THORNDIKE'S CATS

A Guide to Reinforcement Learning

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Contents

Chapter 1

Probability Theory

The objective of *reinforcement learning* is to design agents that interact with their environment in some optimal sense by training them to take actions that maximise a notion of cumulative reward. These environments are often stated in the form of a *Markov decision process*, a fundamental concept in the theory of *dynamic stochastic control*. Therefore, in order to understand reinforcement learning, we require a solid grounding in *probability theory*.

1.1 Random Variables

For the below, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and (S, Σ) be a measurable space.

Definition 1.1.1 (Random Variable)**.** A *random variable* is a function $X : \Omega \to S$ such that for each $B \in \Sigma$, $X^{-1}(B) \in \mathcal{F}$.

Definition 1.1.2 (Probability Distribution)**.** The *distribution* of X is the *pushforward measure* $X_*\mathbb{P} : \Sigma \to [0, 1]$ given by:

$$X_*\mathbb{P}(B) = \mathbb{P}(X^{-1}(B)) = \mathbb{P}(X \in B)$$

We say that $X \sim \mathcal{D}$ iff $\forall B \in \Sigma$, $X_*\mathbb{P}(B) = \mathcal{D}(B)$. This definition is broadly a generalisation of the concept of a cumulative distribution function, acting on sets rather than fixed intervals.

To understand properly the related concept of probability mass and density functions, we require notions of σ-finiteness and absolute continuity of measures.

Definition 1.1.3. A measure μ on a measurable space $(\Omega, \mathcal{F})$ is σ-*finite* if, for each $F \in \mathcal{F}$ with $\mu(F) = \infty$, F can be written as a countable union of sets in $\mathcal{F}$, each with finite measure under μ.

Definition 1.1.4. Given measures μ, ν on $(\Omega, \mathcal{F})$, we say that μ is *absolutely continuous* with respect to ν and write $\mu \ll \nu$, if for each $F \in \mathcal{F}$, $\nu(A) = 0 \Rightarrow \mu(A) = 0$.

Radon-Nikodym Theorem. *Suppose μ, ν are two σ-finite measures on $(\Omega, \mathcal{F})$ with $\mu \ll \nu$. Then there exists a unique[1] measurable function $f : \Omega \to [0, \infty)$ such that for each $F \in \mathcal{F}$,*

$$\mu(F) = \int_F f d\nu$$

f is called the Radon-Nikodym derivative and is denoted $\frac{d\mu}{d\nu}$.

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[1]ν-almost everywhere

1.1.1   Discrete Random Variables

A discrete random variable is a random variable with codomain $(\mathbb{Z}, \mathcal{P}(\mathbb{Z}), \#)$ where $\#$ is the *counting measure.*

Definition 1.1.5 (Probability Mass Function)**.** The *probability mass function* of a distribution is given by the Radon-Nikodym derivative of the pushforward measure with respect to the counting measure. Namely any measurable function $f : \mathbb{Z} \to [0,1]$ such that $\forall Z \subseteq \mathbb{Z}$,

$$X_*\mathbb{P}(Z) = \int_Z f d\#$$

Theorem 1.1.1. *Every probability distribution of a discrete random variable admits a probability mass function.*

Proof. First observe that, should the derivative exist its range must be limited from $[0, \infty)$ to the range of the pushforward measure $[0,1]$ as the integral is summing values in the function range. To show existence, it suffices, by the Radon-Nikodym theorem, to show that every discrete distribution is absolutely continuous with respect to the counting measure. Suppose that $\#(Z) = 0$ then $Z = \varnothing$ and by definition of a measure, $X_*\mathbb{P}(\varnothing) = 0$. $\qquad \square$

Definition 1.1.6 (Cumulative Distribution Function)**.** The *cumulative distribution function* of a distribution $F : \mathbb{Z} \to [0,1]$, with mass function f, is given by the distribution function of an interval:

$$F(x) = X_*\mathbb{P}([0, \max\{0, x\}]) = \int_{[0, \max\{0, x\}]} f d\# = \sum_{i=0}^{\max\{0,x\}} f(x)$$

1.1.2   Continuous Random Variables

A continuous random variable is a random variable with codomain $(\mathbb{R}, L(\mathbb{R}), \lambda)$ where λ is the *Lebesgue measure* and $L(\mathbb{R})$ the *Lebesgue σ-algebra* over $\mathbb{R}$.

Definition 1.1.7 (Probability Density Function)**.** The *probability density function* of a distribution is given by the Radon-Nikodym derivative of the pushforward measure with respect to the Lebesgue measure. Namely any measurable function $f : \mathbb{R} \to [0,1]$ such that $\forall R \in L(\mathbb{R})$,

$$X_*\mathbb{P}(R) = \int_R f d\lambda$$

Note that there is no such theorem analogous to 1.1.1 since the probability of a random variable being a member of a null set[2] is not necessarily zero.

1.2   Expected Value and Conditioning

While not required immediately, it will be useful to define this concept for future reference.

Definition 1.2.1. (Expected Value) The *expected value* of a random variable $X : \Omega \to S$ is given by:

$$\mathbb{E}(X) = \int_\Omega X d\mathbb{P}$$

Recall the definition of the indicator function which will be required later.

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[2]A set with a zero Lebesgue measure.

Definition 1.2.2. (Indicator Function) For a set A, the *indicator function* of A is given by:

$$\mathbb{1}_A(x) := \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Definition 1.2.3. (Conditional Expectation) Suppose $\mathcal{H} \subseteq \mathcal{F}$ is a sub-σ-algebra. The *conditional expectation* of X with respect to $\mathcal{H}$ is any $\mathcal{H}$-measurable function $\mathbb{E}(X \mid \mathcal{H}) : \Omega \to S$ such that for every $H \in \mathcal{H}$,

$$\int_H \mathbb{E}(X \mid \mathcal{H})\mathrm{d}\mathbb{P} = \int_H X \mathrm{d}\mathbb{P}$$

Conditional expectation exists uniquely[3] in L^1 space and possibly all L^p spaces but its existence and uniqueness in further generality requires investigation. Finally, we have the definition of conditional probability with respect to a σ-algebra.

Definition 1.2.4. (Conditional Probability) The conditional probability of $F \in \mathcal{F}$ with respect to $\mathcal{H} \subseteq \mathcal{F}$ is given by:

$$\mathbb{P}(F \mid \mathcal{H}) := \mathbb{E}(\mathbb{1}_F \mid \mathcal{H})$$

In other words, any $\mathcal{H}$-measurable function $f : \Omega \to S$ such that for every $H \in \mathcal{H}$:

$$\int_H f \mathrm{d}\mathbb{P} = \int_H \mathbb{1}_F \mathrm{d}\mathbb{P} = \int_F \mathrm{d}\mathbb{P} = \mathbb{P}(F)$$

Definition 1.2.5. (Kolmogorov Conditional Probability) The conditional probability of $F \in \mathcal{F}$ with respect to $H \in \mathcal{F}$ is given by:

$$\mathbb{P}(F \mid H) := \frac{\mathbb{P}(F \cap H)}{\mathbb{P}(H)}$$

1.3 Stochastic Processes

Conceptually, we think of a stochastic process as sequence of random variables, each affected in some way by a number of the previous. However, in order to define this idea properly, we require a notion of the events that have occurred up to a point in the sequence. We call this a *filtration*.

Definition 1.3.1. (Filtration) A *filtration* of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a collection of σ-algebra $\{\mathcal{F}_t : t \in T\}$ such that for each $t, s \in T$, if $s \leq t$ then $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$.

Definition 1.3.2. (Stochastic Process) Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and (S, Σ) a measurable space. A *stochastic process* is a collection or sequence $\{X_t : t \in T\}$ of random variables, mapping these spaces. The set T is usually, though not necessarily, totally ordered.

We say a stochastic process is *adapted* to a filtration $\{\mathcal{F}_t : t \in T\}$ if each of its X_t is measurable with respect to its corresponding $\mathcal{F}_t$.

Definition 1.3.3. (Generated σ-algebra) Given a set, Ω and a collection of subsets $F \subseteq \mathcal{P}(\Omega)$, we define the *$\sigma$-algebra generated by F*, denoted $\sigma(F)$, as the collection of all subsets of Ω that can be formed by a countable number of unions, intersections or complements of elements of F.

Definition 1.3.4. (Natural Filtration) Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, (S, Σ) a measurable space and $\{X_t : t \in T\}$ a stochastic process. The *natural filtration* of the stochastic process is given by:

$$\mathcal{F}_t = \sigma(\{X_s^{-1}(B) : B \in \Sigma, t \in T, s \leq t\})$$

Lemma 1.3.1. *Every stochastic process is adapted to its natural filtration.*

Proof. This is trvial as X_t will always be measurable with respect to a σ-algebra generated by a collection that includes its preimage. $\qquad\square$

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[3]Almost surely.

1.4 The Markov Property

The *Markov property*, gives a formal definition to the idea of a memoryless process. That is, one whose current state is only informed by its direct predecessor rather than all that came before it.

Definition 1.4.1. (Markov Property) Suppose $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in T}, \mathbb{P})$ is a filtered probability space, (S, Σ) is a measurable space and $\{X_t\}_{t \in T}$ is a stochastic process adapted to the filtration. The stochastic process is a *Markov process* or have the *Markov property* if, for each $A \in \Sigma$ and $s, t \in T$ with $s < t$,

$$\mathbb{P}(X_t \in A \mid \mathcal{F}_t) = \mathbb{P}(X_t \in A \mid \sigma(X_s^{-1}(B) : B \in \Sigma))$$

1.4.1 Markov Chains

In reinforcement learning, we will always be in discrete time and so do not always require the generality of the above definitions, especially in the case of the Markov property. Therefore, we present some simplified definitions, only relevant to the case of countable index sets T. We call such Markov processes: *Markov chains*.

Lemma 1.4.1. *Suppose $T \subseteq \mathbb{N}$. Then the Markov property is equivalent to:*

$$\mathbb{P}\left(X_t \in A \mid \bigcap_{s < t} X_t\right) = \mathbb{P}(X_t \in A \mid X_t)$$

Definition 1.4.2. (Markov Chain) Suppose $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in T}, \mathbb{P})$ is a filtered probability space, (S, Σ) is a measurable space, $T \subseteq \mathbb{N}$ and $\{X_t\}_{t \in T}$ is a stochastic process adapted to the filtration. The stochastic process, is a *Markov chain* if...

Chapter 2

Markov Decision Processes

2.1 Introduction

The *Markov decision process* (MDP) is a fundamental object of study within reinforcement learning. Most RL problems will be presented as the task of solving an MDP. In the case of *deep* reinforcement learning, for example, *artificial neural networks* are used to solve an MDP.

Definition 2.1.1. A *Markov decision process* is a 4-tuple (S, A, P_a, R_a) where:

1. S is the set of states, called the *state space*.

2. A is the set of actions, called the *action space*.

3. $P_a(s, s') = \mathbb{P}(s_{t+1} = s' \mid (s_t = s) \cap (a_t = a))$ is the probability that action a in state s at time t will lead to state s' at time $t + 1$.

4. $R_a(s, s')$ is the immediate reward, or expected immediate reward, received after transitioning from state s to state s', due to action a.

2.1.1   Variable Spaces

In general, the underlying measure space $(\Omega, \mathcal{F}, \mathbb{P})$ of an experiment is rarely explicitly characterized or even characterizable and so we will denote the set of all random variables with discrete space domain as $\mathcal{L}$.

Definition 2.1.2 (The Distribution Space)**.** The *distribution space* of a discrete distribution $\mathcal{D}$ is given by the set of all discrete random variables with distribution $\mathcal{D}$, denoted:

$$\mathcal{L}(\mathcal{D}) = \{X : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{Z}, \mathcal{P}(\mathbb{Z}), \#) : X \sim \mathcal{D}\} = \{X \in \mathcal{L} : X \sim \mathcal{D}\}$$

2.2   Uniform Draw Representation

For the below, we require the notion of a continuous random variable. The domain measure space is $([0, 1], L[0, 1], \lambda)$, the unit interval with the Lebesgue σ-algebra and the Lebesgue measure. The definition is analogous to the discrete case.

Definition 2.2.1. A random variable $U : \Omega \to [0, 1]$ is *uniformly distributed* if for each measurable subset $Z \subseteq [0, 1]$, $\mathbb{P}(U \in Z) = \lambda(Z)$.

Note that this implies that $\forall a, b, k \in [0, 1]$ with $a \leq b$,

$$\mathbb{P}(k \in [a, b]) = \mathbb{P}(k \in (a, b]) = \mathbb{P}(k \in [a, b)) = \mathbb{P}(k \in (a, b)) = b - a$$

Definition 2.2.2 (Uniform Draw Variable)**.** For a uniform random variable U, the associated *uniform draw variable* of a distribution with c.d.f. F is given by $F^{-1}(U)$.

Proposition 2.2.1. $F^{-1}(U)$ *has the distribution associated with* F.

Proof. It suffices to show that the variable has the density that defines the cumulative distribution function.

$$\begin{aligned}
\mathbb{P}(F^{-1}(U) = k) &= \mathbb{P}(\min\{k \in \mathbb{Z} : U \leq F(k)\} = k) \\
&= \mathbb{P}(F(k-1) < U \leq F(k)) \\
&= \mathbb{P}(U \in (F(k-1), F(k)]) = F(k) - F(k-1) = f(k)
\end{aligned}$$

$\square$

Definition 2.2.3. A function $q : [0, 1] \to [0, 1]$ is called a *mean involution* if for any uniformly distributed random variable U, $\mathbb{P}(F^{-1}(q(U)) = k) = f(k)$.

2.3   Poisson Distribution

We define the Poisson distribution as the limit of a binomial distribution.

2.3.1   Poisson Limit Theorem

Definition 2.3.1. Suppose $X \in \mathcal{L}$. We say that $X \sim \mathrm{Bin}(n, p)$ with $n \in \mathbb{Z}$ and $p \in [0, 1]$ if:

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

Suppose that $X \in \mathcal{L}$ has a binomial distribution with n trials and $\mathbb{E}(X) = \lambda \in \mathbb{R}_{\geq 0} < n$. Then the probability of success of each trail is λ/n and as the number of trials tends to infinity, the random variable X becomes a infinite number of infinitesimally probable trials in some fixed time period, whose number of successes is expected to be λ. This is the *Poisson distribution* and its p.m.f. is given by the limit of the binomial as $n \to \infty$.

Proposition 2.3.1.

$$\lim_{n \to \infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} = e^{-\lambda}\frac{\lambda^k}{k!}$$

Definition 2.3.2 (Poisson Distribution)**.** Suppose $X \in \mathcal{L}$. We say that $X \sim \mathcal{P}(\lambda)$ with $\lambda \in \mathbb{R}_{\geq 0}$ if:

$$\mathbb{P}(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}$$

2.4 Poisson Shocks

2.4.1 Common Shock Model

2.4.2 Noble Shock Model

Definition 2.4.1. Suppose that $H \sim \mathcal{P}(\lambda_H)$, $A \sim \mathcal{P}(\lambda_A)$ are independent random variables and let $F^{-1}(U), F^{-1}(q(U)) \sim \mathcal{P}(\lambda_X)$ be uniform draw variables, independent of H and A. The common shock variable is then given by:

$$H + F^{-1}(U) \cap A + F^{-1}(q(U))$$

Proposition 2.4.1. *dd*

Proof.

$$\mathbb{P}(H + F^{-1}(U) \cap A + F^{-1}(q(U)))$$
$$= \sum_{k=0}^{i} \mathbb{P}((H = k) \cap (F^{-1}(U) = i - k) \cap (A + F^{-1}(q(U)) = j))$$
$$= \sum_{l=0}^{j}\sum_{k=0}^{i} \mathbb{P}((H = k) \cap (F^{-1}(U) = i - k) \cap (A = l) \cap (F^{-1}(q(U)) = j - l))$$
$$= \sum_{l=0}^{j}\sum_{k=0}^{i} \mathbb{P}(H = k)\mathbb{P}(A = l)\mathbb{P}((F^{-1}(U) = i - k) \cap (F^{-1}(q(U)) = j - l))$$

$\square$

Chapter 3

The Beautiful Game

3.1 Random Variables *or Measurable Functions*

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,1]}, \mathbb{P})$ be a filtered probability space and let the goals scored by the home team and away team respectively at time $t \in [0,1]$ be $H_t, A_t : \Omega \to \mathbb{N}$. Note that time t is given by the proportion of time elapsed so that for example: half time, or 45 minutes, is given by $t = 0.5$. We assume the goal variables are *homogeneous Poisson counting processes.*

Definition 3.1.1. A *homogeneous Poisson counting process* $\{N_t : t \geq 0\}$ is a stochastic process with

 i. $N_t \geq 0$

 ii. independent increments

 iii. $N_t \sim \lambda_N t$

We will typically adopt a matrix notation for the *joint distributions* of H_1 and A_1 so that:

$$\alpha_{ij} = \mathbb{P}((H_1 = i) \cap (A_1 = j))$$

3.1.1 Empirical Probabilities

The aggregation of information and expertise can be found in the odds offered on betting exchanges for various events including, and particularly, football. The definition below is included for reference as it may feature in a future analysis of efficient markets.

Definition 3.1.2. Consider a bid-ask market M and suppose that b is the bid price and a the ask. Then the quoted spread Q is given by: $Q(M) = 2(a-b)/(a+b)$.

Efficient Market Hypothesis. *Let $h, a, d : [0,1] \to \mathbb{R}$ be the average of the back and lay odds, on a home win, away win and draw outcomes. Then,*

$$i. \quad \mathbb{P}(H_1 > A_1) = \frac{1}{h} \qquad ii. \quad \mathbb{P}(H_1 = A_1) = \frac{1}{d} \qquad iii. \quad \mathbb{P}(A_1 > H_1) = \frac{1}{a}$$

3.2 Distributions *or Measures of Measurable Preimages*

We describe the joint distributions of the variables, in both dependent and independent contexts, under the assumptions that they follow Poisson distributions.

3.2.1 Independent Poisson

We first the consider the independent case characterised by the following equivalent statements:

    i. $\mathbb{P}((H_1 = i) \cap (A_1 = j)) = \mathbb{P}(H_1 = i)\mathbb{P}(A_1 = j)$ ii. $\alpha_{ij} = \alpha_{i0}\alpha_{0j}$

Joint Distribution Matrix (IP1).

$$u : \mathbb{R}^2_{\geq 0} \to M_n(\mathbb{R}) \qquad\qquad u_{ij}(\lambda_H, \lambda_A) = e^{-(\lambda_H + \lambda_A)}\frac{\lambda_H^i \lambda_A^j}{i!j!}$$

Derivation. Trivial from the Poisson p.d.f. and independence assumption.   □

3.2.2 Dependent Poisson (Bivariate)

To consider the bivariate case, we make use of the *decomposability* of distributions.

Definition 3.2.1. A distribution $\mathcal{D}$ is *decomposable* if, for any random variable Z, there exist two independent, identically distributed variables X and Y such that $Z = X + Y$.

Theorem 3.2.1. *The Poisson distribution is decomposable.*

    Therefore,

Joint Distribution Matrix (IP1).

$$u : \mathbb{R}^2_{\geq 0} \to M_n(\mathbb{R}) \qquad\qquad u_{ij}(\lambda_H, \lambda_A) = e^{-(\lambda_H + \lambda_A)}\frac{\lambda_H^i \lambda_A^j}{i!j!}$$

Derivation. Trivial from the Poisson p.d.f. and independence assumption.   □

Chapter 4

4.1   Introduction

Under the assumption that the full-time result market is efficient, we attempt to infer from the odds, the parameters of independent and bivariate Poisson distributions. These distributions are then used to find value in side markets with less liquidity and, hopefully, less liquidity.

4.1.1   Variables

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,1]}, \mathbb{P})$ be a filtered probability space and let the goals scored by the home team and away team respectively at time $t \in [0,1]$ be $H_t, A_t : \Omega \to \mathbb{N}$. Note that time t is given by the proportion of time elapsed so that for example: half time, or 45 minutes, is given by $t = 0.5$. We assume the goal variables are *homogeneous Poisson counting processes*.

Definition 4.1.1. A *homogeneous Poisson counting process* $\{N_t : t \geq 0\}$ is a stochastic process with:

   i. $N_t \geq 0$

   ii. independent increments

   iii. $N_t \sim \lambda_N t$

Definition 4.1.2. A *model* is a matrix of the form $\alpha_{ij} = \mathbb{P}((H_1 = i) \cap (A_1 = j))$.

4.1.2   Exchange Odds

Definition 4.1.3. Consider a bid-ask market M and suppose that b is the bid price and a the ask. Then the quoted spread Q is given by: $Q(M) = 2(a - b)/(a + b)$.

Efficient Market Hypothesis. *Let $h, a, d : [0, 1] \to \mathbb{R}$ be the average of the back and lay odds, on a home win, away win and draw outcomes. Then,*

$$i. \quad \mathbb{P}(H_1 > A_1) = \frac{1}{h} \qquad ii. \quad \mathbb{P}(H_1 = A_1) = \frac{1}{d} \qquad iii. \quad \mathbb{P}(A_1 > H_1) = \frac{1}{a}$$

4.2   Independent Poisson Model

4.2.1   Distribution

Joint Distribution Matrix (Independent).

$$\alpha_{ij} : \mathbb{R}^2_{\geq 0} \to M_n(\mathbb{R}) \qquad \qquad \alpha_{ij} = e^{-(\lambda_H + \lambda_A)} \frac{\lambda_H^i \lambda_A^j}{i! j!}$$

Derivation.

$$\alpha_{ij} = \mathbb{P}((H_1 = i) \cap (A_1 = j))$$
$$= \mathbb{P}(H_1 = i)\mathbb{P}(A_1 = j)$$
$$= e^{-\lambda_H}\frac{\lambda_H^i}{i!}e^{-\lambda_A}\frac{\lambda_A^j}{j!}$$
$$= e^{-(\lambda_H + \lambda_A)}\frac{\lambda_H^i \lambda_A^j}{i!j!}$$

$\square$

4.2.2   Full-time Result Probabilities

Home Win

$$\mathbb{P}(H_1 > A_1) = e^{-(\lambda_H + \lambda_A)} \sum_{i > j \geq 0} \frac{\lambda_H^i \lambda_A^j}{i!j!}$$
$$= e^{-(\lambda_H + \lambda_A)}\left(\lambda_H + \frac{\lambda_H^2 \lambda_A}{2} + \frac{\lambda_H^2}{2} + \frac{\lambda_H^3 \lambda_A^2}{12} + \frac{\lambda_H^3 \lambda_A}{6} + \frac{\lambda_H^3}{6} + \frac{\lambda_H^4 \lambda_A^3}{144} + \frac{\lambda_H^4 \lambda_A^2}{48}\right.$$
$$\left. + \frac{\lambda_H^4 \lambda_A}{24} + \frac{\lambda_H^4}{24} + \frac{\lambda_H^5 \lambda_A^4}{2880} + \frac{\lambda_H^5 \lambda_A^3}{720} + \frac{\lambda_H^4 \lambda_A^2}{240} + \frac{\lambda_H^4 \lambda_A}{120} + \frac{\lambda_H^5}{120} + \dots\right)$$

Draw

$$\mathbb{P}(H_1 = A_1) = e^{-(\lambda_H + \lambda_A)} \sum_{i \geq 0} \frac{(\lambda_H \lambda_A)^i}{(i!)^2}$$
$$= e^{-(\lambda_H + \lambda_A)}\left(1 + \lambda_H \lambda_A + \frac{(\lambda_H \lambda_A)^2}{4} + \frac{(\lambda_H \lambda_A)^3}{36} + \frac{(\lambda_H \lambda_A)^4}{576} + \frac{(\lambda_H \lambda_A)^5}{14400} + \dots\right)$$

4.2.3   Away Win

$$\mathbb{P}(A_1 > H_1) = e^{-(\lambda_A + \lambda_H)} \sum_{j > i \geq 0} \frac{\lambda_H^i \lambda_A^j}{i!j!}$$
$$= e^{-(\lambda_A + \lambda_H)}\left(\lambda_A + \frac{\lambda_A^2 \lambda_H}{2} + \frac{\lambda_A^2}{2} + \frac{\lambda_A^3 \lambda_H^2}{12} + \frac{\lambda_A^3 \lambda_H}{6} + \frac{\lambda_A^3}{6} + \frac{\lambda_A^4 \lambda_H^3}{144} + \frac{\lambda_A^4 \lambda_H^2}{48}\right.$$
$$\left. + \frac{\lambda_A^4 \lambda_H}{24} + \frac{\lambda_A^4}{24} + \frac{\lambda_A^5 \lambda_H^4}{2880} + \frac{\lambda_A^5 \lambda_H^3}{720} + \frac{\lambda_A^4 \lambda_H^2}{240} + \frac{\lambda_A^4 \lambda_H}{120} + \frac{\lambda_A^5}{120} + \dots\right)$$

4.3   Bivariate Poisson Model

4.3.1   Distribution

Joint Distribution Matrix (Independent)**.**

$$\alpha_{ij} : \mathbb{R}_{\geq 0}^3 \to M_n(\mathbb{R}) \qquad\qquad \alpha_{ij} = e^{-(\lambda_H + \lambda_A)}\frac{\lambda_H^i \lambda_A^j}{i!j!}$$

Derivation.

$$\alpha_{ij} = \mathbb{P}((H_1 + X_1 = i) \cap (A_1 + X_1 = j))$$

$$= \sum_{k=0}^{i} \mathbb{P}((H_1 = k) \cap (X_1 = i - k) \cap (A_1 + X_1 = j))$$

$$= \sum_{l=0}^{j} \sum_{k=0}^{i} \mathbb{P}((H_1 = k) \cap (X_1 = i - k) \cap (A_1 = l) \cap (X_1 = j - l))$$

$$= \sum_{l=0}^{j} \sum_{k=0}^{i} \mathbb{P}(H_1 = k) \mathbb{P}(X_1 = i - k) \mathbb{P}(A_1 = l) \mathbb{P}(X_1 = j - l)$$

$$= \sum_{l=0}^{j} \mathbb{P}(A_1 = l) \sum_{k=0}^{i} \mathbb{P}(H_1 = k) \mathbb{P}(X_1 = i - k) \mathbb{P}(X_1 = j - l)$$

$$= e^{-(\lambda_H + \lambda_A)} \frac{\lambda_H^i \lambda_A^j}{i! j!}$$

$\square$