Introduction to statistics

Learning the basics of probability and statistics

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Summary:

- Probability concepts
- Discrete distributions.
- Continuous distributions.
- Calculations on the Normal distribution.
- Convergence
- Inference

Motivation

Figure 1: Dados apontam . . . (data shows . . .)

Basic concepts of probability:

Sample space Ω

It's the set of all the possible outcomes of a experiment, denoted by S or Ω

Event

It's a subset of the sample space.

Basic concepts of probability:

Probability (Definition):

Given a experiment with a sample space Ω and a class of events $\mathcal{A}$, the probability denoted by $\mathbb{P}$ is a function which has $\mathcal{A}$ as domain and associate a numerical value between $[0,1]$ as image.

Probability properties:

1. $\mathbb{P}(\Omega) = 1$ and $\mathbb{P}(\emptyset) = 0$
2. $0 \leq \mathbb{P}(A) \leq 1$, for every event A
3. For any sequence of mutually exclusive events $A_1, A_2, \ldots$ that's events that $A_i \bigcap A_j$ when $i \neq j$ we have that:

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

Basic concepts of probability:

Event independence:

Two events are independent when the occurrence of the first does not affect the probability of ocurrence of the second.
Two events A and B are independent if:

$$\mathbb{P}(A \bigcap B) = \mathbb{P}(A)\mathbb{P}(B)$$

Conditional Events:

The probability of a event A to occur given that the event B occurred is:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \bigcap B)}{\mathbb{P}(B)}$$

Basic concepts of probability:

Bayes theorem:

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}$$

General case:

$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(B|A_i)\mathbb{P}(A_i)}{\sum_{j=1}^{n} \mathbb{P}(B|A_j)\mathbb{P}(A_j)}$$

Bayes example (from Veritasium):

You are felling sick, so you go to the doctor, there you run a battery of tests. After getting the results you tested positive for a rare disease (affects 0.1% of the population), the test will correctly identify that you have it 99% of the times.
What's the chances that you actually have the disease? 99%?

Bayes example Solution

Let's denote the event of you have the disease H (stands for hypothesis, the prior) and the test been positive denoted by E (stands for evidence), so we have: $\mathbb{P}(H) = 0.001$ and $\mathbb{P}(E|H) = 0.99$

$$\mathbb{P}(H|E) = \frac{\mathbb{P}(E|H)\mathbb{P}(H)}{\mathbb{P}(E)} = \frac{\mathbb{P}(E|H)\mathbb{P}(H)}{\mathbb{P}(H)\mathbb{P}(E|H) + \mathbb{P}(H^C)\mathbb{P}(E|H^C)} =$$

$$= \frac{0.99 \cdot 0.001}{0.001 \cdot 0.99 + 0.999 \cdot 0.01} = 0.09 = 9\%$$

What if you test again and it's also positive? You can just take the posterior probability we just calculated and use as a prior:

$$= \frac{0.99 \cdot 0.09}{0.09 \cdot 0.99 + 0.91 \cdot 0.01} = 0.907 \approx 91\%$$

- Awesome video: A visual guide to Bayesian thinking

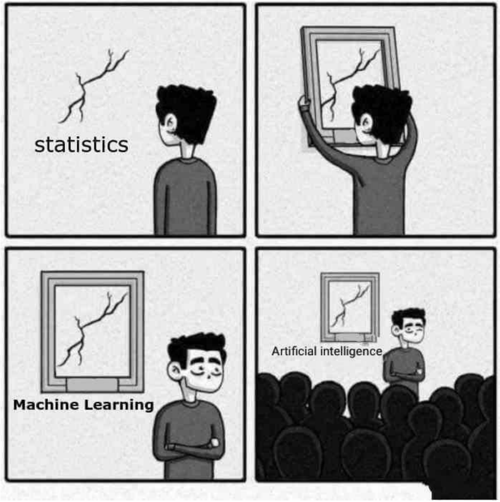


Figure 2: Credits: sandserifcomics

Random Variable (RV)

Consider a experiment with a sample space Ω associated with it. A function that maps each element $\omega \in \Omega$ to a Real number such that $[w \leq X]$ it's called random variable (RV) $(X : \Omega \rightarrow \mathbb{R})$

- Example: Imagine a experiment that consist of 3 consecutive fair coin tosses, so the sample space of this experiment is: S = {(H,H,H), (H,H,T) , . . . (T,T,T)} . Now we want to create a random variable X that counts the number of heads in each outcome, so X((H,H,H)) = 3 and X((H,H,T)) = 2.

Random Variable:

Probability Mass Function (PMF):

$$f_X(x) = \mathbb{P}[X = x] = \mathbb{P}[\{\omega \in \Omega : X(\omega) = x\}]$$

Probability Density Function (PDF)

$$\mathbb{P}[a \leq X \leq b] = \int_a^b f(x)dx$$

Cumulative Distribution Function (CDF)

$$F_X(x) = \mathbb{P}[X \leq x]$$

Expectation:

- Discrete : $\mathbb{E}[X] = \sum x\mathbb{P}(X = x)$
- Continuous: $\mathbb{E}[X] = \int_{-\infty}^{\infty} xf(x)dx$

Variance:

$$\mathbb{V}[X] = \sigma_X^2 = \mathbb{E}[X^2] - \mathbb{E}^2[X]$$

Sample mean:

$$\overline{X_n} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Sample variance and standard deviation:

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

Standard deviation $= s$

Discrete distributions

Bernoulli:

Consider a experiment with has two possible outcomes: success (X=1, with probability p) or failure (X=0), this random variable is called Bernoulli, the PMF is:

$$\mathbb{P}(X = k) = p^k(1-p)^{1-k}$$

Binomial:

Now consider a Bernoulli experiment conducted n times, let X be the random variable that represents the number of successes, X is called Binomial, the PMF is:

$$\mathbb{P}(X = k) = \binom{n}{k} p^k(1-p)^{n-k}$$

Discrete distributions

Geometric:

Again consider a Bernoulli experiment conducted n times, but the first n-1 are failures and the last nth is a success. Let X be number of tries , which is called Geometric, the PMF is:

$$\mathbb{P}(X = k) = (1-p)^k p$$

- A important property is that Geometric distribution is the only discrete distribution that is **memoryless**.

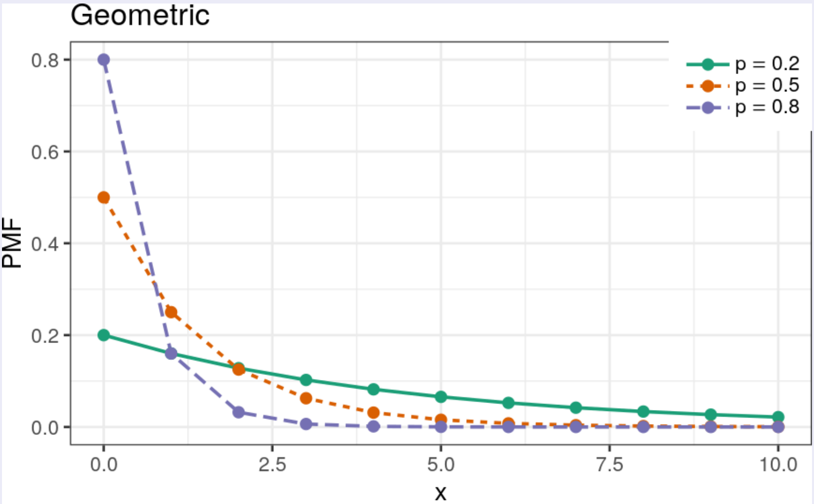
Poisson:

A random variable which value can assume 0,1,2 ... is called Poisson with $\lambda > 0$ parameter if your PMF is:

$$\mathbb{P}(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

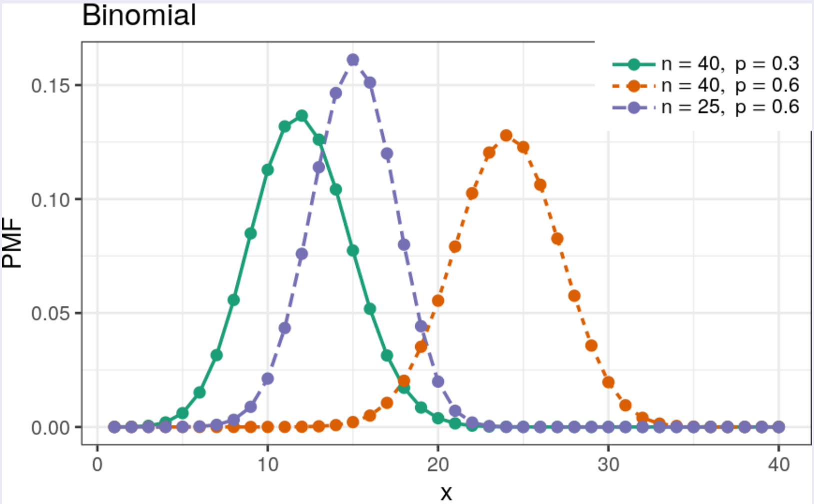
Discrete distributions plots

Geometric:



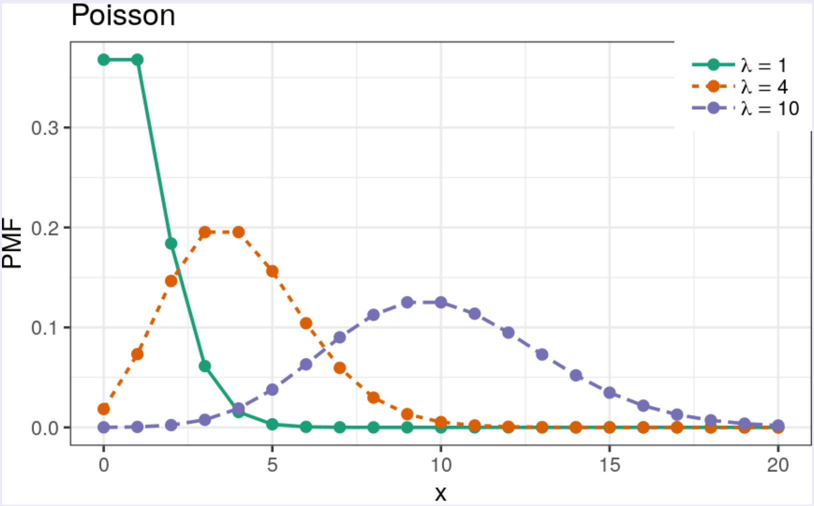
Discrete distributions plots

Binomial:



Discrete distributions plots

Poisson:



Continuous distributions

Normal (or Gaussian, bell curve):

A continuous real random variable is called Normal with $\sigma^2 > 0$ (squared scale), $\mu \in \mathbb{R}$ (location) parameters if your PDF is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$$

- The normal function is a example of Liouville's theorem, an probability cannot be analytically calculated, only be numeric methods.
- Fun facts: the half inside the exponential is for the variance to be 1, and the $\sqrt{2\pi}$ is for the integral in the whole support to become 1.

Continuous distributions

Exponential

A continuous **positive** random variable is called Exponential with $\lambda > 0$ (rate or inverse scale) parameter if your PDF is:

$$f(x) = \lambda e^{-\lambda x}$$

Important property: Exponential and Geometric (discrete) distribution are the only distributions that are **memoryless**.

Memoryless property:

$$\mathbb{P}[X > x + y \mid X > y] = \mathbb{P}[X > x]$$

So no matter how much time has passed it's like the process is starting from beginning.

Continuous distributions

Pareto

A continuous $x \in [x_m, \infty)$ random variable is called pareto with $x_m > 0$ (scale) and $\alpha > 0$ (shape) parameters if your PDF is:

$$f(x) = \frac{\alpha x_m^\alpha}{x^{\alpha+1}}$$

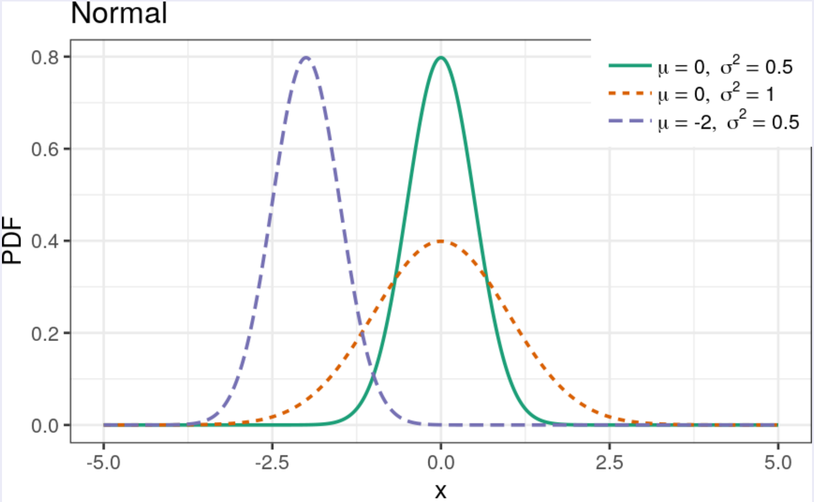
Zipf is the discrete distribution of pareto
Pareto is a **heavy tailed** distribution: It means it goes to zero slower (than exponential).

Pareto principle (80-20 law):

The pareto principle states that 80% of results is caused by 20% of the effects, for example wealth distribution, software bugs etc ...
It's a particular pareto distributed values when $\alpha \approx 1.161$

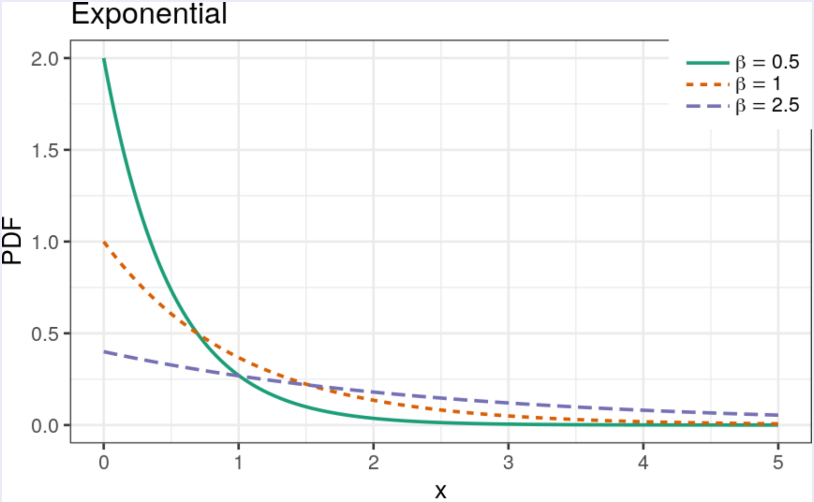
Continuous distributions plots

Normal:



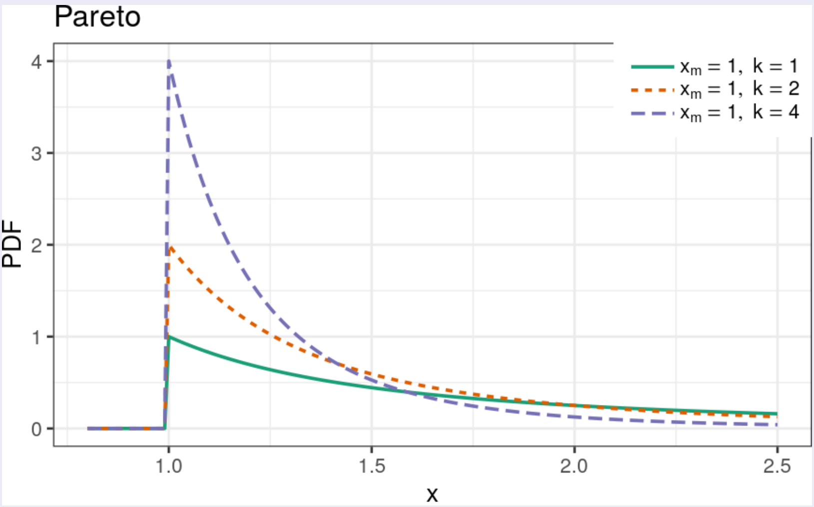
Continuous distributions plots

Exponential:



Continuous distributions plots

Pareto:



Calculations on the Normal distribution

Given a Normal distributed values, how to calculate the probability on it?

With normal distribution we usually use a standard normal (where $\mu = 0, \sigma = 1$) cumulative table and standardize the values.

- How to standardize the values: Given $X \sim N(\mu, \sigma^2)$

$$z = \frac{x - \mu}{\sigma} \quad \text{or} \quad z = \frac{x - \overline{x}}{s}$$

z is called **z score** and is **standard normal** distributed.

- Standard cumulative $\Phi(x)$:

$\Phi(x) = \mathbb{P}(z \leq x)$ also $\Phi(-x) = 1 - \Phi(x)$

$\Phi(x)$ values can we found in a table or using NORMSDIST function in Excel or in Python using stats.norm.cdf function from SciPy.

68-95-99.7 rule:

The 68-95-99.7 rule also know as the empirical rule is a shorthand to remember the percentage of Normal distributed values that lie within arround the mean with a width of 1,2,3 standard deviations.

$$\mathbb{P}(\mu - 1\sigma \leq X \leq \mu + 1\sigma) \approx 0.6827$$
$$\mathbb{P}(\mu - 2\sigma \leq X \leq \mu + 2\sigma) \approx 0.9545$$
$$\mathbb{P}(\mu - 3\sigma \leq X \leq \mu + 3\sigma) \approx 0.9973$$

We don't have to memorize this values, we can calculate it:

$$\mathbb{P}(\mu - 1\sigma \leq X \leq \mu + 1\sigma) = \mathbb{P}(-1\sigma \leq X - \mu \leq 1\sigma) =$$

$$\mathbb{P}\left(-1 \leq \frac{X-\mu}{\sigma} \leq 1\right) = \mathbb{P}(-1 \leq z \leq 1) = \Phi(1) - \Phi(-1) \approx 0.6827$$

68-95-99.7 rule: Chart

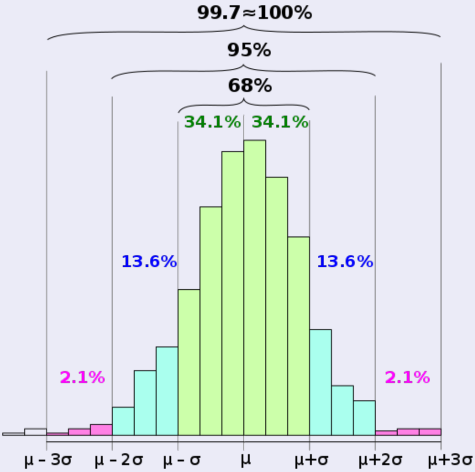
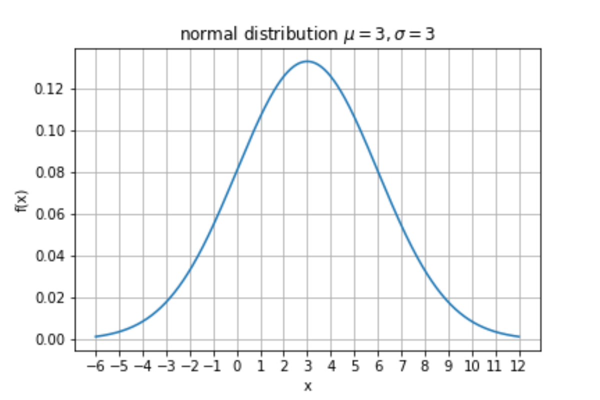


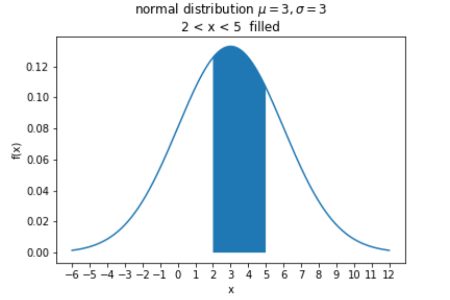
Figure 3: Credits: Wikipedia: Empirical rule histogram

Calculations on the Normal distribution: Example from Ross:

X is a normal random variable with parameters: $\mu = 3$ and $\sigma^2 = 9$,
Calculate: (a) $\mathbb{P}(2 < X < 5)$ (b) $\mathbb{P}(X > 0)$ (c) $\mathbb{P}(|X - 3| > 6)$



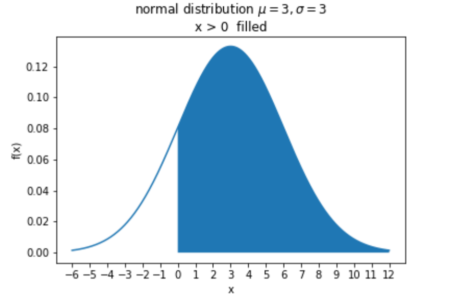
Calculations on the Normal distribution: Example from Ross: part a



$$\mathbb{P}(2 < X < 5) = \mathbb{P}\left(\frac{2-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{5-\mu}{\sigma}\right) = \mathbb{P}\left(\frac{-1}{3} < Z < \frac{2}{3}\right) =$$

$$\Phi\left(\frac{2}{3}\right) - \Phi\left(\frac{-1}{3}\right) = \Phi\left(\frac{2}{3}\right) - \left[1 - \Phi\left(\frac{1}{3}\right)\right] \approx 0.3779$$

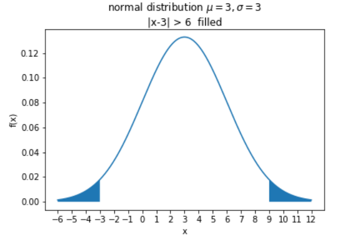
Calculations on the Normal distribution: Example from Ross: part b



$$\mathbb{P}(X > 0) = \mathbb{P}\left(\frac{X-\mu}{\sigma} > \frac{-\mu}{\sigma}\right) = \mathbb{P}(Z > -1) = 1 - \Phi(-1) =$$

$$= \Phi(1) \approx 0.8413$$

Calculations on the Normal distribution: Example from Ross: part c



$$\mathbb{P}(|X-3|>6) = \mathbb{P}(6 < X-3 < -6) = \mathbb{P}(9 < X < -3) =$$

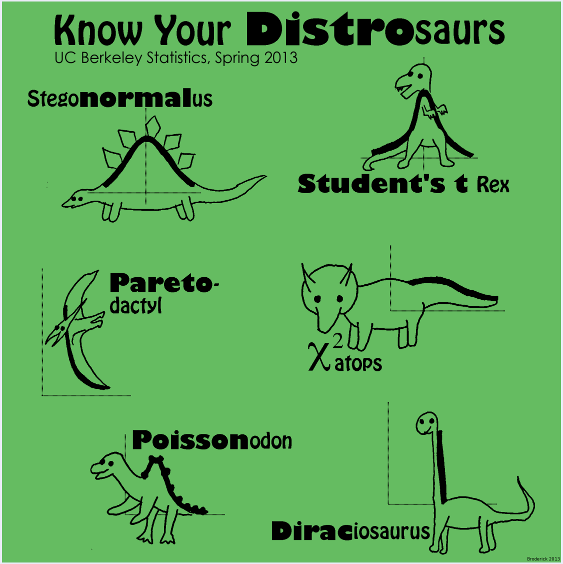
$$\mathbb{P}(X>9)+\mathbb{P}(X<-3) = \mathbb{P}\left(\frac{X-\mu}{\sigma} > \frac{9-\mu}{\sigma}\right)+\mathbb{P}\left(\frac{X-\mu}{\sigma} < \frac{-3-\mu}{\sigma}\right)$$

$$= \mathbb{P}(Z>2)+\mathbb{P}(Z<-2) = 2[1-\Phi(2)] \approx 0.0456$$

Assumptions on distribution choice

Figure 4: Credits: Portal data science

Assumptions on distribution choice



Assumptions on distribution choice

In order to know which distribution of your data values understanding the nature of the problem is fundamental.
Is your values a result from counting? So is it discrete ? Or continuous? Which values are possible?

Discrete distributions:

- Bernoulli: boolean result, example: coin toss, second turn (with only 2 candidates) election.
- Binomial: Number of “success” results given a permanent experiment runs. Example: From 20 devices after a long time what’s the probability of 15 of them has a kind of defect.
- Geometric: Number of failures until the first success. Example: The probability of winning the lottery is 1 in 1 million, What’s the probability of winning it after 3 tries?
- Poisson: Example: Number of cars on the road.

Assumptions on distribution choice

Continuous distributions:

- Normal: No restriction on possible values (positive and negative values are valid). Example: The height of children of the same sex and age.
- chi-squared: Only positive values, unlike normal is not symmetric.
- Exponential: Only positive values, describes the time until failure.
- Pareto: Only positive values and bigger and x_m. Example: Size of gold mines, very few big mines and a lot of small ones.

order statistics and quantiles

- Given $X_1, X_2, \cdots, X_n$ values from the same distribution, let:

$X_{(1)}$ the smallest value from $X_1, X_2, \cdots, X_n$ (minimum)
$X_{(2)}$ 2th smallest value from $X_1, X_2, \cdots, X_n$
$X_{(j)}$ jth smallest value from $X_1, X_2, \cdots, X_n$
$X_{(n)}$ the **biggest** value from $X_1, X_2, \cdots, X_n$ (maximum)

- q-quantiles are values that partition the values into q subsets of (almost) equal sizes. For instance: q=2 we have the median, 4 the quartiles , 100 percentile and so on ...
- Why use quantiles? Why use median instead of the mean? Because Order statistics is a **robust statistic** which means it is not affected by outliers.

boxplots:

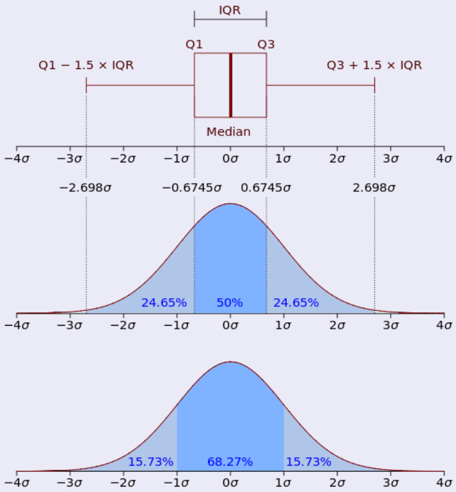
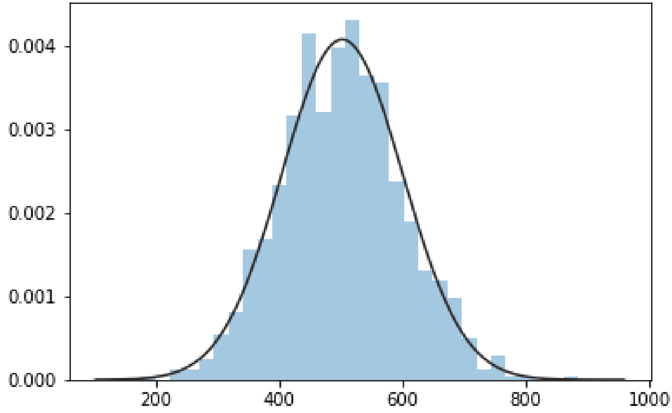


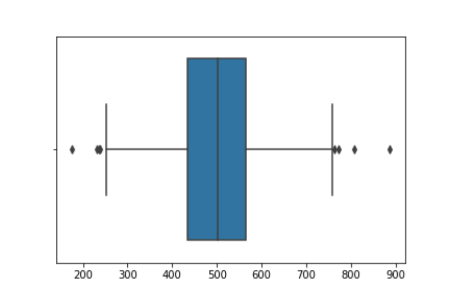
Figure 5: Credits: Wikipedia: Boxplot vs PDF

Boxplots example:

Let X_n be a sequence of normal distributed random variable with $\mu = 500$ and $\sigma = 100$ we have n=1000 samples, results:



Boxplots example:



count	1000.000000
mean	501.933206
std	97.921594
min	175.873266
25%	435.240969
50%	502.530061
75%	564.794388
max	885.273149

Convergence

In statistics there some types of convergence, the main ones are: Let $\{X_1, X_2, \cdots\}$ be a sequence of identically distributed random variables.

1. In Probability: $X_n \xrightarrow{p} Y$:

$$(\forall \varepsilon > 0) \quad \lim_{n\to\infty} \mathbb{P}(|X_n - Y| > \varepsilon) = 0$$

2. In distribution (weakly, in law): $X_n \xrightarrow{D} Y$

$$\lim_{n\to\infty} F_{X_n}(x) = F_Y(y)$$

3. Almost sure (strongly) : $X_n \xrightarrow{as} Y$

$$\mathbb{P}\left(\lim_{n\to\infty} X_n = Y\right) = 1$$

Convergence

Law of large numbers (LLN):

Let $\{X_1, X_2, \cdots\}$ be a sequence of identically distributed random variables and $\mathbb{E}[X] = \mu$

Weak (WLLN)

$\overline{X_n} \xrightarrow{p} \mu \quad n \to \infty$

Strong (SLLN)

$\overline{X_n} \xrightarrow{as} \mu \quad n \to \infty$

In words: The sample mean converge to the (theoretical) expected value as the sample size increases.

Convergence

Central Limit Theorem (CLT)

Let $\{X_1, X_2, \cdots\}$ be a sequence of identically distributed random variables and $\mathbb{E}[X] = \mu$ and $\mathbb{V}[X] = \sigma^2$
The CLT states that:

$$\overline{X_n} \xrightarrow{D} \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

After some transformations we have:

$$\frac{\sqrt{n}(\overline{X_n} - \mu)}{\sigma} \xrightarrow{D} \mathcal{N}(0, 1)$$

Inference:

Inference is the process to deduce (estimate) the properties of underlying probability distribution from the data values.

Maximum likelihood estimator (MLE):

One of the most popular estimator method is the maximum likelihood estimator, it consists of finding the parameters that maximizes the likelihood function (via derivatives).

Q-Q plot:

Q-Q (quantile-quantile) plot: Is a plot where data quantiles are ploted in one axis and the theoretical quantiles of the fitted distribution on the other axis and a linear regression of the points. This plot can be used to provide a assessment of the "goodness of fit" and maybe find out the data outliers on the data.

Inference process

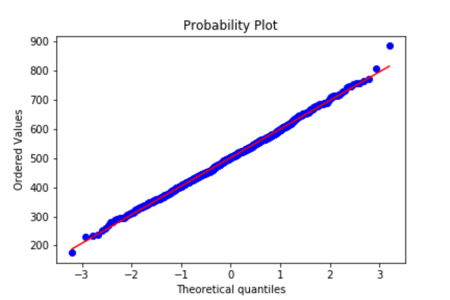
Given some data the inference process goes like this:

1. Choose a distribution (based on the nature of the problem).
2. Fit the distribution (estimate the parameters).
3. Create a q-q plot to judge the goodness of fit, if some outliers are found they can be identified (and maybe left out).

If the line still not good try starting again with a different distribution.

Inference example:

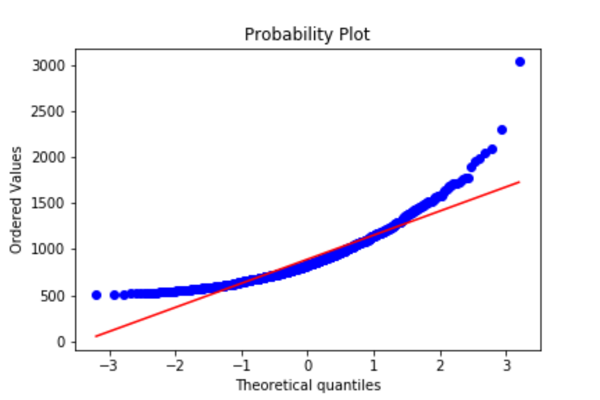
Using the same data from the boxplot example, the Q-Q plot:



The plot has created using stats.probplot from SciPy.
Using stats.norm.fit we can estimate (fit) the parameters (assuming normal distribution), we get: loc=501.93 and scale=97.87 we generated random values using loc=500,scale=100n=1000 (the fit could get better with a bigger sample)

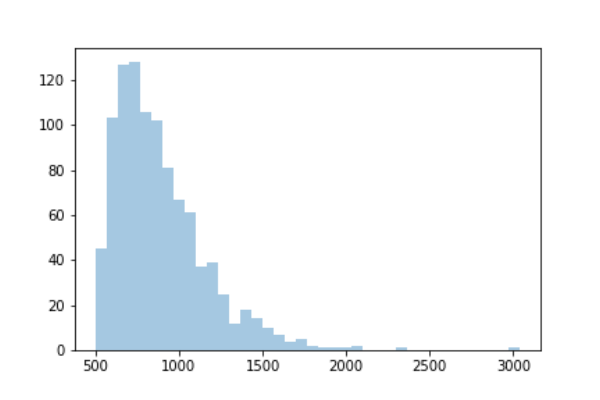
Inference example 2:

With new data (n=1000), the Q-Q plot:



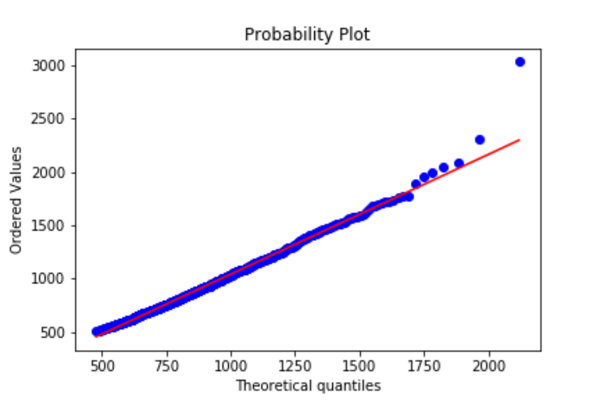
Looking at the line we can clearly see that the line did not fit the points, the data probably is not normal distributed.

Inference example 2: histogram:



the values is not symmetric, one good guest is that it's a chi squared distribution.

Inference example 2: Q-Q plot assuming chi squared:



Using stats.chi2.fit we can estimate (fit) the parameters: df=5.5,loc=460,scale=74, we generated random values using df=4,loc=500,scale=100.

Further reading:

To study those topics in depth, here are some awesome references:

Getting started:

- Podcast: (pt) Pizza de Dados
- Book: The Lady Tasting Tea: (pt) Uma Senhora Toma Chá.
- /r/dataisbeautiful

Studying material:

- (pt) Portal Action
- The Probability and Statistics Cookbook
- Havard Statistics 110: Probability
- Statistics and probability
- Random website
- Book: (en) First Course in Probability by Sheldon Ross: (pt) Probabilidade: Um Curso Moderno com Aplicações