <= N; ++u)
    if (state[u] == NOT_FOUND and not dfs(u, o, state)) return {};

  reverse(o.begin(), o.end());

  return o;
}
```

6 Math

6.1 Binomial

```cpp
ll binom(ll n, ll k) {
  if (k > n) return 0;
  vll dp(k + 1, 0);
  dp[0] = 1;
  for (ll i = 1; i <= n; i++)
    for (ll j = k; j > 0; j--) dp[j] = dp[j] + dp[j - 1];
  return dp[k];
}
```

6.2 Count Divisors Range

```cpp
vl divisors(MAX, 0);
void count_divisors_range() {
  for (ll i = 1; i <= MAX; i++) {
    for (ll j = 1; j * i <= MAX; j++) divisors[i * j]++;
  }
}
```

6.3 Count Divisors

```cpp
ll count_divisors(ll num) {
  ll count = 1;
  for (int i = 2; (ll)i * i <= num; i++) {
    if (num % i == 0) {
      int e = 0;
      do {
        e++;
        num /= i;
      } while (num % i == 0);
      count *= e + 1;
    }
  }
  if (num > 1) {
    count *= 2;
  }
  return count;
}
```

6.4 Factorization With Sieve

```cpp
map<ll, ll> factorization_with_sieve(ll n, const vl& primes) {
  map<ll, ll> fact;

  for (ll d : primes) {
    if (d * d > n) break;

    ll k = 0;
    while (n % d == 0) {
      k++;
      n /= d;
    }

    if (k) fact[d] = k;
  }

  if (n > 1) fact[n] = 1;
  return fact;
}
```

6.5 Factorization

```cpp
map<ll, ll> factorization(ll n) {
  map<ll, ll> ans;
  for (ll i = 2; i * i <= n; i++) {
    ll count = 0;
    for (; n % i == 0; count++, n /= i)
      ;
    if (count) ans[i] = count;
  }
  if (n > 1) ans[n]++;
  return ans;
}
```

6.6 Fast Doubling - Fibonacci

The Doubling Method can be seen as an improvement to the matrix exponentiation method to find the N-th Fibonacci number.
Time: $O(\log N)$.

```cpp
template <typename T>
class FastDoubling {
 public:
  vector<T> sts;
  T a, b, c, d;
  int mod;

  FastDoubling(int mod = 1e9 + 7) : sts(2), mod(mod) {}

  T fib(T x) {
    fill(all(sts), 0);
    a = 0, b = 0, c = 0, d = 0;
    fast_doubling(x, sts);
    return sts[0];
  }

  void fast_doubling(T n, vector<T>& res) {
    if (n == 0) {
      res[0] = 0;
      res[1] = 1;
      return;
    }
    fast_doubling(n >> 1, res);

    a = res[0];
    b = res[1];
    c = (b << 1) - a;

    if (c < 0) c += mod;

    c = (a * c) % mod;
    d = (a * a + b * b) % mod;
    if (n & 1) {
      res[0] = d;
      res[1] = c + d;
    } else {
      res[0] = c;
      res[1] = d;
    }
  }
};
```

6.7 Fast Exp Iterative

```
ll fast_exp_it(ll a, ll n, ll mod = LLONG_MAX) {
  a %= mod;
  ll res = 1;

  while (n) {
    if (n & 1) (res *= a) %= mod;

    (a *= a) %= mod;
    n >>= 1;
  }

  return res;
}
```

6.8 Fast Exp

```
ll fast_exp(ll a, ll n, ll mod = LLONG_MAX) {
  if (n == 0) return 1;
  if (n == 1) return a;

  ll x = fast_exp(a, n / 2, mod) % mod;

  return ((x * x) % mod * (n & 1 ? a : 1ll)) % mod;
}
```

6.9 Fast Fourier Transform (FFT)

Time: $O(N \cdot \log N)$

```
using cd = complex<double>;
const double PI = acos(-1);

void fft(vector<cd>& a, bool invert) {
  int n = a.size();

  for (int i = 1, j = 0; i < n; i++) {
    int bit = n >> 1;
    for (; j & bit; bit >>= 1) j ^= bit;
    j ^= bit;

    if (i < j) swap(a[i], a[j]);
  }

  for (int len = 2; len <= n; len <<= 1) {
    double ang = 2 * PI / len * (invert ? -1 : 1);
    cd wlen(cos(ang), sin(ang));
    for (int i = 0; i < n; i += len) {
      cd w(1);
      for (int j = 0; j < len / 2; j++) {
        cd u = a[i + j], v = a[i + j + len / 2] * w;
        a[i + j] = u + v;
        a[i + j + len / 2] = u - v;
        w *= wlen;
      }
    }
  }

  if (invert) {
    for (cd& x : a) x /= n;
  }
}

void fft_2d(vector<vector<cd>>& V, bool invert) {
  for (int i = 0; i < V.size(); i++) fft(V[i], invert);
  for (int i = 0; i < V.front().size(); i++) {
    vector<cd> col(V.size());
    for (int k = 0; k < V.size(); k++) col[k] = V[k][i];
    fft(col, invert);
    for (int k = 0; k < V.size(); k++) V[k][i] = col[k];
  }
}
```

6.10 GCD

The Euclidean algorithm allows to find the greatest common divisor of two numbers a and b in $O(\log \cdot \min(a, b))$.

```
ll gcd(ll a, ll b) { return b ? gcd(b, a % b) : a; }
```

6.11 Integer Mod

```
const ll INF = 1e18;
const ll mod = 998244353;
template <ll MOD = mod>

struct Modular {
  ll value;
  static const ll MOD_value = MOD;

  Modular(ll v = 0) {
    value = v % MOD;
    if (value < 0) value += MOD;
  }
  Modular(ll a, ll b) : value(0) {
    *this += a;
    *this /= b;
  }

  Modular& operator+=(Modular const& b) {
    value += b.value;
    if (value >= MOD) value -= MOD;
    return *this;
  }
  Modular& operator-=(Modular const& b) {
    value -= b.value;
    if (value < 0) value += MOD;
    return *this;
  }
  Modular& operator*=(Modular const& b) {
    value = (ll)value * b.value % MOD;
    return *this;
  }

  friend Modular mexp(Modular a, ll e) {
    Modular res = 1;
    while (e) {
      if (e & 1) res *= a;
      a *= a;
      e >>= 1;
    }
    return res;
  }
  friend Modular inverse(Modular a) { return mexp(a, MOD - 2); }

  Modular& operator/=(Modular const& b) { return *this *= inverse(b); }
  friend Modular operator+(Modular a, Modular const b) { return a += b; }
  Modular operator++(int) { return this->value = (this->value + 1) % MOD; }
  Modular operator++() { return this->value = (this->value + 1) % MOD; }
  friend Modular operator-(Modular a, Modular const b) { return a -= b; }
  friend Modular operator-(Modular const a) { return 0 - a; }
  Modular operator--(int) {
    return this->value = (this->value - 1 + MOD) % MOD;
  }

  Modular operator--() { return this->value = (this->value - 1 + MOD) % MOD; }
  friend Modular operator*(Modular a, Modular const b) { return a *= b; }
  friend Modular operator/(Modular a, Modular const b) { return a /= b; }
  friend std::ostream& operator<<(std::ostream& os, Modular const& a) {
    return os << a.value;
  }
  friend bool operator==(Modular const& a, Modular const& b) {
    return a.value == b.value;
  }
  friend bool operator!=(Modular const& a, Modular const& b) {
    return a.value != b.value;
  }
};
```

6.12 Is prime

$O(\sqrt{N})$

```
bool isprime(ll n) {
  if (n < 2) return false;
  if (n == 2) return true;
  if (n % 2 == 0) return false;
  for (ll i = 3; i * i <= n; i += 2)
    if (n % i == 0) return false;
  return true;
}
```

6.13 LCM

Calculating the least common multiple (commonly denoted LCM) can be reduced to calculating the GCD with the following simple formula: $\operatorname{lcm}(a,b) = (a \cdot b)/\gcd(a,b)$
Thus, LCM can be calculated using the Euclidean algorithm with the same time complexity:

```
ll lcm(ll a, ll b) { return a / gcd(a, b) * b; }
```

6.14 Euler phi $\varphi(n)$

Computes the number of positive integers less than n that are co-primes with n, in $O(\sqrt{N})$.

```
ll phi(ll n) {
  if (n == 1) return 1;

  auto fs = factorization(n);
  auto res = n;

  for (auto [p, k] : fs) {
    res /= p;
    res *= (p - 1);
  }

  return res;
}
```

6.15 Sieve

```
vl sieve(ll N) {
  bitset<MAX + 1> sieve;
  vl ps{2, 3};
  sieve.set();

  for (ll i = 5, step = 2; i <= N; i += step, step = 6 - step) {
    if (sieve[i]) {
      ps.push_back(i);

      for (ll j = i * i; j <= N; j += 2 * i) sieve[j] = false;
    }
  }
  return ps;
}
```

6.16 Sum Divisors

```
ll sum_divisors(ll num) {
  ll result = 1;

  for (int i = 2; (ll)i * i <= num; i++) {
    if (num % i == 0) {
      int e = 0;
      do {
        e++;
        num /= i;
      } while (num % i == 0);

      ll sum = 0, pow = 1;
      do {
        sum += pow;
        pow *= i;
      } while (e-- > 0);
      result *= sum;
    }
  }
  if (num > 1) {
    result *= (1 + num);
  }
  return result;
}
```

6.17 Sum of difference

Function to calculate sum of absolute difference of all pairs in array: $\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}|A_i - A_j|$

```
ll sum_of_diference(vl& arr, ll n) {
  sort(all(arr));

  ll sum = 0;
  for (ll i = 0; i < n; i++) {
    sum += i * arr[i] - (n - 1 - i) * arr[i];
  }

  return sum;
}
```

7 Problems

7.1 Kth Digit String (CSES)

Time: $O(\log_{10} K)$.
Space: $O(1)$.

```
ll kth_digit_string(ll k) {
  if (k < 10) return k;

  ll c = 180, i = 2, u = 10, r = 0, ans = -1, m;
  for (k -= 9; k > c; i++, u *= 10) {
    k -= c;
    c /= i;
    c *= 10 * (i + 1);
  }

  if ((m = k % i))
    r++;
  else
    m = i;

  ll tmp = (k / i) + r + u - 1;
  for (m = i + 1 - m; m--; tmp /= 10) ans = tmp % 10;

  return ans;
}
```

7.2 Longest Common Substring (LONGCS - SPOJ)

Time: $N = \sum_{i=1}^{k}|S_i|$; $O(N \cdot \log N)$

```
int lcs_ks_strings(vector<string>& sts, int k) {
  vector<int> fml;
  string t;
  for (int i = 0; i < k; i++) {
    t += sts[i];
    for (int j = 0; j < sts[i].size(); j++) fml.push_back(i);
  }

  suffix_array sf(t);
  sf.lcp.insert(sf.lcp.begin(), 0);

  int l = 0, r = 0, cnt = 0, lcs = 0, n = sf.sa.size();
  vector<int> fr(k + 1);
  multiset<int> mst;
  while (l < n) {
    while (r < n and cnt < k) {
      mst.insert(sf.lcp[r]);
      if (!fr[fml[sf.sa[r]]]++) cnt++;
      r++;
    }
    mst.erase(mst.find(sf.lcp[l]));
    if (mst.size() and cnt == k) lcs = max(lcs, *mst.begin());
    fr[fml[sf.sa[l]]]--;
    if (!fr[fml[sf.sa[l]]]) cnt--;
    l++;
  }

  return lcs;
}
```

7.3 Substring Order II (CSES)

Time: $O(M)$
$M = 2 \cdot N - 1$
$N = |S|$

```
// ALLOWS REPETITIONS
string kth_smallest_substring(const string& s, ll k) {
  /* uses /strings/suffix-automaton.cpp
  add 'cnt' and 'nmb' to state struct with (0, -1);
      => for new states 'not cloned': cnt = 1

  create 'order' vector to iterate by length in decreasing
  vector<pair<int, int>>: {len, id}
      => for each new state add to 'order' vector

  to do not allow repetitions:
      => remove: kth+=s.size, sort(order) for(l, p : order)
      => add: st[clone].cnt = 1 (sa_extend)
  */
  string ans;
  k += s.size();
  SuffixAutomaton sa(s);

  sort(all(order), greater<pair<int, int>>());
  // count and mark how many times a substring of a state occurs
  for (auto& [l, p] : order) sa.st[sa.st[p].link].cnt += sa.st[p].cnt;

  auto dfs = [&](auto&& self, int u) {
    if (sa.st[u].nmb != -1) return;
    sa.st[u].nmb = sa.st[u].cnt;
    for (int i = 0; i < 26; ++i) {
      if (sa.st[u].next[i]) {
        self(self, sa.st[u].next[i]);
        sa.st[u].cnt += sa.st[sa.st[u].next[i]].cnt;
      }
    }
  };
  dfs(dfs, 0);

  int u = 0;
  while (sa.st[u].nmb < k) {
    k -= sa.st[u].nmb;
    for (int i = 0; i < 26; i++) {
      if (sa.st[u].next[i]) {
        int v = sa.st[u].next[i];
        if (sa.st[v].cnt < k)
          k -= sa.st[v].cnt;
        else {
          ans.push_back(i + 'a');
          u = v;
          break;
        }
      }
    }
  }

  return ans;
}
```

8 Strings

8.1 Aho-Corasick

The Aho-Corasick algorithm allows us to quickly search for multiple patterns in a text. The set of pattern strings is also called a *dictionary*. We will denote the total length of its constituent strings by m and the size of the alphabet by k.
build: $O(m \cdot k)$
occurrences: $O(|s| + ans)$

```
const int K = 26;
struct Vertex {
  char pch;
  int next[K];
  bool check = false;
  int p = -1, lnk = -1, out = -1, ps = -1, d = 0;

  Vertex(int p = -1, char ch = '$') : p(p), pch(ch) {
    fill(begin(next), end(next), -1);
  }
};

class AhoCorasick {
 public:
  int sz = 0;  // number of strings added
  vector<Vertex> t;
```

```
AhoCorasick() : t(1) {}

void add_string(string const& s) {
  int v = 0, ds = 0;
  for (char ch : s) {
    int c = ch - 'a';
    if (t[v].next[c] == -1) {
      t[v].next[c] = t.size();
      t.emplace_back(v, ch);
    }
    v = t[v].next[c];
    t[v].d = ++ds;
  }
  t[v].check = true;
  t[v].ps = sz++;
}

void build() {
  queue<int> qs;
  qs.push(0);
  while (qs.size()) {
    auto u = qs.front();
    qs.pop();

    if (!t[u].p or t[u].p == -1)
      t[u].lnk = 0;
    else {
      int k = t[t[u].p].lnk;
      int c = t[u].pch - 'a';
      while (t[k].next[c] == -1 and k) k = t[k].lnk;
      int ts = t[k].next[c];
      if (ts == -1)
        t[u].lnk = 0;
      else
        t[u].lnk = ts;
    }

    if (t[t[u].lnk].check)
      t[u].out = t[u].lnk;
    else
      t[u].out = t[t[u].lnk].out;

    for (auto v : t[u].next)
      if (v != -1) qs.push(v);
  }
}

void occurrences(string const& s, vector<vector<int>>& res) {
  // to just "count" replace 'res' vector with an int
  res.resize(sz);
  for (int i = 0, v = 0; i < s.size(); i++) {
    int c = s[i] - 'a';
    while (t[v].next[c] == -1 and v) v = t[v].lnk;
    int ts = t[v].next[c];
    if (ts == -1)
      continue;
    else
      v = t[v].next[c];
      int k = v;
      while (t[k].out != -1) {
        k = t[k].out;
        res[t[k].ps].emplace_back(i - t[k].d + 1);
      }
      if (t[v].check) res[t[v].ps].emplace_back(i - t[v].d + 1);
    }
  }
};
```

8.2 Edit Distance

Returns the minimum number of operations (insert, delete, replace) to transform string a into string b.
Time: $O(M * N)$

```
int min_value(int x, int y, int z) { return min(min(x, y), z); }

int edit_distance(string str1, string str2) {
  int n = (int)str1.size(), m = (int)str2.size();
  int dp[m + 1][n + 1];

  for (int i = 0; i <= m; i++)
    for (int j = 0; j <= n; j++)
      if (i == 0)
        dp[i][j] = j;
      else if (j == 0)
        dp[i][j] = i;
      else if (str1[i - 1] == str2[j - 1])
        dp[i][j] = dp[i - 1][j - 1];
      else
        dp[i][j] = 1 + min_value(dp[i][j - 1], dp[i - 1][j], dp[i - 1][j - 1])
    ;

  return dp[m][n];
}
```

8.3 LCP with Suffix Array

For a given string s we want to compute the longest common prefix (LCP) of two arbitrary suffixes with position i and j. In fact, let the request be to compute the LCP of the suffixes $p[i]$ and $p[j]$. Then the answer to this query will be $\min(lcp[i], lcp[i+1], \ldots, lcp[j-1])$. Thus the problem is reduced to the RMQ.
Time: $O(N)$.

```
vector<int> lcp_suffix_array(string const& s, vector<int> const& p) {
  int n = s.size();
  vector<int> rank(n, 0);
  for (int i = 0; i < n; i++) rank[p[i]] = i;

  int k = 0;
  vector<int> lcp(n - 1, 0);
  for (int i = 0; i < n; i++) {
    if (rank[i] == n - 1) {
      k = 0;
      continue;
    }
    int j = p[rank[i] + 1];
    while (i + k < n && j + k < n && s[i + k] == s[j + k]) k++;
    lcp[rank[i]] = k;
    if (k) k--;
  }
  return lcp;
}
```

8.4 Manacher

Given string s with length n. Find all the pairs (i, j) such that substring $s[i \ldots j]$ is a palindrome. String t is a palindrome when $t = t_{rev}$ (t_{rev} is a reversed string for t).
Time: $O(N)$

```
vi manacher(string s) {
  string t;
  for (auto c : s) t += string("#") + c;
  t = t + '#';

  int n = t.size();
  t = "$" + t + "^";

  vi p(n + 2);
  int l = 1, r = 1;
  for (int i = 1; i <= n; i++) {
    p[i] = max(0, min(r - i, p[l + (r - i)]));
    while (t[i - p[i]] == t[i + p[i]]) p[i]++;
    if (i + p[i] > r) {
      l = i - p[i], r = i + p[i];
    }
    p[i]--;
  }

  return vi(begin(p) + 1, end(p) - 1);
}
```

8.5 Rabin Karp

```
vector<int> rabin_karp(string const& s, string const& t) {
  const int p = 31;
  const int m = 1e9 + 9;
  int S = s.size(), T = t.size();

  vector<long long> p_pow(max(S, T));
  p_pow[0] = 1;
  for (int i = 1; i < (int)p_pow.size(); i++) p_pow[i] = (p_pow[i - 1] * p) %
    m;

  vector<long long> h(T + 1, 0);
  for (int i = 0; i < T; i++)
    h[i + 1] = (h[i] + (t[i] - 'a' + 1) * p_pow[i]) % m;
  long long h_s = 0;
  for (int i = 0; i < S; i++) h_s = (h_s + (s[i] - 'a' + 1) * p_pow[i]) % m;

  vector<int> occurrences;
  for (int i = 0; i + S - 1 < T; i++) {
    long long cur_h = (h[i + S] + m - h[i]) % m;
    if (cur_h == h_s * p_pow[i] % m) occurrences.push_back(i);
  }

  return occurrences;
}
```

8.6 Suffix Array Optimized - O(n)

Suffix Array: sa
Rank for LCP: rnk
LCP: lcp
Time: $O(N)$.

```
// @brunomaletta
struct suffix_array {
  string s;
  int n;
  vector<int> sa, cnt, rnk, lcp;
  rmq<int> RMQ;  // /data-structures/rmq.cpp

  bool cmp(int a1, int b1, int a2, int b2, int a3 = 0, int b3 = 0) {
    return a1 != b1 ? a1 < b1 : (a2 != b2 ? a2 < b2 : a3 < b3);
  }

  template <typename T>
  void radix(int* fr, int* to, T* r, int N, int k) {
    cnt = vector<int>(k + 1, 0);
    for (int i = 0; i < N; i++) cnt[r[fr[i]]]++;
    for (int i = 1; i <= k; i++) cnt[i] += cnt[i - 1];
    for (int i = N - 1; i + 1; i--) to[--cnt[r[fr[i]]]] = fr[i];
  }

  void rec(vector<int>& v, int k) {
    auto &tmp = rnk, &m0 = lcp;
    int N = v.size() - 3, sz = (N + 2) / 3, sz2 = sz + N / 3;
    vector<int> R(sz2 + 3);
    for (int i = 1, j = 0; j < sz2; i += i % 3) R[j++] = i;

    radix(&R[0], &tmp[0], &v[0] + 2, sz2, k);
    radix(&tmp[0], &R[0], &v[0] + 1, sz2, k);
    radix(&R[0], &tmp[0], &v[0] + 0, sz2, k);

    int dif = 0;
    int l0 = -1, l1 = -1, l2 = -1;
    for (int i = 0; i < sz2; i++) {
      if (v[tmp[i]] != l0 or v[tmp[i] + 1] != l1 or v[tmp[i] + 2] != l2)
        l0 = v[tmp[i]], l1 = v[tmp[i] + 1], l2 = v[tmp[i] + 2], dif++;
      if (tmp[i] % 3 == 1)
        R[tmp[i] / 3] = dif;
      else
        R[tmp[i] / 3 + sz] = dif;
    }

    if (dif < sz2) {
      rec(R, dif);
      for (int i = 0; i < sz2; i++) R[sa[i]] = i + 1;
    } else
      for (int i = 0; i < sz2; i++) sa[R[i] - 1] = i;
    for (int i = 0, j = 0; j < sz2; i++)
      if (sa[i] < sz) tmp[j++] = 3 * sa[i];
    radix(&tmp[0], &m0[0], &v[0], sz, k);
    for (int i = 0; i < sz2; i++)
      sa[i] = sa[i] < sz ? 3 * sa[i] + 1 : 3 * (sa[i] - sz) + 2;

    int at = sz2 + sz - 1, p = sz - 1, p2 = sz2 - 1;
    while (p >= 0 and p2 >= 0) {
      if ((sa[p2] % 3 == 1 and
           cmp(v[m0[p]], v[sa[p2]], R[m0[p] / 3], R[sa[p2] / 3 + sz])) or
          (sa[p2] % 3 == 2 and
           cmp(v[m0[p]], v[sa[p2]], v[m0[p] + 1], v[sa[p2] + 1],
               R[m0[p] / 3 + sz], R[sa[p2] / 3 + 1])))
        sa[at--] = sa[p2--];
      else
        sa[at--] = m0[p--];
    }
    while (p >= 0) sa[at--] = m0[p--];
    if (N % 3 == 1)
      for (int i = 0; i < N; i++) sa[i] = sa[i + 1];
  }

  suffix_array(const string& s_)
    : s(s_), n(s.size()), sa(n + 3), cnt(n + 1), rnk(n), lcp(n - 1) {
    vector<int> v(n + 3);
    for (int i = 0; i < n; i++) v[i] = i;
    radix(&v[0], &rnk[0], &s[0], n, 256);
    int dif = 1;
    for (int i = 0; i < n; i++)
      v[rnk[i]] = dif += (i and s[rnk[i]] != s[rnk[i - 1]]);
    if (n >= 2) rec(v, dif);
    sa.resize(n);

    for (int i = 0; i < n; i++) rnk[sa[i]] = i;
    for (int i = 0, k = 0; i < n; i++, k -= !!k) {
      if (rnk[i] == n - 1) {
        k = 0;
        continue;
      }
      int j = sa[rnk[i] + 1];
      while (i + k < n and j + k < n and s[i + k] == s[j + k]) k++;
      lcp[rnk[i]] = k;
    }
    RMQ = rmq<int>(lcp);
  }

  int query(int i, int j) {
    if (i == j) return n - i;
    i = rnk[i], j = rnk[j];
    return RMQ.query(min(i, j), max(i, j) - 1);
  }
};
```

8.7 Suffix Array

Let s be a string of length n. The i-th suffix of s is the substring $s[i \dots n-1]$.
A suffix array will contain integers that represent the starting indexes of the all the suffixes of a given

string, after the aforementioned suffixes are sorted.
Time: $O(N \log N)$.

```
vector<int> sort_cyclic_shifts(string const& s) {
  int n = s.size();
  const int alphabet = 128;

  vector<int> p(n), c(n), cnt(max(alphabet, n), 0);
  for (int i = 0; i < n; i++) cnt[s[i]]++;
  for (int i = 1; i < alphabet; i++) cnt[i] += cnt[i - 1];
  for (int i = 0; i < n; i++) p[--cnt[s[i]]] = i;
  c[p[0]] = 0;
  int classes = 1;
  for (int i = 1; i < n; i++) {
    if (s[p[i]] != s[p[i - 1]]) classes++;
    c[p[i]] = classes - 1;
  }

  vector<int> pn(n), cn(n);
  for (int h = 0; (1 << h) < n; ++h) {
    for (int i = 0; i < n; i++) {
      pn[i] = p[i] - (1 << h);
      if (pn[i] < 0) pn[i] += n;
    }
    fill(cnt.begin(), cnt.begin() + classes, 0);
    for (int i = 0; i < n; i++) cnt[c[pn[i]]]++;
    for (int i = 1; i < classes; i++) cnt[i] += cnt[i - 1];
    for (int i = n - 1; i >= 0; i--) p[--cnt[c[pn[i]]]] = pn[i];
    cn[p[0]] = 0;
    classes = 1;
    for (int i = 1; i < n; i++) {
      pair<int, int> cur = {c[p[i]], c[(p[i] + (1 << h)) % n]};
      pair<int, int> prev = {c[p[i - 1]], c[(p[i - 1] + (1 << h)) % n]};
      if (cur != prev) ++classes;
      cn[p[i]] = classes - 1;
    }
    c.swap(cn);
  }

  return p;
}

vector<int> suffix_array(string s) {
  s += "$";
  vector<int> p = sort_cyclic_shifts(s);
  p.erase(p.begin());
  return p;
}
```

8.8 Suffix Automaton

```
class SuffixAutomaton {
 public:
  struct state {
    int len, link;
    array<int, 26> next;
  };

  vector<state> st;
  int sz = 0, last;

  SuffixAutomaton(const string& s) : st(s.size() << 1) {
    sa_init();
    for (auto v : s) sa_extend((int)(v - 'a'));
  }

  void sa_init() {
    st[0].len = 0;
    st[0].link = -1;
    sz++;
    last = 0;
  }

  void sa_extend(int c) {
    int cur = sz++;
    st[cur].len = st[last].len + 1;
    int p = last;
    while (p != -1 && !st[p].next[c]) {
      st[p].next[c] = cur;
      p = st[p].link;
    }
    if (p == -1)
      st[cur].link = 0;
    else {
      int q = st[p].next[c];
      if (st[p].len + 1 == st[q].len)
        st[cur].link = q;
      else {
        int clone = sz++;
        st[clone].len = st[p].len + 1;
        st[clone].link = st[q].link;
        st[clone].next = st[q].next;
        while (p != -1 && st[p].next[c] == q) {
          st[p].next[c] = clone;
          p = st[p].link;
        }
        st[q].link = st[cur].link = clone;
      }
    }
    last = cur;
  }

  // longest common substring O(N)
  int lcs(const string& T) {
    int v = 0, l = 0, best = 0;
    for (int i = 0; i < T.size(); i++) {
      while (v && !st[v].next[T[i] - 'a']) {
        v = st[v].link;
        l = st[v].len;
      }
      if (st[v].next[T[i] - 'a']) {
        v = st[v].next[T[i] - 'a'];
        l++;
      }
      best = max(best, l);
    }
    return best;
  }
};
```

8.9 Suffix Tree (CP Algo - freopen)

Build: $O(N)$
Memory: $O(N \cdot k)$
k = alphabet length

```
const int aph = 27; // add $ to final of string
const int N = 2e5 + 31;
class SuffixTree {
public:
    string a;
    vector<array<int, aph>> t;
    vector<int> l, r, p, s, dst;
    int tv, tp, ts, la, b;

    SuffixTree(const string& str, char bs = 'a') : a(str), t(N), l(N),
        r(N, str.size() - 1), p(N), s(N), dst(N), b(bs) {
        build();
    }

    void ukkadd(int c) {
    suff:;
        if (r[tv] < tp) {
            if (t[tv][c] == -1) {
                t[tv][c] = ts; l[ts] = la;
                p[ts++] = tv; tv = s[tv]; tp = r[tv] + 1; goto suff;
            }
            tv = t[tv][c]; tp = l[tv];
        }
        if (tp == -1 || c == a[tp] - b) tp++; else {
            l[ts + 1] = la; p[ts + 1] = ts;
            l[ts] = l[tv]; r[ts] = tp - 1; p[ts] = p[tv];
            t[ts][c] = ts + 1; t[ts][a[tp] - b] = tv; l[tv] = tp;
            p[tv] = ts; t[p[ts]][a[l[ts]] - b] = ts; ts += 2;
            tv = s[p[ts - 2]]; tp = l[ts - 2];
            while (tp <= r[ts - 2]) {
                tv = t[tv][a[tp] - b];
                tp += r[tv] - l[tv] + 1;
            }
            if (tp == r[ts - 2] + 1) s[ts - 2] = tv; else s[ts - 2] = ts;
            tp = r[tv] - (tp - r[ts - 2]) + 2; goto suff;
        }
    }

    void build() {
        ts = 2; tv = 0; tp = 0;
        s[0] = 1; l[0] = -1; r[0] = -1; l[1] = -1; r[1] = -1;
        for (auto& arr : t) { arr.fill(-1); } t[1].fill(0);
        for (la = 0; la < (int)a.size(); ++la) ukkadd(a[la] - b);
    }
};
```

8.10 Z Function

Suppose we are given a string s of length n. The Z-function for this string is an array of length n where the i-th element is equal to the greatest number of characters starting from the position i that coincide with the first characters of s.
Time: $O(N)$

```
vector<int> z_function(string s) {
  int n = s.size();
  vector<int> z(n);
  int l = 0, r = 0;
  for (int i = 1; i < n; i++) {
    if (i < r) {
      z[i] = min(r - i, z[i - l]);
    }
    while (i + z[i] < n && s[z[i]] == s[i + z[i]]) {
      z[i]++;
    }
    if (i + z[i] > r) {
      l = i;
      r = i + z[i];
    }
  }
  return z;
}
```

9 Trees

9.1 LCA Binary Lifting (CP Algo)

The algorithm described will need $O(N \cdot \log N)$ for preprocessing the tree, and then $O(\log N)$ for each LCA query.

```
ll n, l;
vector<ll> adj[MAX];

ll timer;
vector<ll> tin, tout;
vector<vector<ll>> up;

void dfs(ll v, ll p) {
  tin[v] = ++timer;
  up[v][0] = p;
  for (ll i = 1; i <= l; ++i) up[v][i] = up[up[v][i - 1]][i - 1];

  for (ll u : adj[v]) {
    if (u != p) dfs(u, v);
  }

  tout[v] = ++timer;
}

bool is_ancestor(ll u, ll v) { return tin[u] <= tin[v] && tout[u] >= tout[v];
    }

ll lca(ll u, ll v) {
  if (is_ancestor(u, v)) return u;
  if (is_ancestor(v, u)) return v;
  for (ll i = l; i >= 0; --i) {
    if (!is_ancestor(up[u][i], v)) u = up[u][i];
  }
  return up[u][0];
}

void preprocess(ll root) {
  tin.resize(n);
  tout.resize(n);
  timer = 0;
  l = ceil(log2(n));
  up.assign(n, vector<ll>(l + 1));
  dfs(root, root);
}
```

9.2 LCA SegTree (CP Algo)

The algorithm can answer each query in $O(\log N)$ with preprocessing in $O(N)$ time.

```
struct LCA {
  vector<ll> height, euler, first, segtree;
  vector<bool> visited;
  ll n;

  LCA(vector<vector<ll>>& adj, ll root = 0) {
    n = adj.size();
    height.resize(n);
    first.resize(n);
    euler.reserve(n * 2);
    visited.assign(n, false);
    dfs(adj, root);
    ll m = euler.size();
    segtree.resize(m * 4);
    build(1, 0, m - 1);
  }

  void dfs(vector<vector<ll>>& adj, ll node, ll h = 0) {
    visited[node] = true;
    height[node] = h;
    first[node] = euler.size();
    euler.push_back(node);
    for (auto to : adj[node]) {
      if (!visited[to]) {
        dfs(adj, to, h + 1);
        euler.push_back(node);
      }
    }
  }

  void build(ll node, ll b, ll e) {
    if (b == e) {
      segtree[node] = euler[b];
    } else {
      ll mid = (b + e) / 2;
      build(node << 1, b, mid);
      build(node << 1 | 1, mid + 1, e);
      ll l = segtree[node << 1], r = segtree[node << 1 | 1];
      segtree[node] = (height[l] < height[r]) ? l : r;
    }
  }

  ll query(ll node, ll b, ll e, ll L, ll R) {
    if (b > R || e < L) return -1;
    if (b >= L && e <= R) return segtree[node];
    ll mid = (b + e) >> 1;

    ll left = query(node << 1, b, mid, L, R);
    ll right = query(node << 1 | 1, mid + 1, e, L, R);
    if (left == -1) return right;
    if (right == -1) return left;
    return height[left] < height[right] ? left : right;
  }

  ll lca(ll u, ll v) {
    ll left = first[u], right = first[v];
    if (left > right) swap(left, right);
    return query(1, 0, euler.size() - 1, left, right);
  }
};
```

9.3 LCA Sparse Table

The algorithm described will need $O(N)$ for preprocessing, and then $O(1)$ for each LCA query.
0 indexed!

```
typedef vector<vl> vl2d;
#define all(a) a.begin(), a.end()
#define len(x) (int)x.size()

template <typename T>
struct SparseTable {
  vector<T> v;
  ll n;
  static const ll b = 30;
  vl mask, t;

  ll op(ll x, ll y) { return v[x] < v[y] ? x : y; }
  ll msb(ll x) { return __builtin_clz(1) - __builtin_clz(x); }
  SparseTable() {}
  SparseTable(const vector<T>& v_) : v(v_), n(v.size()), mask(n), t(n) {
    for (ll i = 0, at = 0; i < n; mask[i++] = at |= 1) {
      at = (at << 1) & ((1 << b) - 1);
      while (at and op(i, i - msb(at & -at)) == i) at ^= at & -at;
    }
    for (ll i = 0; i < n / b; i++)
      t[i] = b * i + b - 1 - msb(mask[b * i + b - 1]);
    for (ll j = 1; (1 << j) <= n / b; j++)
      for (ll i = 0; i + (1 << j) <= n / b; i++)
        t[n / b * j + i] =
          op(t[n / b * (j - 1) + i], t[n / b * (j - 1) + i + (1 << (j - 1))]);
  }
  ll small(ll r, ll sz = b) { return r - msb(mask[r] & ((1 << sz) - 1)); }
  T query(ll l, ll r) {
    if (r - l + 1 <= b) return small(r, r - l + 1);
    ll ans = op(small(l + b - 1), small(r));
    ll x = l / b + 1, y = r / b - 1;
    if (x <= y) {
      ll j = msb(y - x + 1);
      ans = op(ans, op(t[n / b * j + x], t[n / b * j + y - (1 << j) + 1]));
    }
    return ans;
  }
};

struct LCA {
  SparseTable<ll> st;
  ll n;
  vl v, pos, dep;

  LCA(const vl2d& g, ll root) : n(len(g)), pos(n) {
    dfs(root, 0, -1, g);
    st = SparseTable<ll>(vector<ll>(all(dep)));
  }

  void dfs(ll i, ll d, ll p, const vl2d& g) {
    v.emplace_back(len(dep)) = i, pos[i] = len(dep), dep.emplace_back(d);
    for (auto j : g[i])
      if (j != p) {
        dfs(j, d + 1, i, g);
        v.emplace_back(len(dep)) = i, dep.emplace_back(d);
      }
  }

  ll lca(ll a, ll b) {
    ll l = min(pos[a], pos[b]);
    ll r = max(pos[a], pos[b]);
    return v[st.query(l, r)];
  }
  ll dist(ll a, ll b) {
    return dep[pos[a]] + dep[pos[b]] - 2 * dep[pos[lca(a, b)]];
  }
};
```

9.4 Tree Flatten

```
vll tree_flatten(ll root) {
  vl pre;
  pre.reserve(N);

  vll flat(N);
  ll timer = -1;
  auto dfs = [&](auto&& self, ll u, ll p) -> void {
    timer++;
    pre.push_back(u);
    for (auto [v, w] : adj[u])
      if (v != p) {
        self(self, v, u);
      }
    flat[u].second = timer;
  };
  dfs(dfs, root, -1);
  for (ll i = 0; i < (ll)N; i++) flat[pre[i]].first = i;
  return flat;
}
```

9.5 Tree Isomorph

Checks whether two tree are isomorph. The function *thash*() returns the hash of the tree (using centroids as special vertices). Two trees are isomorph if their hash are the same.

```
map<vector<int>, int> mphash;

struct tree {
  int n;
  vector<vector<int>> g;
  vector<int> sz, cs;

  tree(int n_) : n(n_), g(n_), sz(n_) {}

  void dfs_centroid(int v, int p) {
    sz[v] = 1;
    bool cent = true;
    for (int u : g[v])
      if (u != p) {
        dfs_centroid(u, v), sz[v] += sz[u];
        if (sz[u] > n / 2) cent = false;
      }
    if (cent and n - sz[v] <= n / 2) cs.push_back(v);
  }
  int fhash(int v, int p) {
    vector<int> h;
    for (int u : g[v])
      if (u != p) h.push_back(fhash(u, v));
    sort(h.begin(), h.end());
    if (!mphash.count(h)) mphash[h] = mphash.size();
    return mphash[h];
  }
  ll thash() {
    cs.clear();
    dfs_centroid(0, -1);
    if (cs.size() == 1) return fhash(cs[0], -1);
    ll h1 = fhash(cs[0], cs[1]), h2 = fhash(cs[1], cs[0]);
    return (min(h1, h2) << 30) + max(h1, h2);
  }
  void add(int a, int b) {
    g[a].emplace_back(b);
    g[b].emplace_back(a);
  }
};
```

10 Settings and macros

10.1 short-macro.cpp

```
#include <bits/stdc++.h>

using namespace std;

#ifdef DEBUG
#include "./settings-and-macros/debug.cpp"
```

```
#else
#define dbg(...)
#endif

typedef long long ll;
typedef pair<int, int> ii;

#define all(x) x.begin(), x.end()
#define vin(vt) for (auto &e : vt) cin >> e

auto solve() {  }

int main() {
    ios_base::sync_with_stdio(0);
    cin.tie(0);

    ll t = 1;
    //cin >> t;

    while (t--) solve();

    return 0;
}
```

10.2 macro.cpp

```
#include <bits/stdc++.h>
#include <ext/pb_ds/assoc_container.hpp>
#include <ext/pb_ds/tree_policy.hpp>

using namespace __gnu_pbds;
#define ordered_set tree<int, null_type, less<int>, rb_tree_tag,
    tree_order_statistics_node_update>

using namespace std;

#ifdef DEBUG
#include "./settings-and-macros/debug.cpp"
#else
#define dbg(...)
#endif

typedef long long ll;
typedef pair<int, int> pii;
typedef pair<ll, ll> pll;
typedef vector<int> vi;
typedef vector<ll> vl;
typedef vector<pii> vii;
typedef vector<pll> vll;

#define fst first
#define snd second
#define all(x) x.begin(), x.end()
#define len(vt) (int)vt.size()
#define vin(vt) for (auto &e : vt) cin >> e
#define LSOne(S) ((S) & -(S))
#define MSOne(S) (1ull << (63 - __builtin_clzll(S)))
#define fastio ios_base::sync_with_stdio(0); \
                cin.tie(0); \
                cout.tie(0)

const vii dir4 {{1,0},{-1,0},{0,1},{0,-1}};

auto solve() {  }

int main() {
    fastio;

    ll t = 1;
    //cin >> t;

    while (t--) solve();

    return 0;
}
```

11 Theoretical guide

11.1 Notable Series

1. Sum of the first n naturals:

$$S_n = \sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

2. Sum of the squares of the first n naturals:

$$S_n = \sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

3. Sum of the cubes of the first natural n:

$$S_n = \sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

4. Sum of the first n odd numbers:

$$S_n = \sum_{i=1}^{n} 2i - 1 = 1 + 3 + 5 + \cdots + (2n-1) = n^2$$

11.2 Number of Different Substrings

$$\sum_{i=0}^{n-1}(n - p[i]) - \sum_{i=0}^{n-2} \text{lcp}[i] = \frac{n^2+n}{2} - \sum_{i=0}^{n-2} \text{lcp}[i]$$

11.3 Exponent With Module

If a and m are coprime, then

$$a^n \equiv a^{n \mod \phi(m)} \mod m$$

Generally, if $n \geq \log_2 m$, then

$$a^n \equiv a^{\phi(m)+[n \mod \phi(m)]} \mod m$$

11.4 String Matching with FFT

We are given two strings, a text T and a pattern P, consisting of lowercase letters. We have to compute all the occurrences of the pattern in the text.
We create a polynomial for each string ($T[i]$ and $P[i]$ are numbers between 0 and 25 corresponding to the 26 letters of the alphabet):

$$A(x) = a_0x^0 + a_1x^1 + \cdots + a_{n-1}x^{n-1}, \quad n = |T|$$

with

$$a_i = \cos(\alpha_i) + i\sin(\alpha_i), \quad \alpha_i = \frac{2\pi T[i]}{26}$$

And

$$B(x) = b_0x^0 + b_1x^1 + \cdots + b_{m-1}x^{m-1}, \quad m = |P|$$

with

$$b_i = \cos(\beta_i) - i\sin(\beta_i), \quad \beta_i = \frac{2\pi P[m-i-1]}{26}$$

Notice that with the expression $P[m-i-1]$ explicitly reverses the pattern.
The $(m-1+i)$th coefficients of the product of the two polynomials $C(x) = A(x)\cdot B(x)$ will tell us, if the pattern appears in the text at position i.
If there isn't a match, then at least a character is different, which leads that one of the products $a_{i+1}\cdot b_{m-1-j}$ is not equal to 1, which leads to the coefficient $c_{m-1+i} \neq m$.

11.4.1 Wildcards

This is an extension of the previous problem. This time we allow that the pattern contains the wildcard character $*$, which can match every possible letter.
We create the exact same polynomials, except that we set $b_i = 0$ if $P[m-i-1] = *$. If x is the number of wildcards in P, then we will have a match of P in T at index i if $c_{m-1+i} = m - x$.

11.5 Pick's Theorem

Pick's Theorem expresses the area of a polygon, all whose vertices are lattice (integers) points in a coordinate plane, in terms of the number of lattice points inside the polygon and the number of lattice points on the sides (boundaries) of the polygon.

$$A = I + \frac{B}{2} - 1$$

- A: area of the polygon
- I: points inside the polygon
- B: points on the sides (boundaries)

It is possible to easily calculate the number of points on the sides of a side AB.
Consider $x = (x_1 - x_2)$ and $y = (y_1 - y_2)$. If $x = 0$ or $y = 0$, the answer is 1D and trivial (i.e. $x+1$ or $y+1$). Otherwise, the answer is $gcd(a,b)+1$.

11.6 Modular Multiplicative Inverse

A modular multiplicative inverse of an integer a is an integer x such that $a\cdot x$ is congruent to 1 modular some modulus m. To write it in a formal way:

$$a\cdot x \equiv 1 \mod m.$$

Euler's theorem, which states that the following congruence is true if a and m are co-primes:

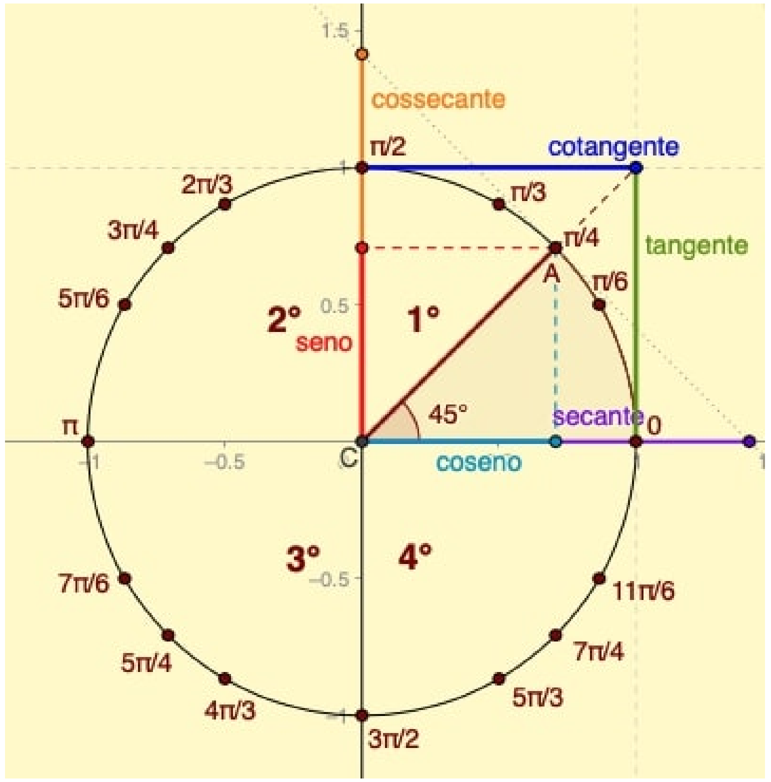
$$a^{\phi(m)} \equiv 1 \mod m$$

Multiply both sides of the above equations by a^{-1}, and we get:

- For an arbitrary (but coprime) modulus m: $a^{\phi(m)-1} \equiv a^{-1} \mod m$
- For a prime modulus m: $a^{m-2} \equiv a^{-1} \mod m$

From these results, we can easily find the modular inverse using the binary exponentiation algorithm, which works in $O(\log m)$ time.

11.7 Unit Circle


1.5
cossecante
π/2
cotangente
2π/3
π/3
3π/4
π/4
tangente
A
π/6
5π/6
0.5
2°
seno
1°
45°
secante
π
0
-1
-0.5
C
coseno
1
3°
4°
-0.5
7π/6
11π/6
5π/4
7π/4
4π/3
5π/3
3π/2