Chordal Sparse Matrices and Semidefinite Programming

Theo Diamandis
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Semidefinite Programs (SDPs) have PSD constraints

Primal Problem: variable $Z \in \mathbf{S}^n$

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\begin{array}{ll}
\text{maximize} & \mathbf{Tr}\,(GZ) \\
\text{s.t.} & \mathbf{Tr}(F_i Z) + c_i = 0, \quad i = 1, \ldots m \\
 & Z \succeq 0,
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Dual problem: variable $x \in \mathbf{R}^m$

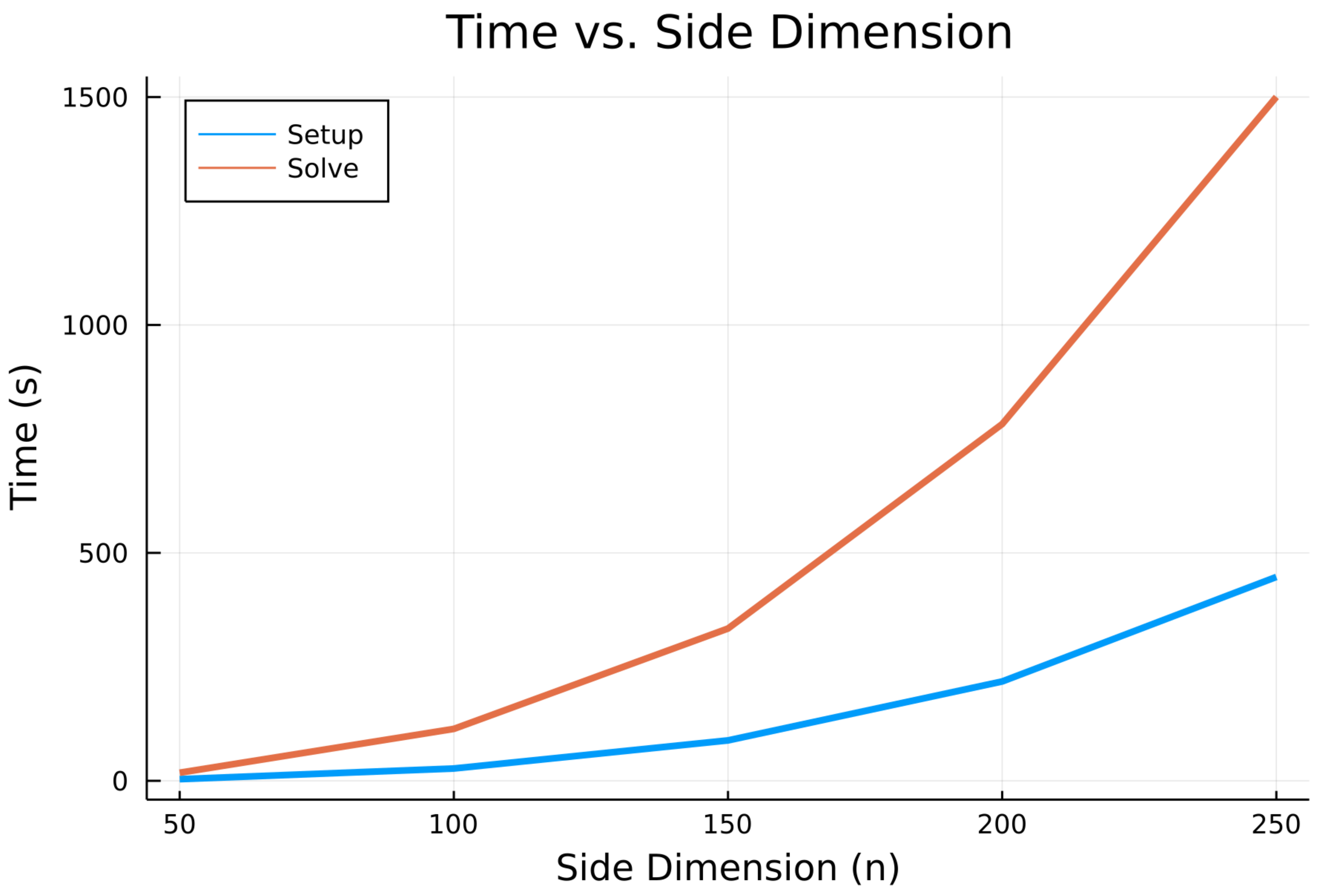
$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{s.t.} & -\sum_i x_i F_i - G \succeq 0 \end{array}$$

SDPs are a powerful modeling tool

- Signal Proc & Control
    - Phase Retrieval (e.g. imaging)
    - Polynomial controller design
- Stats/ML
    - Outlier Detection (e.g. ellipsoidal peeling)
    - Experiment Design
    - Factor Analysis
    - Low-rank matrix completion & decomp.
    - Neural network robustness verification
- Circuit Design
- Portfolio Optimization
- Combinatorial Optimization
- Mechanical structure optimization
- Operations Research
    - Facility Location
    - Scheduling
- Robust optimization
- Geometric data processing & CV

SDPs are plagued by scaling issues

Per iteration cost is (approx) cubic in the side dimension



But we can exploit sparsity

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Only use, for example, 2% of entries of Z

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Sparse

Goal: get rid of the "dense" $n \times n$ PSD constraint → reduce # variables by 98%

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Goal: get rid of the "dense" $n \times n$ PSD constraint → reduce # variables by 98%

Need: a check that a sparse matrix can be "completed" to be a PSD matrix

Chordal Decomposition: check $Z \succeq 0$ via sub matrices

$$
\begin{array}{cc}
 & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} &
\begin{bmatrix}
\bullet & \bullet & \bullet & & \\
\bullet & \bullet & \bullet & & \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
 & & \bullet & \bullet & \bullet \\
 & & \bullet & \bullet & \bullet
\end{bmatrix}
\end{array}
\succeq 0
\quad \Longleftrightarrow \quad
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$$O(5^3) = C \cdot 125$$

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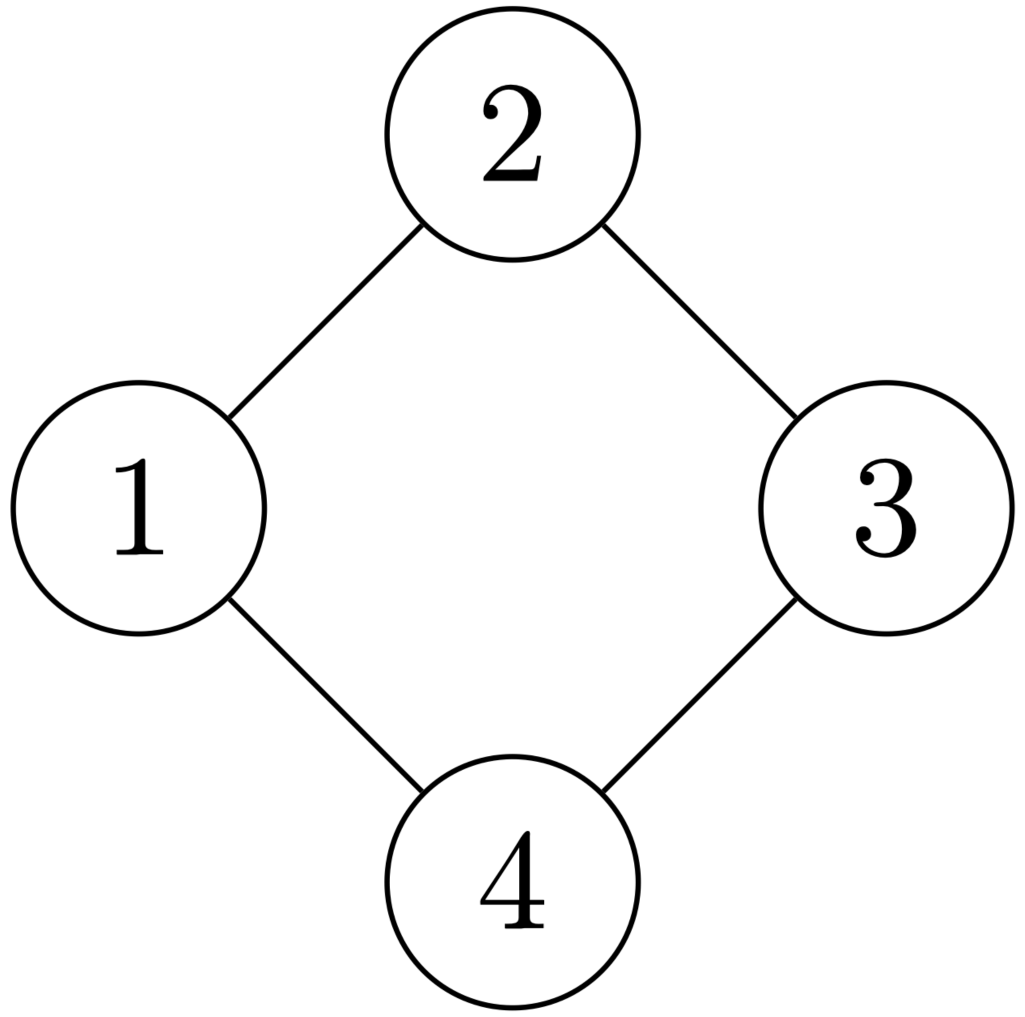
$O(5^3) = C \cdot 125$

$O(2 \cdot 3^3) = C \cdot 54$

Detour: Chordal Graph Theory

Sparsity patterns = undirected graphs

$$\begin{bmatrix} 11 & 21 & \mathbf{0} & 41 \\ 21 & 22 & 32 & \mathbf{0} \\ \mathbf{0} & 32 & 33 & 43 \\ 41 & \mathbf{0} & 43 & 44 \end{bmatrix} \Longrightarrow \begin{array}{cc} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} \bullet & \bullet & & \bullet \\ \bullet & \bullet & \bullet & \\ & \bullet & \bullet & \bullet \\ \bullet & & \bullet & \bullet \end{bmatrix} \end{array} \Longleftrightarrow$$



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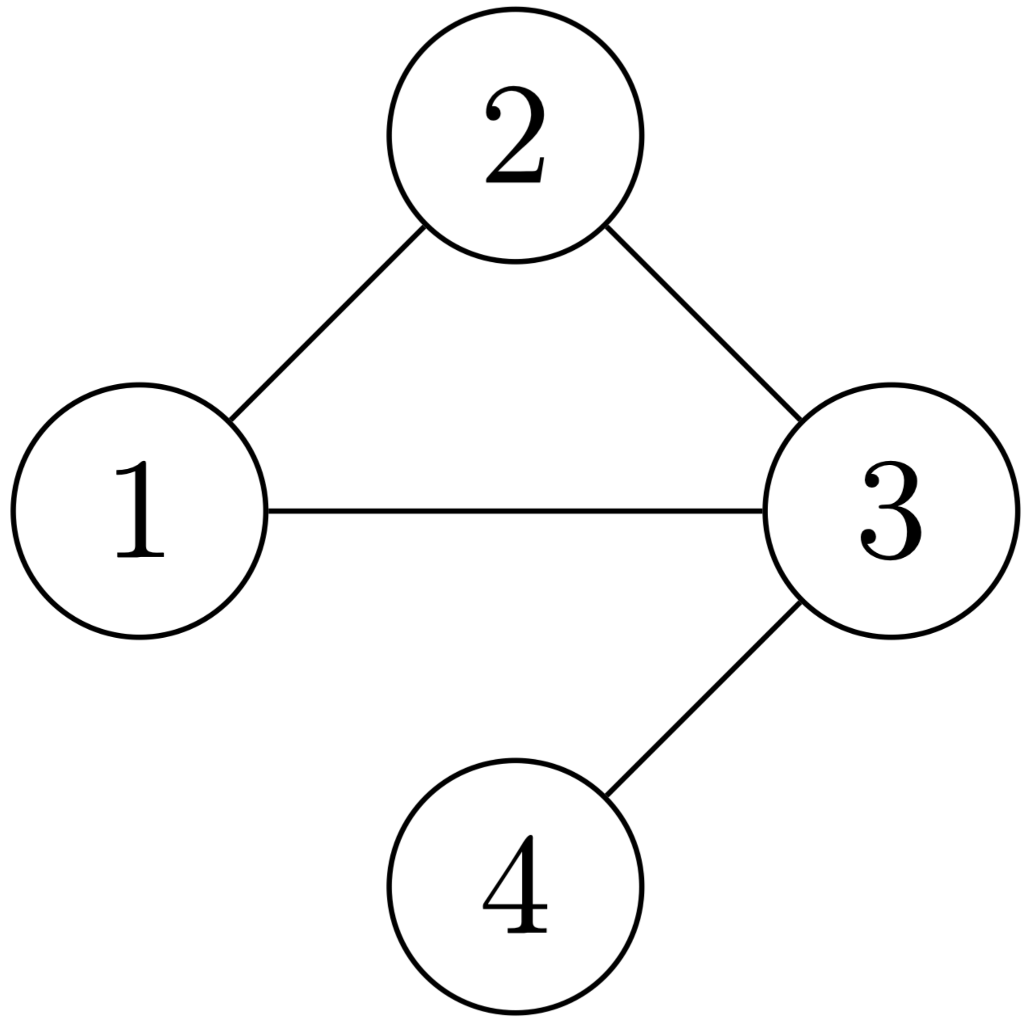

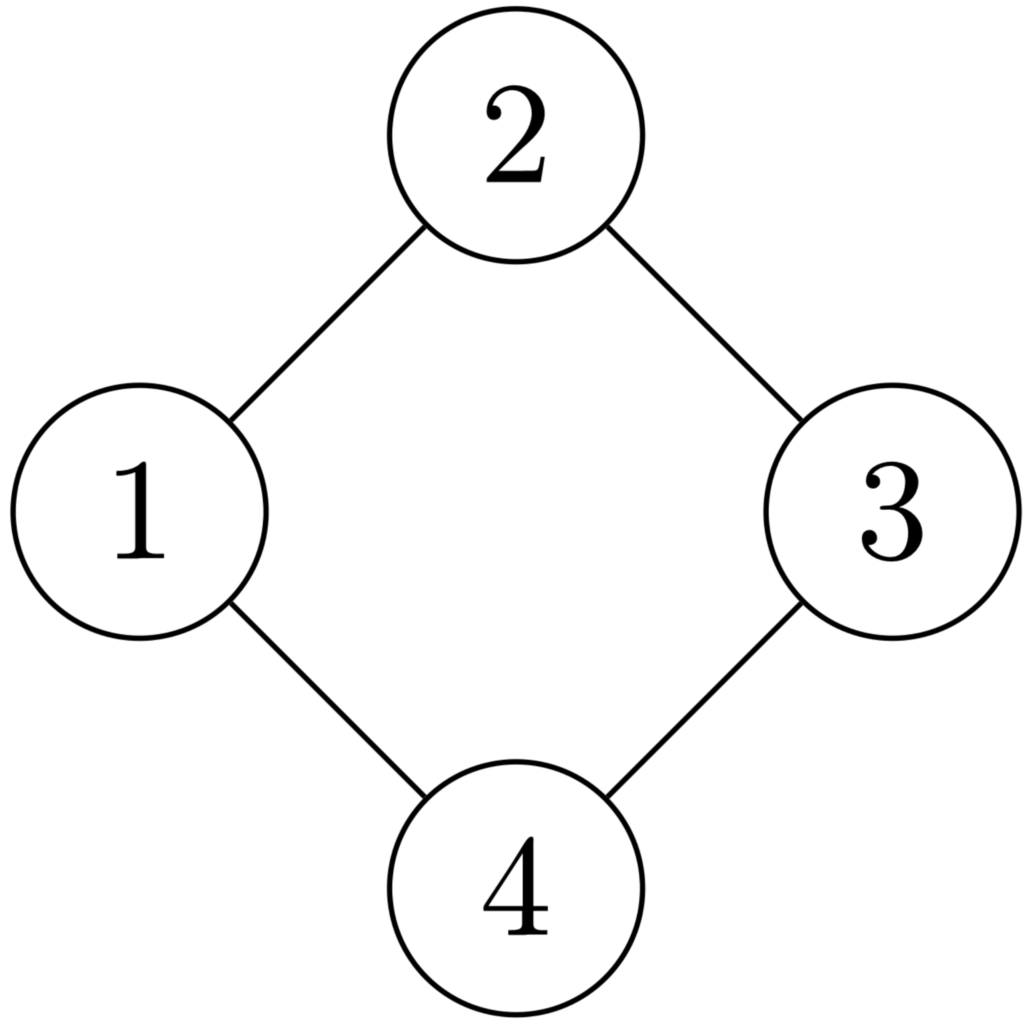

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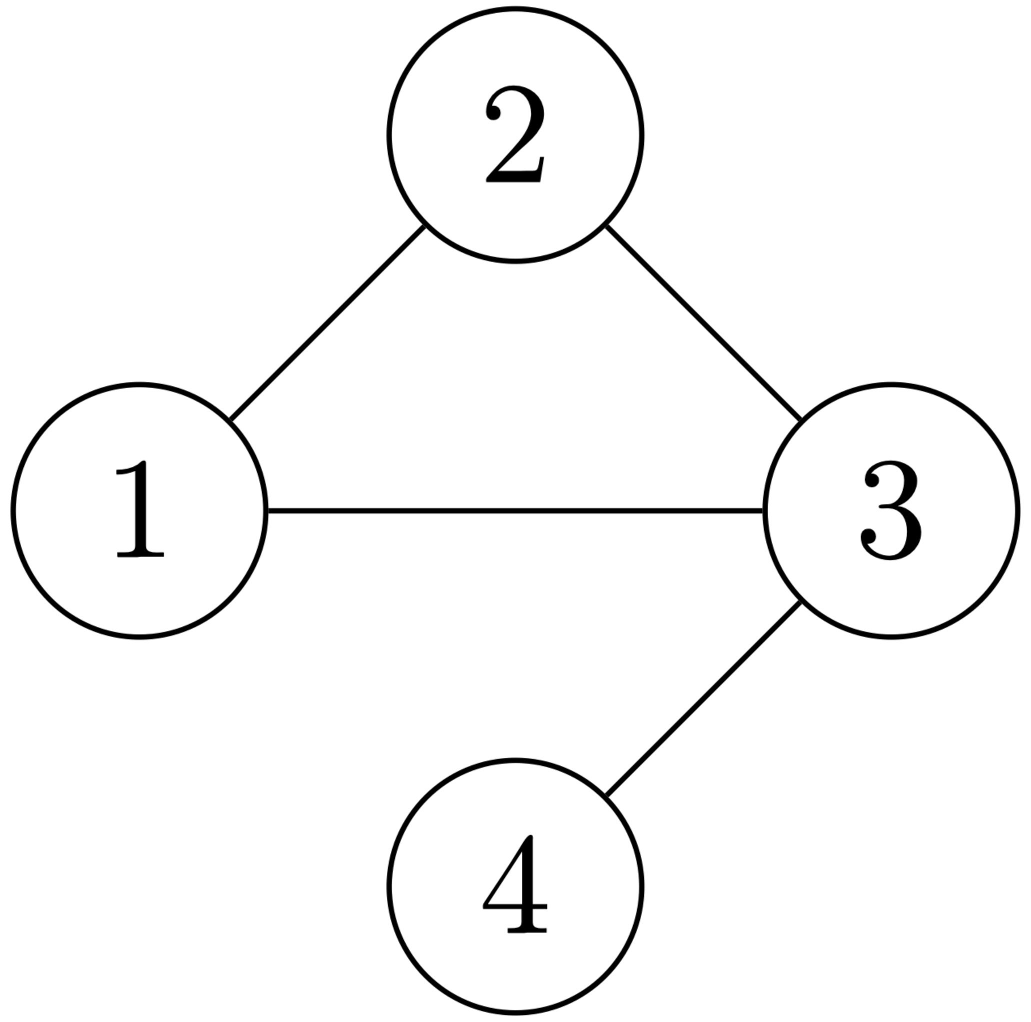

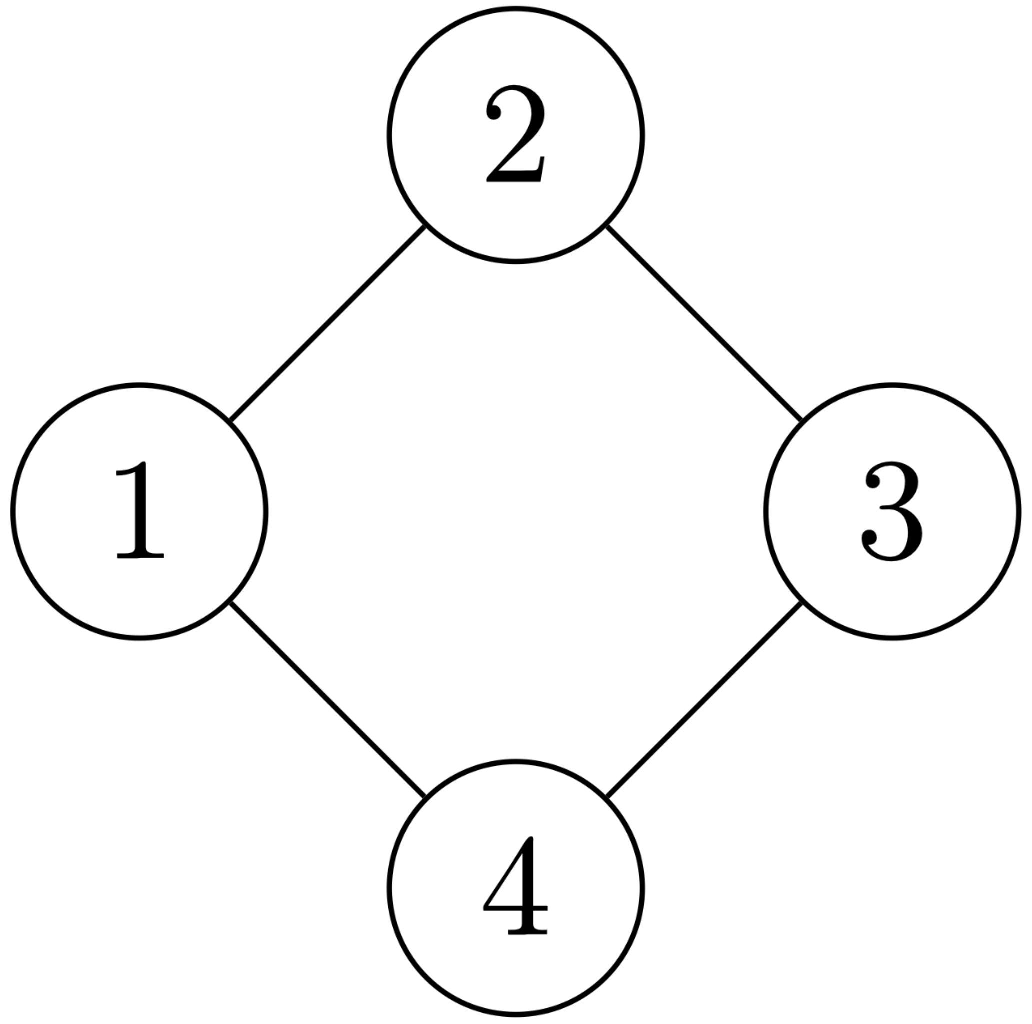

Definition: A graph is *chordal* if all cycles of length > 3 have a chord

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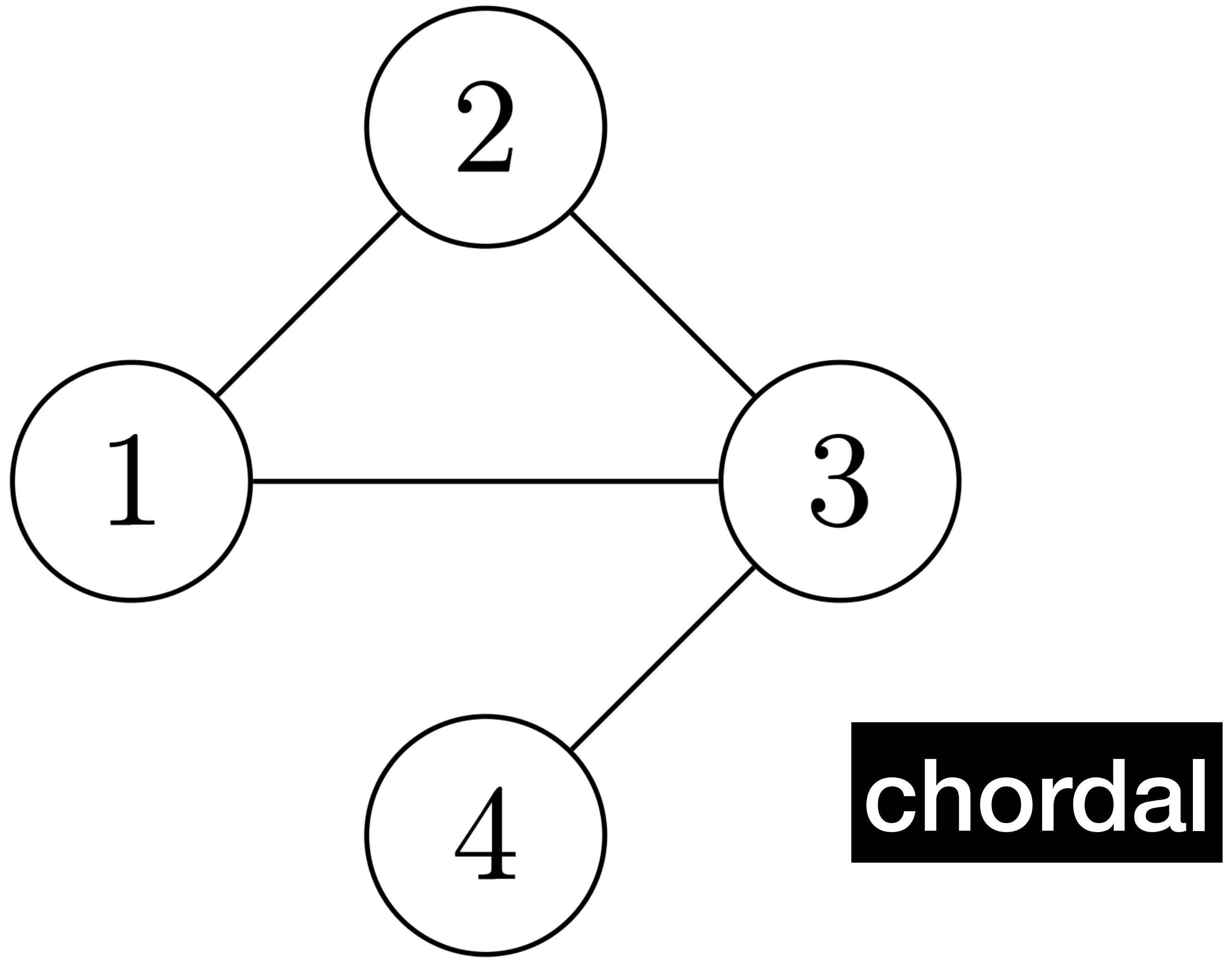

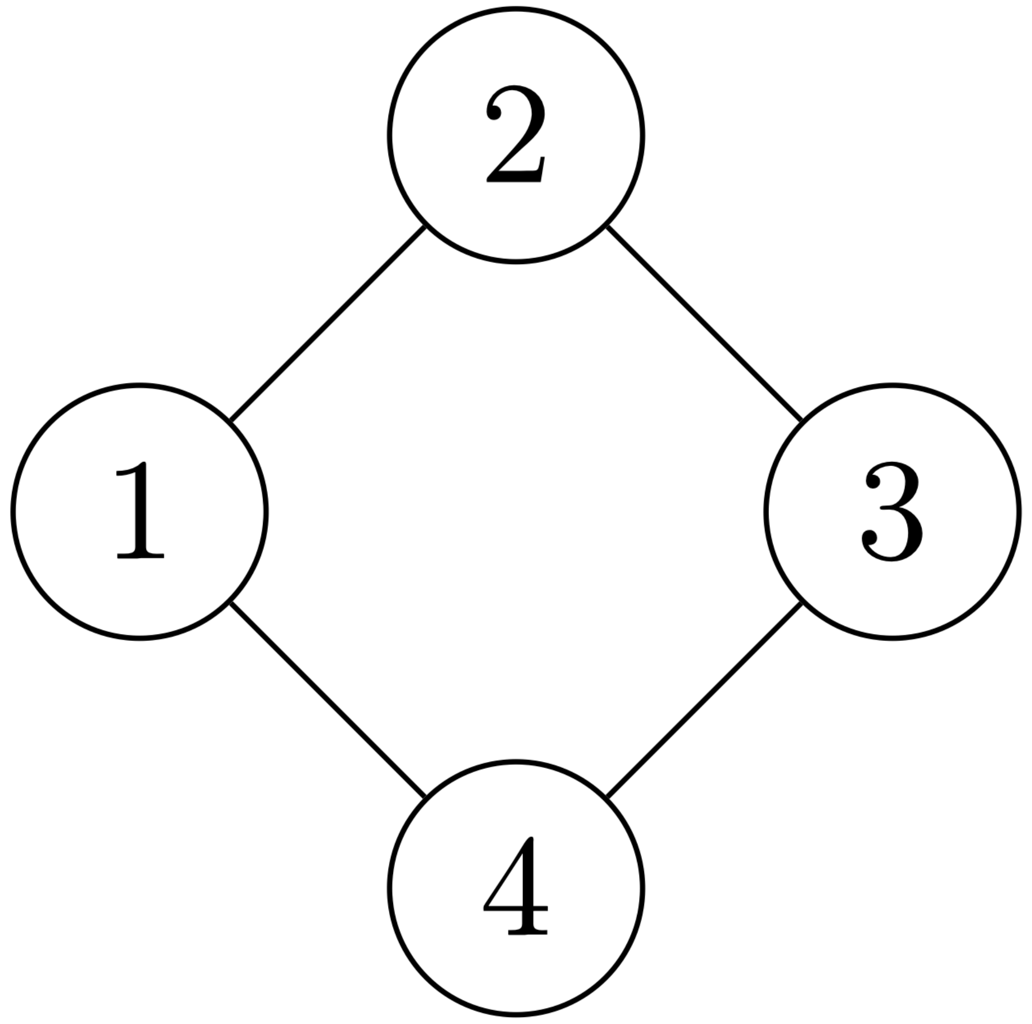

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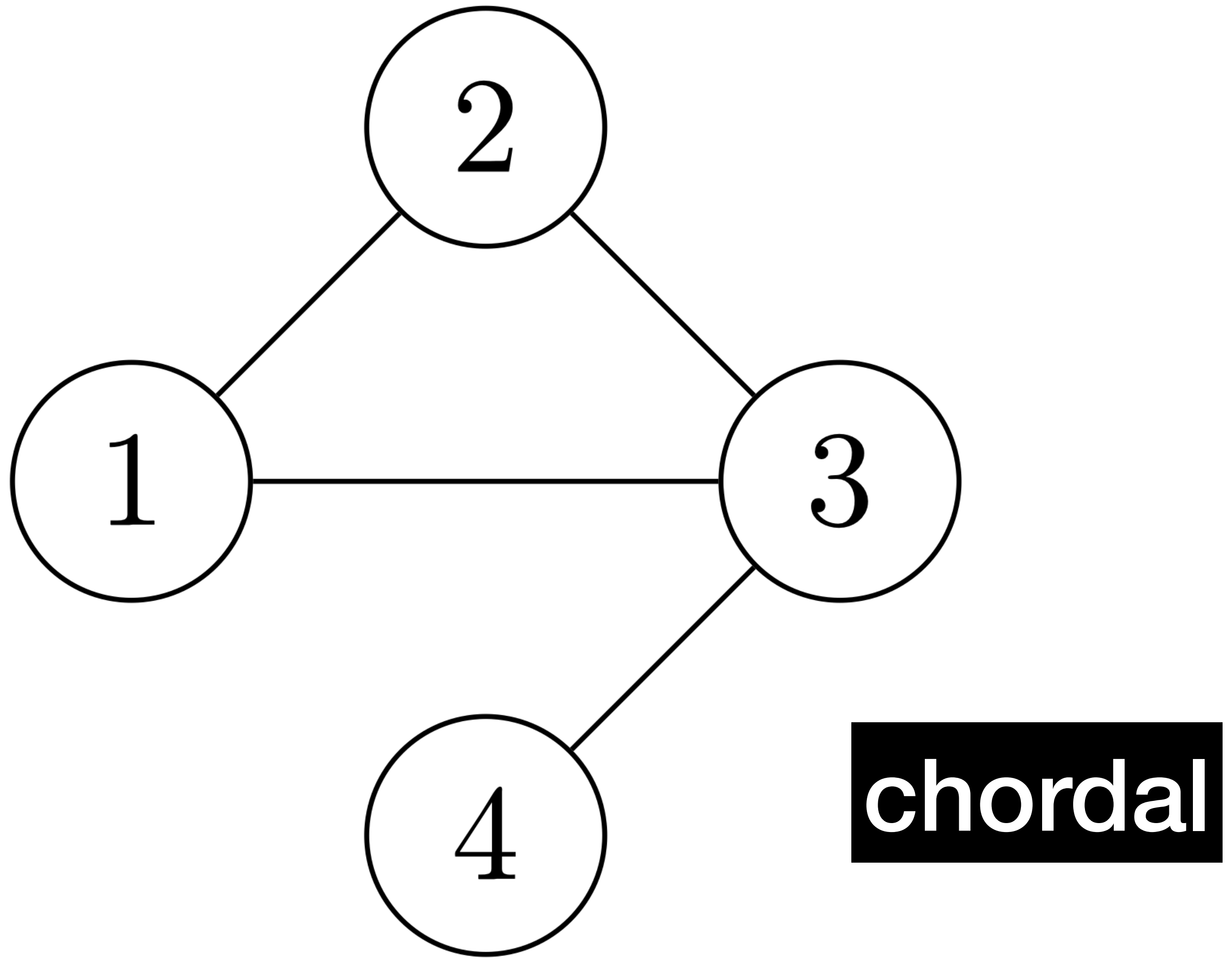

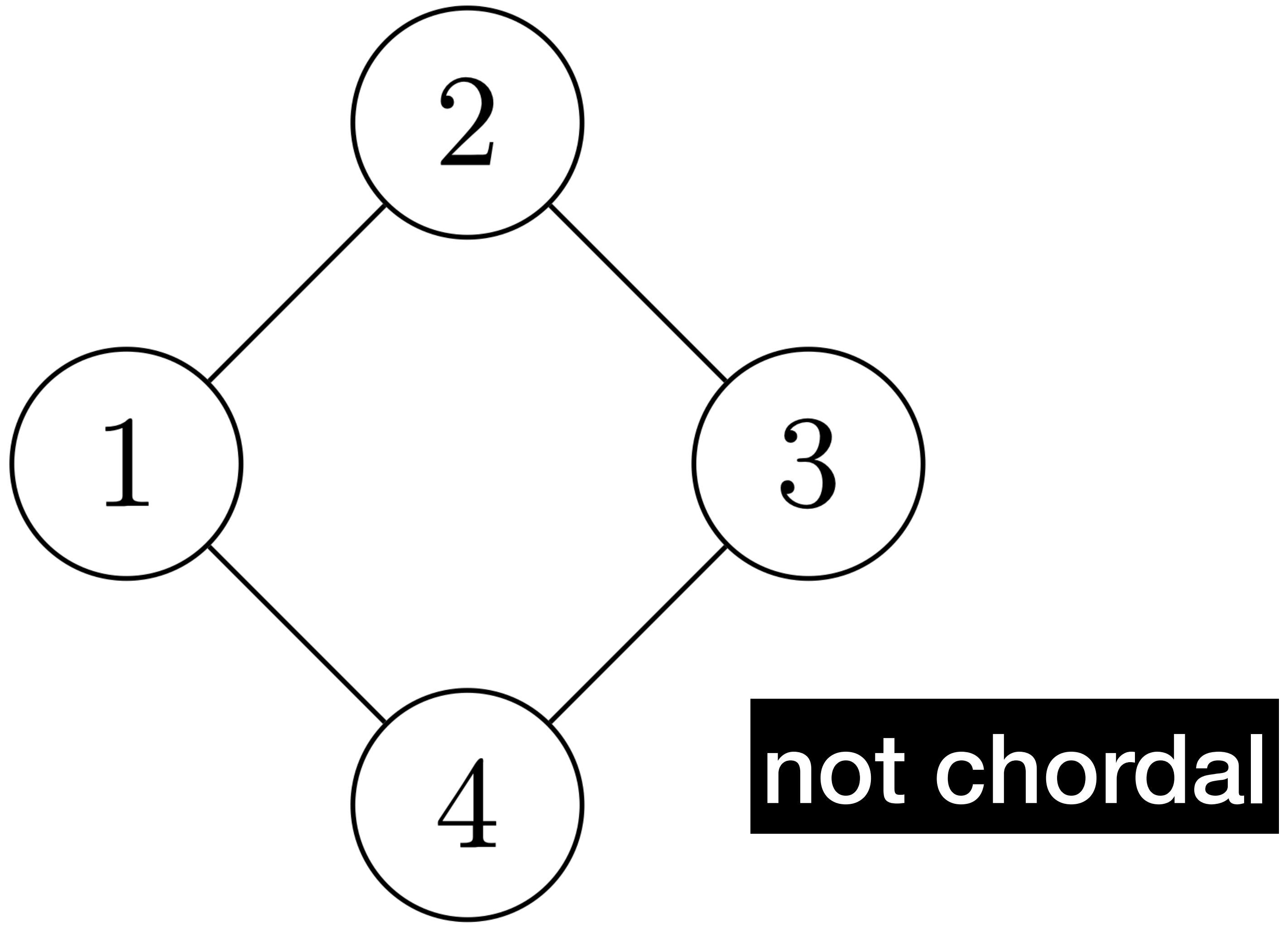

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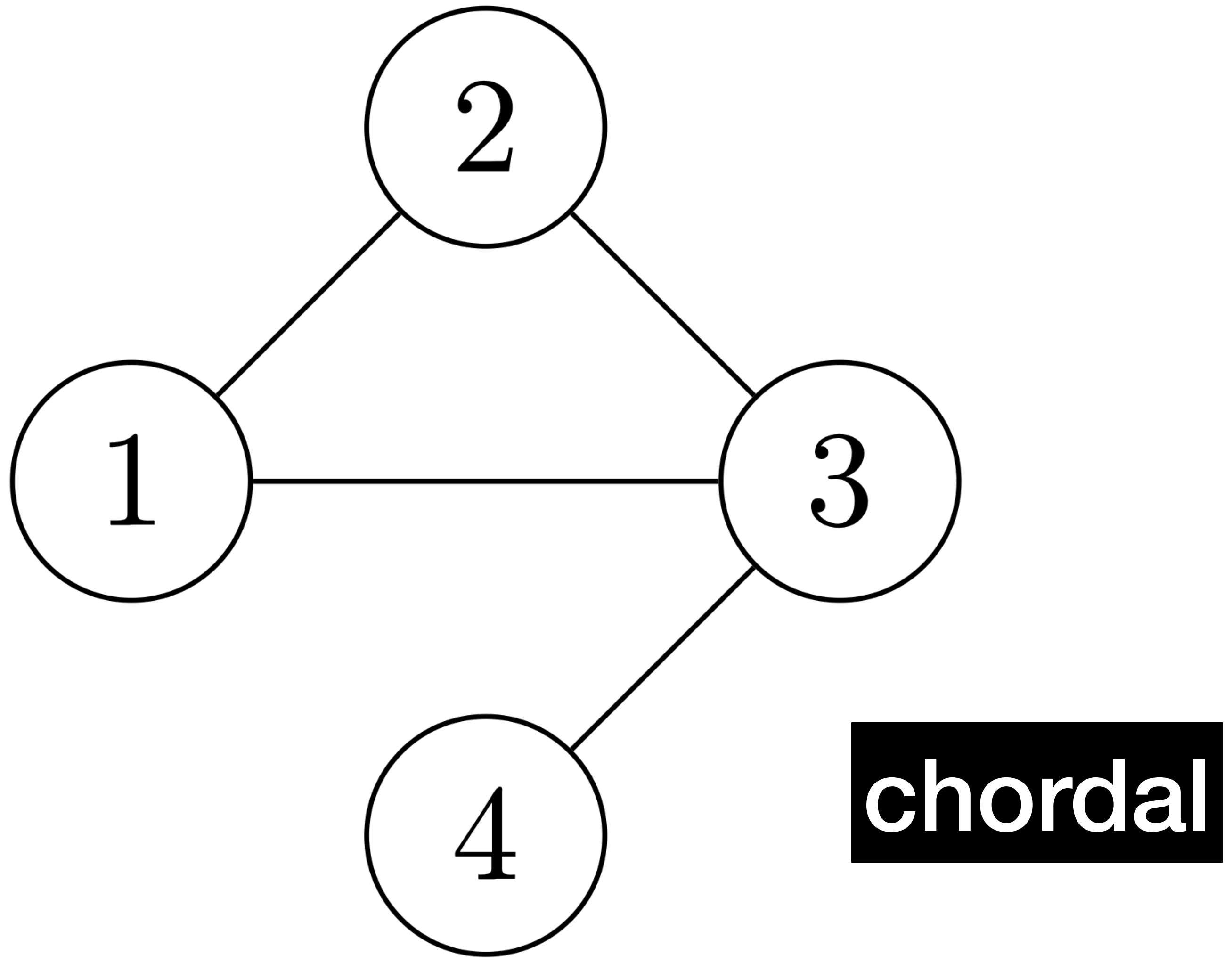

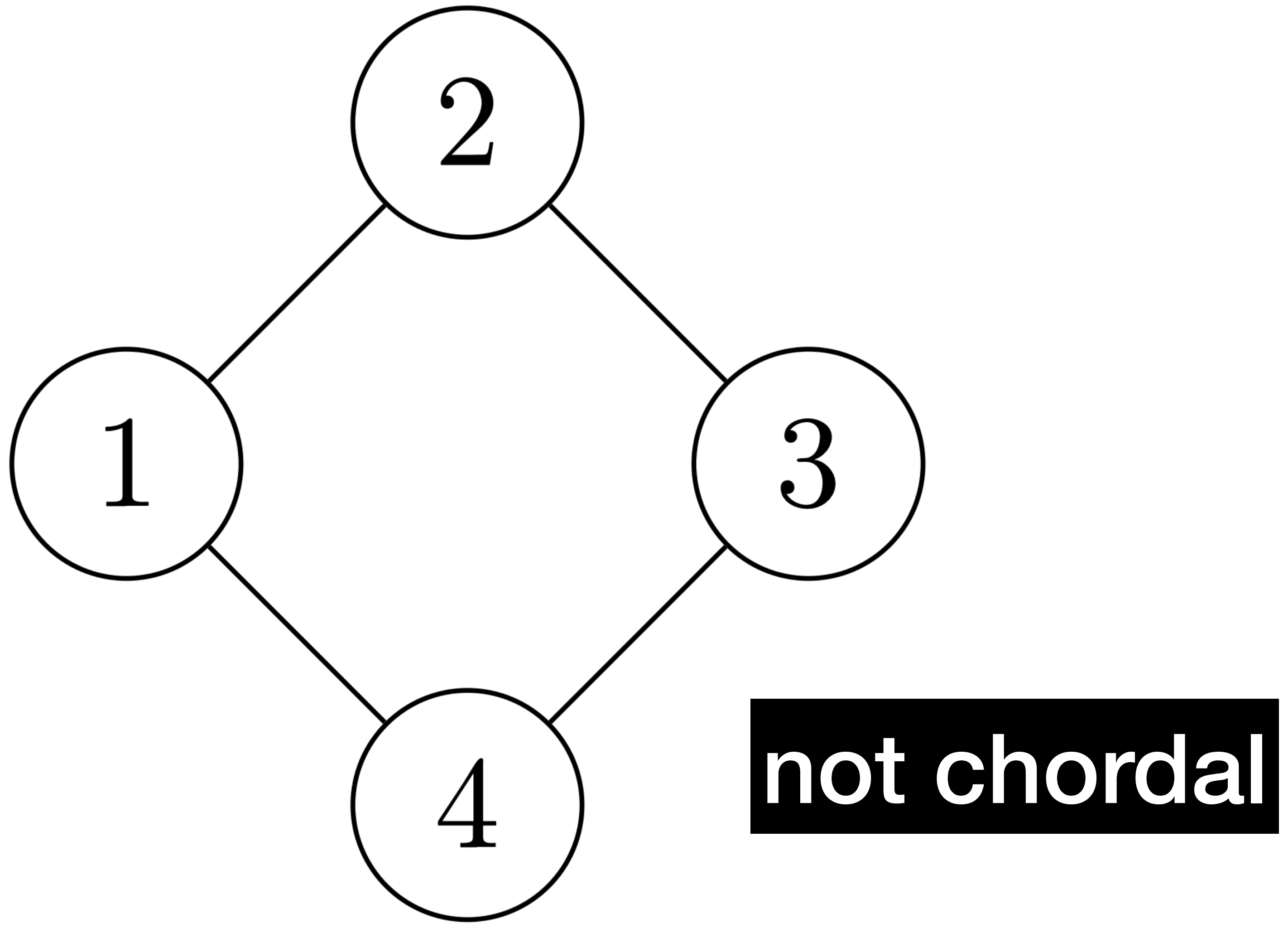

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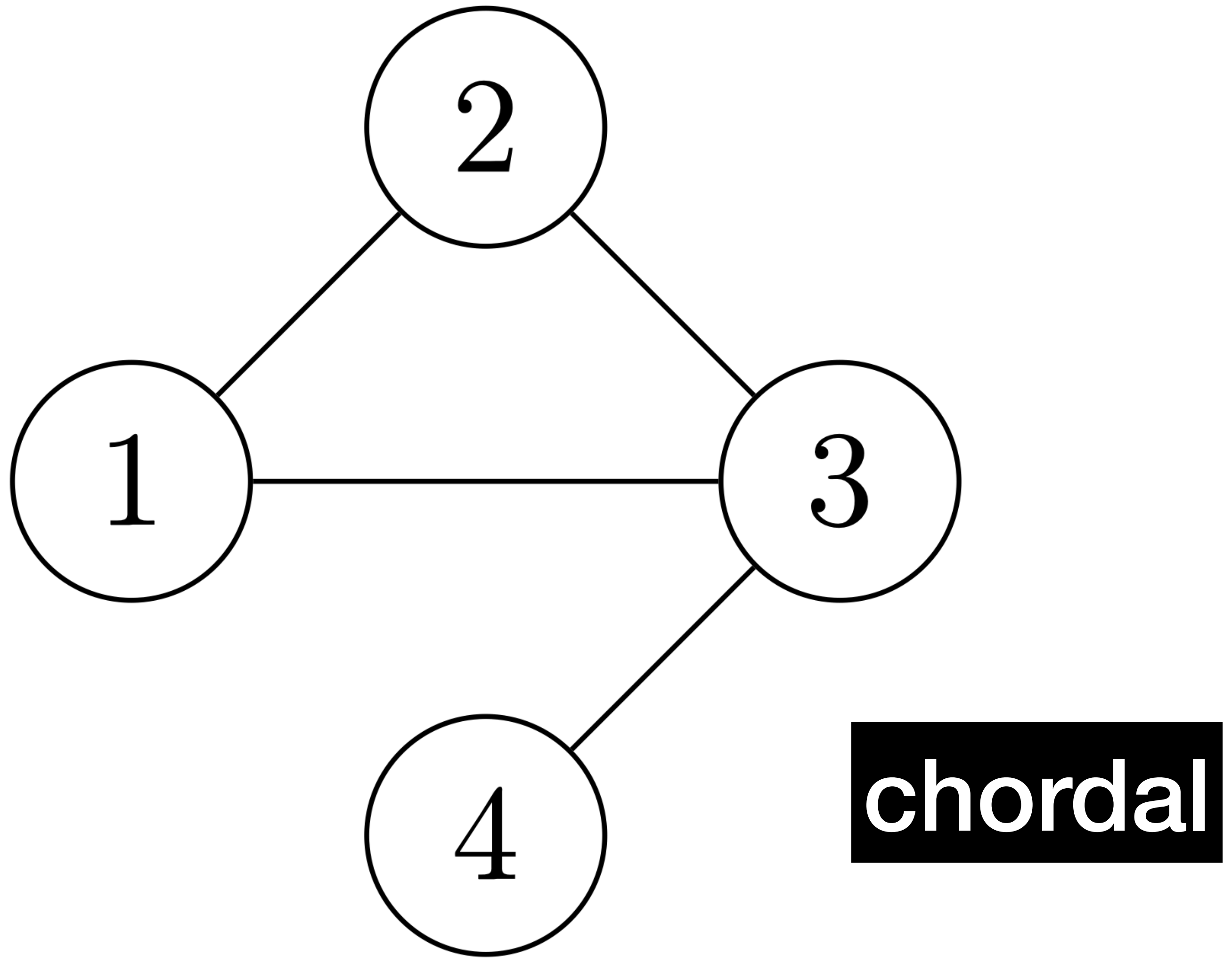

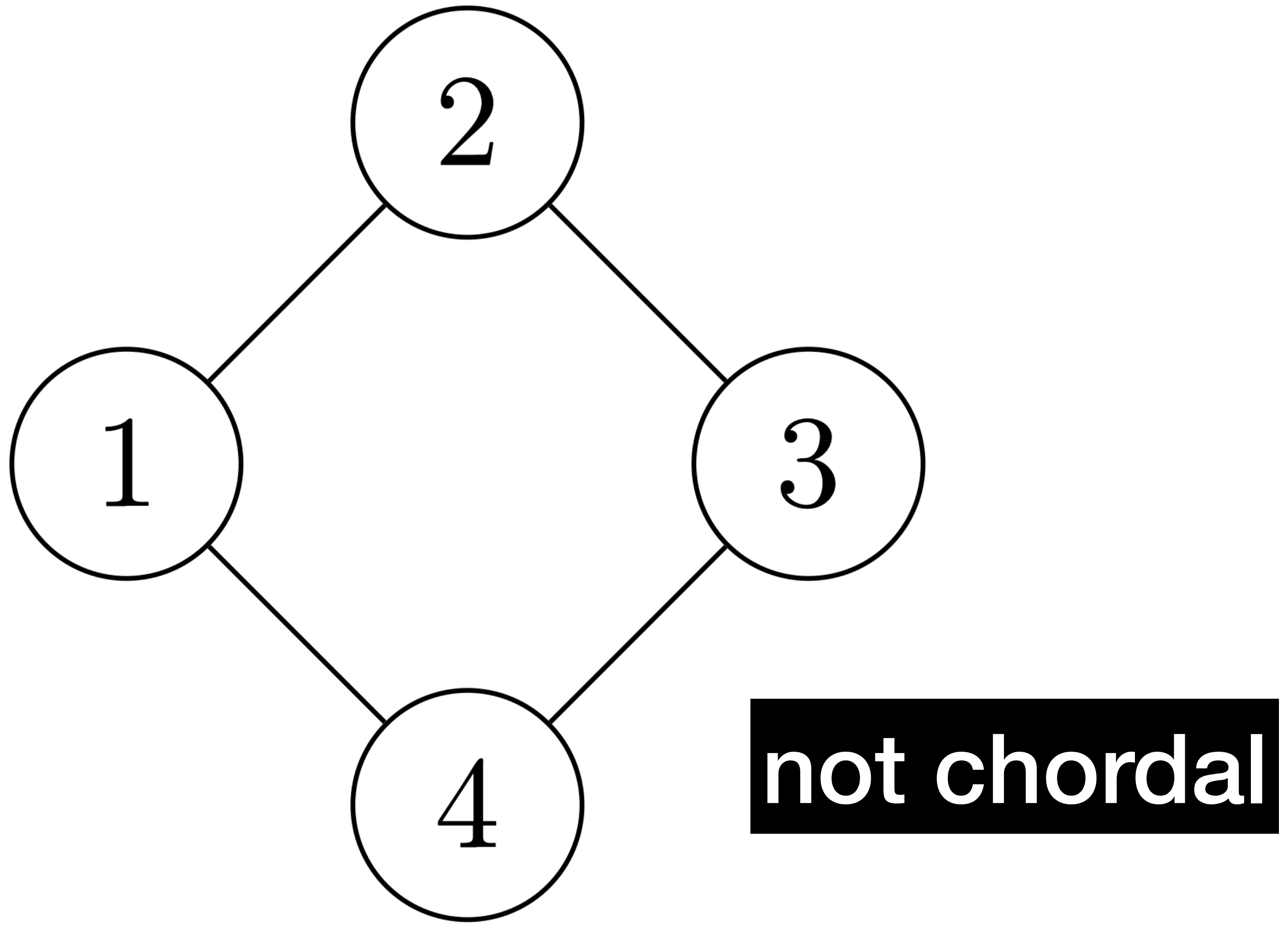

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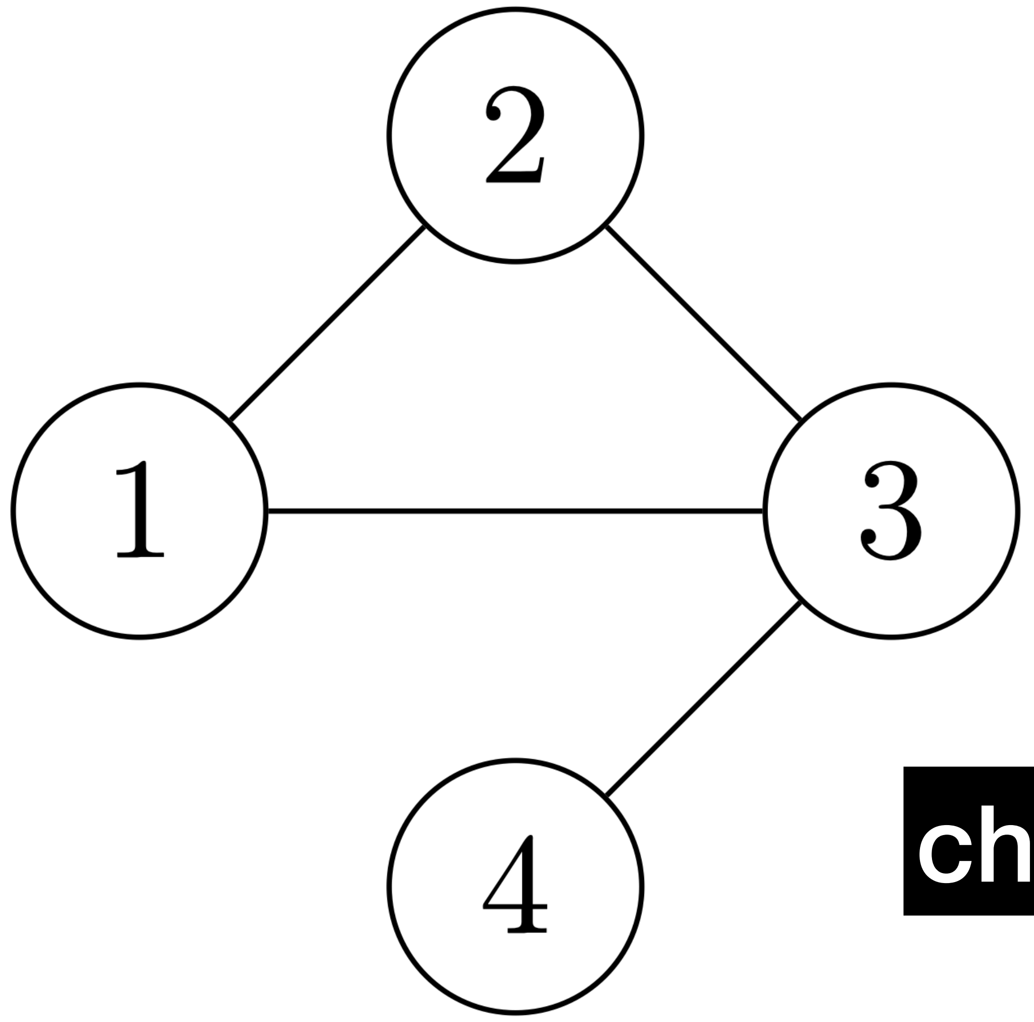
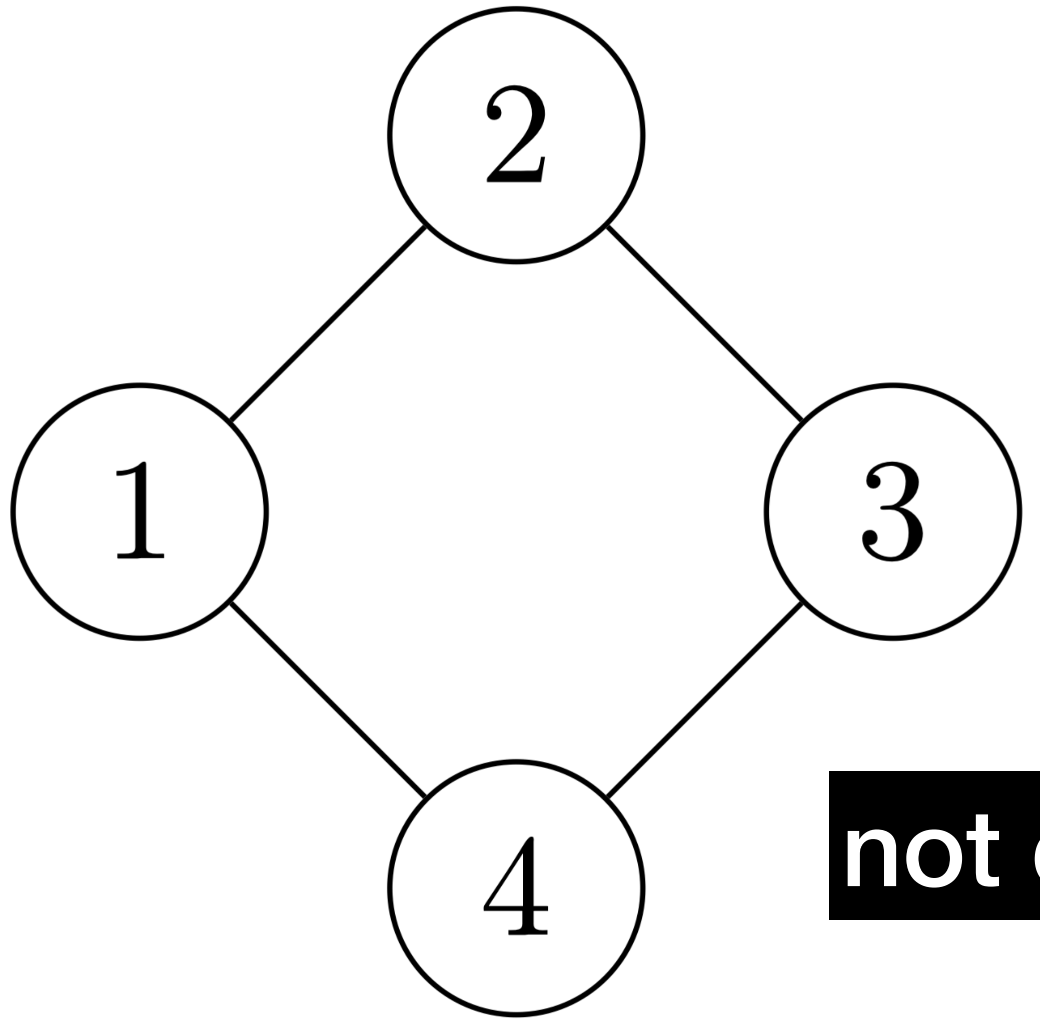


chordal



not chordal

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Definition: A *clique* is a set of vertices that induces a maximal complete subgraph

Examples (Code Demo)

Example sparsity patterns:

- Banded
- Arrow
- Random

- Think about which are Chordal

Check if Z is PSD by looking at sub matrices
[Grone et al. 1984]

Theorem: A sparse matrix Z **with a chordal sparsity pattern** can be PSD completed if and only if

$$Z_{C_p} \succeq 0, \qquad p = 1, \ldots, P$$

where Z_{C_p} is the sub matrix of Z indexed by the clique C_p.

Applied to SDPs, reduces PSD constraint size

Matrices F_i, G have a chordal aggregate sparsity pattern

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Iteration cost is now $O\left(\Sigma_{p=1}^{P} |C_p|^3\right)$ vs $O(n^3)$

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See COSMO.jl & Chordal.jl

Iteration cost is now $O\left(\Sigma_{p=1}^{P} |C_p|^3\right)$ vs $O(n^3)$

Use similar approach for the dual problem
[Agler et al. 1988]

Theorem: A sparse matrix S **with a chordal sparsity pattern** is PSD if and only if there exist matrices $S_p \succeq 0$ such that

$$S = \sum_{p=1}^{P} T_p S_p T_p^T$$

where T_p is a selector matrix:

$$[T_p]_{ij} = \begin{cases} 1 & C_p(i) = j, \\ 0 & \text{otherwise} \end{cases}$$

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Algorithmic Details

These constructions lead to two new cones:

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Sparse PSD-completable cone (primal problem):

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- We want to compute projections and gradients/Hessians of barriers

Reformulating $\rightarrow$ faster FO & IP algorithms

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Reformulating → faster FO & IP algorithms

- For first order algorithms (e.g., ADMM), we project onto the PSD cone
  - Chordal decomposition → break up large PSD projection into P small ones

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  - Chordal decomposition $\rightarrow$ break up large PSD projection into P small ones
- For interior point algorithms, must compute gradient and/or Hessian of barrier
  - Chordal decomposition $\rightarrow$ efficiently compute these [ADV13, VA15]

Chordal Decomposition Steps

1. Input Sparsity Pattern
2. Chordal Extension
3. Find (& merge) cliques
4. Form SDP

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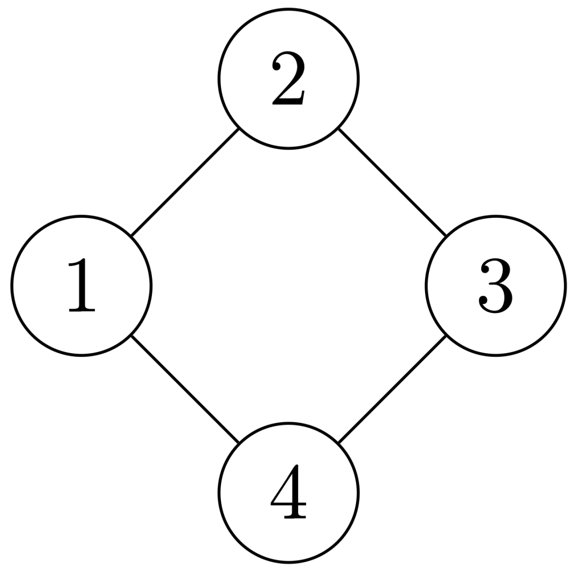

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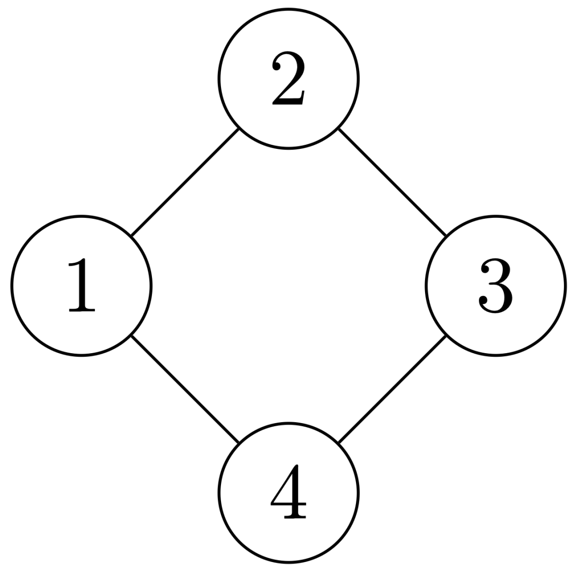

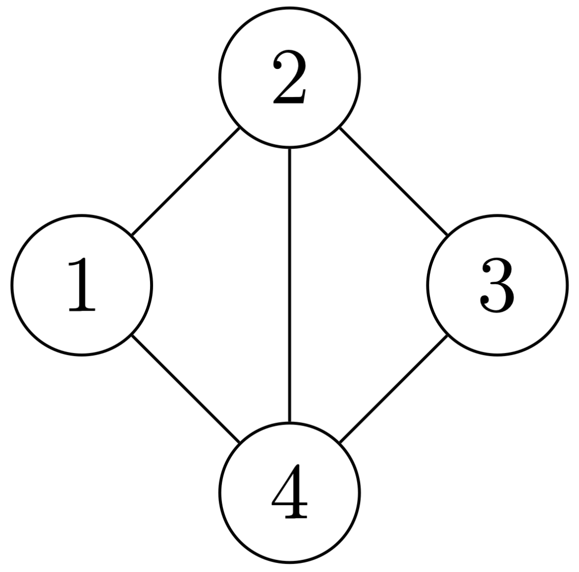

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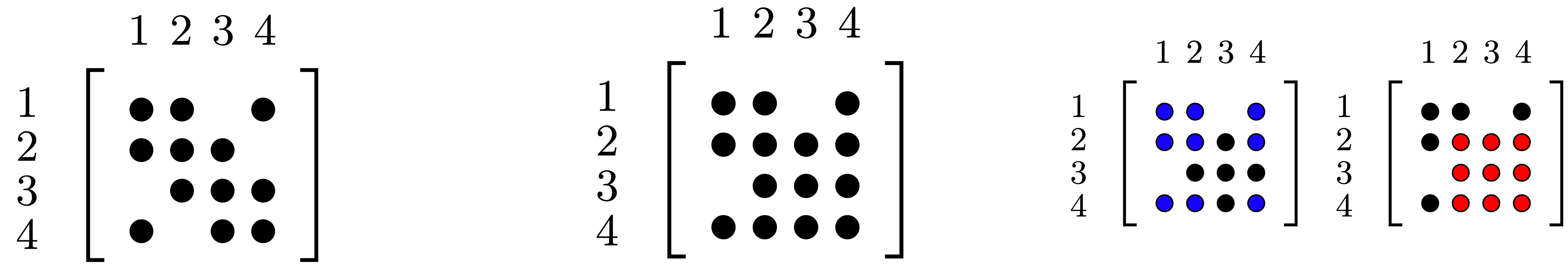

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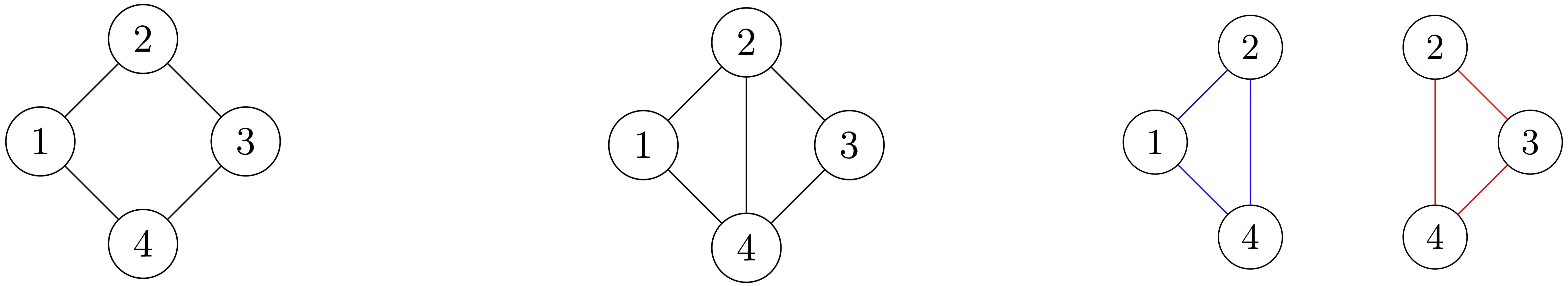
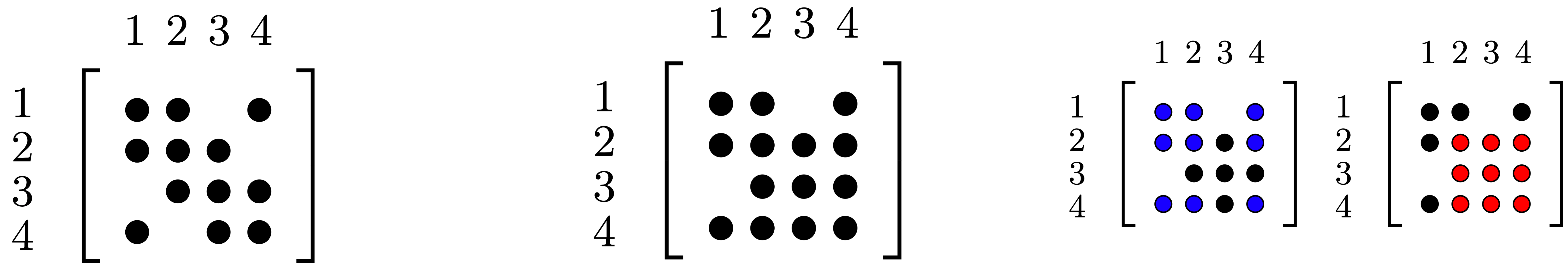
4. Form SDP



Chordal Decomposition Steps



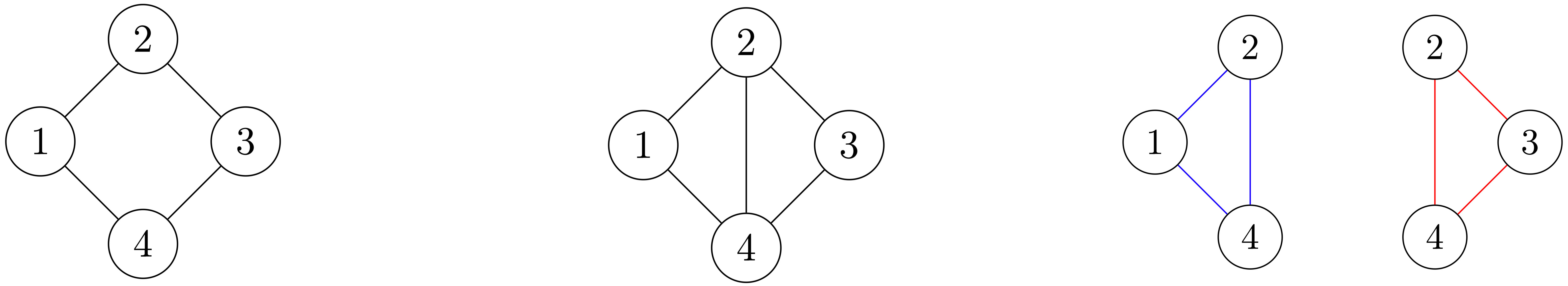
1 Input Sparsity Pattern

2 Chordal Extension

3 Find (& merge) cliques

4 Form SDP

Reordering matters!



Chordal Decomposition Steps

$$\begin{array}{c|c} & \begin{matrix}1 & 2 & 3 & 4\end{matrix} \\ \begin{matrix}1\\2\\3\\4\end{matrix} & \begin{bmatrix}\bullet & \bullet & & \bullet \\ \bullet & \bullet & \bullet & \\ & \bullet & \bullet & \bullet \\ \bullet & & \bullet & \bullet\end{bmatrix}\end{array}$$

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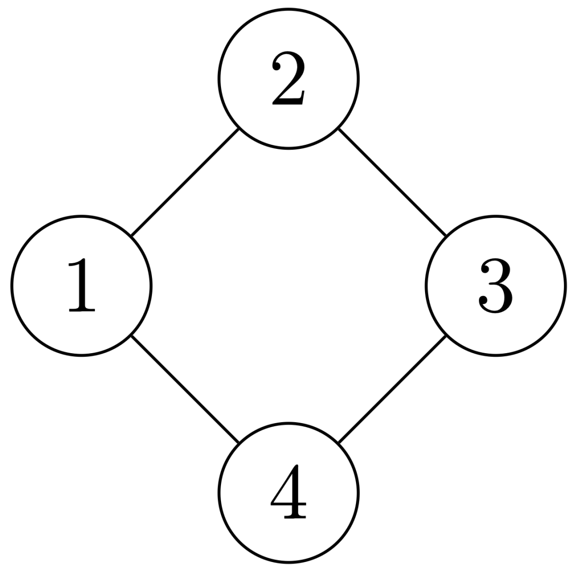

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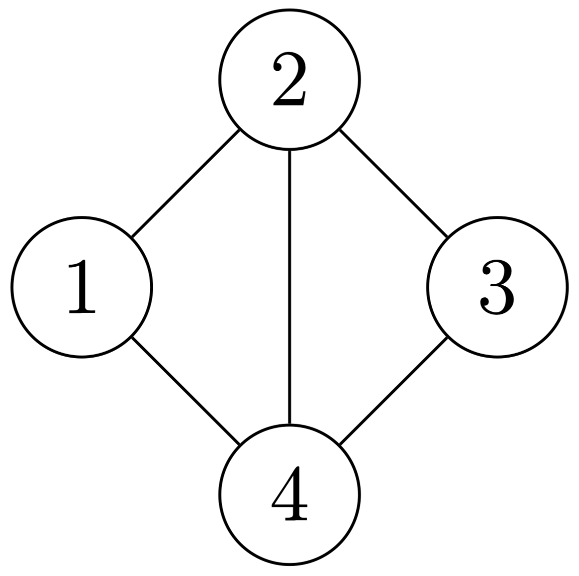
2 Chordal Extension

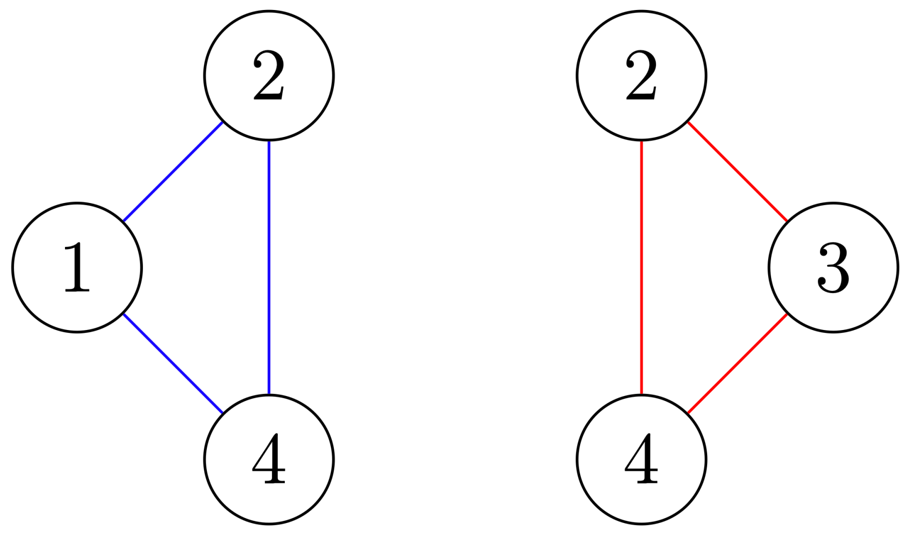
Reordering matters!



$$\begin{array}{c|c} & \begin{matrix}1 & 2 & 3 & 4\end{matrix} \\ \begin{matrix}1\\2\\3\\4\end{matrix} & \begin{bmatrix}\color{blue}\bullet & \color{blue}\bullet & & \color{blue}\bullet \\ \color{blue}\bullet & \color{blue}\bullet & \bullet & \color{blue}\bullet \\ & \bullet & \bullet & \bullet \\ \color{blue}\bullet & \color{blue}\bullet & \bullet & \color{blue}\bullet\end{bmatrix}\end{array} \quad \begin{array}{c|c} & \begin{matrix}1 & 2 & 3 & 4\end{matrix} \\ \begin{matrix}1\\2\\3\\4\end{matrix} & \begin{bmatrix}\bullet & \bullet & & \bullet \\ \bullet & \color{red}\bullet & \color{red}\bullet & \color{red}\bullet \\ & \color{red}\bullet & \color{red}\bullet & \color{red}\bullet \\ \bullet & \color{red}\bullet & \color{red}\bullet & \color{red}\bullet\end{bmatrix}\end{array}$$

3 Find (& merge) cliques

[Michael Garstka et al. 2020]



4 Form SDP

Chordal Decomposition Steps

$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} \bullet & \bullet & & \bullet \\ \bullet & \bullet & \bullet & \\ & \bullet & \bullet & \bullet \\ \bullet & & \bullet & \bullet \end{bmatrix} \end{array}$$

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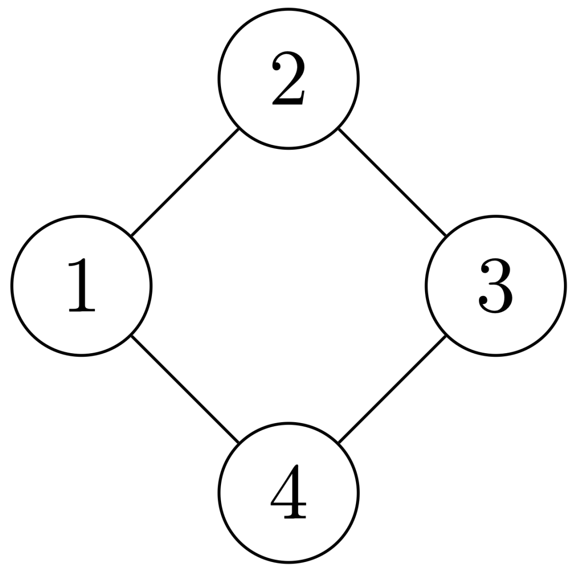

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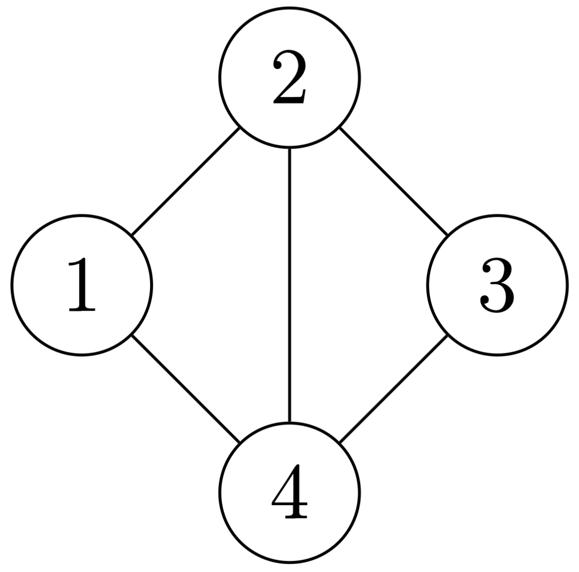

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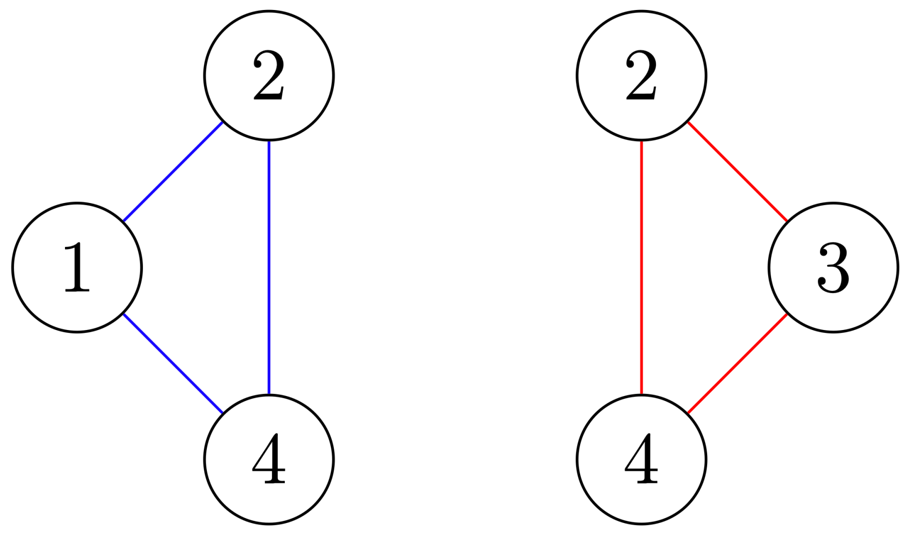
3 Find (& merge) cliques

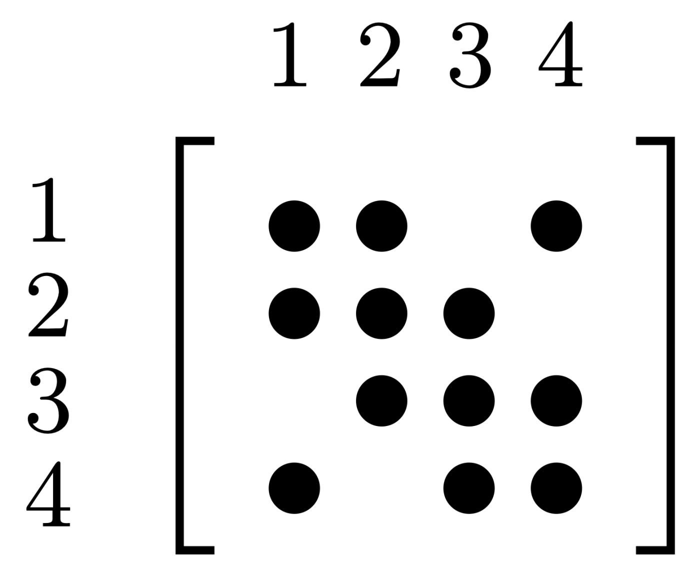
[Michael Garstka et al. 2020]

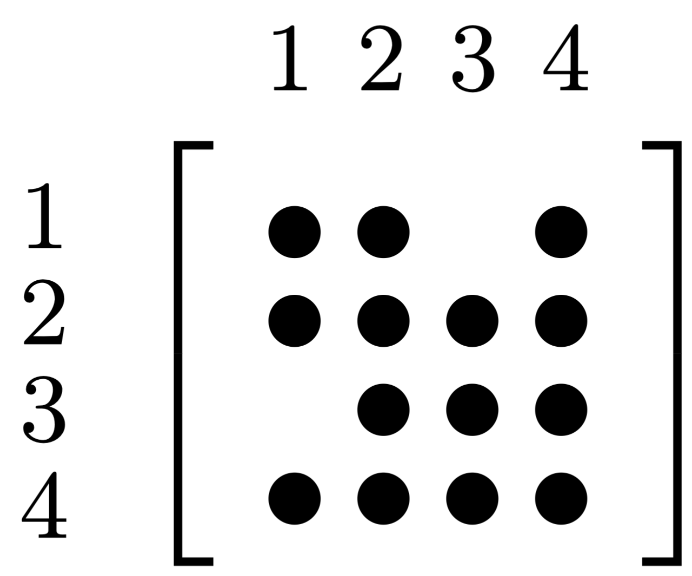


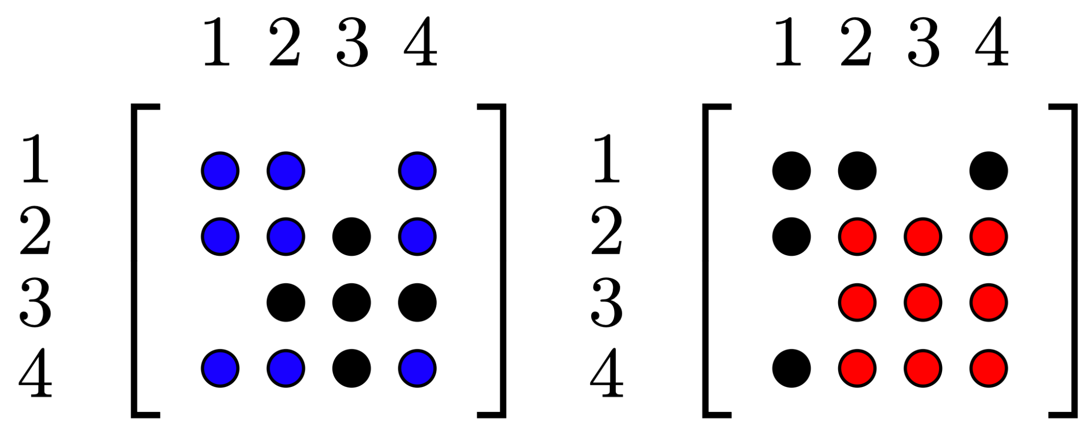
4 Form SDP

5 Solve SDP

Chordal Decomposition Steps







1 Input Sparsity Pattern

2 Chordal Extension

Reordering matters!

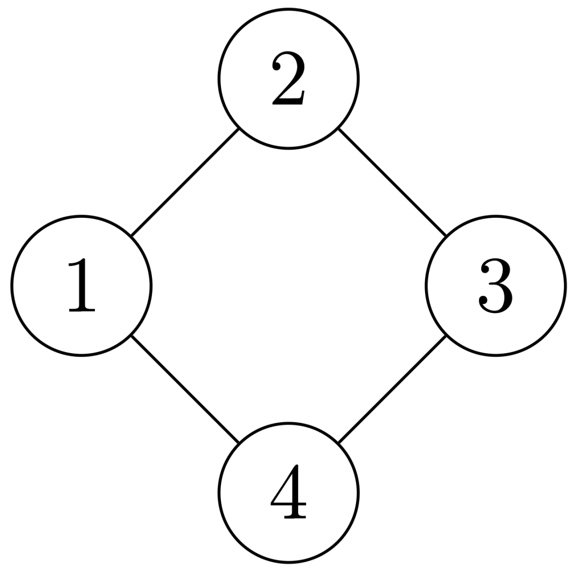
3 Find (& merge) cliques

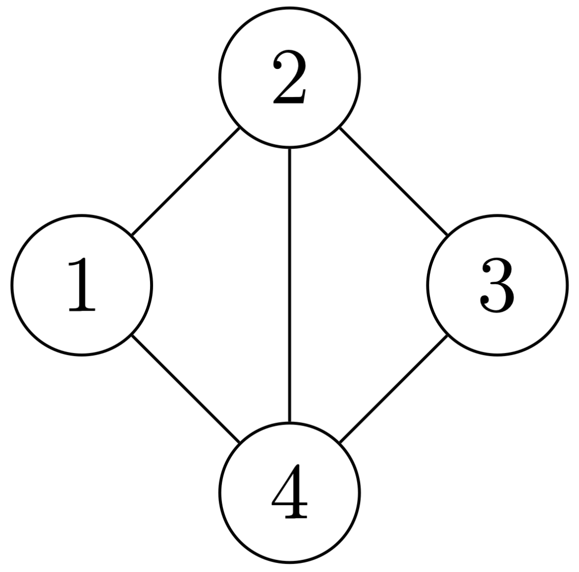
[Michael Garstka et al. 2020]

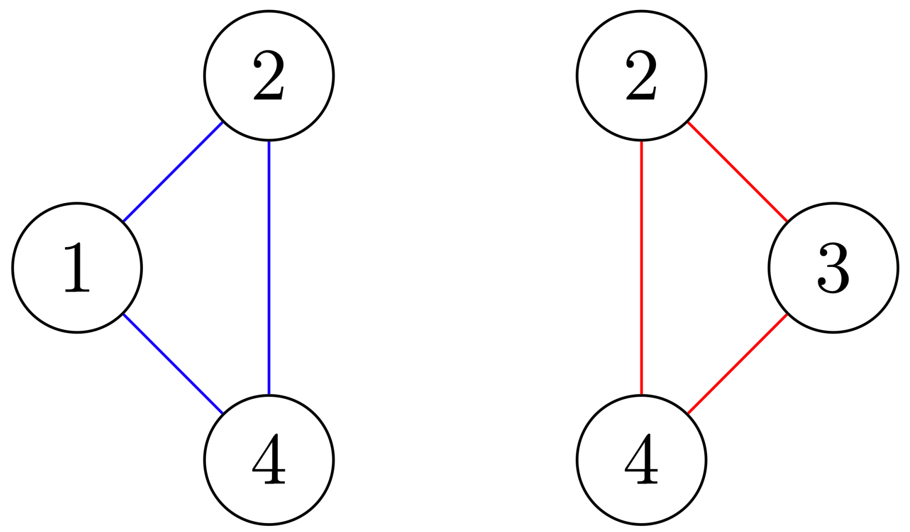
4 Form SDP

5 Solve SDP

6 [primal] Reconstruct Z







We can reconstruct the full Z if needed

Our optimization variable Z is only partially specified

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Two main approaches:

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1. Maximum determinant completion [VA15]

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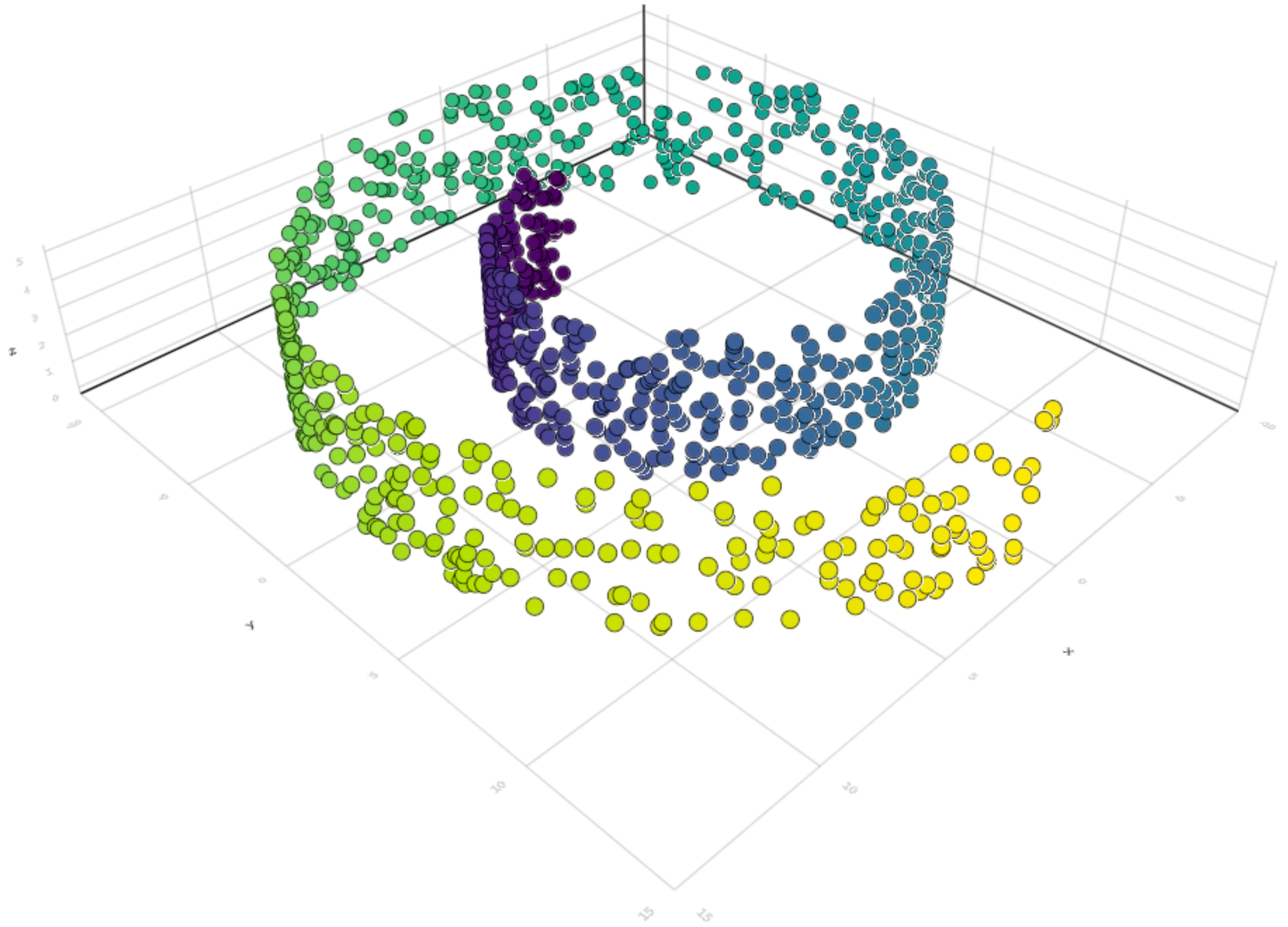
$$r = \max_{p} \mathbf{rank}(Z_{C_p})$$

Numerical Example

Example: Maximum Variance Unfolding

Preserve local distances + maximize (global) variance



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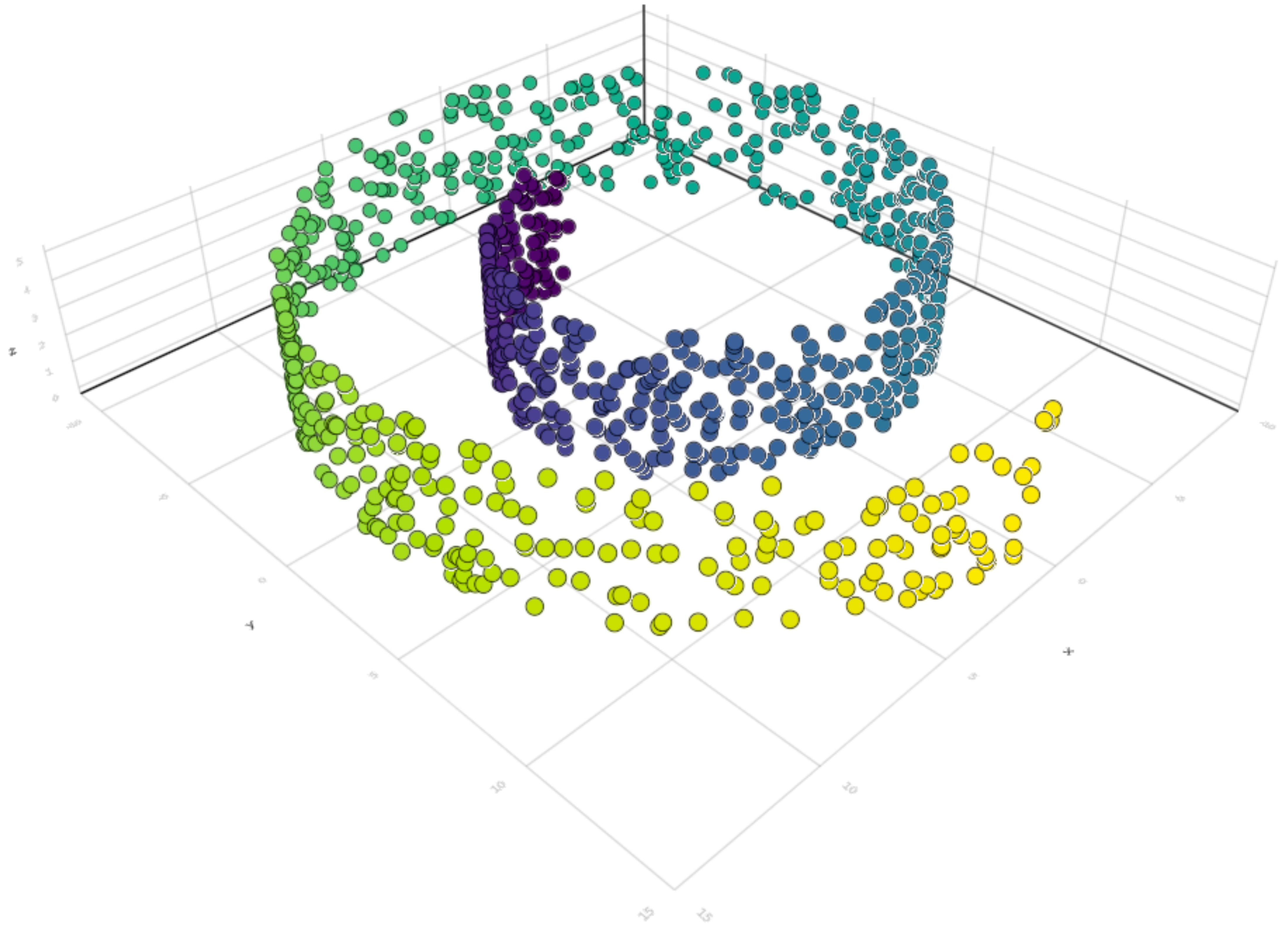

Data: $x_i \in \mathbb{R}^{d_x}$ with pairwise distances $D_{ij} = \|x_i - x_j\|^2$

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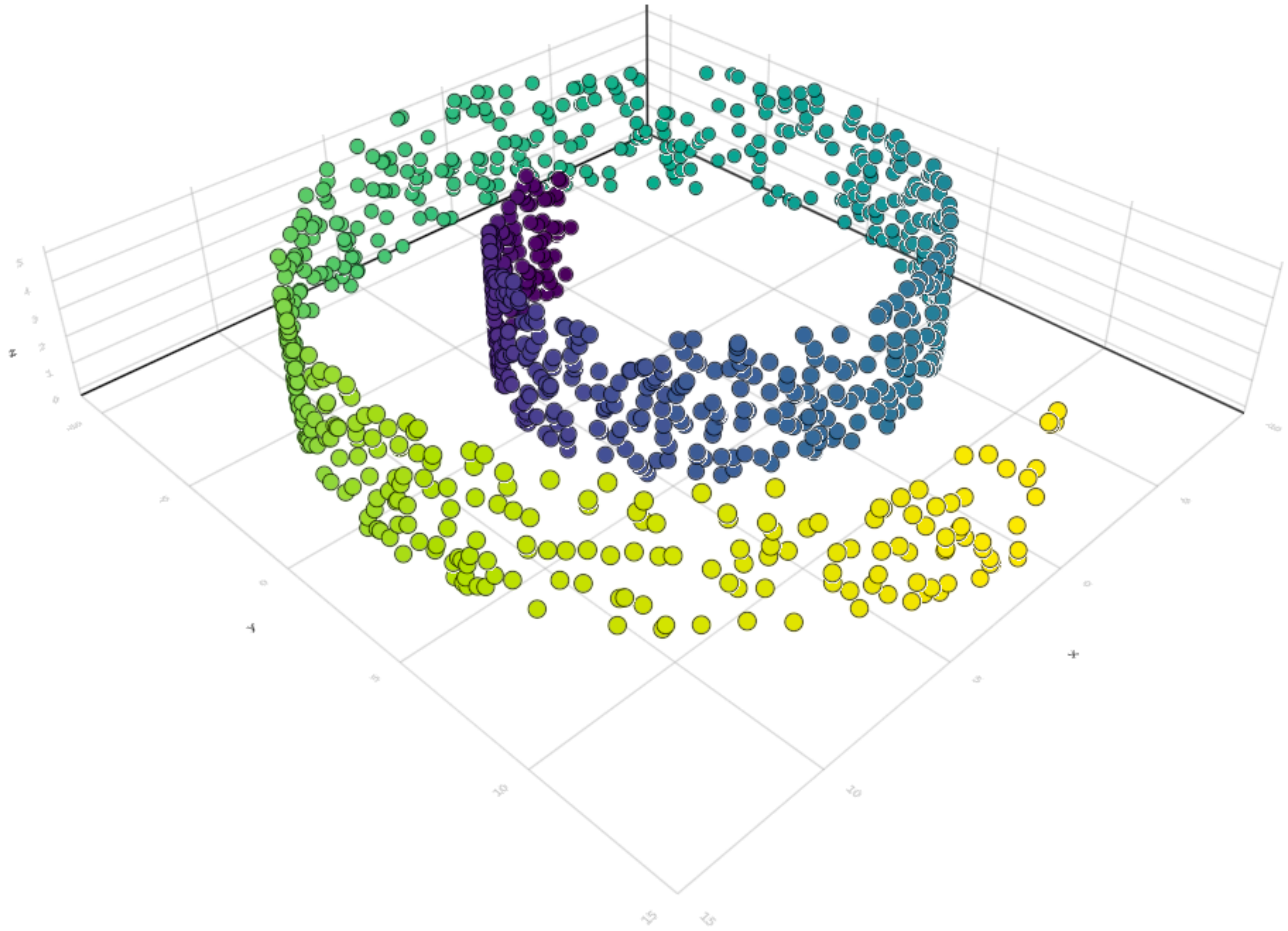

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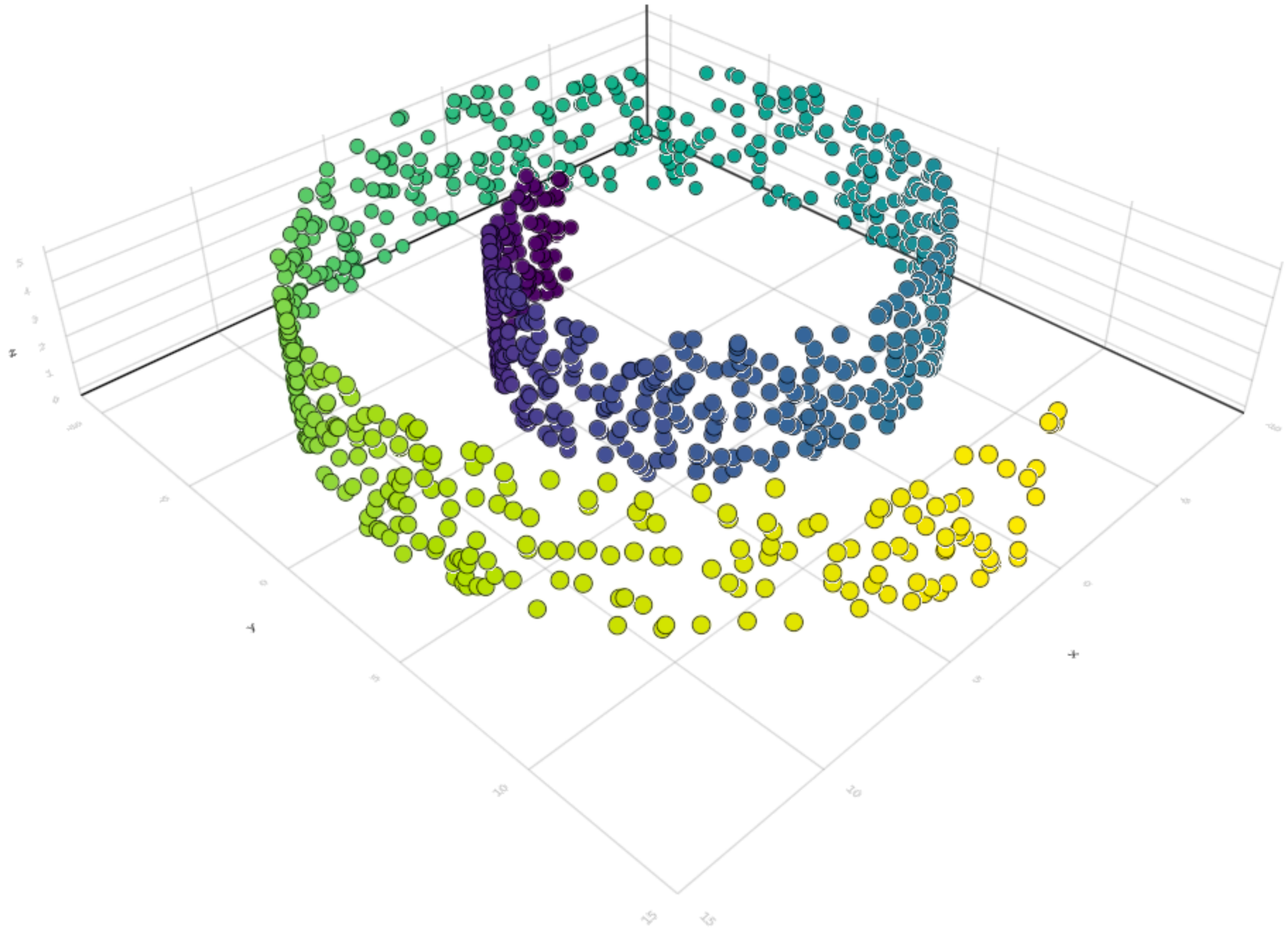

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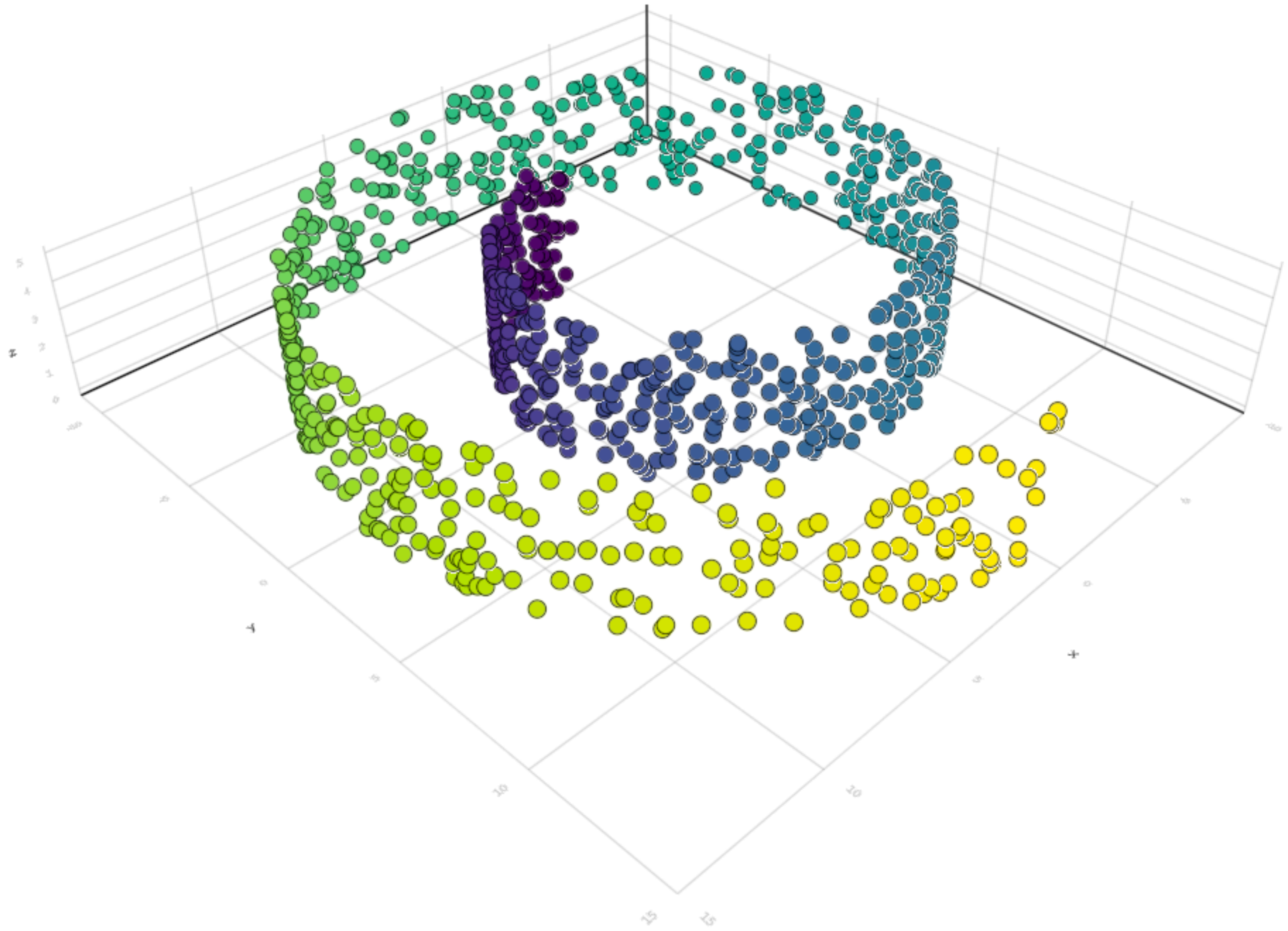

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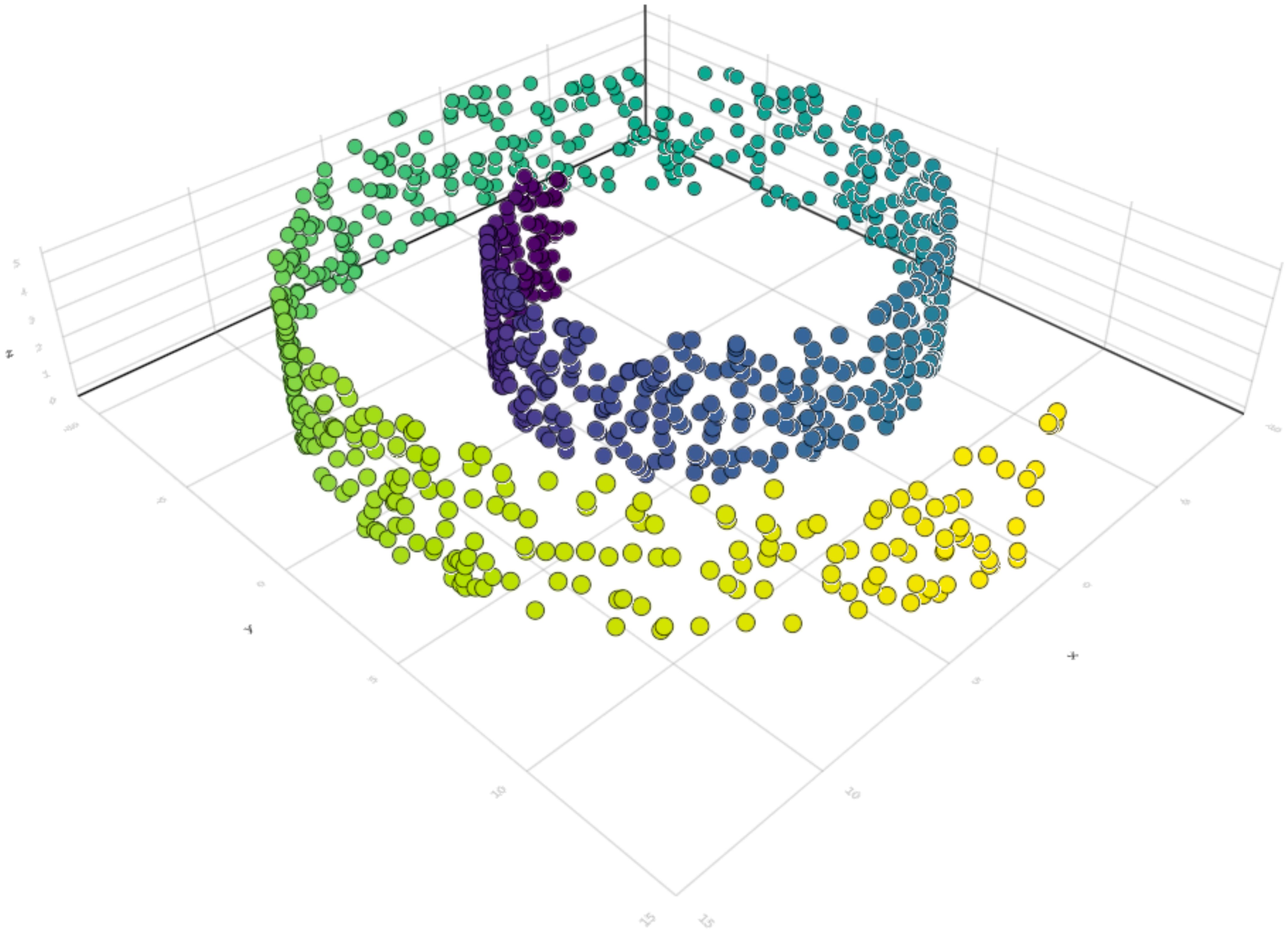

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Variance

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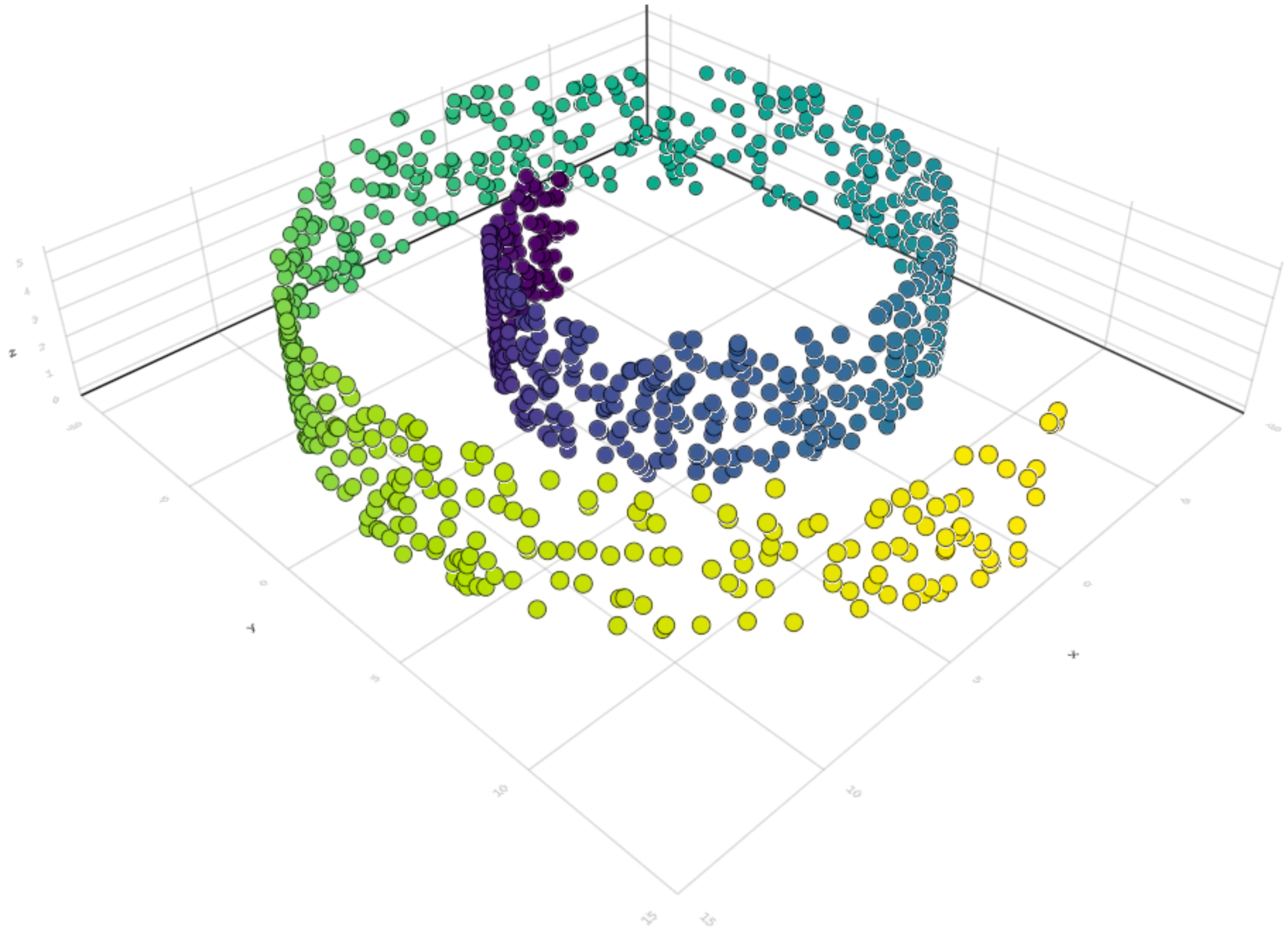

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Variance

Preserve kNN distances

Example: Maximum Variance Unfolding

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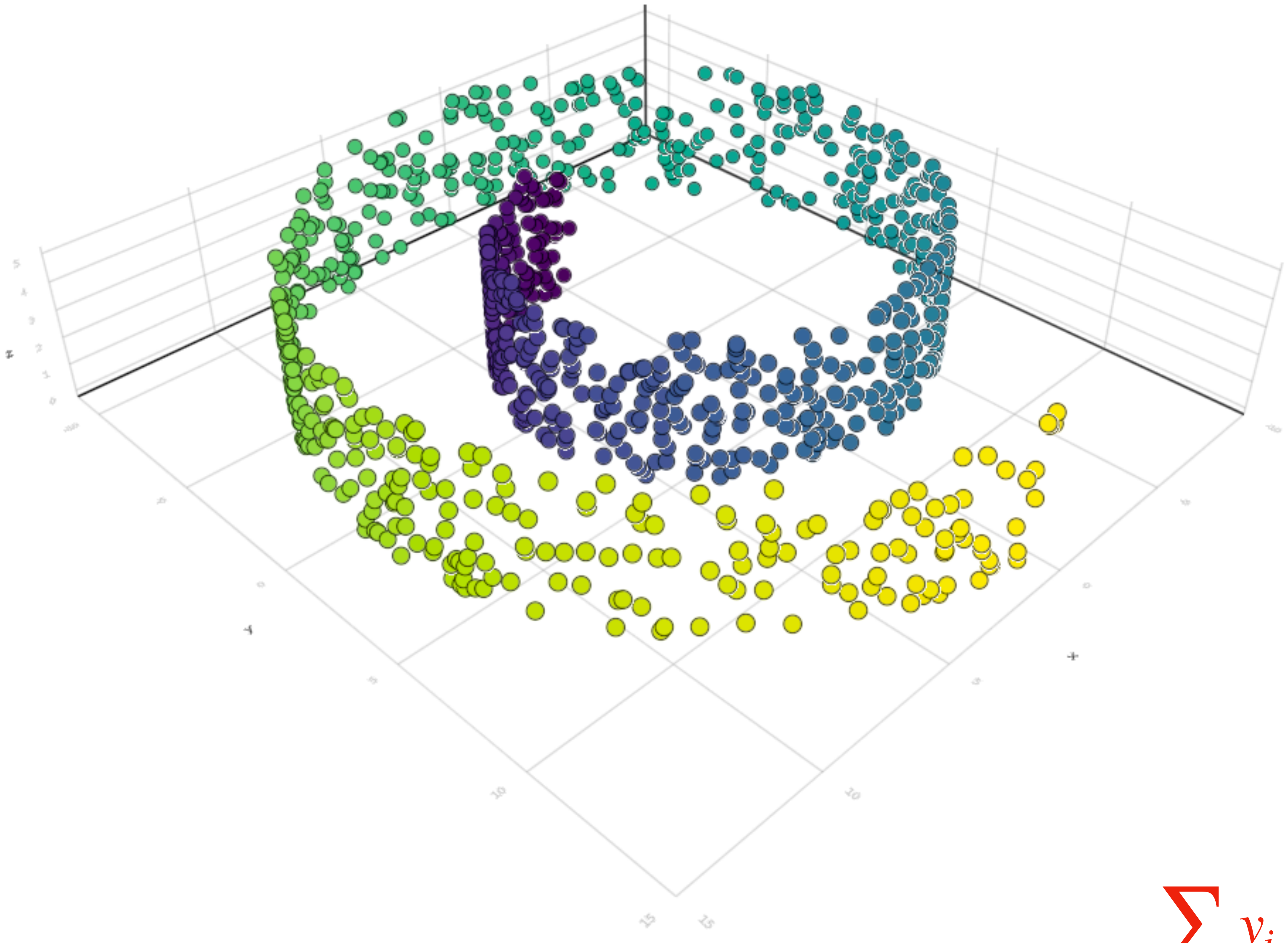

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$\sum y_i = 0$

Preserve kNN distances

MVU Sparsity Pattern

N = 250, k = 6

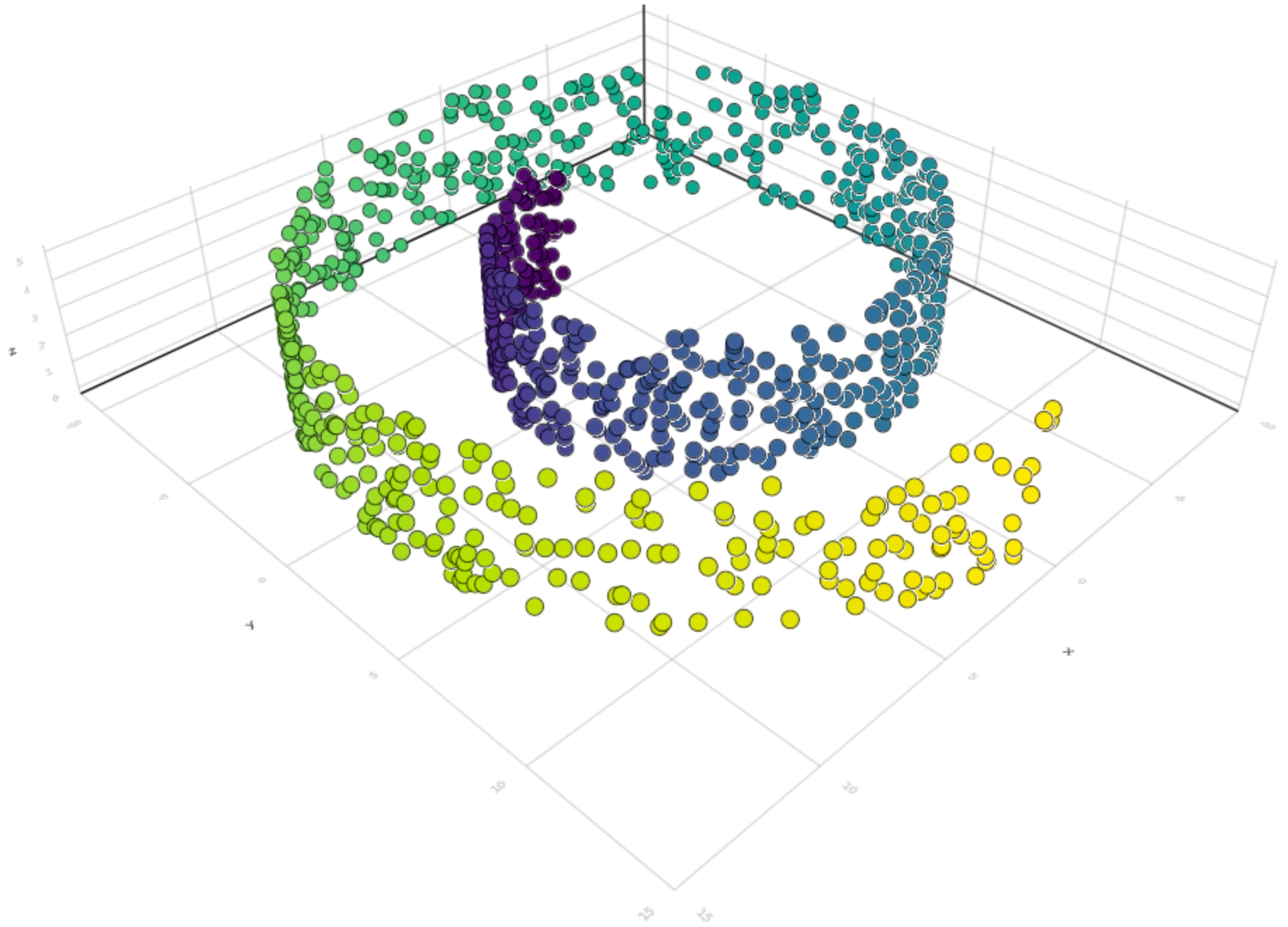
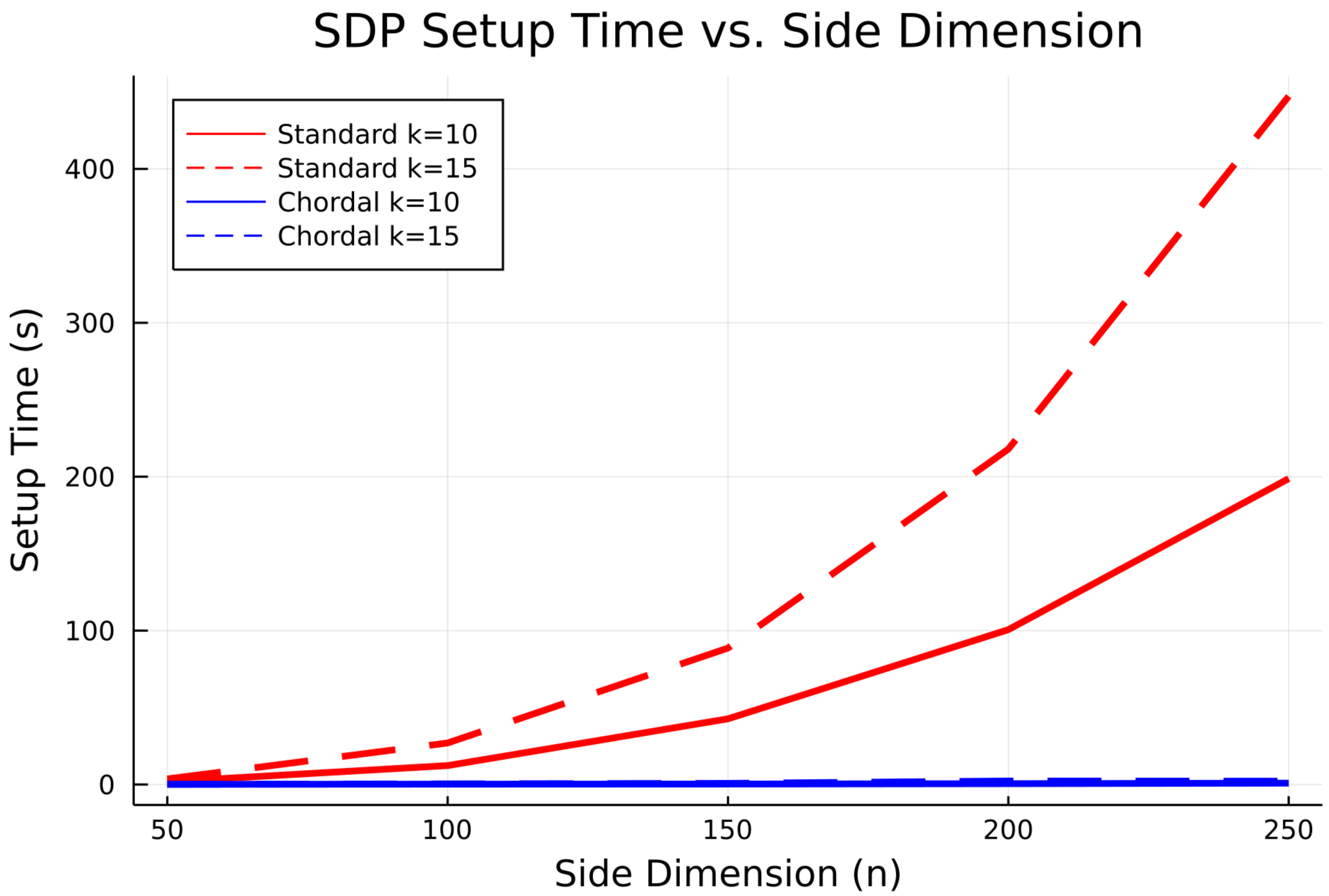
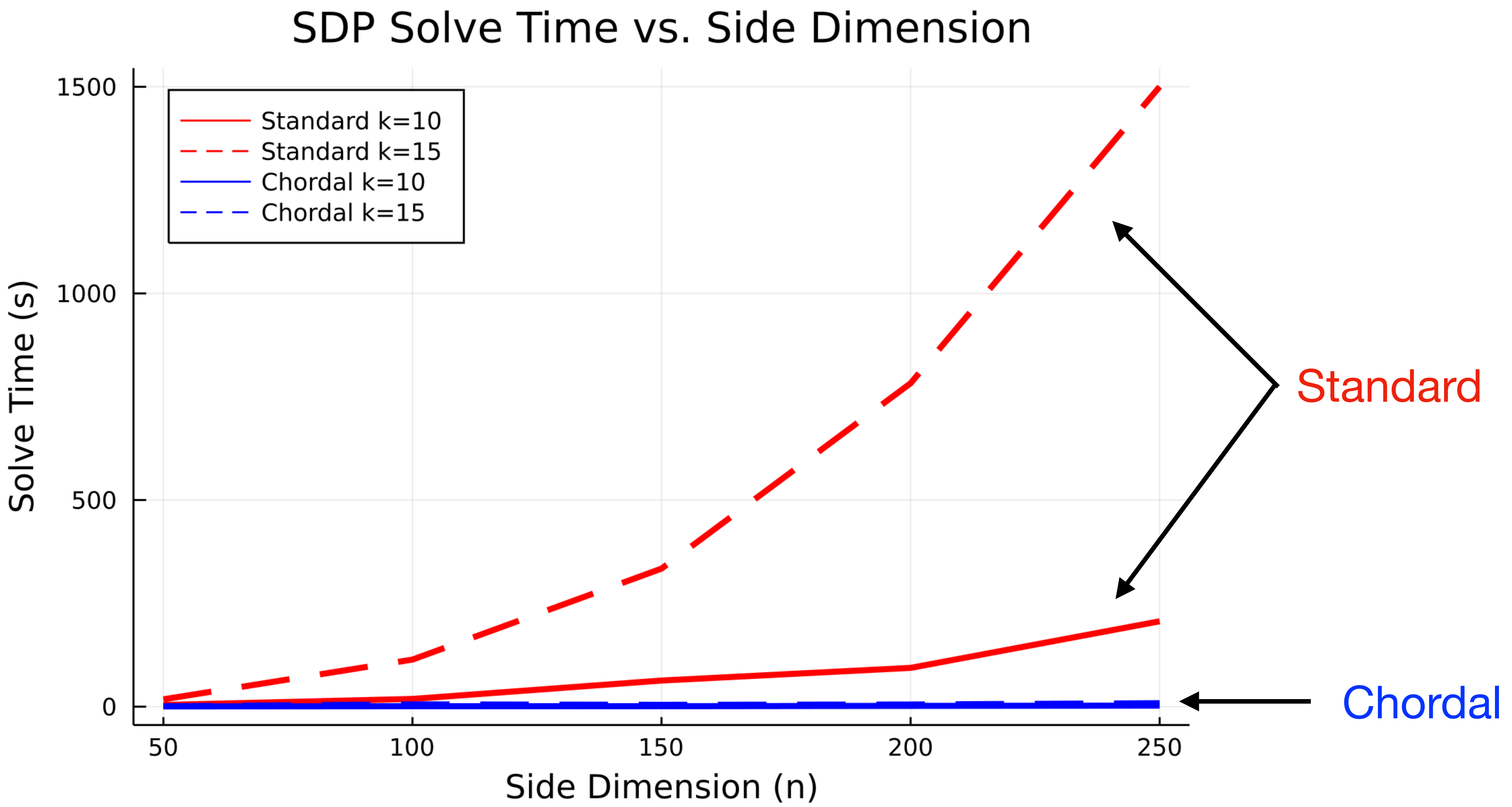


Figure 3: *Aggregate sparsity pattern for MVU with $n = 250$ and $k = 6$, which has 6487 nonzeros (left) and its chordal extension, which has 7382 nonzeros (right).*

Chordal decomposition *dramatically* speeds up SDPs





Up to 175x faster on maximum variance unfolding!

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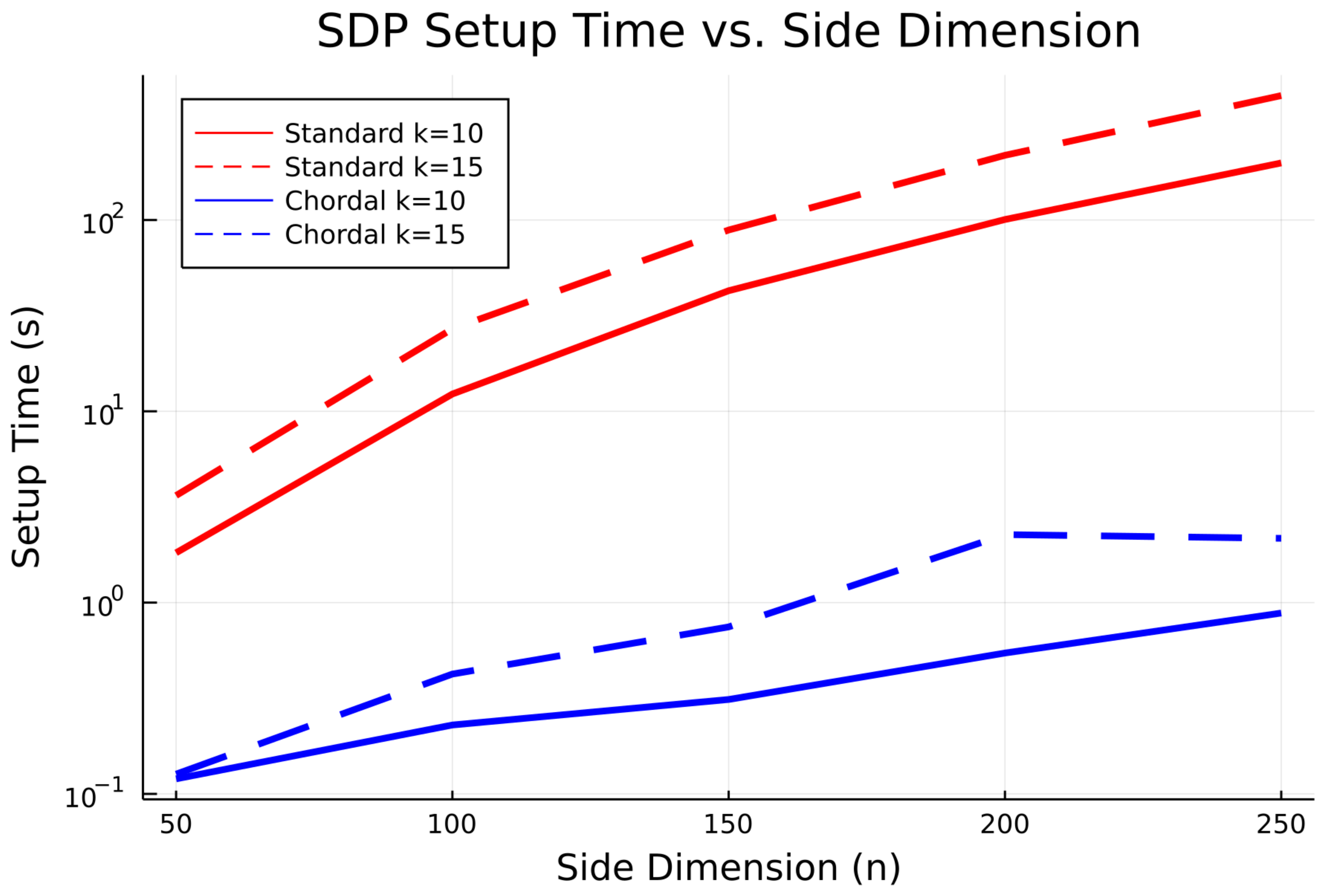

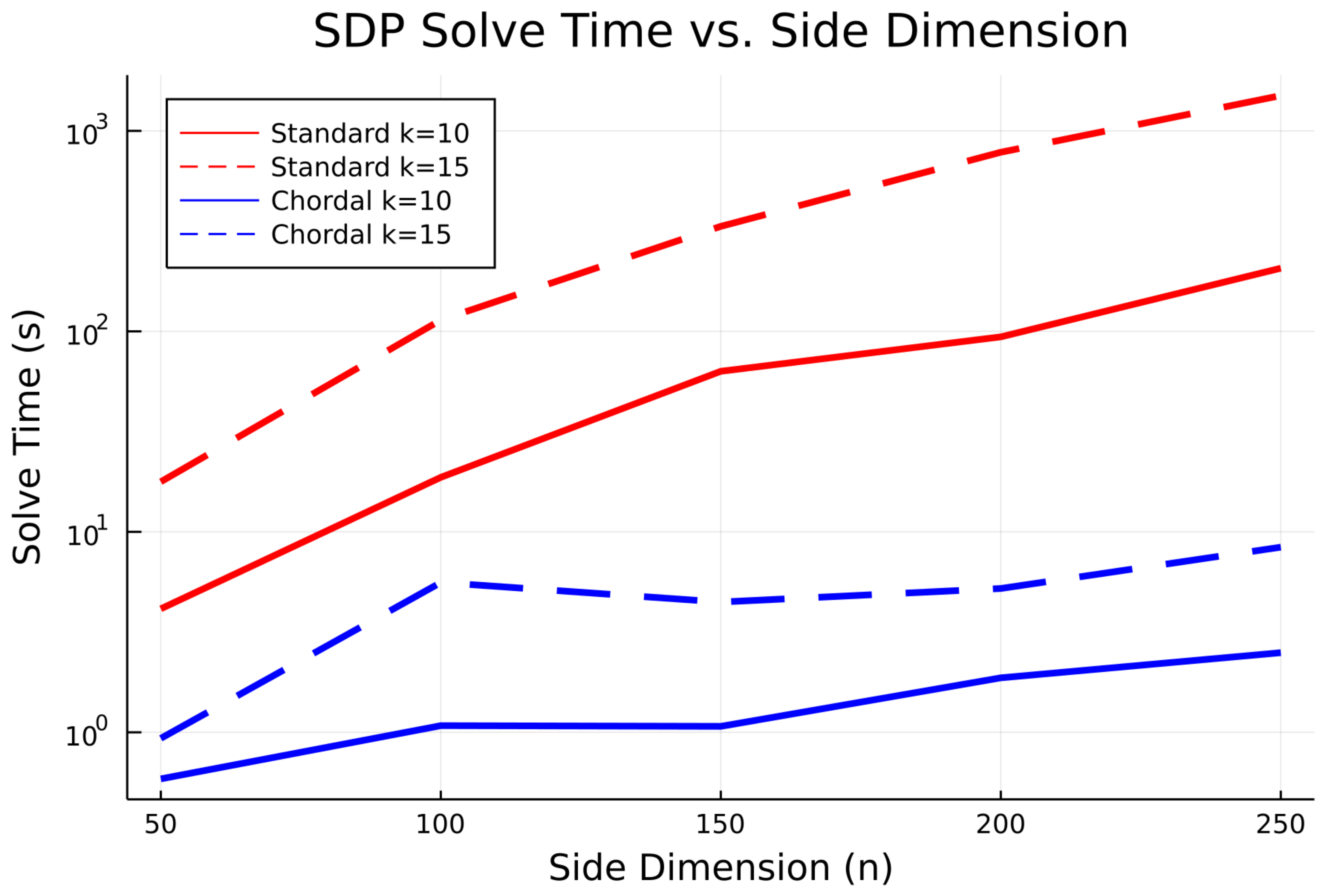

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Wrap up

Chordal.jl has several functions for sparse matrices

Survey: [Vandenberghe and Andersen 2015]

- Chordal decomposition of SDPs (with JuMP.jl) [Fukuda et al. 2001]
- Elimination trees, clique trees
- Clique graphs & merging [Garstka et al. 2020]
- Maximum determinant & minimum rank PSD completion [Sun 2015]
- Chordality tests [Tarjan and Yannakakis 1984]
- Euclidean distance matrix completion
- Several examples

Project Ideas

- Use Maximum Variance Unfolding to embed and then classify MNIST
  - Q: Does error propagate up the clique tree? Can we see this in accuracy?
- Investigate scaling of decomposed vs. standard sparse SDPs
  - Big issue in SDP solvers (especially first order ones)
- Implement a pure Julia sparse super nodal Cholesky
- Use these techniques for discrete optimization problems (see Ch 7 in [VA15])
- Integrate with Jump.jl (maybe not for this class, but great open source contribution!)

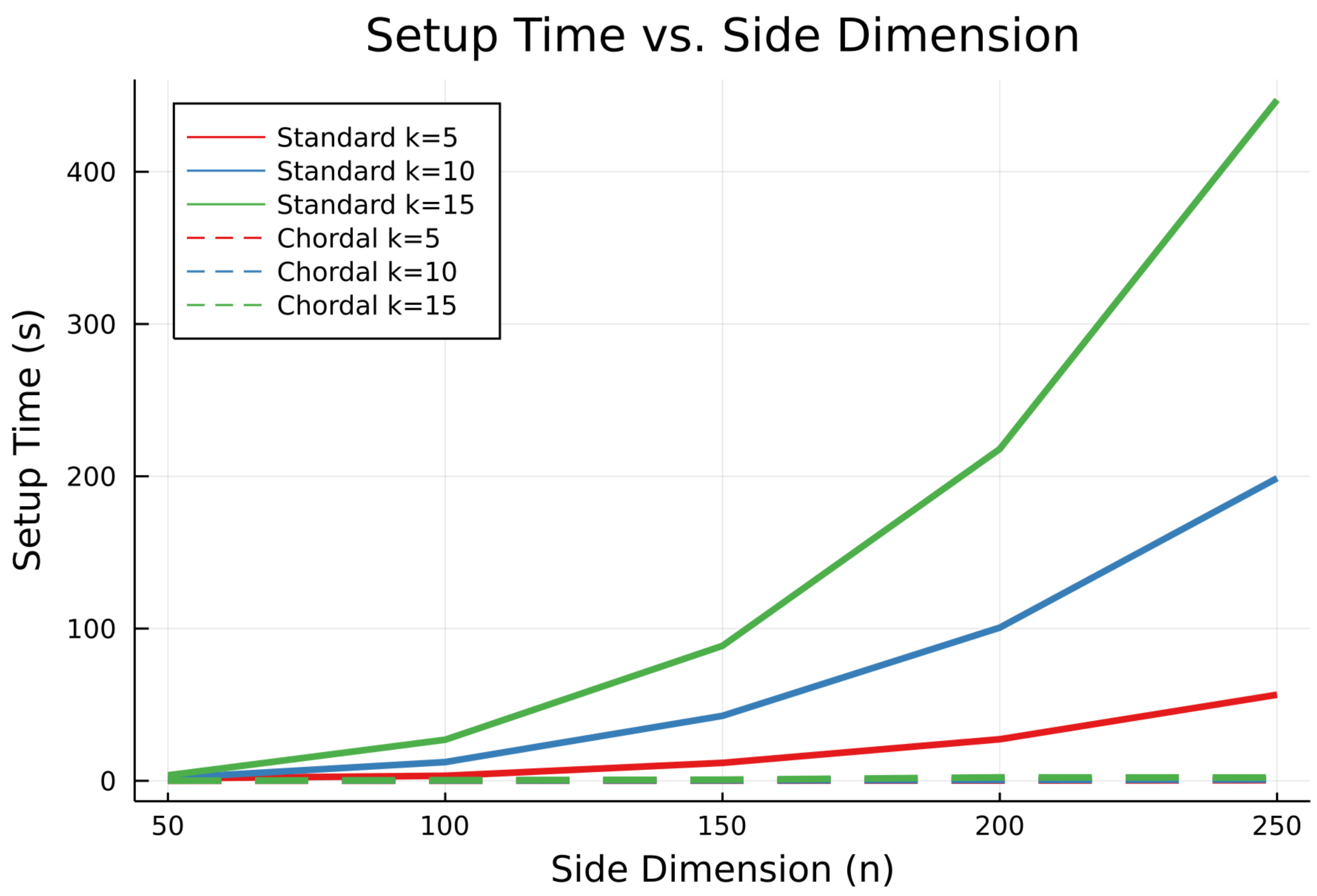
References

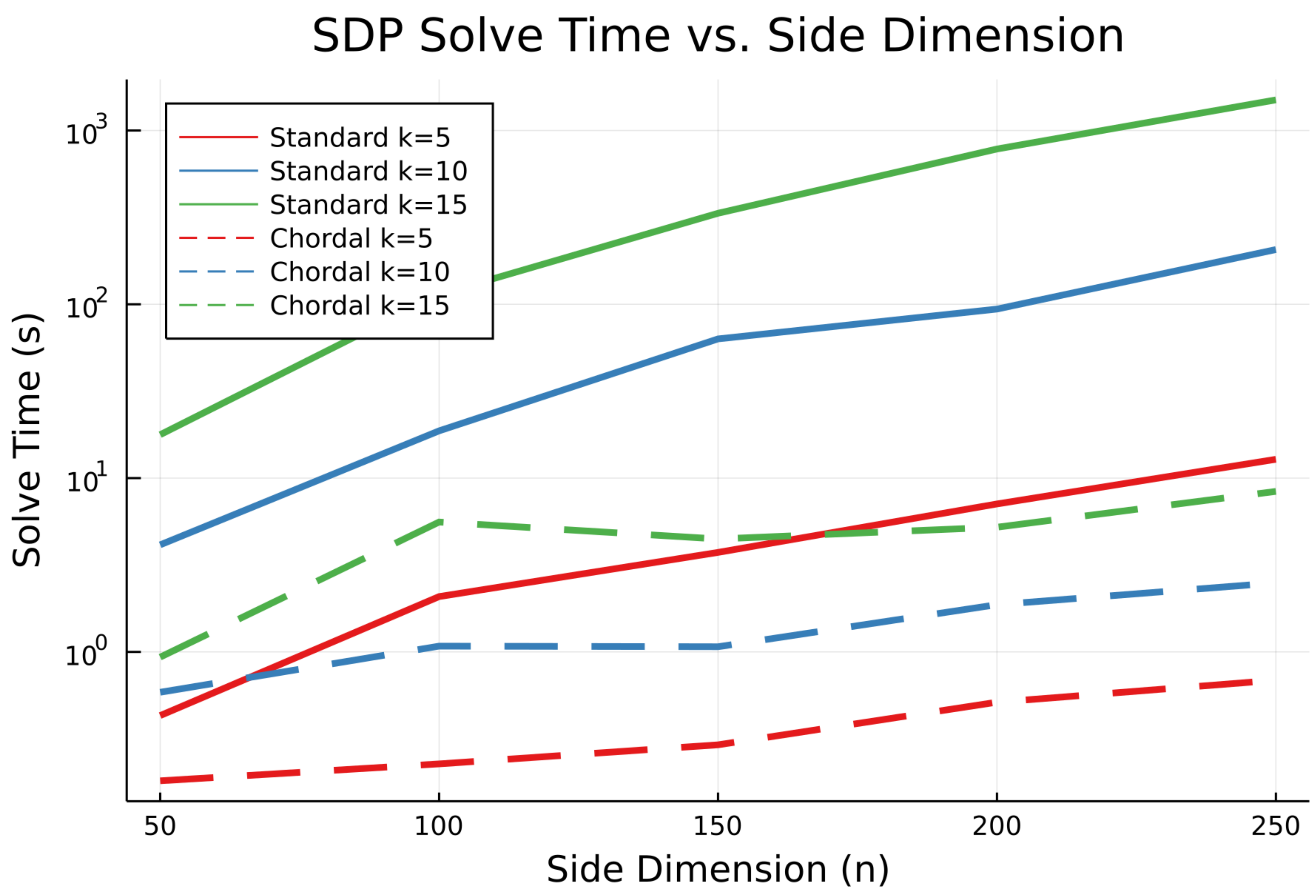
- Martin Andersen et al. “Logarithmic barriers for sparse matrix cones.” In: Optimization Methods and Software 28.3 (2013), pp. 396-423
- Mituhiro Fukuda et al. “Exploiting sparsity in semidefinite programming via matrix completion I: General framework.” In: SIAM Journal on Optimization 11.3 (2001), pp. 647–674.
- Michael Garstka, Mark Cannon, and Paul Goulart. “A clique graph based merging strategy for decomposable SDPs.” In: IFAC-PapersOnLine 53.2 (2020), pp. 7355–7361.
- Yifan Sun. “Decomposition methods for semidefinite optimization.” PhD thesis. UCLA, 2015.
- Robert E Tarjan and Mihalis Yannakakis. “Simple linear-time algorithms to test chordality of graphs, test acyclicity of hypergraphs, and selectively reduce acyclic hypergraphs.” In: SIAM Journal on computing 13.3 (1984), pp. 566–579.
- Lieven Vandenberghe and Martin S Andersen. “Chordal graphs and semidefinite optimization.” In: Foundations and Trends in Optimization 1.4 (2015), pp. 241–433.

Appendix

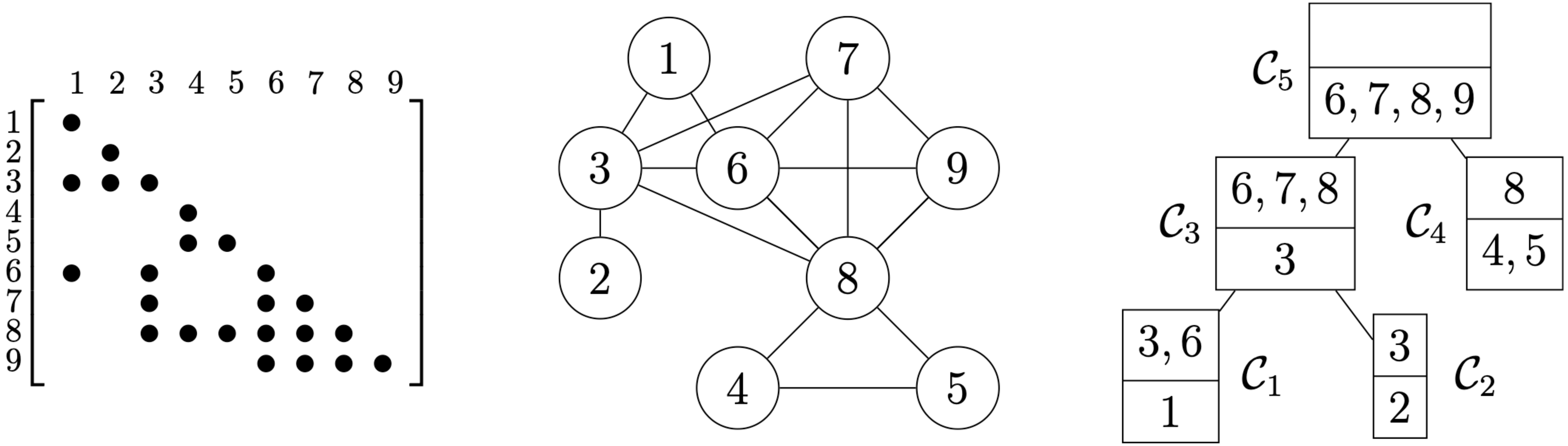
Chordal decomposition *dramatically* speeds up SDPs

(Primal form SDP)





Up to 200x faster on maximum variance unfolding!



1 2 3 4 5 6 7 8 9
$\mathcal{C}_5$
6, 7, 8, 9
$\mathcal{C}_3$
6, 7, 8
3
$\mathcal{C}_4$
8
4, 5
3, 6
1
$\mathcal{C}_1$
3
2
$\mathcal{C}_2$

Definition 4.1 (Running intersection property)**.**
For each pair of cliques $\mathcal{C}_i, \mathcal{C}_j \in \mathcal{B}$, the intersection $\mathcal{C}_i \cap \mathcal{C}_j$ is contained in all the cliques on the path in the clique tree connecting $\mathcal{C}_i$ and $\mathcal{C}_j$.

This property is also referred to as the *clique-intersection property* in [NFF+03] and the *induced subtree property* in [VA+15]. For a given chordal graph, a clique tree can be computed using the algorithm described in [PS90]. The clique tree for an example sparsity pattern is shown in Figure 1(c).

In a *post-ordered* clique tree the descendants of a node are given consecutive numbers, and a suitable post-ordering can be found via depth-first search. For a clique $\mathcal{C}_\ell$ we refer to the first clique encountered on the path to the root as its *parent clique* $\mathcal{C}_{\text{par}}$. Conversely $\mathcal{C}_\ell$ is called the *child* of $\mathcal{C}_{\text{par}}$. If two cliques have the same parent clique we refer to them as *siblings*.