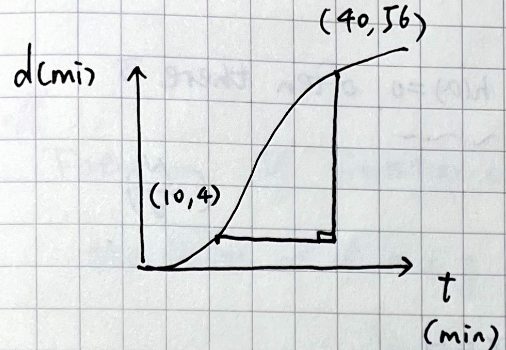
MTH 132. Michigan State University.

Chapter 1.

1.5

1. Motivation



Q_1:

From Lansing to Detroit.

Interval [10, 40] average velosity?

$$\frac{\text{How far}}{\text{How long}} = \frac{52\text{min}}{30\text{min}}$$

Q_2: At what time speed greatest?

Around min 25 our speed greatest

[take two consecutive points that were very close together]

2. ☆ Definition: The slope of the secant line of $y = f(x)$ through $P(x_1, f(x_1))$ $Q(x_2, f(x_2))$ is given by.

$$m = \frac{f(x_1) - f(x_2)}{x_1 - x_2}$$

① a function

② two specified point(s)

Alternatively, this is referred to as average rate of change

Interval $[x_1, x_2]$

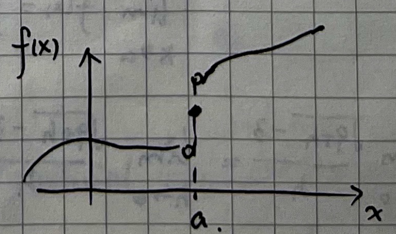
3. ☆ tagent line. (a secant line through two very close points)

4. limit: suppose f(x) is defined when x is near number a,

then we write $\lim\limits_{x \to a} f(x) = L$

Left-hand limit: $\lim\limits_{x \to a^-} f(x) = L$.

Right-hand limit: $\lim\limits_{x \to a^+} f(x) = L$



$\lim\limits_{x \to a^-} f(x) \neq \lim\limits_{x \to a^+} f(x)$ ☆

Theorem.

$\lim_{x\to a} f(x) = L$ if and only if ($\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$)

5 vertical asymptotes.

For function of form $\frac{g(x)}{h(x)}$ where $g(a) \neq 0$ and $h(a) = 0$, often there is vertical asymptotes

$\left(\frac{N}{0}\right)$

Eg: Find $y = \frac{x^2+2x+1}{x+x^2}$ vertical asymptotes.

let $x + x^2 = 0 \Rightarrow x = 0$ or -1. $x^2+2x+1 = 0 \Rightarrow x = -1$

1.6

So $x=0$ is what we want.

6 limit laws.

Theorem. $\lim_{x\to a} f(x)$ & $\lim_{x\to a} g(x)$ exist.

1. $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

2. $\lim_{x\to a} [cf(x)] = c \lim_{x\to a} f(x)$

3. $\lim_{x\to a} [f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot$

4. $\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided $\lim_{x\to a} g(x) \neq$

5. $\lim_{x\to a} c = c$

6. $\lim_{x\to a} x = a$

7. $\lim_{x\to a} x^n = a^n$

8. $\lim_{x\to a} [f(x)^n] = [\lim_{x\to a} f(x)]^n$

Theorem. If f is a polynomial, a rational function, or a root and a is in the domain of f then.

$$\lim_{x\to a} f(x) = f(a).$$

Eg: $\lim_{h\to 0} \frac{\sqrt{9+h}-3}{h} = \lim_{h\to 0} \frac{\sqrt{9+h}-3}{h} \cdot \frac{\sqrt{9+h}+3}{\sqrt{9+h}+3} = \lim_{h\to 0} \frac{9+h-9}{9(\sqrt{9+h}+3)} = \lim_{h\to}$

Eg.: Evaluate. $\lim_{x\to 0} \frac{\sin x}{x} = 1.$

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \cos x = 1.$$

☆ Theorem: If $f(x) \le g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$$

☆ (Squeeze Theorem). If $f(x) \le g(x) \le h(x)$ when x is near a (except possibly at a)

and. $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L.$ then

$\lim_{x\to a} g(x)$ exist. and $\lim_{x\to a} g(x) = L.$

Eg: $\lim_{x\to 0} [x^2\cos(\frac{2}{x})]$. use Squeeze Thm to find

$\because -1 \le \cos u \le 1 \Rightarrow -x^2 \le x^2 \cos u \le x^2 \Rightarrow -x^2 \le x^2\cos(\frac{2}{x}) \le x^2$

$\lim_{x\to 0}(-x^2) = 0$ $\qquad$ $\lim_{x\to 0} x^2 = 0$

So by the Squeez Thm we have $\lim_{x\to 0}[x^2\cos(\frac{2}{x})] = 0.$

1.8

☆ Definition: A function f is continuous at a number a if and only if $\lim_{x\to a} f(x) = f(a).$

① $f(a)$ to be defined

② $\lim_{x\to a} f(x)$ existed

☆ Theorem:

Essentially all the function we love (polynomial, rational, root, triginometric) are continuous everywhere in their domains.

☆ Hard to understand!

Theorem. If f and g are continuous at a & c is constant, then the following are also continuous at a.

1. $f+g$ 2. $f-g$. 3. cf. 4. $f \cdot g$. 5 f/g, if $g(a) \neq 0$

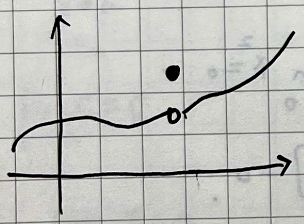
Theorm. if g is continuous at a and f is continuous at $g(a)$. then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

Eg: Determine $g(x) = \frac{\sqrt{1-x^2}}{x}$ is where is continuous?

Solve: continuous in its domain.

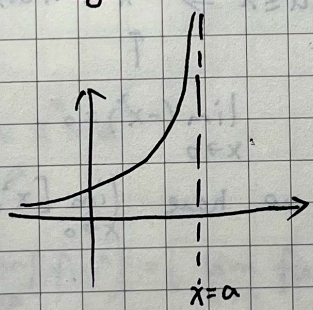
$$\begin{cases} x \neq 0 \\ 1-x^2 \geq 0 \Rightarrow -1 \leq x \leq 1 \end{cases} \Rightarrow [-1,0) \cup (0,1]$$

☆ ~~Some cases of discontinuity~~



removed.

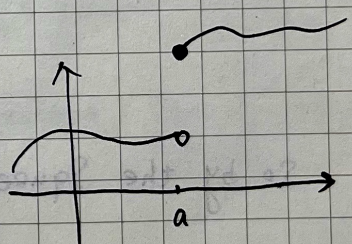
(0/0)



vertical asymptotes

(N/0)

infinite



jump

(piecewise)

Eg: $f(x) = \frac{x^2+5x+6}{x^2-3x-10}$ find discontinuities and classify.

$f(x) = \frac{(x+3)(x+2)}{(x-5)(x+2)}$ $x=5$ or $x=-2$.

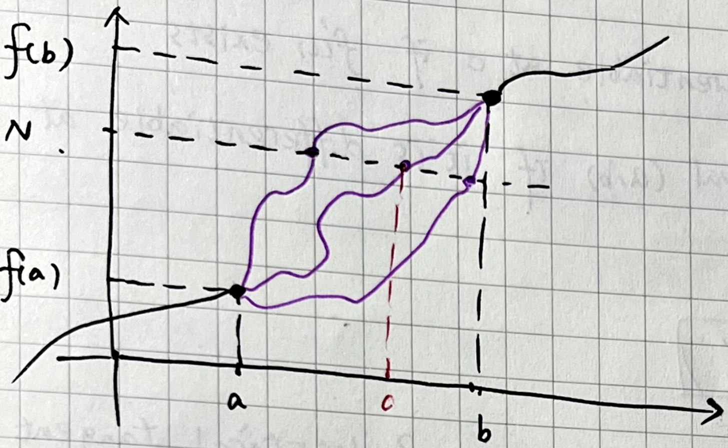
$x=5 \downarrow \frac{N}{0}$ vertical asymptotes

$x=-2 \downarrow \frac{0}{0}$ removed.

Intermediate Value Theorem (IVT)

Suppose that f is continuous on closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$.

Then there exist a number $c \in (a, b)$ such that $f(c) = N$.



Eg: Use IVT to show there is a root of function $f(x) = x^4 + x - 3$ on interval $(1,2)$.

Pf: ~~f~~ Since f is polynomial

$\therefore f$ continuous $(-\infty, +\infty) \rightarrow$ continuous $(1,2)$

$\because f(1) = -1 \quad f(2) = 15$. $\because -1 < 0 < 15$. by the IVT. $\Rightarrow$ there is $c \in (1,2)$ where $f(c) = 0$

Chapter 2.

2.1. Derivative.

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x).$$

2.2. Derivative function

notation (Leibniz). $\frac{df}{dx}$ or $\frac{d}{dx} f(x)$ or $f'(x)$.

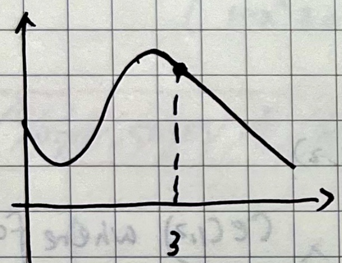
derivative $x = c$ is $\left.\frac{df}{dx}\right|_{x=c}$

Definition: A function f is differentiable at c if $f'(c)$ exists

It is differentiable on the interval (a,b) if it is differentiable at every number in (a,b)

Some cases of non-differentiable

1. Jagged Edge.

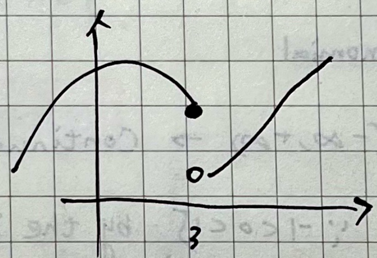


as $x \to 3^-$ slope $\to 0$

as $x \to 3^+$ slope $\to -1$.

left & right hand limit do not agree the limit will not exist

2. Not Continuous



$\lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$

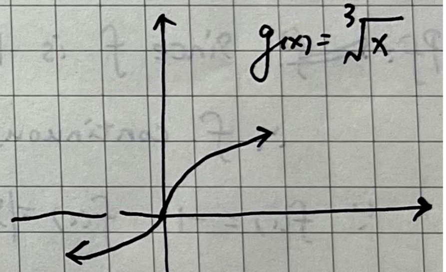
$\Downarrow$

$f(x) \to$ not continuous

$\Downarrow$

non-differentiable

3. vertical tangent



tangent line ne[ar 0] is vertical.

undefined !

Powers

$f(x)=c \quad f'(x)=0$

$f(x)=mx+b \quad f'(x)=m$

$f(x)=x^n \quad f'(x)=nx^{n-1}$

$$\begin{cases} \frac{1}{x^n}=x^{-n} \\ \sqrt[n]{x}=x^{\frac{1}{n}} \end{cases}$$

$$\frac{d}{dx}(cf(x)) = (cf(x))' = cf'(x).$$

$$\frac{d}{dx}(f(x) \pm g(x)) = (f(x) \pm g(x))' = f'(x) \pm g'(x).$$

$$\frac{d}{dx}(f(x)g(x)) = (f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

2.4. $\frac{d}{dx}(\sin x) = \cos x \quad (a)$

$\frac{d}{dx}(\cos x) = -\sin x \quad (b)$

$$\frac{d}{dx}(\sin x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}.$$

$$= \lim_{h \to 0} \sin x \left(\frac{\cos h - 1}{h}\right) + \cos x \frac{\sin h}{h}.$$

$$= \sin x \left[\lim_{h \to 0} \frac{\cos h - 1}{h}\right] + \cos x \left[\lim_{h \to 0} \frac{\sin h}{h}\right]$$

☆ ☆

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = \lim_{h \to 0} (-\sin h) = 0$$

$$\Rightarrow \frac{d}{dx}(\sin x) = \cos x$$

$$\lim_{h \to 0} \frac{\sin h}{h} = \lim_{h \to 0} \cos h = 1$$

Same pf of $\frac{d}{dx}(\cos x) = -\sin x$!

Put to memory:

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \boxed{\sec^2 x}$$

$$\frac{d}{dx}(\sec(x)) = \frac{d}{dx}\left(\frac{1}{\cos x}\right)$$

$$= \frac{0\cdot\cos x - 1\cdot(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{\tan x}{\cos x}$$

$$= \boxed{\tan x \cdot \sec x}$$

$$\frac{d}{dx}(\csc(x)) = \frac{d}{dx}\left(\frac{1}{\sin x}\right)$$

$$= \frac{0\cdot\sin x - 1\cdot\cos x}{\sin^2 x} = \frac{-\cos x}{\sin^2 x}$$

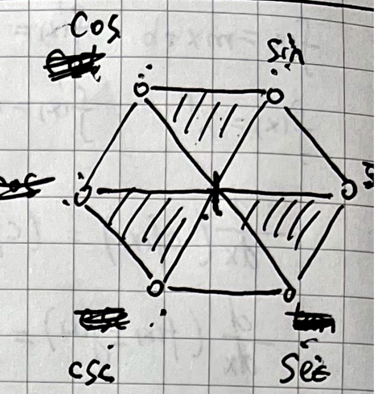
$$= \frac{-\cot x}{\sin x} = \boxed{-\cot x\cdot\csc x}$$

$$\frac{d}{dx}(\cot(x)) = \frac{d}{dx}\left(\frac{1}{\tan x}\right) = \frac{0\cdot\tan x - 1\cdot\sec^2 x}{(\tan x)^2}$$

$$= -\sec^2 x\cdot\cot^2 x = -\frac{1}{\cos^2 x}\cdot\frac{\cos^2 x}{\sin^2 x}$$

$$\cancel{=\sec} \quad = -\frac{1}{\sin^2 x} = \boxed{-\csc^2 x}$$

Tips:



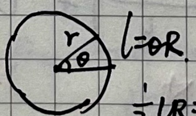
$$\begin{cases} \cos\cdot\sec = 1 \\ \sin\cdot\csc = 1 \\ \tan\cdot\cot = 1 \\ \sin^2 + \cos^2 = 1 \\ \cot^2 + 1 = \csc^2 \\ 1 + \tan^2 = \sec^2 \end{cases}$$

Theorem: $\lim_{x\to 0}\frac{\sin x}{x} = 1.\quad \lim_{x\to 0}\frac{\cos x - 1}{x} = 0$

Pf:

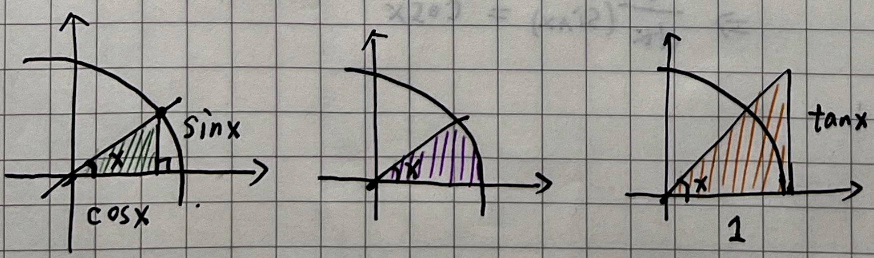
(1). Recall: Area of $\triangle$: $\frac{1}{2}bh$

Area of sector: $\frac{1}{2}\theta r^2$.



$l = \theta R$

$\frac{1}{2}lR = \frac{1}{2}\theta r^2$



Area:

▲ ≤ ▲ ≤ ▲

$$\frac{1}{2}\sin x\cos x \le \frac{1}{2}x \le \frac{1}{2}(1)\cdot\tan x$$

$$\cos x\cdot\sin x \le x \le \tan x$$

$$\cos x\cdot\sin x \le x \le \frac{\sin x}{\cos x}$$

$$\cos x \le \frac{x}{\sin x} \le \frac{1}{\cos x}$$

⇓

Squeeze theorem

$\lim_{x\to 0}\cos x = 1$

$\lim_{x\to 0}\frac{1}{\cos x} = 1$

$\Rightarrow \lim_{x\to 0}\frac{x}{\sin x} = 1$

$\therefore \lim_{x\to 0}\frac{\sin x}{x} = 1.$

Theorem: If $\lim_{x\to a} f(x)=0$.

$$\boxed{\lim_{x\to a}\frac{\sin(f(x))}{f(x)}=1 \quad \text{or.} \quad \lim_{x\to a}\frac{f(x)}{\sin(f(x))}=1}$$

Eg: Find $\lim_{x\to 0}\frac{\sin(5x)}{3x}. = \lim_{x\to 0}\frac{\sin(5x)}{5x}\cdot\frac{5}{3}=\frac{5}{3}$

Find. $\lim_{t\to 0}\frac{\tan(2t^2)}{4t} = \lim_{t\to 0}\frac{\sin(2t^2)}{\cos(2t^2)}\cdot\frac{1}{4t} = \lim_{t\to 0}\frac{\sin(2t^2)}{2t^2}\cdot\frac{t}{2}\cdot\frac{1}{\cos(2t^2)} = 1\cdot 0\cdot\frac{1}{1}=0$.

2.5. chain rule.

$[f(g(x))]' = f'_g\cdot g'_x$ let $g(x)=u$

$$\frac{dF}{dx}=\frac{dF}{du}\cdot\frac{du}{dx}$$

2.6. ☆ consider curve $x^3+y^3=6xy$.

(a) Find $\frac{dy}{dx}$?

$x^3+y^3=6xy$

$3x^2+3y^2\cdot y'=6y+6xy' \Rightarrow x^2+y^2\cdot\frac{dy}{dx}=2y+2x\frac{dy}{dx}$

$\Rightarrow (y^2-2x)\frac{dy}{dx}=2y-x^2$

$\therefore \frac{dy}{dx}=\frac{2y-x^2}{y^2-2x}$

(b). Find equation of tagent line at point $(3,3)$?

since $\left.\frac{dy}{dx}\right|_{\substack{x=3\\y=3}} = \frac{2\times 3-3^2}{3^2-2\times 3}=\frac{6-9}{9-6}=-1$.

so $y-3=-1\times(x-3) \Rightarrow y=-x+6$.

(c) At what point in first quadrant is the tangent line horizontal?

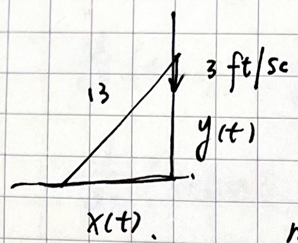
$\frac{dy}{dx}=0 \Rightarrow 2y=x^2 \Rightarrow y=x^2/2$.

$x^3+y^3=6xy$

$x^3+\frac{1}{8}x^6=6x\cdot\frac{1}{2}x^2=3x^3 \Rightarrow \frac{1}{8}x^6=2x^3 \Rightarrow x^3=16 \Rightarrow x=\sqrt[3]{16}$

$\approx 2.5198\ldots$

2.8. application 1.



A 13 foot ladder is leaning against a wall, and it is sliding down, with the top of ladder moving towards down along the wall 3 ft/s. How fast is the bottom of the ladder moving away from the wall when the bottom is 5 ft from the base of the wall?

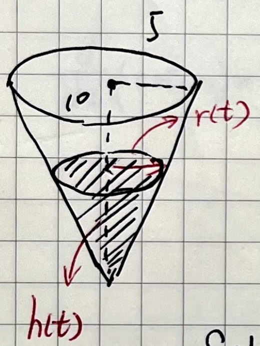
Solve: easy to have: $x^2(t) + y^2(t) = 13^2$

$\therefore\ 2x(t)x'(t) + 2y(t)y'(t) = 0 \Rightarrow x(t)x'(t) + y(t)(-3) = 0$

$\therefore\ x'(t) = \dfrac{3y(t)}{x(t)}.$

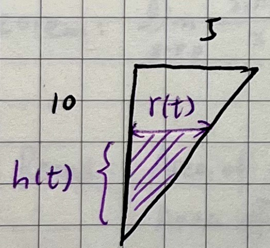
when $x(t) = 5$ $\quad y(t) = 12$ $\quad$ so $\boxed{x'(t) = \frac{36}{5}}$.

application 2.



A cone-shaped tank is filling with water at a constant rate of 9 ft³/min. The tank is 10 ft tall, and has a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?

Solve:



$\because\ \dfrac{r(t)}{h(t)} = \dfrac{5}{10} \Rightarrow 2r(t) = h(t).$

$\therefore\ V(t) = \frac{1}{3}h(t)\pi r(t)^2$

$V(t) = \frac{\pi}{3}h(t)\cdot\left(\frac{h(t)}{2}\right)^2 = \frac{\pi}{12}h(t)^3$

according problem: $V'(t) = 9.$

$V'(t) = \frac{\pi}{4}h(t)^2\cdot h'(t) = 9.$

$h(a)^2\cdot h'(a) = \frac{36}{\pi} \qquad h(a) = 6$

so $\boxed{h'(a) = \frac{1}{\pi}}.$

NO

Date

Chapter 3.

3.1 Application of differentiation

3.1. Absolute Mins & Maxs

Definition: Let c be a number in the domain D of a function f. Then $f(c)$ is the

- absolute (global) maximum value of f on D if $f(c) \geq f(x)$ for all x in D
- absolute (global) minimum value of f on D if $f(c) \leq f(x)$ for all x in D
- local maximum value of f on D if $f(c) \geq f(x)$ for all x near c
- local minimum value of f on D if $f(c) \leq f(x)$ for all x near c

Theorem: Extreme Value Theorem (EVT). If f is continuous on a closed interval $[a,b]$, then f attains an absolute maximum value and an absolute minimum in $[a,b]$.

☆ Definition: A critical number of a function f is a number c in the domain of f such that either $f'(c)=0$ or $f'(c)$ DNE.

3.2

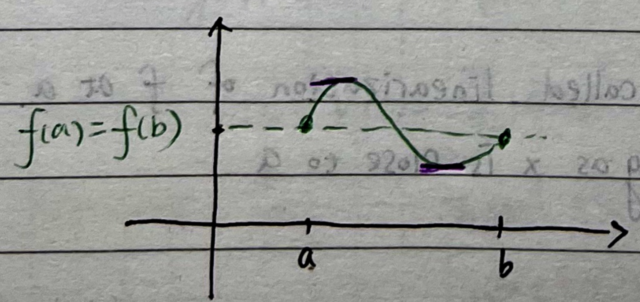
☆ (Rolle's Theorem). Let $f(x)$ be a function which satisfies the following three properties:

(1) $f(x)$ is continuous on the interval $[a,b]$.

(2) $f(x)$ is differentiable on (a,b)

(3) $f(a)=f(b)$

Then there is a number c in (a,b) such that $f'(c)=0$



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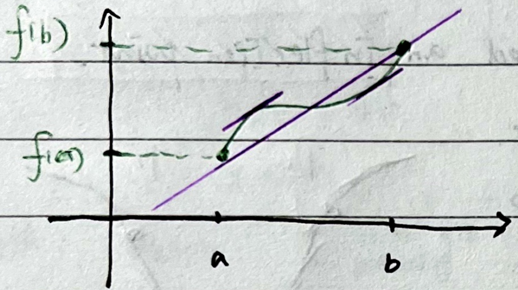
(Mean Value Theorem) (MVT). Let $f(x)$ be a function which satisfy the following two properties: (1) $f(x)$ is continuous on interval $[a,b]$

(2) $f(x)$ is differential on (a,b)

Then there is a number c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$



corallary 1: If $f'(x)=0$ for all x in interval (a,b). then $f(x)$ must be constant in (a,b)

Corallary 2: If $f'(x)=g'(x)$. for all x in interval (a,b). then $f(x)=g(x)+C$

3.3.

Theorem: if $f'(x)>0$ on (a,b), then $f(x)$ is increasing on (a,b)

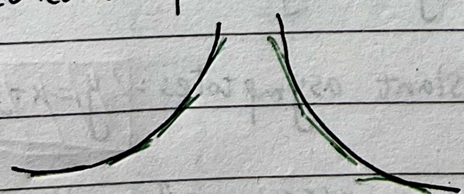
if $f'(x)<0$ on (a,b), then $f(x)$ is decreasing on (a,b)

Definition: f lies above all tangent line. $\leftrightarrow$ concave up.

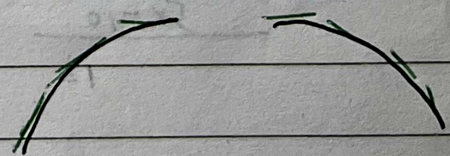
f lies down all tangent line $\leftrightarrow$ Concave down.

Picture:

concave up



concave down



Theorem: If $f''(x)>0$. for all x in I. then $f(x)$ concave up on I

If $f''(x)<0$ for all x in I. then $f(x)$ concave down on I.

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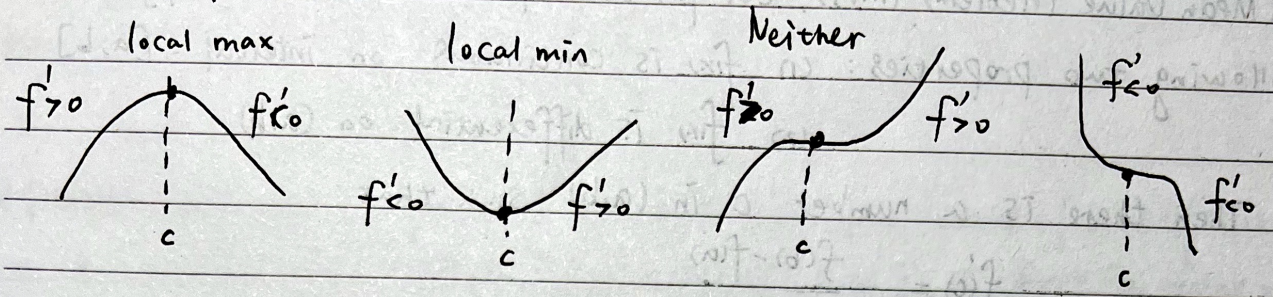
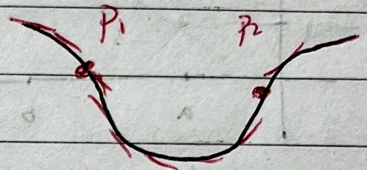
Date

Theorem: (Picture)



Definition: Point P on curve $y=f(x)$ is called an $\boxed{\text{inflection point}}$.
if f is continuous there and either.
(a) $f''>0 \to f''<0$ (b). $f''<0 \to f''>0$



3.4. horizontal asymptotes (Nothing to record)

3.5. Slant asymptotes

Definition: The function $y=f(x)$ has the slant asymptotes. $y=mx+b$ if
$\lim_{x\to\infty}[f(x)-(mx+b)]=0$ or if $\lim_{x\to-\infty}[f(x)-(mx+b)]=0$

Eg1: Find the slope of asymptotes of $y=\frac{x^2+3x+2}{x-2}$

Solution:

$$\begin{array}{r} x+5 \\ x-2\,\overline{)\,x^2+3x+2} \\ \underline{x^2-2x} \\ 5x+2 \\ \underline{5x-10} \\ 12 \end{array}$$

$\therefore y = x+5 + \frac{12}{x-2}$

$\lim_{x\to\infty}[y-(x+5)]=0$

$\therefore$ slant asymptotes: $\boxed{y=x+5}$

Eg2: Consider function $f(x)=\frac{x^2-2x+4}{x-2}$ $f'(x)=\frac{x(x-4)}{(x-2)^2}$ $f''(x)=\frac{8}{(x-2)^3}$.

(a). domain of $f(x)$
(b) Find x & y intercepts
(c) Vertical / horizontal / slant asymptotes
(d). Increasing / decreasing & classify critical points
(e). Concave up / down. inflection pt
(f) Sketch $f(x)$

Solution:

(a). $x \neq 2 \Rightarrow (-\infty, 2) \cup (2, +\infty)$

(b). y-int. $x=0$ x-int. $y=0$

$y = \frac{4}{-2} = -2$ $x^2 - 2x + 4 = 0$ does not exist.

$(0, -2)$ $\Delta = 4 - 4 \times 1 \times 4 = -12 < 0$

(c). VA: $y = \dfrac{x^2 - 2x + 4}{x - 2}$. $\left(\frac{N}{0}\right)$.

$$\begin{cases} x^2 - 2x + 4 \neq 0 \\ x - 2 = 0 \end{cases} \Rightarrow x = 2.$$

HA: none. $\left(\lim\limits_{x \to \infty} y = \lim\limits_{x \to \infty} (2x - 2) = \infty\right)$.

SA:

$$\begin{array}{r} x \\ x-2 \overline{\big) x^2 - 2x + 4.} \\ \underline{x^2 - 2x } \\ 4 \end{array}$$

$\therefore y = x + \frac{4}{x-2}$. $\lim\limits_{x \to \infty} \left(x + \frac{4}{x-2} - x\right) = 0$

$\therefore$ slant asymptotes: $y = x$.

(d). $f'(x) = \dfrac{x(x-4)}{(x-2)^2}$

$\nearrow \searrow \quad \searrow \nearrow$

f' $(+)$ $(-)$ $(-)$. $(+)$.

0   2   4

f is inc $(-\infty, 0) \cup (4, +\infty)$ $x=0$ local max

f is dec $(0, 2) \cup (2, 4)$. $x=4$ local min.

(e). $f''(x) = \dfrac{8}{(x-2)^3}$

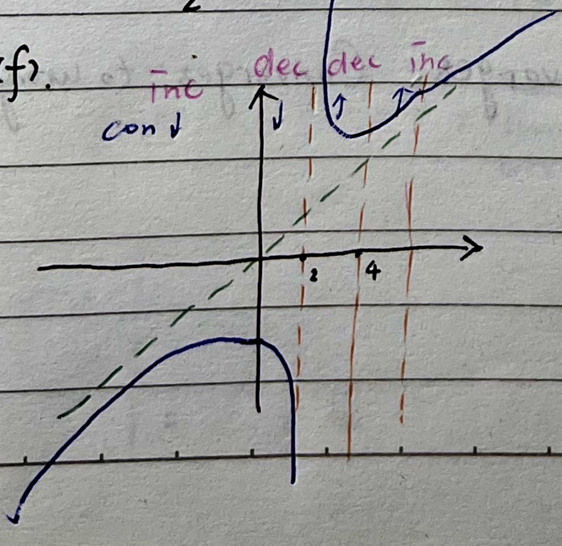
f is con $\downarrow$ $(-\infty, 2)$.

f'' $(-)$. $(+)$

2

f is con $\uparrow$ $(2, +\infty)$.

(f).



So final we got the picture.

NO

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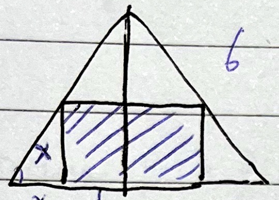
3.7. Optimize question

Eg: Find the max area of a rectangle inscribed in a equilateral traingle of side length 6 and one side of the rectangle lies along the base of triangle

Solution:



$\frac{x}{\sqrt{3}}$ $\quad$ $\cancel{2\sqrt{3}}$ 3

$3(\cancel{2\sqrt{3}} - \frac{x}{\sqrt{3}})$.

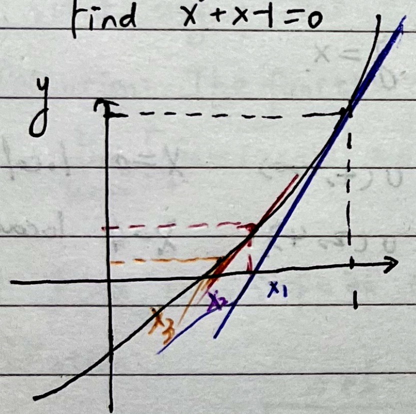
$S = x \cdot \left(\frac{6}{\cancel{2\sqrt{3}}} - \frac{2x}{\sqrt{3}}\right) = \frac{6}{\cancel{2\sqrt{3}}} x - \frac{2x^2}{\sqrt{3}}$

$\frac{dS}{dx} = -\frac{4}{\sqrt{3}} x + \frac{6}{\cancel{2\sqrt{3}}} \Rightarrow x = \frac{3\sqrt{3}}{2}$ $\quad x \in [0, 3\sqrt{3})$

~~$S = 12\sqrt{3} - \frac{2\times 9}{\sqrt{3}} = 6\sqrt{3}$~~ $\quad \therefore \boxed{S = \frac{9}{2}\sqrt{3}}$

3.8 Newton's Method

Eg: Find $x^5 + x + 1 = 0$



Algebraically becomes:

x_1.

$y - f(x_1) = f'(x_1)(x - x_1)$

$0 - f(x_1) = f'(x_1)(x - x)$

$\Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$y - f(x_2) = f'(x_2)(x - x_2)$

$0 - f(x_2) = f'(x_2)(x - x_2)$ $\quad$ ......

Theorem: If x_1 is the initial guess of some root of $f(x)$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \quad (n \geq 1)$$

anti-

Eg: Horizontal tangent line, Never Converges, Converges to wrong point

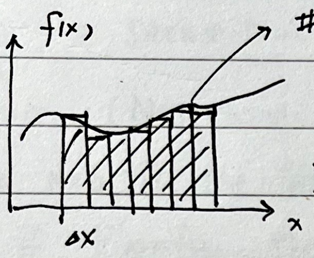
3.9. Antiderivative

Definition: A function F is called an antiderivative of f on an interval I if $F'(x)=f$ for all x in I

Theorem: If F is anti-derivative of f then $f=F'(x)+C$.

Chapter 4 & 5

4.1



Area $\approx f(x_0^*)\cdot\Delta x+f(x_1^*)\cdot\Delta x+\cdots+f(x_{n-1}^*)\Delta x$

Def: upper sum, lower sum, left sum, right sum
$\downarrow$ Choose max; $\downarrow$ Choose min; $\downarrow$ Choose left; $\downarrow$ choose right

Theorem: $\sum_{i=1}^{n} i = 1+2+\cdots+n=\frac{n(n+1)}{2}$

$$\sum_{i=1}^{n} i^2=1^2+2^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$$

4.2

Definition: If f is continuous on $[a,b]$ then

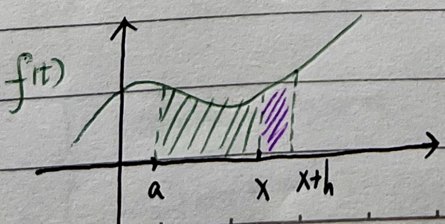
$$\boxed{\int_a^b f(x)dx=\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\cdot\Delta x}$$

4.3. Theorem: If f is continuous on $[a,b]$, then the function g defined by

$$g(x)=\int_a^x f(t)dt \qquad a\leq x\leq b$$

is continuous on $[a,b]$ and differentiable on (a,b) & $g'(x)=f(x)$

Pf: $g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=\lim_{h\to 0}\frac{\int_a^{x+h} f(t)dt-\int_a^x f(t)dt}{h}$



$$=\lim_{h\to 0}\frac{\int_x^{x+h} f(t)dt}{h}\approx\lim_{h\to 0}\frac{h\cdot f(x+h)}{h}=\lim_{h\to 0} f(x+h)$$

$= f(x)$. QED

NO

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Eg: Find derivative of $H(x)=\int_{-1}^{x^3} \sin(2t^3)\,dt$

<Sol>. let $u=x^3$.

$$H(u)=\int_{-1}^{u} \sin(2t^3)\,dt \Rightarrow \frac{dH}{du}=\sin(2u^3).$$

$$\frac{dH}{dx}=\frac{dH}{du}\cdot\frac{du}{dx}$$

$$=\sin(2u^3)\cdot 3x^2$$

$$=\sin\left(2(x^3)^3\right)\cdot 3x^2 = 3x^2\sin(2x^9)$$

Theorem: If f is continuous on $[a,b]$, then

$$\boxed{\int_a^b f(x)\,dx = F(b)-F(a)}$$

where F is any antiderivative of f. $(F'=f)$

Pf: Consider $F(u)=\int_a^u f(x)\,dx$ $F(u)$ is an antiderivative of $f(x)$. So the most general antiderivative of f is given by $F(u)+c$. (according to last theorem)

Consider $\int_a^u f(x)dx = F(u)+c$

$$0=\int_a^a f(x)\,dx = F(a)+c \Rightarrow c=-F(a)$$

$$\therefore \int_a^u f(x)\,dx = F(u)-F(a)$$

Theorem: $\int [f(x)+g(x)]\,dx = \int f(x)dx + \int g(x)dx$

and

$$\int kf(x)dx = k\int f(x)dx$$

☆ $\int f(x)\,dx = F(x) \Leftrightarrow F'(x)=f(x)$

Xin Feng Zhi Pin 8mmx28

NO

Date

Eg: (a) $F'(x) = \frac{1}{\sqrt{x}} + \cos(x)$.

$$\int \left(\frac{1}{\sqrt{x}} + \cos x\right) dx = \int \frac{1}{\sqrt{x}}\, dx + \int \cos x\, dx = 2 \cdot x^{\frac{1}{2}} + \sin x + c$$

(b) $F'(x) = \sec(x) \cdot (\tan(x) + \sec(x))$

$$\int \sec x\,(\tan x + \sec x)\, dx = \int (\sec x \tan x + \sec^2 x)\, dx$$

$$= \sec x + \tan x + c$$

Theorem: (Net change formula). The net change of a quantity $F(x)$ from $x=a$ to $x=b$ is the integral of its derivative from a to be. That is, if $F'(x) = f(x)$, then $\boxed{F(b) - F(a) = \int_a^b F'(x)\, dx = \int_a^b f(x)\, dx}$

In other words:

$$F(b) = F(a) + \int_a^b F'(x)\, dx = F(a) + \int_a^b f(x)\, dx$$

Theorem:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1)$$

$$\int \frac{1}{x} dx = \ln|x| + c$$

$$\int e^x dx = e^x + c$$

$$\int a^x dx = \frac{a^x}{\ln a} + c$$

$$\int \sin x\, dx = -\cos x + c$$

$$\int \cos x\, dx = \sin x + c$$

$$\int \sec^2 x\, dx = \tan x + c$$

$$\int \csc^2 x\, dx = -\cot x + c$$

$$\int \sec x \tan x\, dx = \sec x + c$$

$$\int \csc x \cot x\, dx = -\csc x + c$$

$$\int \frac{1}{x^2+1} dx = \arctan x + c$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$$

$$\int \sinh x\, dx = \cosh x + C$$

$$\int \cosh x\, dx = \sinh x + C$$

Date

5.1

Definition: The average value of a function $f(x)$ on interval $[a,b]$ is defined to be:

$$\boxed{f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx}$$

remark:

$$\frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{b-a}\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\cdot\frac{b-a}{n} = \lim_{n\to\infty}\sum_{i=1}^{n} y_i\cdot\frac{1}{n}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} y_i$$

$$= \lim_{n\to\infty}\frac{1}{n}(y_1+y_2+y_3+\dots+y_n)$$

Theorem: If $u=g(x)$ is a differentiable function whose range is I and $f(x)$ is continuous on I, then

$du=g'(x)dx$

$$\boxed{\int f(g(x))\cdot g'(x)\,dx = \int f(u)du}$$

Eg: $I=\int x^2\sqrt{x^3+1}\,dx$

Solution: let $u=x^3+1$

$du=3x^2\cdot dx$

So $I=\int x^2\sqrt{u}\,dx=\int\sqrt{u}\cdot\frac{1}{3}du$

$=\frac{1}{3}\int u^{1/2}du=\frac{1}{3}\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}=\frac{1}{3}\times\frac{u^{3/2}}{\frac{3}{2}}+c$

$=\frac{2u^{3/2}}{9}+c=\frac{2}{9}(x^3+1)^{3/2}+c$

Eg: $I=\int\frac{dt}{(1-3t)^5}$

Solution: let $\varphi=1-3t$

$d\varphi=-3\,dt$

So $I=\int\frac{1}{\varphi^5}\cdot-\frac{1}{3}d\varphi$

$=-\frac{1}{3}\int\varphi^{-5}d\varphi$

$=-\frac{1}{3}\times\frac{1}{-5+1}\times\varphi^{-5+1}+c$

$=\frac{1}{12}\times\varphi^{-4}+c=\frac{1}{12\varphi^4}+c$

$=\frac{1}{12(1-3t)^4}+c$

Theorem: If $g'(x)$ is continuous on $[a,b]$ and f is continuous on range of $u=g(x)$

then $$\boxed{\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du}$$

(1). **Eg:** $I=\int_0^1 \sqrt[3]{1+7x}\,dx$

Solution: let $u=1+7x$ $0\le x\le 1$

$0\le 7x\le 7$

$du=7\,dx$ $1\le 7x+1\le 8 \Rightarrow 1\le u\le 8.$

$1\le \sqrt[3]{7x+1}\le 2$

So $I=\int_1^8 u^{\frac{1}{3}}\cdot\frac{1}{7}\,du$

$=\frac{1}{7}\int_1^8 u^{\frac{1}{3}}\,du = \frac{1}{7}\times\frac{u^{\frac{1}{3}+1}}{\frac{1}{3}+1}\Big|_1^8 = \frac{1}{7}\times\frac{u^{\frac{4}{3}}}{\frac{4}{3}}\Big|_1^8$

$=\frac{1}{7}\times\frac{3}{4}\times u^{\frac{4}{3}}\Big|_1^8 = \frac{3}{28}\times\left[8^{\frac{4}{3}}-1^{\frac{4}{3}}\right] = \frac{3}{28}\times[16-1] = \frac{45}{28}$

(2). **Eg:** $I=\int_0^{\sqrt{\pi}} x\cos(x^2)\,dx$

Solution: let $u=x^2 \rightarrow du=2x\,dx \rightarrow x\,dx=\frac{1}{2}du$

$0\le x\le\sqrt{\pi} \rightarrow 0\le x^2\le\pi \rightarrow 0\le u\le\pi$

So $I=\int_0^{\pi}\cos u\cdot\frac{1}{2}du = \frac{1}{2}\int_0^{\pi}\cos u\,du = \frac{1}{2}\sin u\Big|_0^{\pi} = 0$

Theorem: $\begin{cases} f(x)=f(-x) \rightarrow \text{even} \\ f(x)+f(-x)=0 \rightarrow \text{odd.}\end{cases}$

(a) If f is even then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

(b) If f is odd then $\int_{-a}^{a} f(x)\,dx = 0$

Eg: $I=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2 x\sec^2 x\,dx$

Solution: $u=\tan x \rightarrow du=\sec^2 x\,dx$

$-\frac{\pi}{4}\le x\le\frac{\pi}{4} \Rightarrow -1\le\tan x\le 1 \Rightarrow -1\le u\le 1$

So $I=\int_{-1}^{1} u^2\,du = 2\int_0^1 u^2\,du = 2\times\frac{1}{3}u^3\Big|_0^1 = \frac{2}{3}.$

Eg: $I=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(1+x+x^2\tan x)\,dx$

Solution: $\varphi(x)=1+x+x^2\tan x$

$\varphi(-x)=1-x-x^2\tan x$ $\rightarrow$ $(x+x^2\tan x)$ is a odd function

$$\text{So } I=\int_{-\pi/4}^{\pi/4}1\cdot dx+\underbrace{\int_{-\pi/4}^{\pi/4}(x+x^2\tan x)\,dx}_{=0}=\frac{\pi}{2}.$$

Theorem: If $f(x)$ and $g(x)$ are continuous function on $[a,b]$ where $f(x)\geq g(x)$ for all x in $[a,b]$, then the area of the region ~~on~~ in between the graphs of $y=f(x)$ and $y=g(x)$ between $x=a$ and $x=b$ is given by:

$$\boxed{\cancel{\text{Area}=\int_a^b(f(x)-g(x))\,dx}}\rightarrow \text{general } \int_a^b|f(x)-g(x)|\,dx$$

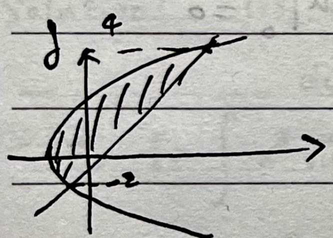
Eg: Find enclosed area by the line $y=x-1$ & parabola $y^2=2x+6$

Solution:

$$\begin{cases} y=x-1 \rightarrow x=y+1 \\ y^2=2x+6 \rightarrow x=\frac{1}{2}y^2-3 \end{cases} \Rightarrow \frac{1}{2}y^2-3=y+1 \Rightarrow (y-4)(y+2)=0$$

$$\frac{1}{2}y^2-y-4=0 \qquad y=4 \text{ or } y=-2$$

$$y^2-2y-8=0$$



$$\int_{-2}^{4}(\text{bigger } x)-(\text{smaller } x)\,dy$$

$$=\int_{-2}^{4}\left[(y+1)-\left(\frac{1}{2}y^2-3\right)\right]dy$$

$$=\left[\frac{1}{2}y^2+y-\frac{1}{6}y^3+3y\right]\Big|_{-2}^{4}$$

$$=\left[8+4-\frac{32}{3}+12\right]-\left[2-2+\frac{4}{3}-6\right]$$

$$=\left[24-\frac{32}{3}\right]+6-\frac{4}{3}=30-\frac{36}{3}=18.$$