MTH 133.

NO

Date

Chapter 6.

6.1 Inverse function.

Definition: Let f be a one-to-one function with domain A and range B. Then its inverse function f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y \qquad \text{for any } y \text{ in } B$$

Cancellation:

$$f^{-1}(f(x)) = x \qquad \text{for every } x \text{ in } A$$

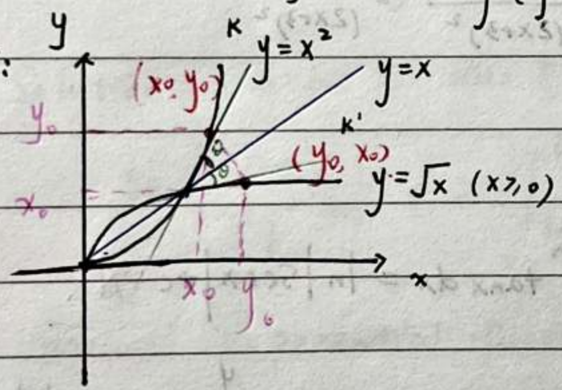
$$f(f^{-1}(x)) = x \qquad \text{for every } x \text{ in } B$$

Theorem: $f(x)$ continuous $\Rightarrow$ $f^{-1}(x)$ also continuous.

Theorem: If f is a one-to-one differentiable function, with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$ then the inverse function is differentiable at a and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

Pf:



since: $K \cdot K' = 1$

$\therefore f'(x)\Big|_{x=x_0} \cdot [f^{-1}(x)]'\Big|_{x=y_0} = 1.$

$\therefore f'(x_0) \cdot [f^{-1}(y_0)]' = 1$

$\therefore f'(x_0) \cdot [f^{-1}(f(x_0))]' = 1$

Eg: $f(x) = x^3+2$. find $(f^{-1})'(3)$.

Solution:

Since $y = x^3+2$

so $y-2 = x^3$

so $\sqrt[3]{y-2} = x$

so $f^{-1}(x) = \sqrt[3]{x-2}$.

$f'(x) = 3x^2$

$$[f^{-1}(3)]' = \frac{1}{[f(f^{-1}(3))]'} = \frac{1}{[f(1)]'} = \frac{1}{f'(1)} = \frac{1}{3}$$

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NO

Date

Eg: $f(x)=\frac{4x+1}{2x+3}$ 1) Find inverse of f 2) Find $[f^{-1}(1/3)]'$

Solution:

1). $y=\frac{4x+1}{2x+3}$

$x=\frac{4y+1}{2y+3}$

$2xy+3x=4y+1$

$(2x-4)y=1-3x$

$y=\frac{1-3x}{2x-4}$

$f^{-1}(x)=\frac{1-3x}{2x-4}$

2). $[f^{-1}(1/3)]' = \frac{1}{[f(f^{-1}(1/3))]'}$

$= \frac{1}{[f(0)]'} = \frac{1}{\frac{10}{9}} = \frac{9}{10}$

$f^{-1}(1/3) = \frac{1-3\times\frac{1}{3}}{2\times\frac{1}{3}-4} = 0$

$f'(x) = \frac{4\times(2x+3)-(4x+1)\cdot 2}{(2x+3)^2}$

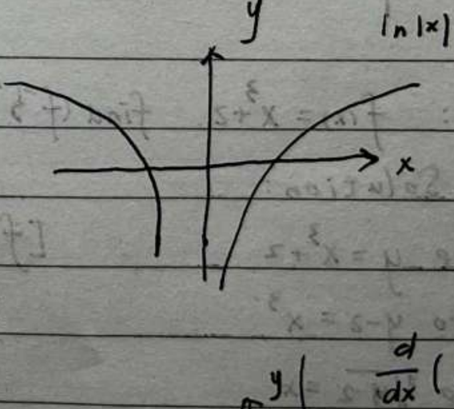
$= \frac{8x+12-8x-2}{(2x+3)^2} = \frac{10}{(2x+3)^2}$

6.2. The natural Logarithmic function:

$\ln x = \int_0^x$ ~~fixed~~ $\frac{1}{t}dt \quad (x>0)$

$\frac{d}{dx}\ln|x| = \frac{1}{x} \qquad \int \frac{1}{x}dx = \ln|x|+c \qquad \int \tan x\, dx = \ln|\sec x|+c$ ☆

Pf: $\int \frac{\sin x}{\cos x}dx$

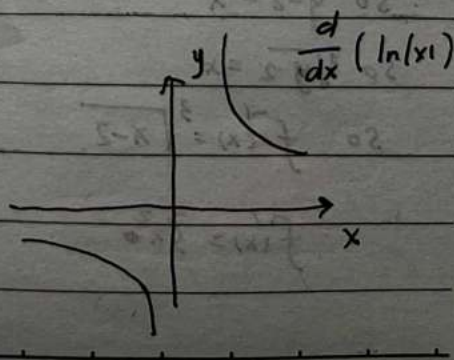


let $u=\cos x$

$du=-\sin x\, dx$.

so $\int \frac{-du}{u} = -\ln|u|+c = -\ln|\cos x|+c$

$= \ln|\cos x|^{-1}+c = \ln|\sec x|+c.$



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Date

Eg: Find derivative $y=\frac{(\sin^2 x)(x+3)^4}{(5x-8)^{10}}$

Solution: $\ln y=\ln\frac{\sin^2 x\cdot(x+3)^4}{(5x-8)^{10}}=2\ln\sin x+4\ln(x+3)-10\ln(5x-8)$

take derivative:

$$\frac{1}{y}\cdot y'=2\cdot\frac{\cos x}{\sin x}+\frac{4}{x+3}-\frac{10}{5x-8}\times 5$$

$$\therefore y'=y\left(\frac{2}{\tan x}+\frac{4}{x+3}-\frac{50}{5x-8}\right)\Rightarrow y'=\frac{\sin^2 x(x+3)^4}{(5x-8)^{10}}\left(2\cot x+\frac{4}{x+3}-\frac{50}{5x-8}\right)$$

Eg: Find derivative of $y=t^2+3\ln(5\ln t)$

Solution: $y'=2t+3\cdot\frac{1}{5\ln t}\cdot\frac{5}{t}=2t+\frac{3}{t\ln t}$

Eg: Evaluate the integral $\int_9^{10}\frac{1}{t\ln t}\,dt$

Solution: $u=\ln t\Rightarrow du=\frac{1}{t}dt$. $9\le t\le 10\Rightarrow 2\ln 3\le u\le\ln 10$

So: $\int_9^{10}\frac{1}{\ln t}\frac{1}{t}dt=\int_{\ln 9}^{\ln 10}\frac{1}{u}du=\ln u\Big|_{\ln 9}^{\ln 10}=\ln\left(\ln\frac{10}{9}\right)$

6.3 natural exponential function

$e^x=\exp(x)\iff e^x=y\iff x=\ln y\ \therefore e^{\ln x}=x\quad \ln(e^x)=x$

Eg: Evaluate integral $\int 4e^x\sqrt{3+e^x}\,dx$

Solution: let $u=e^x\Rightarrow du=e^x dx$

So $\int 4e^x\sqrt{3+u}\,dx=4\int\sqrt{3+u}\,du$ let $p=3+u\Rightarrow dp=du$

$$=4\int p^{\frac{1}{2}}dp=4\cdot\frac{1}{\frac{1}{2}+1}\cdot p^{\frac{1}{2}+1}=4\cdot\frac{2}{3}\cdot p^{\frac{3}{2}}=\frac{8}{3}p^{\frac{3}{2}}=\frac{8}{3}(3+u)^{\frac{3}{2}}$$

$$=\frac{8}{3}(3+e^x)^{\frac{3}{2}}+C$$

$\therefore I = \frac{x^2}{3}e^{3x} - \frac{2}{3}\left[\frac{x}{3}e^{3x} - \int \frac{1}{3}e^{3x}dx\right] = \frac{e^{3x}}{3}\left(x^2 - \frac{2x}{3} + \frac{2}{9}\right) + C$ ✗

6.5. Exponential Growth and Decay

Theorem: The only solution of the differential equation $\frac{dy}{dt} = ky$ are the exponential functions:

$$y(t) = y(0) \cdot e^{kt}$$

Pf: $\frac{dy}{dt} = y(0) \cdot e^{kt} \cdot k = ky$.

So: $\frac{dP}{dt} = kP$ (or $\frac{1}{P} \cdot \frac{dP}{dt} = k$) $\Leftrightarrow P(t) = P_0 \cdot e^{kt}$.

$\boxed{\text{half-life} = \frac{\ln(\frac{1}{2})}{k}}$

Definition: Newton's law of cooling

$T(t)$ temperature of object

$$\frac{dT}{dt} = -k(T - T_s)$$

T_s temperature of surrounding

$\Leftrightarrow \boxed{T(t) = T_s + (T_0 - T_s)e^{-kt}}$ ✓ $T(0) = T_0$

Pf: $\frac{dT}{dt} = -k(T - T_s)$

$\Leftrightarrow \frac{1}{T - T_s}\, dT = -k\, dt$

$\Leftrightarrow \int \frac{1}{T - T_s}\, dT = \int -k\, dt$

$\Leftrightarrow \ln(T - T_s) = -kt + C$ when $t = 0$ $T(0) = T_0$

$\therefore \ln(T_0 - T_s) = 0 + C$

$\therefore \ln(T - T_s) = -kt + \ln(T_0 - T_s)$

$\ln\left(\frac{T - T_s}{T_0 - T_s}\right) = -kt \Leftrightarrow T - T_s = (T_0 - T_s)e^{-kt}$

$\therefore T(t) = T_s + (T_0 - T_s)e^{-kt}$

NO

Date

6.6 Inverse Trigonometric Function

① $\sin^{-1}$ has domain $[-1,1]$ range $[-\pi/2, \pi/2]$

$$\sin^{-1}x = y \iff \sin y = x$$

$\arcsin(\sin x) = x$ for $x \in [-\pi/2, \pi/2]$

$\sin(\arcsin x) = x$ for $x \in [-1,1]$

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \quad (-1<x<1)$$

Review:

$(\tan x)' = \sec^2 x$

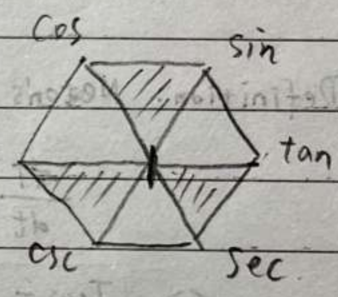
$(\sec x)' = \tan x \cdot \sec x$

$(\csc x)' = -\cot x \cdot \csc x$

$(\cot x)' = -\csc^2 x$

Pf: $y = \arcsin x \iff \sin y = x \iff \cos y \cdot y' = 1 \iff y' = \frac{1}{\cos y}$

$\therefore y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$



② $\cos^{-1}$ has domain $[-1,1]$ range $[0,\pi]$

$$\cos^{-1}x = y \iff \cos y = x$$

$\cos(\arccos(x)) = x$ for $x \in [-1,1]$

$\arccos(\cos x) = x$ for $x \in [0,\pi]$

$$\frac{d}{dx}(\arccos x) = \frac{-1}{\sqrt{1-x^2}}$$

Pf: $y = \arccos x \iff \cos y = x \iff -\sin y \cdot y' = 1$

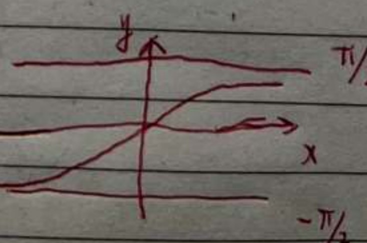
$\therefore y' = \frac{-1}{\sin y} = \frac{-1}{\sqrt{1-\cos^2 y}} = \frac{-1}{\sqrt{1-x^2}}$

③ $\tan^{-1}$ has domain range $(-\frac{\pi}{2}, \pi/2)$

$$\tan^{-1}x = y \iff \tan y = x$$

The line $y = \pi/2$ and $y = -\pi/2$ horizontal asymptotes of graph $\tan^{-1}$

$\lim_{x\to\infty} \tan^{-1}x = \pi/2 \qquad \lim_{x\to-\infty} \tan^{-1}x = -\pi/2$



$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

Pf $y = \arctan x \iff \tan y = x \iff \sec^2 y \cdot y' = 1$

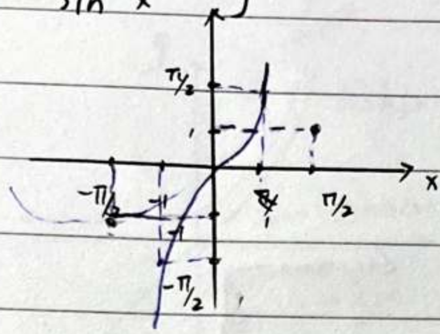
$\therefore y' = \frac{1}{\sec^2 x} = \frac{1}{1+\tan^2 x} = \frac{1}{1+x^2}$

$\sin\frac{\pi}{2}=1$ $\cos\frac{\pi}{2}=0$ $\cos 0=1$

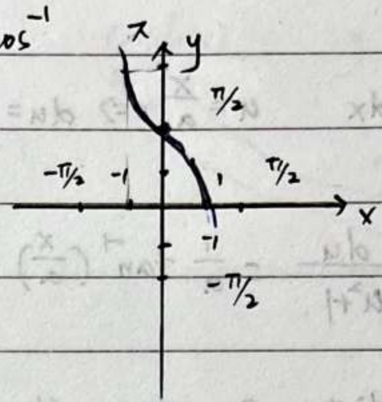
$\arcsin 1=\frac{\pi}{2}$ $\arccos 0=\frac{\pi}{2}$ $\arccos 1=0$

Pictures

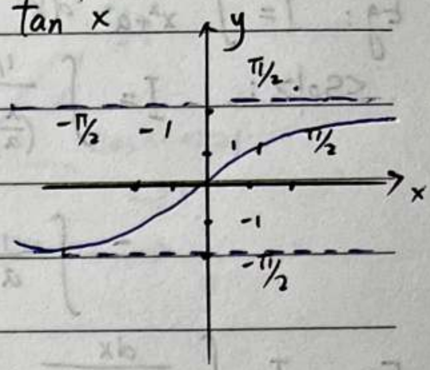
$\sin^{-1}x$



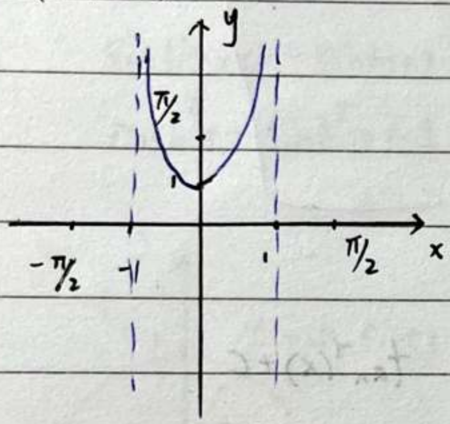
$\cos^{-1}$



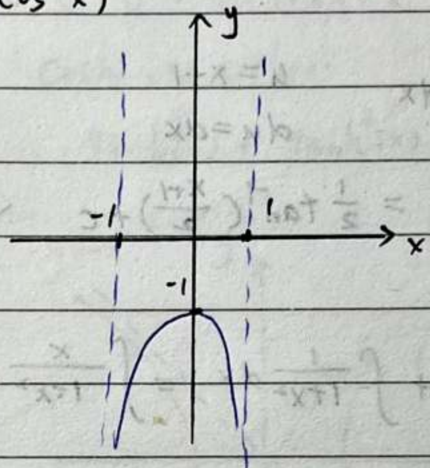
$\tan^{-1}x$



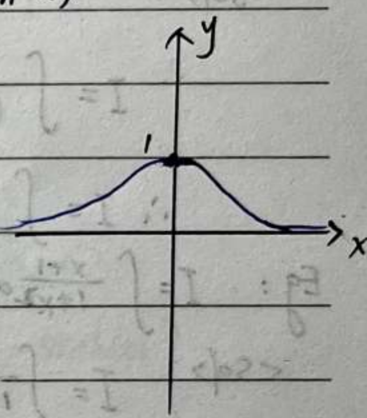
$(\sin^{-1}x)'$



$(\cos^{-1}x)'$



$(\tan^{-1}x)'$



☆ $y=\csc^{-1}x$ $(|x|\geq 1)$ $\Leftrightarrow$ $\csc y=x$ and $y\in(0,\frac{\pi}{2}]\cup(\pi,\frac{3\pi}{2}]$

$y=\sec^{-1}x$ $(|x|\geq 1)$ $\Leftrightarrow$ $\sec y=x$ and $y\in(0,\frac{\pi}{2}]\cup(\pi,\frac{3\pi}{2}]$

$y=\cot^{-1}x$ $(x\in\mathbb{R})$ $\Leftrightarrow$ $\cot y=x$ and $y\in(0,\pi)$

☆ $(\csc^{-1}x)'=\frac{-1}{x\sqrt{x^2-1}}$ $(\sec^{-1}x)'=\frac{1}{x\sqrt{x^2-1}}$ $(\cot^{-1}x)'=\frac{-1}{1+x^2}$

$\int\frac{1}{\sqrt{1-x^2}}dx=\sin^{-1}(x)+C$ $\int\frac{1}{x^2+1}dx=\tan^{-1}(x)+C$

Pf: $y=\csc^{-1}x\Leftrightarrow\csc y=x$

$\therefore -\cot y\cdot\csc y\cdot y'=1$

$\therefore y'=\frac{-1}{\csc y\cdot\cot y}$

$=\frac{-1}{x\sqrt{x^2-1}}$

Pf: $y=\sec^{-1}x\Leftrightarrow\sec y=x$

$\therefore \sec y\tan y\cdot y'=1$

$\therefore y'=\frac{1}{\sec y\cdot\tan y}$

$=\frac{1}{x\sqrt{x^2-1}}$

Pf: $\cot^{-1}x=y$

$\cot y=x$

$-\csc^2 y\cdot y'=1$

$y'=\frac{-1}{1+x^2}$

NO

Date

Eg: $I = \int \frac{1}{x^2+a^2} \cdot dx$

<Sol>: $I = \int \frac{1/a^2}{(\frac{x}{a})^2+1} dx \qquad u = \frac{x}{a} \Leftrightarrow du = \frac{1}{a} dx$

$= \int \frac{1}{a} \cdot \frac{du}{u^2+1} = \frac{1}{a} \tan^{-1}(\frac{x}{a}) + C$

Eg: $I = \int \frac{dx}{x^2+2x+5}$ (Hint: complete the square)

<Sol>: $x^2+2x+5 = x^2+2x+1+4 = (x+1)^2+4 = (x+1)^2+2^2$

$\therefore I = \int \frac{1}{(x+1)^2+4} dx \qquad u = x+1, \; du = dx$

$\therefore I = \int \frac{du}{u^2+4} = \frac{1}{2} \tan^{-1}(\frac{x+1}{2}) + C$

Eg: $I = \int \frac{x+1}{1+x^2} dx$

<Sol>: $I = \int \frac{x}{1+x^2} dx + \int \frac{1}{1+x^2} dx = \int \frac{x}{1+x^2} dx + \tan^{-1}(x) + C$

$u = 1+x^2 \Leftrightarrow du = 2x\, dx$

$\therefore \int \frac{x}{1+x^2} dx = \int \frac{1}{u} \frac{1}{2} du = \frac{1}{2} \ln|u|$

$\therefore I = \frac{1}{2} \ln|1+x^2| + \tan^{-1}(x) + C$

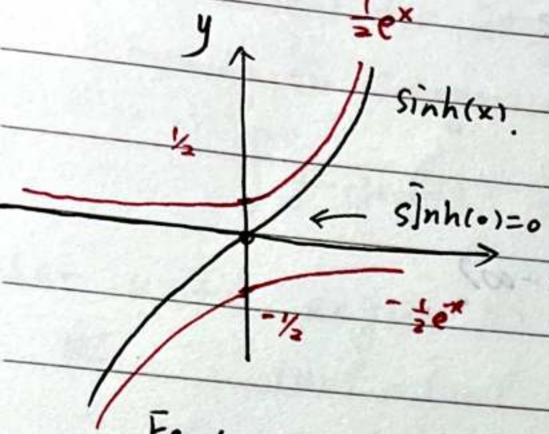
6.7 Hyperbolic Function

$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch}(x) = \frac{1}{\sinh(x)}$

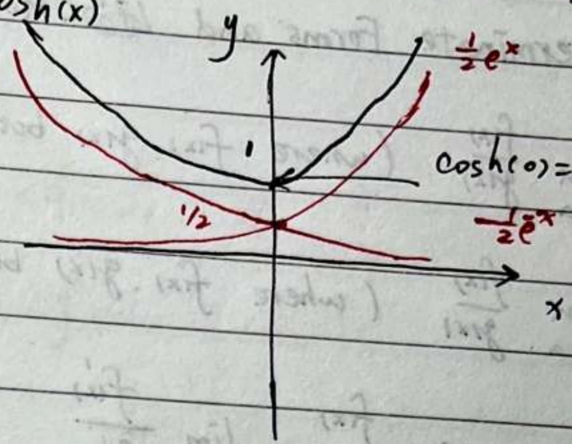
$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech}(x) = \frac{1}{\cosh(x)}$

$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} \qquad \coth(x) = \frac{1}{\tanh(x)}$

Sinh(x)



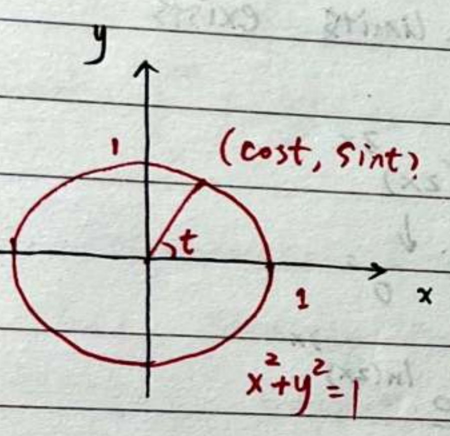
cosh(x)

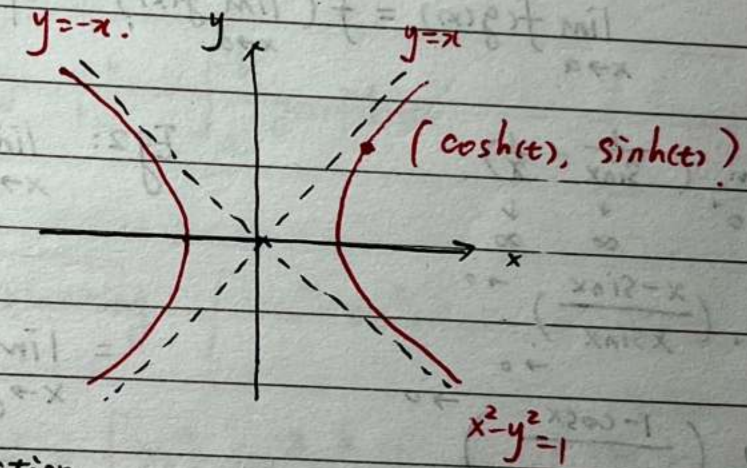


Feature:

$\sinh(-x) = -\sinh(x)$ $\qquad$ $\cosh(-x) = \cosh(x)$

$\cosh^2 x - \sinh^2 x = 1.$ $\qquad$ $\text{sech}^2(x) + \tanh^2(x) = 1.$





derivative of hyperbolic function.

$\frac{d}{dx}(\sinh x) = \cosh(x)$ $\qquad$ $\frac{d}{dx}(\text{csh} x) = -\text{csh}(x)\cdot \cot(x).$

$\frac{d}{dx}(\cosh(x)) = \sinh(x)$ $\qquad$ $\frac{d}{dx}(\text{sech} x) = -\text{sech}(x)\tanh(x)$

$\frac{d}{dx}(\tanh x) = \text{sech}^2 x$ $\qquad$ $\frac{d}{dx}(\coth x) = -\text{csh}^2(x)$

Date

6.8. Interminate Forms and L'Hospital's Rule

$\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ (where $f(x)$, $g(x)$ both $\to 0$).

or $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ (where $f(x)$, $g(x)$ both $\to \infty$ or $-\infty$).

then. $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$.

Calculus 1 theorem: if $f(x)$ is continuous then

$\lim\limits_{x\to a} f(g(x)) = f(\lim\limits_{x\to a} g(x))$ if the limit exists

Eg1. $\lim\limits_{x\to 0^+} (\underset{\infty}{\frac{1}{\sin x}} - \underset{\infty}{\frac{1}{x}})$.

$= \lim\limits_{x\to 0^+} \left(\frac{x-\sin x}{x\sin x}\right)$ $\to 0$ / $\to 0$

$= \lim\limits_{x\to 0^+} \left(\frac{1-\cos x}{\sin x + x\cos x}\right)$ $\to 0$ / $\to 0$

$= \lim\limits_{x\to 0^+} \left(\frac{\sin x}{\cos x + \cos x - x\sin x}\right) = 0$. $\to 0$ / $\to 2$

Eg: $\lim\limits_{x\to \infty} (1+\frac{1}{x})^x$

$= \lim\limits_{x\to \infty} e^{x\ln(1+\frac{1}{x})} = e^{\lim\limits_{x\to\infty} x\ln(1+\frac{1}{x})}$

for $\lim\limits_{x\to \infty} \underset{\infty}{x} \underset{0}{\ln(1+\frac{1}{x})} = \lim\limits_{x\to\infty} \frac{\ln(1+\frac{1}{x})}{1/x}$ $\to 0$ / $\to 0$

$= \lim\limits_{x\to\infty} \frac{\frac{1}{1+1/x}(-\frac{1}{x^2})}{(-1/x^2)} = \lim\limits_{x\to\infty} \frac{1}{1+1/x} = 1$

$\therefore \lim\limits_{x\to\infty} (1+\frac{1}{x})^x = e$.

Eg2: $\lim\limits_{x\to 0^+} (2x)^{3x}$ $\to 0^0$

$= \lim\limits_{x\to 0^+} e^{\ln(2x)^{3x}}$

$= \lim\limits_{x\to 0^+} e^{3x\cdot\ln(2x)}$

$= e^{\lim\limits_{x\to 0^+} 3x\cdot\ln(2x)}$

$\lim\limits_{x\to 0_+} \underset{0}{3x} \underset{-\infty}{\ln(2x)} = 3\lim\limits_{x\to 0_+} \frac{\ln(2x)}{1/x}$ $\to -\infty$ / $\to \infty$

$= 3\lim\limits_{x\to 0_+} \frac{\frac{1}{2x}\cdot 2}{-\frac{1}{x^2}} = 3\lim\limits_{x\to 0_+}(-x)$

$= 0$

$\therefore \lim\limits_{x\to 0^+} (2x)^{3x} = e^0 = 1$

NO

Date

Chapter 7.

7.1. Technique for integration.

Formula for integration by parts:

$$\int f(x) g'(x)\, dx = f(x) g(x) - \int g(x) f'(x)\, dx$$ ☆

Let $u = f(x)$ $v = g(x)$. $\Rightarrow \int u\,dv = uv - \int v\,du$.

Pf: $(uv)' = (uv)'$.

$\Leftrightarrow (uv)' = u'v + uv'$

$\Leftrightarrow \int (uv)'\,dx = \int u'v\,dx + \int uv'\,dx$ Simply & nice proof!

$\Leftrightarrow uv = \int u'v\,dx + \int uv'\,dx$

$\Leftrightarrow \int u\,dv = uv - \int v\,du$.

Remark: $\int_a^b f(x) g'(x)\, dx = f(x) g(x) \Big|_a^b - \int_a^b g(x) f'(x)\, dx$

7.2. Trigonometric Integral

7.2.1. Strategy for evaluating $\int \sin^m x \cos^n x\, dx$

(a) $n = 2k+1$.

$$\int \sin^m x \cdot \cos^{2k+1} x\, dx = \int \sin^m x \cdot (\cos^2 x)^k \cos x\, dx = \int \sin^m x\, (1-\sin^2 x)^k\, d\sin x$$

Then subsitude $u = \sin x \Leftrightarrow du = \cos x\, dx$

(b). $m = 2k+1$

$$\int \sin^{2k+1} x \cos^n x\, dx = \int (\sin^2 x)^k \cdot \sin x \cdot \cos^n x\, dx = \int (1-\cos^2 x)^k \cos^n x\, (-d\cos x)$$

Then subsitude $u = \cos x \Rightarrow du = -\sin x\, dx$

(c) both even

$\sin^2 x = \frac{1-\cos 2x}{2}$. $\cos^2 x = \frac{\cos 2x + 1}{2}$ $\sin x \cos x = \frac{1}{2}\sin 2x$.

Date

7.2.4. Strategy for Evaluating $\int \tan^m x \sec^n x\,dx$

(a) $n=2k$.

$$\int \tan^m x \sec^{2k} x\,dx = \int \tan^m x\,(\sec^2 x)^{k-1} \sec^2 x\,dx = \int \tan^m x\,(1+\tan^2 x)^{k-1}\,d(\tan x).$$

Then subsitute $u=\tan x \Rightarrow du=\sec^2 x\,dx$

(b). $m=2k+1$

$$\int \tan^{2k+1} x \sec^n dx = \int (\tan^2 x)^k \sec^{n-1} x \cdot \sec x \tan x\,dx = \int (\sec^2 x-1)^k \sec^{n-1} x\,d(\sec x)$$

Then Subsitute $u=\sec x \Rightarrow du=\tan x \sec x\,dx$

Theorem:

$$\int \tan x\,dx = \ln|\sec x| + c \qquad \int \sec x\,dx = \ln|\sec x+\tan x|+c$$

$$\text{Pf: } \int \sec x \cdot \frac{\sec x+\tan x}{\sec x+\tan x}\,dx = \int \frac{\sec^2 x+\sec x\tan x}{\sec x+\tan x}\,dx = \int \frac{1}{\sec x+\tan x}\cdot d(\sec x+\tan x) = \ln|\sec x+\tan x|+C.$$

(a). $\sin A\cos B = \frac{1}{2}[\sin(A-B)+\sin(A+B)]$

(b) $\sin A\cdot\sin B = \frac{1}{2}[\cos(A-B)-\cos(A+B)]$

(c) $\cos A\cdot\cos B = \frac{1}{2}[\cos(A-B)+\cos(A+B)]$

7.3 Theorem of Trigonometric Subsititutions

$$\begin{cases} \sqrt{a^2-x^2} & x=a\sin\theta & -\frac{\pi}{2}\le\theta\le\frac{\pi}{2} \\ \sqrt{a^2+x^2} & x=a\tan\theta & -\frac{\pi}{2}<\theta<\frac{\pi}{2} \\ \sqrt{x^2-a^2} & x=a\sec\theta & *\,0\le\theta<\frac{\pi}{2} \text{ or } \pi\le\theta<\frac{3\pi}{2} \end{cases}$$

7.4. Integration of Rational Function by Partial Fraction

$$\frac{2x+1}{x(x+1)} = \frac{1}{x}+\frac{1}{x+1} \text{ then integral}$$

NO

Date

Partial Fraction Technique: for the rational function $\frac{f(x)}{g(x)}$

where $\deg(f) < \deg(g)$ denominater 不可约. 二次 terms

1. Factor ~~deminator~~ into linear and irreducible quadratic
2. Express the rational function as a sum of ~~pa~~ partial fraction form $\frac{A}{(ax+b)^i}$ or $\frac{Ax+B}{ax^2+bx+c}$

where i takes all the value less than or equal to the power ~~of~~ of factor

3. Solve for unknown constant using algebra technique ~~for~~

Eg: Evaluate: $I=\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx = \ln|x^2+1| + \arctan(x) - 2\ln|x-1| - \frac{1}{x-1} + C$

$$I = \int \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}\,dx$$

$$-2x+4 = (Ax+B)(x-1)^2 + C(x^2+1)(x-1) + D(x^2+1)$$

$$-2x+4 = (Ax+B)(x^2-2x+1) + C(x^3-x^2+x-1) + D(x^2+1)$$

$$-2x+4 = (Ax^3-2Ax^2+Ax+Bx^2-2Bx+B) + C(x^3-x^2+x-1) + D(x^2+1)$$

$$\therefore -2x+4 = (A+C)x^3 + (B-2A-C+D)x^2 + (A+C-2B)x + \cancel{A+}B-C+D$$

$$\begin{cases} A+C=0 & \Rightarrow A=-C \\ B-2A-C+D=0 & \Rightarrow B-A+D=0 \\ A+C-2B=-2 & \cancel{A-C=4} \\ B-C+D=4 & -C+A=4 \end{cases} \Rightarrow \begin{cases} A=2 \\ B=+1 \\ C=-2 \\ D=\cancel{3}\,1 \end{cases}$$

$-2B=-2 \Rightarrow B=+1.$ $\quad 2A=4$

$A=2 \Rightarrow C=-2$

$+1-(-2)+D=4$

$D=\cancel{3}\,1$

$$\therefore I = \int \frac{2x+1}{x^2+1} \cancel{+} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\,dx = \int \frac{2x+1}{x^2+1}\,dx - 2\ln|x-1| - \frac{1}{x-1} + C.$$

$$\int \frac{2x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx = \ln|x^2+1| + \arctan(x)$$

$u=x^2+1 \Rightarrow du=2x\,dx$ $\quad \arctan x$

Remark: $\deg(f) \geq \deg(g)$. long division

Eg: Evaluate $I = \int \frac{x^5+5x^4+7x^3+x^2+x-2}{x^4+x^2}\,dx = \frac{5}{2}\ln|x^2+1| - 2\tan^{-1}(x) + \frac{2}{x} + \ln|x| + 5x + \frac{1}{2}x^2 + C.$

$$
\begin{array}{r}
x+5 \\
x^4+x^2 \overline{\smash{)}\, x^5+5x^4+7x^3+x^2+x-2} \\
\underline{x^5+x^3 } \\
5x^4+6x^3+x^2+x-2 \\
\underline{5x^4+5x^2 } \\
6x^3-4x^2+x-2
\end{array}
$$

$$\therefore I = \int (x+5) + \frac{6x^3-4x^2+x-2}{x^4+x^2}\,dx$$

$$= \left(\frac{1}{2}x^2+5x\right) + C + \int \frac{6x^3-4x^2+x-2}{x^2(x^2+1)}\,dx$$

$$\frac{6x^3-4x^2+x-2}{x^2(x^2+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x^2} + \frac{D}{x}$$

$$\cancel{6x^3-4x^2+x-2 = (Ax+B)x^2 + c(x^2+1) = Ax^3+(B+C)x^2+C}$$

$$6x^3-4x^2+x-2 = (Ax+B)x^2 + C(x^2+1) + Dx(x^2+1)$$

$$6x^3-4x^2+x-2 = Ax^3+Bx^2+Cx^2+C+Dx^3+Dx$$

$$6x^3-4x^2+x-2 = (A+D)x^3 + (B+C)x^2 + Dx + C$$

$$
\begin{cases}
A+D=6 & \Rightarrow A=5 \\
B+C=-4 & \Rightarrow B=-2 \\
D=1 & \\
C=-2 &
\end{cases}
\Rightarrow \int \frac{5x-2}{x^2+1} - \frac{2}{x^2} + \frac{1}{x}\,dx
$$

$$= \int \frac{5x-2}{x^2+1} + 2x^{-1} + \ln|x|$$

$$= \int \frac{5x}{x^2+1}\,dx - 2\arctan x + \frac{2}{x} + \ln|x|$$

$u = x^2+1 \quad du = 2x\,dx \quad \Rightarrow \frac{5}{2}du = 5x\,dx$

$$= \int \frac{1}{u}\cdot\frac{5}{2}\,du = \frac{5}{2}\ln|x^2+1|.$$

7.8 improper integral.

Type1. (a) If $\int_a^t f(x)dx$ exist for every number $t \geq a$ then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

(b) If $\int_t^b f(x)dx$ exists for every number $t \leq b$ then

$$\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx$$

(c). If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)dx$$

Theorem: $\int_1^\infty \frac{1}{x^p}dx$ is convergent if $p>1$ and divergent if $p \leq 1$

Proof:

Case 1: $p>1 \Rightarrow p-1>0$.

$$\int_1^\infty \frac{1}{x^p}dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^p}dx$$

$$= \lim_{t\to\infty} \left[\frac{1}{-p+1}\cdot x^{-p+1}\right]_1^t$$

$$= \lim_{t\to\infty} \left[\frac{1}{1-p}t^{1-p} - \frac{1}{1-p}\right]$$

$p-1>0 \Rightarrow 1-p<0$ $\quad t\to\infty$ $\quad \frac{1}{1-p}\cdot t^{1-p} \to 0$

$\therefore = \frac{1}{p-1}$. Convergent

case 2: $p \leq 1 \Rightarrow p-1 \leq 0$

$$\int_1^\infty \frac{1}{x^p}dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^p}dx$$

$$= \lim_{t\to\infty}\left[\frac{1}{1-p}t^{1-p} - \frac{1}{1-p}\right]$$

$$= \infty.$$

divergent.

Type 2:

(a) If f is continuous on $[a,b)$ and it is discoantinuous at b then

$$\int_a^b f(x)dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$

(b) If f is continuous on $(a,b]$ and it is discountinous at a then

$$\int_a^b f(x)dx = \lim_{t\to a^+}\int_t^b f(x)dx$$

NO

Date

(c) If f has discountinous at c, where $a<c<b$. and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Theorem: Comparison Theorem

Suppose that f and g are continuous function with $f(x) \geq g(x) \geq 0$ for $x \geq a$

(a) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is Convergent

(b) If $\int_a^\infty g(x)dx$ is divergent, then $\int_b^\infty g(x)dx$ is divergent.

Eg: $I = \int \sqrt{x^2-2x+2}\,dx$

$x^2-2x+1+1 = (x-1)^2+1$ $\qquad u = \tan\theta = x-1$ $\qquad \sec\theta = \sqrt{1+\tan^2\theta}$

$u = x-1 \Rightarrow du = dx$ $\qquad \cancel{x = \tan\theta}$ $\qquad = \sqrt{1+x^2-2x+1} = \sqrt{x^2-2x+2}$

$\therefore I = \int \sqrt{u^2+1}\,du$

$u = \tan\theta \Rightarrow I = \int \sqrt{\tan^2\theta+1}\,\sec^2\theta\,d\theta = \int |\sec\theta|\sec^2\theta\,d\theta$

$du = \sec^2\theta\,d\theta$ $\qquad = \int \sec^3\theta\,d\theta = \int (\tan^2\theta+1)\sec\theta\,d\theta$

$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$= \int \sec\theta \cdot \tan^2\theta\,d\theta + \int \sec\theta\,d\theta$

$\cancel{u = \sec\theta}$ $\qquad \cancel{u = \tan\theta}$ $\qquad u = \tan\theta$ $\qquad dv = \sec\theta\tan\theta\,d\theta$

$\cancel{dv = \tan^2\theta}$ $\qquad \cancel{du = \sec^2\theta\,d\theta}$ $\qquad du = \sec^2\theta\,d\theta$ $\qquad v = \sec\theta$

$$\int \sec^3\theta\,d\theta = \sec\theta\tan\theta - \int \sec^3\theta\,d\theta + \ln|\sec\theta+\tan\theta| + C$$

$$\therefore \int \sec^3\theta\,d\theta = \frac{\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta| + C}{2}$$

$$\therefore I = \frac{1}{2}\left[\sqrt{x^2-2x+2}\,(x-1) + \ln\left|\sqrt{x^2-2x+2}+x-1\right| + C\right]$$

unbounded improper integral.

Ex: $\int_1^2 \frac{1}{\sqrt{x-1}} dx$.

Step 1: value of x need special care with integral $x=1$.

Step 2: replace 1 by k $\int_k^2 \frac{1}{\sqrt{x-1}} dx = 2\sqrt{x-1}\Big|_k^2 = 2-2\sqrt{k-1}$.

Step 3: $\lim_{k\to 1^+} (2-2\sqrt{k-1}) = 2$

~~Fraction Work Review~~

Ex 2: $\int_0^\infty \frac{3}{5x+e^{2x}} dx$

$\frac{3}{5x+e^{2x}} < \begin{cases} \frac{3}{5x} & (p=1. \text{ still divergent}) \text{ pass.} \\ \frac{3}{e^{2x}} & (\text{may worth a good try}) \end{cases}$

$\int_0^\infty \frac{3}{e^{2x}} dx = \int_0^\infty 3e^{-2x} dx = 3\left(-\frac{1}{2}e^{-2x}\right)\Big|_0^\infty = -\frac{3}{2e^{2x}}\Big|_0^\infty$

$= -\frac{3}{2e^\infty} + \frac{3}{2e^0} = \frac{3}{2} \Rightarrow \int_0^\infty \frac{3}{e^{2x}} dx$ ~~is convergent~~

Therefore: $\int_0^\infty \frac{3}{5x+e^{3x}} dx$ ~~is convergent~~.

NO

Date

Fraction Webwork Review

$$Q_6: \quad I = \int \frac{9x^2-31x-8}{(x-3)(x^2-2x-3)}\,dx \longrightarrow F(x)$$

① Factor denominator $(x-3)(x^2-2x-3)$

$$= (x-3)(x-3)(x+1) = (x-3)^2(x+1)$$

② $$F(x) = \frac{A}{x+1} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$$

$$= \frac{A(x-3)^2 + B(x+1)(x-3) + C(x+1)}{(x+1)(x-3)^2}$$

numerator:

$$A(x^2-6x+9) + B(x^2-2x-3) + C(x+1)$$

$$= Ax^2 - 6Ax + 9A + Bx^2 - 2Bx - 3B + Cx + C$$

$$= (A+B)x^2 + (C-2B-6A)x + 9A - 3B + C = 9x^2 - 31x - 8$$

$$\begin{cases} A+B=9 & ① \\ C-2B-6A=-31 & ② \\ 9A-3B+C=-8 & ③ \end{cases}$$

③−② we have

$$9A-3B+\cancel{C}-\cancel{C}+2B+6A=-8+31$$

$$15A-B=23 \quad ④$$

$$B=9-A \quad ⑤$$

$$15A-(9-A)=16A-9=23 \Rightarrow A=\cancel{8}\,2$$

$$\cancel{C+14-96=-31}$$

$$\cancel{C=96-31-14=96-45=51}$$

$$\Rightarrow \begin{cases} A=\cancel{8}\,2 \\ B=+7 \\ C=\cancel{51}\,-5 \end{cases}$$

later just integral

NO

Date

Chapter 11

11.1 Sequence

Def: A sequence is a list of numbers in a definite order

$a_1 \to$ first term $\quad a_n \to$ n th term

$\{a_1, a_2, \cdots a_n\}$ is also denoted $\{a_n\}_{n=1}^{\infty}$

$a_n = f(n)$ whose domain $\{1, 2, 3, \cdots\}$

Def: A sequence $\{a_n\}$ has the limit L and we can write $\lim_{n\to\infty} a_n = L$

if $\lim_{n\to\infty} a_n$ exists, the sequence converges

otherwise, the sequence diverges

Theorem: If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.

Squeeze Theorem for sequence:

If $a_n \le b_n \le c_n$ for $n \geq n_0$ AND $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$

$\Rightarrow \lim_{n\to\infty} b_n = L$.

corallany $\Rightarrow$ If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$

Monotonic Sequence:

$a_n \Rightarrow \begin{cases} \text{increasing if } a_n < a_{n+1} & n \geq 1 \\ \text{decreasing if } a_n > a_{n+1} & n \geq 1. \end{cases}$

Theorem: $\begin{cases} \text{Increasing Sequence Converge} \iff \text{bounded above} \\ \text{Decreasing Sequence Converge} \iff \text{bounded below} \end{cases}$

11.2 Series

Def: We say $\sum_{i=1}^{\infty} a_i$ converges if its partial sum sequence converge

In this case, $S=\sum_{i=1}^{\infty} a_i$ is called the sum ~~of~~ of series

If the sequence is divergent, the series is also divergent

$$\sum_{n=0}^{\infty} ar^n = \begin{cases} \frac{a}{1-r} & \text{if } |r|<1 \\ \text{Divergent} & \text{if } |r|\geq 1 \end{cases}$$

Theorem: ① $\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow \lim_{n\to\infty} a_n = 0$

② $\lim_{n\to\infty} a_n = 0 \not\Longrightarrow \sum_{n=1}^{\infty} a_n$ converges or diverges

③ If $\lim_{n\to\infty} a_n$ doesn't exist OR $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges

$\lim_{n\to\infty} \frac{1}{n} = 0$ $\quad \sum_{n=1}^{\infty} \frac{1}{n}$ is ~~not~~ diverge.

①

Test for Series:

$\left\{\begin{array}{l} \cancel{a_n} \neq 0 \\ \infty \\ DNE \end{array}\right.$

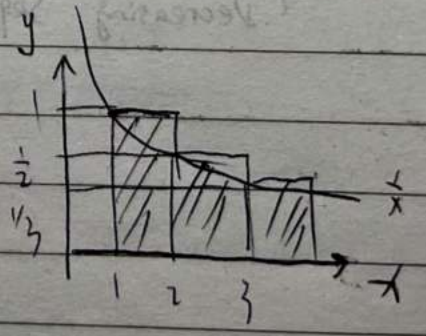
n-th term (Divergence) If $\lim_{n\to\infty} a_n$ ~~DNE or ≠ 0~~ $\Rightarrow \sum a_n$ DIV

②

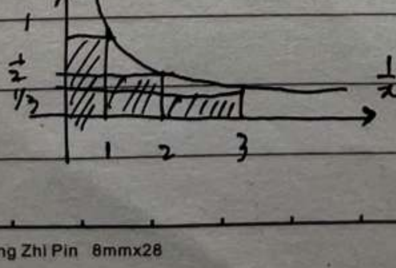
$$\sum_{n=1}^{\infty} \frac{1}{n^p} < \begin{array}{ll} \text{CON} & p>1 \\ \text{DIV} & p\leq 1 \end{array}$$

③

Harmonic Series



$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \geq \int_{1}^{n+1} \frac{1}{x}\,dx = \ln(n+1)$$



④

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Integral Test · Theorem:

☆ f is continuous, positive, decreasing. $a_n = f(n)$

$\int_1^{\infty} f(x)dx$ DIV $\Rightarrow \sum_{n=1}^{\infty} a_n$ DIV ; $\int_1^{\infty} f(x)dx$ CON $\Rightarrow \sum_{n=1}^{\infty} a_n$ CON

11.4 Comparison Test.

Theorem:

$\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ be series such that.

$0 \le a_i \le b_i$ for all $i \geq N$. where $N \geq 1$ is some integer

- If $\sum_{i=1}^{\infty} b_i$ Converge then $\sum_{i=1}^{\infty} a_i$ also Converge
- If $\sum_{i=1}^{\infty} a_i$ diverge then $\sum_{i=1}^{\infty} b_i$ also diverge.

☆ Limit Comparison test:

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c < \infty$, then $\sum a_n$ and $\sum b_n$ both CON or DIV.

☆ Direct Comparison.

11.5 Alternating Series $\sum (-1)^i a_i$

Theorem: Assume $a_i > 0$ & $\lim_{n\to\infty} a_i = 0$ & $a_i > a_{i+1}$

Then the alternating series CON.

Conditionally

Alternating Series Test

11.6

Theorem:

let $\sum_{n=1}^{\infty} a_n$ be a series, if the absolute value series $\sum_{n=1}^{\infty} |a_n|$ CON,

then $\sum_{n=1}^{\infty} a_n$ also CON.

Proof: Since $0 \le p_n = a_n - |a_n| \le 2|a_n|$

$0 \le q_n = |a_n| + a_n \le 2|a_n|$

Because Comparison Test. $\sum p_n$ & $\sum q_n$ both CON.

$\sum a_n = \frac{1}{2}(-\sum p_n + \sum q_n)$ also CON.

Definition:

Absolutely convergent. $\sum_{n=1}^{\infty} |a_n|$ CON

Conditionally Convergent $\sum_{n=1}^{\infty} a_n$ CON. but $\sum_{n=1}^{\infty} |a_n|$ DIV.

Theorem (Ratio Test)

Let $\sum_{i=1}^{\infty} a_i$ be a series with $a_i \neq 0$

Assume $\alpha = \lim_{i \to \infty} \left| \frac{a_{i+1}}{a_i} \right|$ exist (including $L = \infty$)

① if $L < 1$ the series is absolutely converge

② if $L > 1$ (including $L = \infty$) the series diverge

③ if $L = 1$, then the test is inconclusive.

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Proof:

(a) Assume $L<1$. Take r such that $L<r<1$. Then for some N, we have

$$\left|\frac{a_{i+1}}{a_i}\right| \leq r \Rightarrow |a_{i+1}| \leq r|a_i| \quad \text{for all } i \geq N.$$

This implies $|a_i| \leq |a_N| r^{i-N}$ for all $i \geq N$, since $0<r<1$ by the Comparison / geometric series Test. $\sum_{i=N}^{\infty} |a_i|$ converge, so does $\sum_{i=1}^{\infty} a_i$

(b) Assume $L>1$ Take M such that $L>M>1$. Then for some N, we have

$$\left|\frac{a_{i+1}}{a_i}\right| \geq M \Rightarrow |a_{i+1}| \geq M|a_i| \quad \text{for all } i \geq N$$

This implies $|a_i| \geq |a_N| M^{i-N}$ for all $i \geq N$, since $M>1$ by the Comparison / geometric Series Test $\sum_{i=N}^{\infty} |a_i|$ diverge, so does $\sum_{i=1}^{\infty} a_i$

11.8. Power Series.

A power series is an infinite series of the form.

$$\sum_{i=0}^{\infty} a_i (x-x_0)^i = a_0 + a_1(x-x_0) + \cdots +$$

where a_i and x_0 are constant and x is real variable

$$f(x) = 1+x+x^2+x^3+\cdots = \frac{1}{1-x} \quad (-1<x<1) = \sum_{i=0}^{\infty} x^i$$

(according geometric series)

Xin Feng Zhi Pin 8mmx28

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Theorem:

(a) If power series $\sum_{i=0}^{\infty} a_i (x-x_0)^i$ is convergent at some number $x_1 \neq x_0$
Then it is absolutely CON at all number x with $|x-x_0| < |x_1-x_0|$

(b) If power series $\sum_{i=0}^{\infty} a_i (x-x_0)^i$ is divergent at some number $x_2 \neq x_0$, then it is DIV at all number x with $|x-x_0| > |x_2-x_0|$.

Theorem (Radius of Convergence of power series)

Given a power series $\sum_{i=0}^{\infty} a_i (x-x_0)^i$

there is always a number R, $(0 \leq R \leq \infty)$ such that the series

$\sum_{i=0}^{\infty} a_i (x-x_0)^i$ is absolutely CON if $|x-x_0| < R$ ($x = x_0$ when $R=0$)

and is DIV if $|x-x_0| > R$ (no x when $R = \infty$)

This number R is called the radius of convergence of the power series

Note that the series could DIV/CON if $|x-x_0| = R$.

interval of convergence (x_0-R, x_0+R) $[x_0-R, x_0+R]$

open close

$\boxed{\text{write on First: } \frac{1}{1-r} = \sum_{i=0}^{\infty} r^i = 1+r+r^2+r^3+\cdots}$ ✬

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Date

11.9 Representation of Function as power series

Theorem: (Differentiation and integration of power series)

Let power series $\sum_{i=0}^{\infty} a_i (x-x_0)^i$ have a radius of convergence $R>0$. Then for all $|x-x_0|<R$, we have

- Term-by-term differentiation:

$$\frac{d}{dx}\sum_{i=0}^{\infty} a_i (x-x_0)^i = \sum_{i=1}^{\infty} i\, a_i (x-x_0)^{i-1}$$

- Term-by-term integration:

$$\int \sum a_i (x-x_0)^i dx = \sum_{i=0}^{\infty} \frac{a_i}{i+1}(x-x_0)^{i+1} + C$$

Also, the radius of convergence of the resulting power series are both equal to R.

Ex 1: Find power series of $y=\tan^{-1}(x+1)$

<Solution>: $\frac{d}{dx}\left(\tan^{-1}(x+1)\right) = \frac{1}{1+(x+1)^2} = \frac{1}{1-[-(x+1)^2]} \overset{r=-(x+1)^2}{=\!=\!=} \sum_{i=0}^{\infty}\left[-(x+1)^2\right]^i$

$$= \sum_{i=0}^{\infty} (-1)^i (x+1)^{2i}$$

Therefore:

$$f(x) = \tan^{-1}(x+1) = \int \frac{d}{dx}\left(\tan^{-1}(x+1)\right) dx = \int \sum_{i=0}^{\infty} (-1)^i (x+1)^{2i} dx$$

$$= \sum_{i=0}^{\infty} (-1)^i \frac{1}{2i+1}(x+1)^{2i+1} + C \quad \cancel{\text{put } x=0 \Rightarrow \frac{\pi}{4}=}$$

~~Ex 2: Find $\frac{1}{(1-x)^2}$ power series:~~

<Solution>: $\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(\sum_{i=0}^{\infty} x^i\right) = \sum_{i=1}^{\infty} i x^{i-1}$

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Date

Ex3: (Practice in class)

$$f(x) = \tan^{-1}\left(\frac{x}{3}\right)$$

<Solution>:

$$\frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{3}\right)\right) = \frac{1}{1+\left(\frac{x}{3}\right)^2}\cdot\frac{1}{3} = \frac{1}{1+\frac{x^2}{9}}\cdot\frac{1}{3} = \frac{1}{3}\frac{1}{1-\left(-\frac{x^2}{9}\right)} \overset{r=-\frac{x^2}{9}}{=\!=\!=} \frac{1}{3}\sum_{i=0}^{\infty}\left(-\frac{x^2}{9}\right)^i$$

$$= \frac{1}{3}\sum_{i=0}^{\infty}\frac{(-1)^i}{9}x^{2i}$$

$$f(x) = \tan^{-1}\left(\frac{x}{3}\right) = \int \frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{3}\right)\right)dx = \int \frac{1}{3}\sum_{i=0}^{\infty}\left(-\frac{1}{9}\right)^i x^{2i}\,dx = \frac{1}{3}\sum_{i=0}^{\infty}\frac{\left(-\frac{1}{9}\right)^i}{2i+1}x^{2i+1} + C$$

put $x=0 \Rightarrow C=0$ $\quad \therefore f(x) = \sum_{i=0}^{\infty}\frac{(-1)^i}{3^{2i+1}(2i+1)}x^{2i+1}$

11.10. Taylor and Maclaurin Series ← If $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n \qquad |x-a|<R$$

Special case: when $a=0$ it is Maclaurin Series:

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$

Note!! $f(x) = \sum_{i=0}^{\infty}\frac{f^{(i)}(a)}{i!}(x-a)^i$ and $f(x) \approx \sum_{i=0}^{n}\frac{f^{(i)}(a)}{i!}(x-a)^i = T_n(x)$

$T_n(x)$ is the n-th degree Talyor polynomial of f at a

Talyor Remainder Theorem:

If f is $(n+1)$-times differentiable on an open interval I containing a, then for each x in I

$$f(x) = T_n(x) + R_n(x)$$

where T_n is nth degree Taylor polynomial of f at a and where R_n, the Lagrange form of the remainder of order n is given by

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \text{for some } c \in (a,x)$$

$$f(x) = \underbrace{f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n}_{T_n(x)} + \underbrace{\frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} + \cdots}_{R_n(x)}$$

~~Starters~~

~~entrees~~

~~caesar sa~~lad

~~romaine, pancetta, parmesan. cheese,~~ anchovies.

~~Croutions, tossed with caesar~~.

Lagrange form of remainder of order

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} = \qquad c \in (a,x)$$