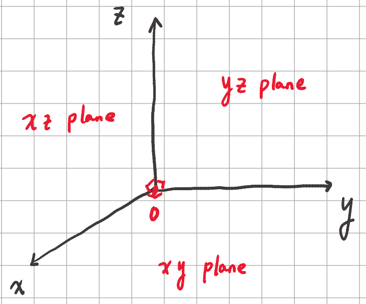
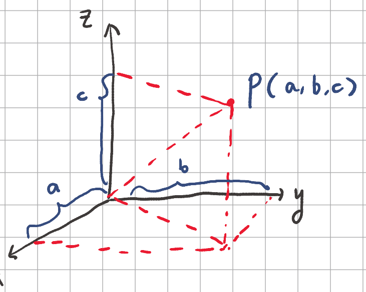
12.1 3D coordinate system



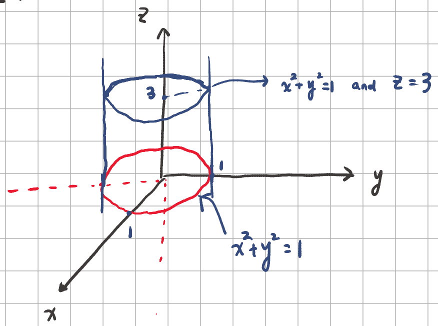
- fixed point O
  $(0,0,0)$
- axis are perpendicular (90°) to each other



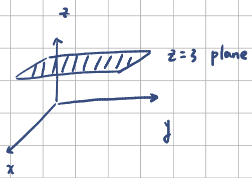
$\boxed{\text{Example}}$: 1 - surface in $\mathbb{R}^3$

Example 2 :

Draw $x^2+y^2=1$ and $z=3$



+



Definition: Distance formular between 2 points in 3D space

$P_1(x_1, y_1, z_1)$ $P_2(x_2, y_2, z_2)$ $\Rightarrow$ $|P_1P_2| = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$

Definition: Equation of a sphere

An equation of a sphere with center $C(h, k, l)$ and radius is r

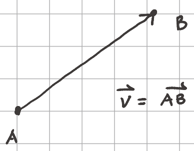
therefore $(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$ when center is $(0,0,0)$ and $r=1$

then equation is $x^2 + y^2 + z^2 = 1$ ← unit sphere

12.2. Vectors

Definition: Vector $(\vec{v})$

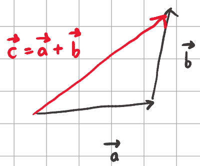
A vector is represented by an arrow or direct line segment



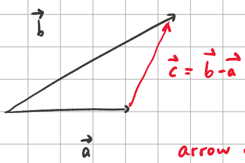
A – initial pt

B – terminal pt

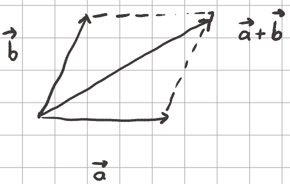
vector add:



vector minus



arrow of c is from front minus behind



Definition: Scalar multiplication

If c is a constant $\vec{v}$ is a vector $\Rightarrow$ $c\vec{v} = c \cdot$ length of $\vec{v}$ and whose

direction is $\begin{cases} \text{same } \vec{v} & c>0 \\ \text{opposite } \vec{v} & c<0 \end{cases}$

Definition:

Components

vector notation $\vec{v}$ where initial pt is the origin and terminal $\langle a_1, a_2, a_3 \rangle$

$$\Rightarrow \quad \vec{v} = \langle a_1, a_2, a_3 \rangle \qquad \text{position vector}$$

$$A(x_1, y_1, z_1) \quad B(x_2, y_2, z_2) \Rightarrow \overrightarrow{AB} = (x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1)$$

property of vector:

$\vec{0} = (0, 0, 0)$ $\qquad$ $\vec{a}, \vec{b}, \vec{c}$ vector $\qquad$ c, d scalar

① $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

② $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$

③ $\vec{a} + \vec{0} = \vec{a}$

④ $\vec{a} + (-\vec{a}) = 0$

⑤ $c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}$

⑥ $(c+d)\vec{a} = c\vec{a} + d\vec{a}$

⑦ $(cd)\vec{a} = c(d\vec{a})$

⑧ $1\vec{a} = \vec{a}$

Standard basis vector in $\mathbb{R}^3$

$\vec{i} = \langle 1, 0, 0 \rangle \qquad \vec{j} = \langle 0, 1, 0 \rangle \qquad \vec{k} = \langle 0, 0, 1 \rangle$

For any vector $\vec{m} = x\vec{i} + y\vec{j} + z\vec{k}$

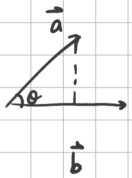
12.3 Dot product

$$\vec{a} = \langle a_1, a_2, a_3 \rangle \qquad \vec{b} = \langle b_1, b_2, b_3 \rangle$$

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Projection (投影)

向量 $\vec{a}$ 在向量 $\vec{b}$ 上的投影计算 记作 $\mathrm{proj}_b \vec{a}$



$$\mathrm{proj}_b \vec{a} = \left(\vec{a} \cdot \frac{\vec{b}}{|\vec{b}|}\right) \frac{\vec{b}}{|\vec{b}|}$$

☆ $\mathrm{orth}_{\vec{a}} \vec{b} = \vec{b} - \mathrm{proj}_{\vec{a}} \vec{b}$

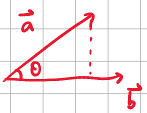
Component (分量)

向量 $\vec{a}$ 在向量 $\vec{b}$ 方向上的投影的大小 记作 $\mathrm{comp}_b \vec{a}$

$$\mathrm{comp}_b \vec{a} = \vec{a} \cdot \frac{\vec{b}}{|\vec{b}|}$$

补充:

$\vec{a}$ 在 $\vec{b}$ 上 $\mathrm{comp}_b \vec{a}$ (标量) 只看大小



$$|\vec{a}| \cos\theta = |\vec{a}| \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \vec{a} \cdot \frac{\vec{b}}{|\vec{b}|}$$

对于 $\mathrm{proj}_b \vec{a}$ 则考虑大小乘以单位向量 $\mathrm{proj}_b \vec{a} = \mathrm{comp}_b \vec{a} \, \frac{\vec{b}}{|\vec{b}|} = \vec{a} \cdot \frac{\vec{b}}{|\vec{b}|} \frac{\vec{b}}{|\vec{b}|}$

12.4 Cross Product

Determinant

① 2D

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

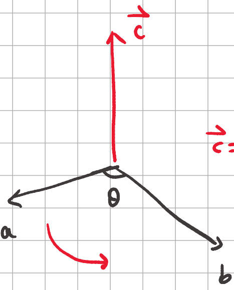
② 3D

$$\begin{vmatrix} \vec{a_1} & \vec{a_2} & \vec{a_3} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (b_2c_3 - b_3c_2)\vec{a_1} - (b_1c_3 - b_3c_1)\vec{a_2} + (b_1c_2 - b_2c_1)\vec{a_3}$$

Definition: Cross Product

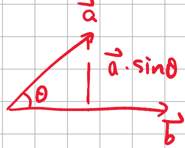
$\vec{a} = \langle a_1, a_2, a_3 \rangle$ $\vec{b} = \langle b_1, b_2, b_3 \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\vec{i} - (a_1b_3 - a_3b_1)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}$$



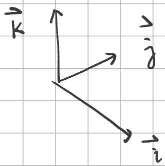
$\vec{c} = \vec{a} \times \vec{b}$

$|\vec{c}| = |\vec{a}||\vec{b}| \sin\theta$



$S = |\vec{a}||\vec{b}| \sin\theta$

Corollary : 2 non-zeor vector $\vec{a}$ & $\vec{b}$ are parallel if $\vec{a} \times \vec{b} = 0$



$$\begin{cases} \vec{i} \times \vec{j} = \vec{k} \\ \vec{j} \times \vec{k} = \vec{i} \\ \vec{k} \times \vec{i} = \vec{j} \end{cases}$$

Cross Product Theorem:

① $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

② $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$

③ $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

④ $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

⑤ $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$

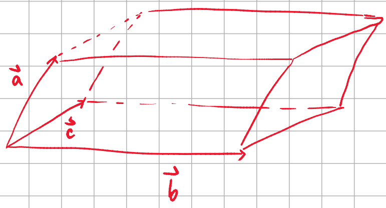
⑥ $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \cdot \vec{c}$

Triple Product

$\vec{a} = \langle a_1, a_2, a_3 \rangle$ $\vec{b} = \langle b_1, b_2, b_3 \rangle$ $\vec{c} = \langle c_1, c_2, c_3 \rangle$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

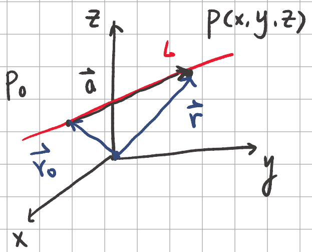
* Volumn of the parallelopiped created from vector $\vec{a}, \vec{b}, \vec{c}$, is tripple product $\vec{a} \cdot (\vec{b} \times \vec{c})$



$$\text{Volumn} = Ah = |\vec{b} \times \vec{c}| \cdot |\vec{a}| \cos\theta$$
$$= |\vec{a} \cdot (\vec{b} \times \vec{c})|$$

12.5 Equation of lines and Planes

Equation of a line-vector version



L - line

$\vec{v}$ - vector paralle to L

P_0 — point on L

P — orbitaring point on L

$\vec{r_0}$ — positive vector of P_0

$\vec{r}$ — positive vector

$\vec{a} = \overrightarrow{P_0P}$

$$\vec{r} = \vec{r_0} + \vec{a}$$

$$= \vec{r_0} + t\vec{v}$$

$\Rightarrow$ $\boxed{\vec{r} = \vec{r_0} + t\vec{v}}$ (t can be any number)

Equation of line

$t>0$ P lies on the positive side of P_0

$t<0$ P lies on the negative side of P_0

$t=0$ P lies exactly as P_0

$\vec{v} = \langle a, b, c \rangle$ $\vec{r} = \langle x, y, z \rangle$ $\vec{r_0} = \langle x_0, y_0, z_0 \rangle$

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tz \rangle \Rightarrow \begin{cases} x(t) = x_0 + ta \\ y(t) = y_0 + tb \\ z(t) = z_0 + tc \end{cases} \quad t \in \mathbb{R}$$

Convert it $\Rightarrow$ $t = \dfrac{x-x_0}{a} = \dfrac{y-y_0}{b} = \dfrac{z-z_0}{c}$

(symmetric equations of L)

Definition:

$\vec{r}(t) = \vec{r_0} + t\vec{v}$ and $\vec{S}(t) = \vec{S_0} + t\vec{\omega}$

$\vec{r}(t) \parallel \vec{s}(t)$ only if $\vec{v} \parallel \vec{\omega}$

For find intersection:

$\vec{r}\langle t\rangle = \langle 1, 0, 2\rangle + t\langle 3, 5, 6\rangle$

$\vec{s}(t) = \langle 3, 0, 6\rangle + s\langle 5, 7, 8\rangle$

$\vec{r}\langle t)$ intersect with $\vec{s}\langle t)$ only

$$\begin{cases} 3t+1 = 5s+3 \\ 5t = 7s \\ 6t+2 = 8s+6 \end{cases} \Rightarrow \text{no solution!}$$

$\frac{3}{5} \neq \frac{5}{7} \neq \frac{6}{8}$ no paralle!

Equation of plane:

If $\vec{n} = \langle a, b, c \rangle$ is a specified vector, $\vec{r_0} = \langle x_0, y_0, z_0 \rangle$ is the position vector from the point $P_0(x_0, y_0, z_0)$ and $\vec{r} = \langle x, y, z \rangle$

then

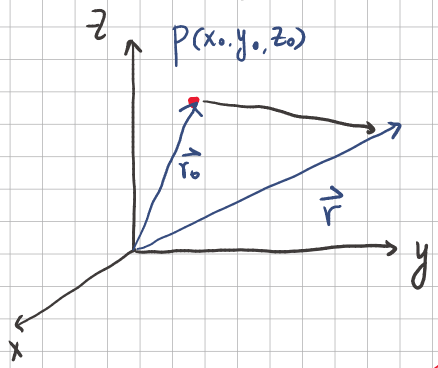
$$\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0$$

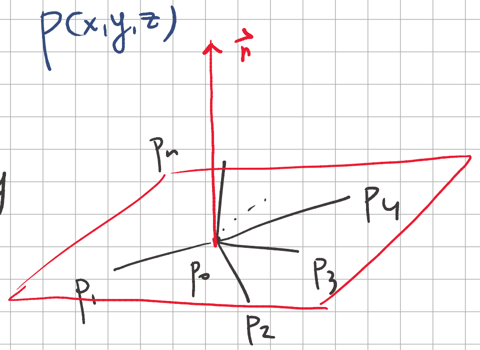
is the set of all vectors with the initial point $\vec{r_0}$ perpendicular to $\vec{n}$. More commonly this is the vector equation for the plane perpendicular to $\vec{n}$ through the point $P_0(x_0, y_0, z_0)$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$\Downarrow$$

$$ax + by + cz = d$$





12.6 Cylinder and Quadric Surfaces
(二次曲面)

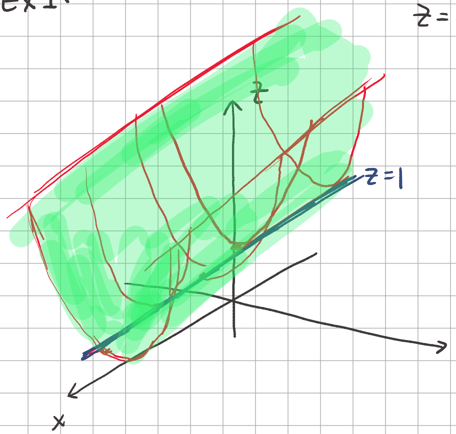
Definition:

(a) In order to sketch a graph of a surface, it is useful to determine the curves of intersection of the surface with planes paralle to the coordinate planes. These curves are called traces

(b) A cylinder is a surface that consists of all lines (called rulings) that are parallel to a given line and pass through a given plane curve.

Ex 1:

$z = y^2 + 1$

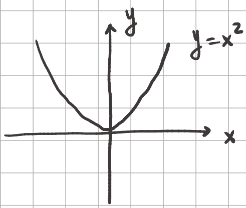


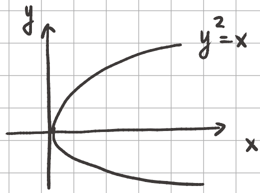
Definition: A quardric surface is a graph of a second-degree equation in three variables x, y, and z. The most general equation is:

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

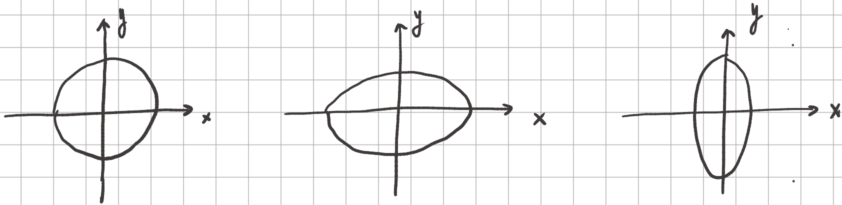
(we primarily work on $\begin{cases} Ax^2 + By^2 + Cz^2 + J = 0 \\ Ax^2 + By^2 + Iz = 0 \end{cases}$

Parabola: The basic form $y=x^2$ or $x=y^2$

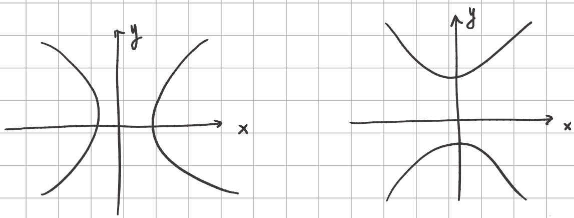




Ellipse: The basic form $x^2+y^2=1$



Hyperbola: The basic form $x^2-y^2=1$ or $y^2-x^2=1$

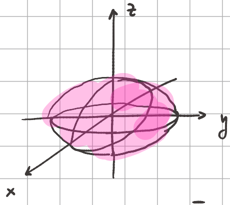


☆ If only two variable, then it is a cylinder

Ellipsoid

椭圆体

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

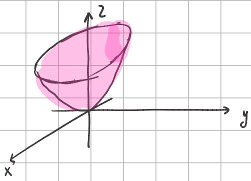


3平方=常

Elliptical Paraboloid

椭圆抛物面

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c}$$

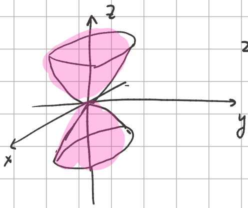


2平方=一次

Elliptical Cone

椭圆锥

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$$

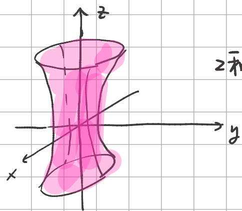


2平方=1方

Hyperboloid of one sheet

单双曲

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

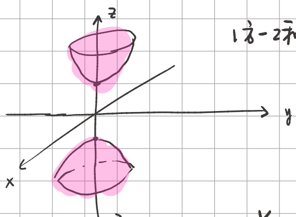


2平方−1方=常

Hyperbolic of two sheets

双双曲

$$\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$



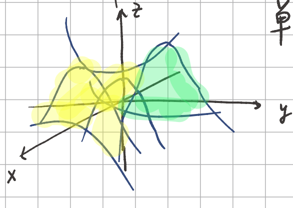
1方−2平方=常

Hyperbolic Paraboloid

双曲抛物面

$$\frac{y^2}{b^2}-\frac{x^2}{a^2}=\frac{z}{c}$$

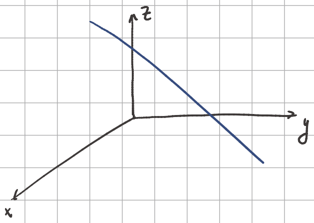
$(c>0)$



单方−单方=一次

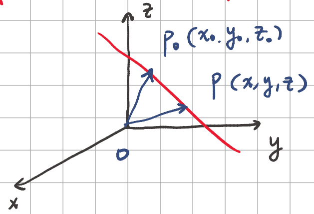
13.1 Vector Functions and Space Curves

Definition: In general, a function is a rule that assigns to each element in the domain an element in the range. A vector-valued function, or vector function is simply a function whose domain is a set of real number and whose range is a set of vectors (primarily in $\mathbb{R}^3$)



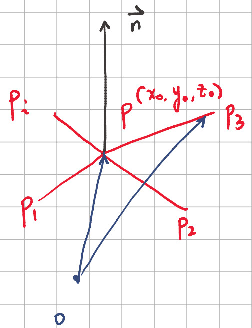
Ex: line $\vec{r}(t) = \langle t, 2t+1, 3-t \rangle$

☆ Recall From chapter 12 the line function and plane function



$\vec{op} = \vec{op_0} + t\vec{a}$ line function

$= (x_0, y_0, z_0) + t\vec{a}$



$\vec{PP_i} = \vec{op_i} - \vec{op} = \vec{r} - \vec{r_0}$

$\vec{PP_i}$ can cover a whole plane

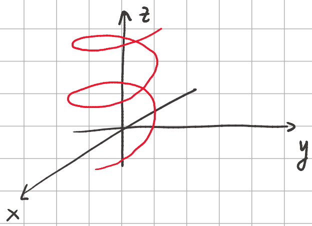
$\vec{PP_i} \cdot \vec{n} = 0 \Rightarrow \vec{n} \cdot (\vec{r} - \vec{r_0}) = 0$ plane function

$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

helix $\vec{r}(t) = \langle 2\cos t, 2\sin t, t \rangle$

螺旋



Definition: If $\vec{r}(t)$ is a vector function (whose vector are in $\mathbb{R}^3$) then

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$$

where f, g and h are real-valued function called component functions of $\vec{r}$

Definition: If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ then $\lim_{t \to a} \vec{r}(t) = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle$

$$\lim_{t \to a} \vec{r}(t) = \vec{r}(a) \iff \vec{r} \text{ is continuous}$$

Definition: If f, g and h are continuous real-valued functions on an interval $I \in R$

then the set C of all point (x, y, z) in space where

$$x = f(t) \qquad y = g(t) \qquad z = h(t)$$

and t varies throughout the interval I is called a space curve.

We can also define planar curves by removing the z component.

Ex: Find a vector function that represent the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + z = 5$

Solution $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow r = 1 \Rightarrow \begin{cases} x = \cos\theta \\ y = \sin\theta \\ z = 5 - \cos\theta \end{cases} \Rightarrow \vec{r}(t) = (\cos t, \sin t, 5 - \cos t)$

$t \in [0, 2\pi]$

$\vec{r} = \langle f(t), g(t), h(t) \rangle$

tangent vector : $\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$

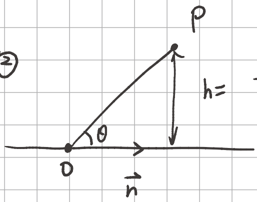
tangent unit vector : $\vec{T}(t) = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$

Integration $\displaystyle\int_a^b \vec{r}(t)\,dt = \langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \rangle$

① $ax+by+cz=d_1$

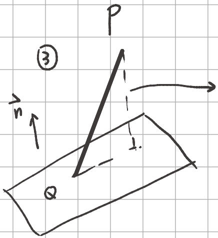
$ax+by+cz=d_2$ $\Rightarrow$ distance $= \dfrac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$.

②



$h = \dfrac{|\vec{OP}\times\vec{n}|}{|\vec{n}|}$

③

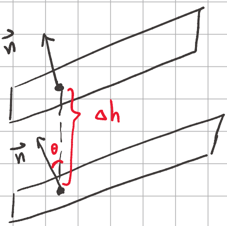


$d = \dfrac{|\vec{PQ}\cdot\vec{n}|}{|\vec{n}|}$

$PQ\cdot\cos\theta = PQ \cdot \dfrac{|\vec{PQ}\cdot\vec{n}|}{|\vec{PQ}||\vec{n}|}$

$= \dfrac{|\vec{PQ}\cdot\vec{n}|}{|\vec{n}|}$

Proof ①



$ax+by+cz=d_1$

$ax+by+c\left(z-\frac{d_1}{c}\right)=0 \Rightarrow \left(0,0,\frac{d_1}{c}\right)$

Similar $ax+by+c\left(z-\frac{d_2}{c}\right)=0 \Rightarrow \left(0,0,\frac{d_2}{c}\right)$

$\therefore \Delta h = \left|\dfrac{d_1-d_2}{c}\right|$ $\Delta h\cdot\cos\theta = \dfrac{|d_1-d_2|}{|c|}\cdot\dfrac{|c|}{\sqrt{a^2+b^2+c^2}} = \dfrac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$

$\vec{n}=(a,b,c)$ $\vec{m}=(0,0,1)$

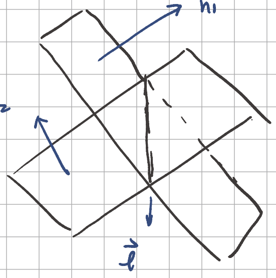
$\cos\theta = \dfrac{\vec{n}\cdot\vec{m}}{|\vec{n}||\vec{m}|} = \dfrac{c}{\sqrt{a^2+b^2+c^2}}$

④ The way to figure out to plane intersect line

method 1:

$a_1x+b_1y+c_1z=0$

$a_2x+b_2y+c_2z=0$



$\vec{n_1}=\langle a_1, b_1, c_1\rangle$

$\vec{n_2}=\langle a_2, b_2, c_2\rangle$

direction of $\vec{l} = \vec{n_1}\times\vec{n_2}$

method 2:

$$\begin{cases} a_1x+b_1y+c_1z=0 \\ a_2x+b_2y+c_2z=0 \end{cases} \Rightarrow \text{Solve } y, z \text{ by } x$$

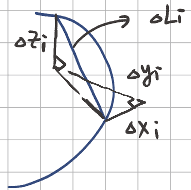
replace x by t

then $\vec{r}\langle t\rangle = \langle t, mt+b, pt+q\rangle = \langle 0, b, q\rangle + t\underbrace{\langle 0, m, p\rangle}_{\text{line direction}}$

Arc length of curve

$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle \quad a\le t\le b$

$$l = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$$



Appendix:

For $\vec{T}$ the unit tangent to a smooth curve, the curvature function is given by

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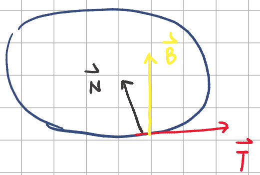
$$\kappa = \left| \frac{d\vec{T}}{dt} \right| \qquad r = \frac{1}{\kappa}$$



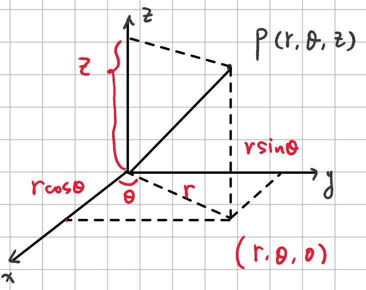
tangent vector $\vec{T} = \frac{d\vec{r}}{dt}$ 速度

normal vector $\vec{N} = \frac{d\vec{T}}{dt}$ 向心加速度

Binormal vector $\vec{B} = \vec{T} \times \vec{N}$

13.2 Cylinder & Spherical coordinate system

Cylinder coordinate (r, θ, z)



$\boxed{\text{Def}^n}$ In the cylindrical coordinate system, a point in the space represented by (r, θ, z) when (r, θ) are polar coordinate in x-y plane, while z is the usual z coordinate

Converting between cylinder and Rectangula coordinate

① To convert from cylindrical (r, θ, z) to rectangula (x, y, z)

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

② To convert from rectangula to cylindrical coordinate

$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{cases}$$

range of $\tan^{-1}(\theta) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Example Describe the surface $z = 2 - 4r^2$
$\underbrace{}_{\text{rectangala format}}$

Since $r^2 = x^2 + y^2$

$$z = 2 - 4(x^2+y^2)$$

$$z = 2 - 4x^2 - 4y^2 \quad [\text{eg : Elliptical paraboloid (shift)}]$$

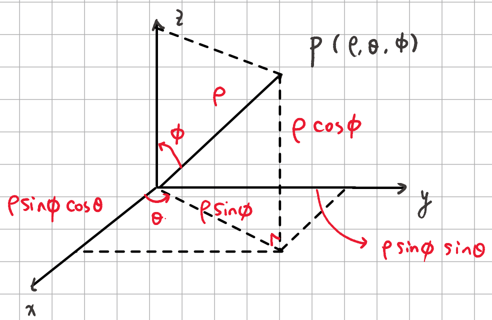
Example $z = y^2 - x^2$ rewrite in cylindrical coordinate

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$z = r^2\sin^2\theta - r^2\cos^2\theta \qquad z = r^2(\sin^2\theta - \cos^2\theta)$$

$$\therefore \quad z = -r^2(\cos^2\theta - \sin^2\theta) \qquad \therefore \; z = -r^2\cos 2\theta$$

Spherical coordinate system (ρ, θ, ϕ)



$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi$

① Spherical to rectangala

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$$

$\cos\phi = \dfrac{z}{\rho} \qquad \sin\theta = \dfrac{y}{\rho\sin\phi}$

② Rectangala to spherical

$$\rho = \sqrt{x^2+y^2+z^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\phi = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$$

$\boxed{\text{Example}}$

Describe $\rho = 4\cos\theta \sin\phi$

$$\rho^2 = 4\rho\underbrace{\sin\phi\cos\theta}_{x}$$

$$x^2+y^2+z^2 = 4x \Rightarrow x^2-4x+4+y^2+z^2=4$$

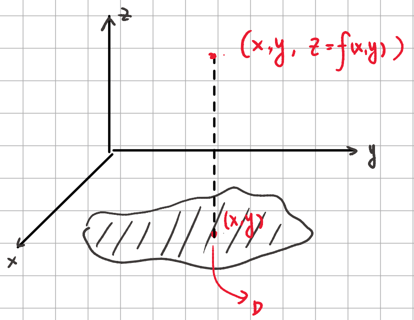
$\therefore (x-2)^2+y^2+z^2=4$ eq: sphere center $(2.0.0)$ and $r=2$

14.1 Function of several variables

Function of two variables $z=f(x.y)$

$\boxed{\text{Def}^n}$ $\underbrace{(x.y)\in \text{set } D}_{\text{input}} \xrightarrow{\text{Output}} f(x.y)$ or $z=f(x.y)$

The graph of f is all points (x,y,z) that satisfy $(x,y)\in D$ and $z=f(x,y)$



$\boxed{\text{Example}}$ Evaluate $f(3,2)$ and find and sketch the arrow diagram

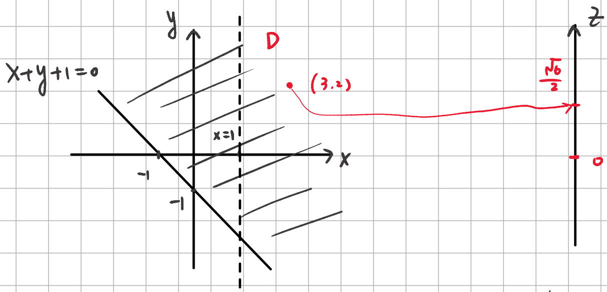
(a) $f(x,y) = \dfrac{\sqrt{x+y+1}}{x-1}$ (b) $f(x,y) = x\ln(y^2-x)$

Sol : (a) Domain : $x+y+1 \geq 0$ and $x \neq 1$

$$\Rightarrow D = \{(x,y) \in \mathbb{R}^2 \mid x+y+1 \geq 0 \text{ and } x \neq 1\}$$

$$f(3,2) = \frac{\sqrt{3+2+1}}{3-1} = \frac{\sqrt{6}}{2}$$

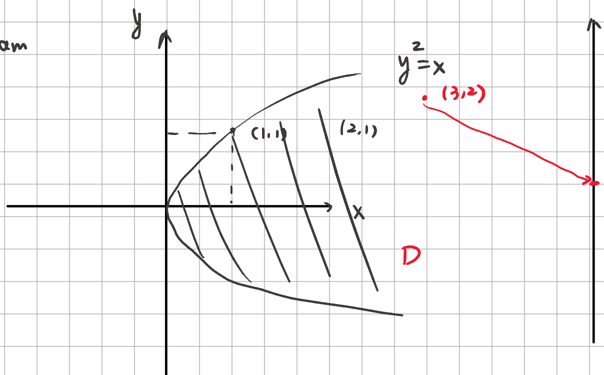
arrow diagram



(b) $y^2 \geq x \Rightarrow D = \{(x,y) \in \mathbb{R}^2 \mid y^2 > x\}$

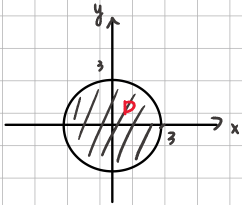
$$f(3,2) = 3\ln(4-3) = 0$$

arrow diagram



Example Find range and domain of

$$g(x,y)=\sqrt{\underbrace{9-x^2-y^2}_{\geq 0}}$$



$$\text{Domain} = D = \{(x,y) \mid x^2+y^2 \leq 9\}$$

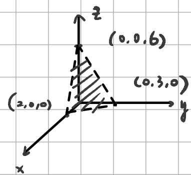
$$\text{Range} = \{z \mid z=\sqrt{9-x^2-y^2},\ (x,y)\in D\}$$

graphs

Def^n The graph of f is all points (x,y,z) that satisfies $(x,y) \in D$ and $z=f(x,y)$

Example Sketch a graph of a function $f(x,y)=6-3x-2y$

$$z=6-3x-2y$$



Take $x=0 \quad y=0 \Rightarrow z=6 \Rightarrow (0,0,6)$

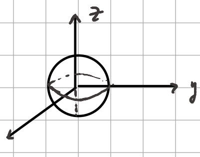
$x=0 \quad z=0 \Rightarrow y=3 \Rightarrow (0,3,0)$

$y=0 \quad z=0 \Rightarrow x=2 \Rightarrow (2,0,0)$

Example Sketch the graph $g(x,y)=\sqrt{9-x^2-y^2}$

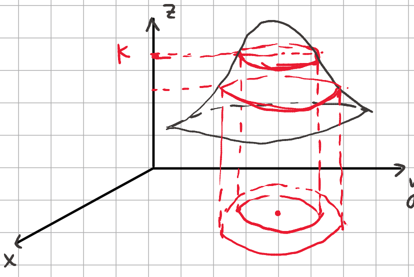
$$z=\sqrt{9-x^2-y^2} \Rightarrow x^2+y^2+z^2=9 \Rightarrow r=3$$

sphere



Level Curves

Def^n The level curve of a function f of two variable are the curves with eq^n $f(x,y)=k$ where k is a constant (in the range of f)



Example Sketch the level curves of $f(x,y)=6-3x-2y$

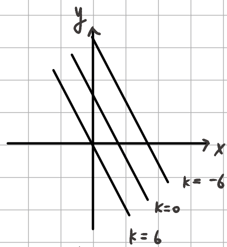
$$f(x,y)=6-3x-2y=k$$

$$\therefore\ y=-\frac{3}{2}x+\frac{6-k}{2}$$

$k=0 \Rightarrow y=-\frac{3}{2}x+3$

$k=-6 \Rightarrow y=-\frac{3}{2}x+6$

$k=6 \Rightarrow y=-\frac{3}{2}x$

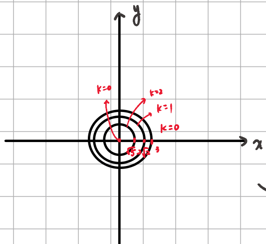


level curves

Example level curve for

$g(x,y)=\sqrt{9-x^2-y^2}$ for $k=0,1,2,3$

$g(x,y)=\sqrt{9-x^2-y^2}=k \Rightarrow x^2+y^2=9-k^2 \quad r=\sqrt{9-k^2}$



level curve

14.3 Partial derivatives

$\boxed{\text{Def}}$ Partial derivative of a function $f(x,y)$ with respect to x is the function f_x (with respect to function f_y) defined by

$$f_x(x,y) = f_x = \frac{\partial f(x,y)}{\partial x} = \frac{\partial f}{\partial x} = D_x f$$

$$f_y(x,y) = f_y = \frac{\partial f(x,y)}{\partial y} = \frac{\partial f}{\partial y} = D_y f$$

$\boxed{\text{Example}}$ $f(x,y) = x^3 + x^2y^3 - 2y^2$

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(2,1)} = 3x^2 + y^3 \cdot 2x \Big|_{(x,y)=(2,1)} = 3\times4 + 1\times2\times2 = 16$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x,y)=(2,1)} = x^2 \cdot 3y^2 - 4y \Big|_{(x,y)=(2,1)} = 4\times3 - 4 = 8$$

$\boxed{\text{Example}}$ $f(x,y) = \sin\left(\frac{x}{1+y}\right)$

<sol> :

$f_x = \cos\left(\frac{x}{1+y}\right) \cdot \frac{1}{1+y}$ Consider y as constant

$f_y = \cos\left(\frac{x}{1+y}\right) \cdot x \cdot \frac{-1}{(1+y)^2}$ Consider x as constant

Second Partial Derivative

$\boxed{\text{Def}^n}$ $f_{xx} = (f_x)_x$ $f_{xy} = (f_x)_y$

$f_{yy} = (f_y)_y$ $f_{yx} = (f_y)_x$

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x\, \partial x}$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\, \partial x}$$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y\, \partial y}$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\, \partial y}$$

$\boxed{\text{Example}}$

$$f(x,y) = x^3 + x^2y^3 - 2y^2$$

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(3x^2 + y^3 \cdot 2x\right) = 6x + 2y^3$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(3x^2 + y^3 \cdot 2x\right) = 2x \cdot 3y^2 = 6xy^2$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(x^2 \cdot 3y^2 - 4y\right) = 3y^2 \cdot 2x = 6y^2x$$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(x^2 \cdot 3y^2 - 4y\right) = x^2 \cdot 6y - 4 = 6x^2y - 4$$

14.4 Tagent Planes and Linear Approximation

$\boxed{\text{Def}^{n}}$: The tangent plane to the surface $z=f(x,y)$ at the point $P(x_0, y_0, z_0)$ is defined to be the plane that contains both tangent lines to C_1 and C_2 the curves of intersection between $z=f(x,y)$ and $x=x_0$ and $y=y_0$ respectively

Theorem: Suppose f has continuous partial derivative. An equation of the tangent plane to the surface $z=f(x,y)$ at the point $P(x_0, y_0, z_0)$ is given by

$$z-z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

Proof: For plane through $P(x_0, y_0, z_0)$

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$$

$$\frac{A}{C}(x-x_0) + \frac{B}{C}(y-y_0) + (z-z_0) = 0 \qquad \left(\text{define } -a = \frac{A}{C} \quad -b = \frac{B}{C}\right)$$

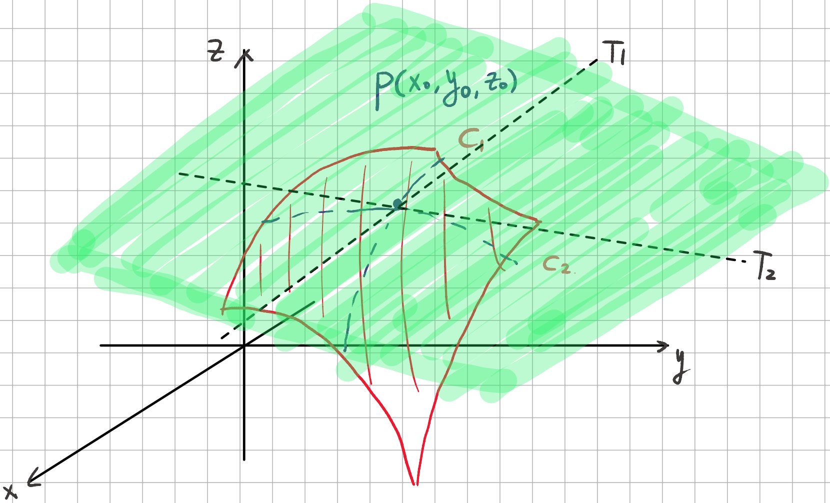
$$\therefore \quad z-z_0 = a(x-x_0) + b(y-y_0)$$

Setting $x=x_0 \Rightarrow \quad z-z_0 = b(y-y_0)$

For $f_y(x_0,y_0) = \dfrac{z-z_0}{y-y_0} \Rightarrow \quad b = f_y(x_0,y_0)$

Same For $f_x(x_0,y_0) = \dfrac{z-z_0}{x-x_0} \Rightarrow \quad a = f_x(x_0,y_0)$

Therefore: $z-z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$



Suppose a surface $S = f(x,y)$ f has continuous partial derivative

$P \equiv (x_0, y_0, z_0)$ $\begin{cases} C_1 - \text{curve determined by } y = y_0 \text{ cross } S \\ C_2 - \text{curve determined by } x = x_0 \text{ cross } S \end{cases}$

T_1 — tangent line of C_1 at P ; T_2 Tangent line of C_2 at P

(*) Tangent plane to the surface S at P is the plane that contains all possible tangent line at P to covers that lies on S.

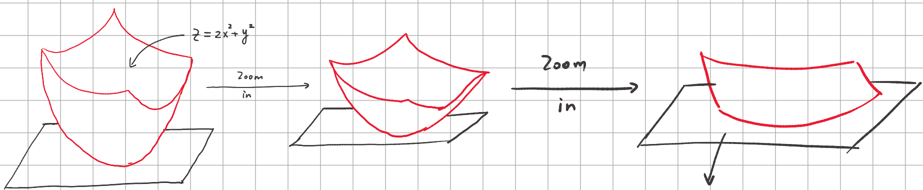
$$0 = f_x (x - x_0) + f_y (y - y_0) + f_z (z - z_0)$$

$\boxed{\text{Def}^n}$ (a) The linear function whose graph is the tangent plane is called the linearization of f at (a,b) and is given by

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

(b) As we stated above this approximates the function near (a,b)

That is $L(x,y) \approx f(x,y)$ so the linearization is sometimes referred to as the linear approximation or the tangent plane approximation of f at (a,b)



$$z = z_0 + f_x(x-x_0) + f_y(y-y_0)$$

$\downarrow$

$\mathcal{L}(x,y)$

$$\mathcal{L}(x,y) = z_0 + f_x(x-x_0) + f_y(y-y_0)$$

Then the function $\mathcal{L}(x,y)$

is called linearization of f at P

$$f(x,y) \approx f(x_0,y_0) + f_x(x-x_0) + f_y(y-y_0)$$

is called linear approximation or tangent plane approximation of f at (x_0,y_0)

16.6 Parameter surface

1D $\vec{r}(t) = \langle t, t^2, t^3 + t^4 \rangle$

2D $\vec{r}(u,v) = \langle u+v, u^2, u+3v \rangle$

$\boxed{\text{Def}^{n}}$: For a vector function $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

defined on a region D in a uv plane, the parameter surface is

given by $x = x(u,v)$ $y = y(u,v)$ $z = z(u,v)$

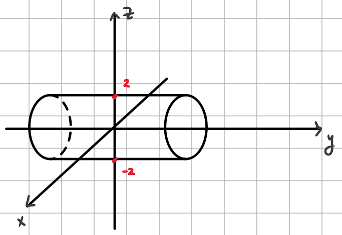
$\boxed{\text{Example}}$

Sketch $\vec{r}(u,v) = 2\cos u\,\vec{i} + v\vec{j} + 2\sin u\,\vec{k}$

$\vec{r}(u,v) = \langle 2\cos u, v, 2\sin u \rangle$

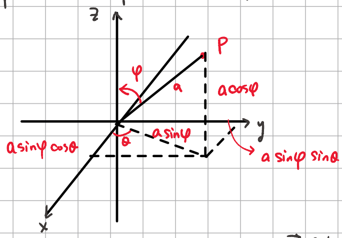
parameter eq^{n} : $\begin{cases} x = 2\cos u \\ y = v \\ z = 2\sin u \end{cases} \Rightarrow \begin{cases} x^2 + z^2 = 4 \\ y = v. \end{cases}$

Since $y = v$. $x^2 + z^2 = 4$, and there has no restriction for v. $\Rightarrow$ cylinder



$\boxed{\text{Example}}$

Find the parametric representation of sphere $x^2 + y^2 + z^2 = a^2$



$\begin{cases} x = a\sin\varphi\cos\theta \\ y = a\sin\varphi\sin\theta \\ z = a\cos\varphi \end{cases}$

$\vec{r}(\phi, \theta) = \langle a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi \rangle$

Tangent plane :

✱ Tangent plane to surface S traced out

$\vec{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v) \rangle$ at point $P_0(u_0, v_0)$ with position

vector $\vec{r}(u_0, v_0)$

✱ tips: For $\vec{r}(t)$, $\vec{r}(s)$ traced

surface S. just $\vec{r}'(t)$, $\vec{r}'(s)$

$\vec{n} = \vec{r}'(t) \times \vec{r}'(s)$

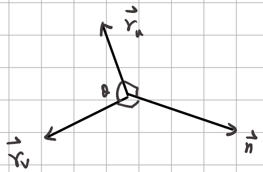
① Find $\vec{r}_u$ & $\vec{r}_v$ at (u_0, v_0)

$$\vec{r}_u = \left\langle \frac{\partial x}{\partial u}(u_0, v_0),\ \frac{\partial y}{\partial u}(u_0, v_0),\ \frac{\partial z}{\partial u}(u_0, v_0) \right\rangle$$

$$\vec{r}_v = \left\langle \frac{\partial x}{\partial v}(u_0, v_0),\ \frac{\partial y}{\partial v}(u_0, v_0),\ \frac{\partial z}{\partial v}(u_0, v_0) \right\rangle$$

② Calculate $\vec{r}_u \times \vec{r}_v$ This is the normal vector to the tangent line

③ Find the equation of the plane



$\boxed{\text{Example}}$ Consider $z = x^2 + y^2$ $\quad z \le 16$

(a) Find the parametric equation

(b) Find tangent plane at point $(1, 2, 5)$

<Solution> (a) $\vec{r}(u,v) = \langle u, v, u^2 + v^2 \rangle \quad u^2 + v^2 \le 16$

(b) $\vec{r}_u = \langle 1, 0, 2u \rangle \Big|_{\substack{u=1 \\ v=2}} = \vec{r}_u = \langle 1, 0, 2 \rangle$

$\vec{r}_v = \langle 0, 1, 2v \rangle \Big|_{\substack{u=1 \\ v=2}} = \vec{r}_v = \langle 0, 1, 4 \rangle$ $\qquad \therefore -2(x-1) - 4(y-2) + z - 5 = 0$

$$\vec{n} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{vmatrix} = (0-2)i - (4-0)j + (1-0)k = \langle -2, -4, 1 \rangle$$

14.5 Multivariable Chain Rules

$\boxed{\text{Theorem}}$: Suppose $z=f(x,y)$ is differentiable with $x=x(t)$ $y=y(t)$

both differentiable, then

$$\frac{\partial f}{\partial t}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}.$$

$\boxed{\text{Example}}$

$$z=x^2+y^2+xy \qquad x=\sin t \qquad y=e^{-t} \qquad \frac{dz}{dt}=?$$

<Solution> : $\dfrac{dz}{dt}=\dfrac{\partial z}{\partial x}\dfrac{dx}{dt}+\dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$

$$=(2x+y)\cos t-(2y+x)e^{-t}=\left(2\sin t+e^{-t}\right)\cos t-\left(2e^{-t}+\sin t\right)e^{-t}$$

⊛ If $F(x,y)=0 \Rightarrow \dfrac{\partial F}{\partial x}\cdot\dfrac{dx}{dx}+\dfrac{\partial F}{\partial y}\cdot\dfrac{dy}{dx}=0$

$$\Rightarrow \frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{dy}{dx}=0 \Rightarrow F_x+F_y\frac{dy}{dx}=0$$

$$\therefore \frac{dy}{dx}=-\frac{F_x}{F_y}$$

The chain rule

Suppose that $z=f(x,y)$ is a differentiable $x=g(s,t)$ $y=h(s,t)$

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

14.6 Directional Derivative and Gradient Vector

Directional Derivative

$\boxed{\text{Def}^n}$ $D_{\vec{u}} f(x_0, y_0)$

The directional derivative of $f(x,y)$ at (x_0, y_0) in the direction of a unit

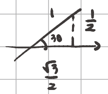
vector $\vec{u} = \langle a, b \rangle$, $D_{\vec{u}} f(x_0,y_0) = \lim_{x \to 0} \dfrac{f(x_0+ha,\ y_0+hb) - f(x_0,y_0)}{h}$ if limit exists

☆ $D_{\vec{u}} f(x,y) = f_x(x,y)\, a + f_y(x,y)\, b$

$\boxed{\text{Example}}$ Find the directional derivative $D_{\vec{u}} f(x,y)$ if $f(x,y) = x^3 - 3xy + y^2$.

and unit vector $\vec{u}$ is given by angle $\theta = \pi/6$. Find $D_{\vec{u}} f(1,2)$

<sol> :



$\therefore \vec{u} = \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$ $\quad f_x = 3x^2 - 3y$ $\quad f_y = -3x + 2y$

$\therefore D_{\vec{u}} f(x,y) = \frac{\sqrt{3}}{2}(3x^2 - 3y) + \frac{1}{2}(-3x + 2y)$

$= \frac{3\sqrt{3}}{2}x^2 - \frac{3\sqrt{3}}{2}y + \frac{-3}{2}x + y = \frac{3\sqrt{3}}{2}x^2 + \frac{-3\sqrt{3}+2}{2}y - \frac{3}{2}x \Big|_{(1,2)} = \frac{13 - 3\sqrt{3}}{2}$

Gradient vector

$\boxed{\text{Def}^n}$ Gradient of $f = \nabla f$

① If f is a function of 2 variables (x,y)

☆ $\nabla f(x,y) = \langle f_x(x,y),\ f_y(x,y) \rangle$

② If f is a function of 3 variables (x,y,z)

☆ $\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle$

Example

$$f(x,y) = \sin x + e^{xy} \qquad \nabla f(0,1)$$

$$\nabla f(x,y) = \langle \cos x + ye^{xy},\ xe^{xy} \rangle \qquad \nabla f(0,1) = \langle 2, 0 \rangle$$

Notice!

☆ $D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u} = \langle f_x, f_y \rangle \cdot \langle a, b \rangle = f_x a + f_y b$

Directional vector = gradient of f • the vector

Example $f(x,y,z) = x \sin yz$ (a) Find ∇f (b) Find $D_{\vec{u}} f$ at $(1,3,0)$ in the direction $\vec{v} = \vec{i} + 2\vec{j} - \vec{k}$

<Sol>:

(a) $\frac{\partial f}{\partial x} = \sin yz \qquad \frac{\partial f}{\partial y} = x \cdot z \cos yz \qquad \frac{\partial f}{\partial z} = xy \cos yz$

$$\therefore \nabla f(x,y,z) = \langle \sin yz,\ xz \cos yz,\ xy \cos yz \rangle$$

(b) $\nabla f(1,3,0) = \langle 0, 0, 3 \rangle \qquad |\vec{v}| = \sqrt{1+4+1} = \sqrt{6} \qquad \therefore \vec{u} = \langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}} \rangle$

☆ We can only use the unit vector

$$D_{\vec{u}} f = \frac{-3}{\sqrt{6}} = \frac{-3}{6} \cdot \sqrt{6} = -\frac{\sqrt{6}}{2}$$

Maximizing the directional derivative

$\boxed{\text{Theorem}}$: Suppose f is a differentiable function of 2 or 3 variables. The maximum value of directional derivative $D_{\vec{u}} f(\vec{x})$ is $|\nabla f(\vec{x})|$ and occurs when $\vec{u}$ has the same direction as the gradient vector $\nabla f(\vec{x})$

$\boxed{\text{Example}}$

(a) $f(x,y) = xe^y$ find the rate of change of f at point $P(2,0)$ in the direction from P to $Q(1/2, 2)$

(b) what is the maximum rate of change

<sol>: (a) $f_x = e^y \quad f_y = xe^y \qquad \nabla f = \langle e^y, xe^y \rangle$

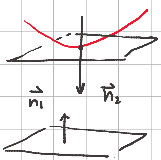
$\overrightarrow{PQ} = \langle -\frac{3}{2}, 2 \rangle \quad |\overrightarrow{PQ}| = \sqrt{\frac{9}{4}+4} = \frac{5}{2} \Rightarrow \vec{u} = \langle -\frac{3}{5}, \frac{4}{5} \rangle$

$\nabla f(2,0) = \langle 1, 2 \rangle \quad D_{\vec{u}} f = \langle 1,2 \rangle \cdot \langle -\frac{3}{5}, \frac{4}{5} \rangle = -\frac{3}{5} + \frac{8}{5} = 1$

(b) maximum $D_{\vec{u}} f(\vec{x}) = |\nabla f(2,0)| = \sqrt{5}$

occars when $\vec{u}$ is the same direction of $\langle 1,2 \rangle$

$\boxed{\text{Example}}$ At what point on the paraboloid $y = 2x^2 + 6z^2$ is its tangent plane parallel to the plane $3y - x + z = -2$?



<sol> $\vec{n_1} = \langle -1, 3, 1 \rangle \qquad f_x = 4x \quad f_y = -1 \quad f_z = 12z$

Find tangent plane:

$$f_x(x-x_0) + f_y(y-y_0) + f_z(z-z_0) = 0$$

$\nabla f = \vec{n}$ of a plane

$\therefore \vec{n_2} = \langle 4x_0, -1, 12z_0 \rangle$

$(x_0, y_0, z_0) = (\frac{1}{12}, \frac{1}{54}, -\frac{1}{36})$

$\therefore \vec{n_1} \parallel \vec{n_2} \Rightarrow \frac{-1}{4x_0} = \frac{3}{-1} = \frac{1}{12z_0} \Rightarrow x_0 = \frac{1}{12}, \ z_0 = -\frac{1}{36} \Rightarrow y_0 = \frac{1}{54} \Rightarrow$

14.7 Maximum and Minimum values

Theorem

If $f(x)$ has local max or min at (a,b) and first partial derivative exists. $f_x(a,b)=0$ and $f_y(a,b)=0$

Def^n

Critical point of f if $\boxed{f_x(x,y)=0 \ \& \ f_y(x,y)=0}$

Second Derivative Test

point (a,b) $\quad D=$ Hessian Matrix $= \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx}f_{yy}-f_{xy}^2$

(a) If $D>0$ & $f_{xx}(a,b)>0 \Rightarrow f(a,b)$ local min

(b) If $D>0$ & $f_{xx}(a,b)<0 \Rightarrow f(a,b)$ local max

(c) If $D<0$. then $f(a,b)$ is Saddle point (鞍点)

$D=0$ or $f_{xx}(a,b)$ not defined

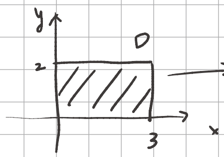
Extreme value Theorem

If f is continuous a closed boundary set on R^2 then f attains an absolute max value $f(x_1,y_1)$ & absolute value minimum value $f(x_2,y_2)$ at (x_1,y_1) & (x_2,y_2)

Example Find absolute max & min

$f(x,y) = x^2 - 2xy + 2y \quad D = \{(x,y) \mid 0 \le x \le 3,\ 0 \le y \le 2\}$

<Sol>:



$\to$ bounded closed $\therefore$ by extreme value theorem

$\therefore$ abs max & min exist

Line	function	min	max
$L_1 = y = 0$	$f(x,0) = x^2$	0	9
$L_2 = x = 0$	$f(0,y) = 2y$	0	4
$L_3 : x = 3$	$f(3,y) = 9 - 4y$	1	9
$L_4 = y = 2$	$f(x,2) = x^2 - 4x + 4$	0	4

$\Rightarrow \begin{cases} \min = 0 \\ \max = 9 \end{cases}$

Critical point $\begin{cases} f_x = 2x - 2y = 0 \\ f_y = -2x + 2 = 0 \end{cases} \Rightarrow (1,1) \Rightarrow f(1,1) = 1$

Absolutely max $f(3,0) = 9$ Abs min $f(0,0) = f(2,2) = 0$

14.8 Lagrange Multipliers

$\boxed{\text{Method of Lagrange Multipliers}}$

To find max & min values of $f(x,y,z)$ subject to the Constant $g(x,y,z)=k$

(assume these extreme values exists , and $\nabla\vec{g}=\vec{0}$ on the surface $g(x,y,z)=k$]

(a) Find all values of x,y,z and λ st

$$\nabla f(x,y,z)=\lambda \nabla g(x,y,z) \quad \text{and } g(x,y,z)=k$$

(b) Evaluate f at all the points (x,y,z) that resulted from step (a)

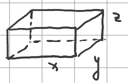
The largest value is the max value of f

The smallest value is the min value of f

$\boxed{\text{Example}}$ A rectangular box without a lid is to be made from $12\,m^2$ of cardboard

Find the maximum volumn of such a box

<sol> :



main function $f(x,y,z)=xyz$

$g(x,y,z)=xy+2xz+2yz=12$

$x^2+x^2+x^2=12$

$x^2=4 \Rightarrow x=\pm 2$

$\nabla f(x,y,z)=(yz,\ xz,\ xy) \quad \nabla g(x,y,z)=(y+2z,\ x+2z,\ 2x+2y)$

$$\nabla f=\lambda\nabla g \Rightarrow \begin{cases} yz=\lambda(y+2z) & ① \\ xz=\lambda(x+2z) & ② \\ xy=\lambda(2x+2y) & ③ \end{cases}$$

pts $(2,2,4)$ or $(-2,-2,-4)$

$x\cdot\frac{1}{2}x=\lambda(x+x)$

$\frac{1}{2}x^2=\lambda\cdot 2x$

$\Rightarrow \lambda=\frac{1}{4}x=\frac{1}{2}$ or $-\frac{1}{2}$

$\frac{①}{②} \Rightarrow \frac{y}{x}=\frac{y+2z}{x+2z} \Rightarrow xy+2yz=xy+2xz \Rightarrow yz=xz$

$\frac{①}{③} \Rightarrow \frac{z}{x}=\frac{y+2z}{2x+2y} \Rightarrow 2xz+2yz=xy+2xz \Rightarrow 2z=x$

$\Rightarrow \boxed{x=y=2z}$

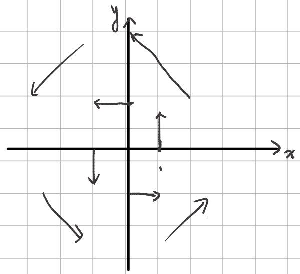
$\frac{②}{③} \Rightarrow \frac{z}{y}=\frac{x+2z}{2x+2y} \Rightarrow 2xz+2yz=xy+2zy \Rightarrow 2z=y$

$f(x,y,z)=xyz=2\cdot 2\cdot 1=4$

16.1 Vector fields

A vector fields is a function whose domain is a set of points in $\mathbb{R}^2$ (or in $\mathbb{R}^3$) whose range is a set of vectors in V_2 (or V_3)

$\boxed{\text{Example}}$ A vector field in $\mathbb{R}^2$ defined by $\vec{F}(x,y) = \langle -y, x \rangle$



(x,y)	$\vec{F}(x,y)$
$(1,0)$	$\vec{F}(1,0) = \langle 0,1 \rangle$
$(0,1)$	$\vec{F}(0,1) = \langle -1,0 \rangle$
$(2,2)$	$\vec{F}(2,2) = \langle -2,2 \rangle$

$\boxed{\text{Gradient field}}$

If f is a scalar function of two or three variables, gradient vector field is $\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle$

$\boxed{\text{Conservative Vector field}}$

$\boxed{\text{Def}^n}$

A conservative vector field (gradient field) is a vector field that is a gradient of a function It has the form $\vec{F} = \nabla f$ for scalar function f

$\boxed{\text{FACT}}$ gradient vector in a vector field $\perp$ level curve

$\boxed{\text{Example}}$ Find the gradient vector field of $f(x,y) = x^2y - y^3$

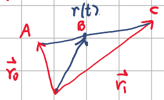
<sol>: $\vec{F} = \nabla f = \langle 2xy, x^2 - 3y^2 \rangle$

16.2 Line Integrals

Remember: (a) A vector representation of a line segment that starts at $\vec{r_0}$ and ends at $\vec{r_1}$

is given by $\vec{r}(t) = (1-t)\vec{r_0} + t\vec{r_1} \quad 0 \le t \le 1$

Pf:



$\vec{r}(t) = \vec{r_0} + \vec{AB}$

$= \vec{r_0} + t\,\vec{AC} = \vec{r_0} + t(\vec{r_1} - \vec{r_0}) = (1-t)\vec{r_0} + t\vec{r_1}$

(b) Arc length function $S(t) = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du$

(c) $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

☆ 若 $\vec{F}(x,y,z) = \nabla f(x,y,z)$

则 $\int_C \vec{F}\, d\vec{r} = \int_C \nabla f\, d\vec{r} = f(r(b)) - f(r(a))$

(d) $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

Theorem 2.2: The arc length of a curve C parametrized by $\vec{r}(t)$ is given by: $\int_C 1\, ds$

Volumn of $E = \iiint_E 1 \cdot dV$ Area of $R = \iint_R 1 \cdot dA$ Length of $C = \int_C 1 \cdot ds$

$\boxed{\text{Def}^n}$ If f is defined on a smooth curve C, then the line integral of f along C is:

$$\int_C f(x,y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Same $\int_C f(x,y,z)\, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \Rightarrow \int_a^b f(\vec{r}(t)) \left|\vec{r}'(t)\right| dt$

16.3 The fundamental Theorem of line Integrals

$\boxed{\text{Example}}$: Calculate the line integrals $\int_C \vec{F}\,d\vec{r}$ from $(0,0)$ to $(1,1)$ where $\vec{F}=\langle 2x, 2y\rangle$

(a) when C is parametrized by $\langle t,t\rangle$ $t\in[0,1]$

$$\begin{cases} x(t)=t \\ y(t)=t \end{cases} \Rightarrow \vec{F}=\langle 2x(t), 2y(t)\rangle$$

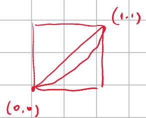
$$\vec{r}(t)=\langle t,t\rangle \Rightarrow d\vec{r}=\langle 1,1\rangle dt$$

$$\int_C \vec{F}\,d\vec{r} = \int_0^1 \langle 2t, 2t\rangle \langle 1,1\rangle dt = \int_0^1 4t\,dt = 2t^2\Big|_0^1 = 2$$

(b) when C is parametrized by $\langle t, t^2\rangle$ $t\in[0,1]$

$$\vec{r}(t)=\langle t,t^2\rangle \qquad d\vec{r}=\langle 1,2t\rangle dt \qquad \vec{F}=\langle 2t, 2t^2\rangle$$

$$\int_C \vec{F}\,d\vec{r} = \int_0^1 \langle 2t, 2t^2\rangle \langle 1, 2t\rangle dt = \int_0^1 2t+4t^3\,dt = t^2+t^4\Big|_0^1 = 2$$



No matter how path you take every line integral will give you same result.

$\boxed{\text{Def}^n}$ Let $\vec{F}$ to be a vector field defined on an open region D in space, and suppose that for any two points A and B in D the line integral : $\int_C \vec{F}\,d\vec{r}$ along the path from A to B in D is the same over all curves. Then the integral is called <u>path independent</u> in D.

$\boxed{\text{Theorem}}$: Suppose $\vec{F}$ is a vector field that is continuous on an open connected Region D. If $\int_C \vec{F}\,d\vec{r}$ is independent of path in D. Then $\vec{F}$ is a conservative vector field on D that is exists a function f such that $\vec{F}=\nabla f$)

How to identify conservative field ?

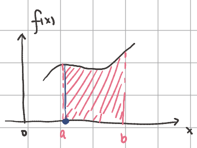
$\nabla\times\vec{F}=0$ or $\int_C \vec{F}\,d\vec{r}=0$

2D : $\nabla\times\vec{F} = \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 0$

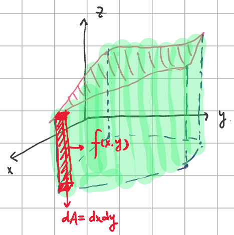
$$3D : \nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

15.1 Double integration

$\int_a^b f(x)\,dx \longrightarrow$



$\displaystyle\int_{y=b_1}^{y=b_2}\int_{x=a_1}^{x=a_2} f(x,y)\,\underbrace{dA}_{dxdy} \longrightarrow$



$\boxed{\text{Def}^n}$

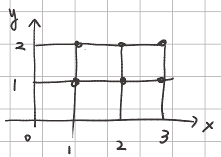
If $f(x,y) \geq 0$ then the volumn V of the solid that lies below $z = f(x,y)$ and above the region R is defined to be $V = \iint_R f(x,y)\,dA$

$\boxed{\text{Example}}$:

Use six rectangle of width and length 1 to approximate the volumn under $6-x-y$

$0 \leq x \leq 3 \qquad 0 \leq y \leq \alpha$

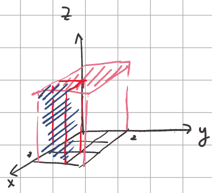
<Sol> $\quad z = 6-x-y \qquad R = \{(x,y) \mid 0 \leq x \leq 3,\ 0 \leq y \leq 2\}$



$V = \iint_R (6-x-y)\,dA$

$= [f(1,1) + f(2,1) + f(3,1) + f(1,2) + f(2,2) + f(3,2)]\,dA$

$= 15$



$\boxed{\text{Def}^n}$

If $f(x,y)$ is integrable then we define $\iint_R f(x,y)\,dA = \lim_{n\to\infty} \Delta A \cdot f(x_{ij}^*, y_{ij}^*)$

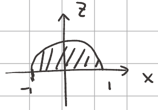
where n — # of sub rectangle. x_{ij}^*, y_{ij}^* · x and y values. ΔA area of sub rectangle

· The average volumn of $f^n(x,y)$ is given by $f_{Avg} = \dfrac{1}{A(R)} \iint_R f(x,y)\,dA$ ($A(R)$. area of region)

$\boxed{\text{Example}}$ If $R\ \{(x,y) \mid -1 \le x \le 1,\ -2 \le y \le 2\}$ evaluate $\iint_R \sqrt{1-x^2}\, dA$

<sol> $\iint_R \sqrt{1-x^2}\, dA = \int_{-2}^{2}\int_{-1}^{1} \sqrt{1-x^2}\, dx\, dy$ Consider $\int_{-1}^{1} \sqrt{1-x^2}\, dx$

$z = \sqrt{1-x^2} \Rightarrow x^2 + z^2 = 1$



$\therefore \int_{-1}^{1} \sqrt{1-x^2}\, dx = \frac{\pi}{2}$ $\therefore \int_{-2}^{2} \frac{\pi}{2}\, dy = \frac{\pi}{2} \cdot 4 = 2\pi$

Properties of double integral.

- $\iint_R f(x,y) + g(x,y)\, dA = \iint_R f(x,y)\, dA + \iint_R g(x,y)\, dA$

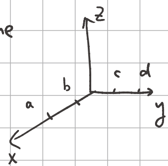
- $\iint_R c f(x,y)\, dA = c \iint_R f(x,y)\, dA$

- If $f(x,y) \ge g(x,y)$ (x,y) in R then $\iint_R f(x,y)\, dA \ge \iint_R g(x,y)\, dA$

15.2 Iterated integrals

$\boxed{\text{Def}^n}$ If $R = [a,b] \times [c,d]$ and $f(x,y)$ is integrable curve R then we define

$$\iint_R f(x,y)\, dA = \int_c^d \left[\int_a^b f(x,y)\, dx \right] dy = \int_a^b \left[\int_c^d f(x,y)\, dy \right] dx$$



$\boxed{\text{Example}}$ Evaluate $\int_0^3 \int_1^2 x^2 y\, dy\, dx$

$$\int_0^3 \left[\int_1^2 x^2 y\, dy \right] dx = \int_0^3 \left(x^2 \cdot \frac{1}{2} y^2 \Big|_1^2 \right) dx = \int_0^3 \left(x^2 \cdot \frac{4-1}{2} \right) dx = \frac{3}{2} \cdot \frac{1}{3} x^3 \Big|_0^3 = \frac{1}{2} x^3 \Big|_0^3 = \frac{27}{2}$$

$\boxed{\text{Fubin's Theorem}}$

If $f(x,y)$ is continuous on the rectangle $R = [a,b] \times [c,d]$, then

$$\iint_R f(x,y)\, dA = \int_c^d \int_a^b f(x,y)\, dx\, dy = \int_a^b \int_c^d f(x,y)\, dy\, dx$$

$\boxed{\text{Theorem}}$ If $f(x,y) = g(x) \cdot h(y)$ and $R = [a,b] \times [c,d]$

then $\displaystyle\iint_R f(x,y)\,dA = \iint_R g(x)\,h(y)\,dA = \left[\int_a^b g(x)\,dx\right] \times \left[\int_c^d h(y)\,dy\right]$

$\boxed{\text{Example}}$ Find average of $z = x\sec^2 y$ over the region R.

$R = \{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le \pi/4\}$

<Sol>. $\displaystyle \text{Avg} = \frac{1}{A(R)} \iint_R f(x,y)\,dx\,dy = \frac{1}{2 \times \frac{\pi}{4}} \int_0^{\frac{\pi}{4}} \int_0^2 x\sec^2 y\,dx\,dy$

$\displaystyle = \frac{4}{\pi} \int_0^{\frac{\pi}{4}} \sec^2 y \cdot dy = \frac{4}{\pi} \int_0^{\frac{\pi}{4}} d(\tan y) = \frac{4}{\pi} \tan y \Big|_0^{\frac{\pi}{4}} = \frac{4}{\pi}$

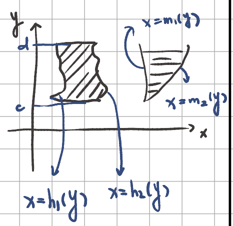
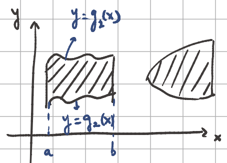
or $\displaystyle \int_0^{\frac{\pi}{4}} \int_0^2 x\sec^2 y\,dx\,dy = \left[\int_0^2 x\,dx\right] \cdot \left[\int_0^{\frac{\pi}{4}} \sec^2 y\,dy\right] = 2$ $\displaystyle 2 \cdot \frac{2}{\pi} = \frac{4}{\pi}$

$\boxed{\text{Example}}$ $\displaystyle \iint_R (x - 3y^2)\,dA$ $R = \{(x,y) \mid 0 \le x \le 2,\ 1 \le y \le 2\}$

<Sol>: $\displaystyle \int_1^2 \int_0^2 (x - 3y^2)\,dx\,dy = \int_1^2 \left[\frac{1}{2}x^2 - 3y^2 x\right]_0^2 dy = \int_1^2 (2 - 6y^2)\,dy = 2y - 2y^3 \Big|_1^2$

$= 2(2 - 8 - 0) = -12$

15.3 Double integral over general region

Type I	Type II
	
$D=\{(x,y) \mid c\le y\le d,\ h_1(c)\le x\le h_2(d)\}$	$D=\{(x,y) \mid a\le x\le b,\ g_1(a)\le y\le g_2(b)\}$

Type I

If $f(x,y)$ is continuous on type I region $D=\{(x,y) \mid c\le y\le d,\ h_1(y)\le x\le h_2(y)\}$

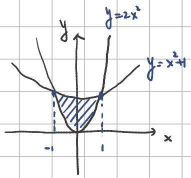
$$\iint_D f(x,y)\,dx\,dy=\int_c^d\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Type II

If $f(x,y)$ is continuous given region $D=\{(x,y) \mid a\le x\le b,\ g_1(a)\le y\le g_1(b)\}$

$$\iint_D f(x,y)\,dx\,dy=\int_{x=a}^{x=b}\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

$\boxed{\text{Example}}$ Evaluate $\iint_D (x+2y)\,dA$ D-region by parabola $y=2x^2$ & $y=1+x^2$



$\begin{cases} y=2x^2 \\ y=x^2+1 \end{cases} \Rightarrow x=\pm1$ Region $D=\{(x,y) \mid -1\le x\le 1,\ 2x^2\le y\le x^2+1\}$

$$\iint_D f(x,y)\,dx\,dy=\int_{-1}^{1}\int_{2x^2}^{x^2+1}(x+2y)\,dy\,dx=\int_{-1}^{1}\Big[xy+y^2\Big]_{2x^2}^{x^2+1}dx$$

$$=\int_{-1}^{1} x\left[x^2+1-2x^2\right]+\left[(x^2+1)^2-(2x^2)^2\right]dx$$

calculation $= x(1-x^2)+\left[x^4+2x^2+1-4x^4\right]=x-x^3+2x^2+1-3x^4$

$=\underline{-3x^4-x^3+2x^2+x+1}$

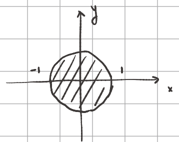
$$=\int_{-1}^{1}(-3x^4-x^3+2x^2+x+1)\,dx=-\frac{3}{5}x^5-\frac{1}{4}x^4+\frac{2}{3}x^3+\frac{1}{2}x^2+x\Big|_{-1}^{1}=-\frac{6}{5}+\frac{4}{3}+2$$

$$=-\frac{18}{15}+\frac{20}{15}+\frac{30}{15}=\frac{32}{15}$$

$\boxed{\text{Example}}$ $\iint_D 3y\,dA$. region bounded by $\sqrt{1-y^2}$ & $-\sqrt{1-y^2}$

$x=\sqrt{1-y^2}$ $x=-\sqrt{1-y^2}$

$x^2+y^2=1\ (x\ge0)$ $x^2+y^2=1\ (x<0)$



$$\iint_D 3y\,dA=\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1+y^2}}3y\,dx\,dy=\int_{-1}^{1}3yx\Big|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}dy=\int_{-1}^{1}6y\sqrt{1-y^2}\,dy$$

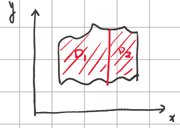
$\underline{t=1-2y^2}$

$dt=-4y\,dy$

$-\frac{3}{2}dt=6y\,dy$

$$\int_0^0-\frac{3}{2}\sqrt{t}\,dt=0$$

Property of integral over Region



· If $f(x,y)$ & $g(x,y)$ are continuous on the bounded region D then the following holds

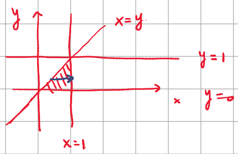
① If $D = D_1 \cup D_2$ then $\iint_D f(x,y)\,dxdy = \iint_{D_1} f(x,y)\,dxdy + \iint_{D_2} f(x,y)\,dxdy$

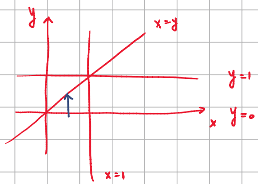
② $\iint_D 1\,dxdy = \text{Area}(D)$

③ if $m \le f(x,y) \le M$, for all $(x,y) \in D$ $m \cdot \text{Area}(D) \le \iint_D f(x,y)\,dxdy \le M \cdot \text{Area}(D)$

$\boxed{\text{Example}}$ Evaluate $\int_0^1 \int_y^1 \frac{\sin x}{x}\,dx\,dy$ by switching the bound

$$\int_0^1 \int_y^1 \Rightarrow \int_{y=0}^{y=1} \int_{x=y}^{x=1}$$





$$\therefore \int_0^1 \int_0^x \frac{\sin x}{x}\,dy\,dx = \int_0^1 \frac{\sin x}{x}\, y \Big|_0^x\,dx$$

$$= \int_0^1 \sin x\,dx = -\cos x \Big|_0^1 = 1 - \cos(1)$$

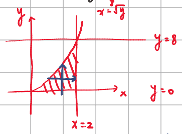
$$\int_0^1 \int_0^x f(x,y)\,dy\,dx$$

$\boxed{0 \le y \le x}$
$0 \le x \le 1$

$\boxed{\text{Example}}$ $\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\,dx\,dy$ by switching the bound

$$\int_{y=0}^{y=8} \int_{x=\sqrt[3]{y}}^{x=2}$$

$$\begin{cases} \sqrt[3]{y} \le x \le 2 \\ 0 \le y \le 8 \end{cases}$$



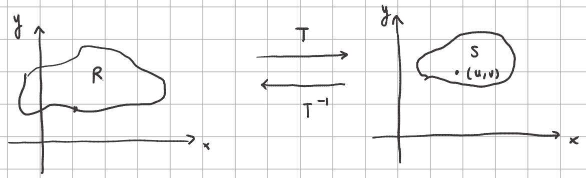
$0 \le y \le x^3$
$0 \le x \le 2$

$$\int_0^2 \int_0^{x^3} e^{x^4}\,dy\,dx = \int_0^2 e^{x^4} y \Big|_0^{x^3} dx = \int_0^2 x^3 e^{x^4}\,dx \qquad \begin{array}{c} t = x^4 \\ \hline\hline dt = 4x^3 dx \end{array}$$

$$\int_0^{16} e^t \tfrac{1}{4}\,dt = \tfrac{1}{4}\left[e^{16} - 1\right]$$

$$\begin{cases} dx \rightarrow \\ dy \uparrow \end{cases}$$

15.10 Changes of variables in double integrals



$$\iint_D f(x,y)\,dx\,dy$$

Transformation :

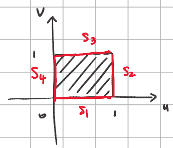
$\boxed{\text{Def}^n}$ A transformation is a function $T(u,v) = (g(u,v), h(u,v))$, in other words, transformation is a function whose domain & range are both subsets of $\mathbb{R}^2$

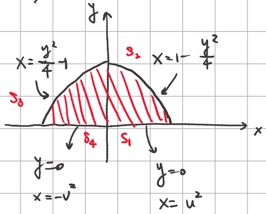
$\boxed{\text{Theorem}}$ If $T(u,v) = (g(u,v), h(u,v))$ is a transformation with g & h are continuous, the T sends boundaries to boundaries

$\boxed{\text{Example}}$. Transformation

A transformation $T(u,v)$ $x = u^2 - v^2$ $y = 2uv$. Find the range of $\{(u,v) \mid 0 \le u \le 1,\ 0 \le v \le 1\}$

<Sol> $T(u,v) = (u^2 - v^2, 2uv)$





S_1: $0 \le u \le 1$, $v = 0$

$x = u^2 \in [0,1]$ $y = 0$

S_2: $u = 1$ $0 \le v \le 1$

$x = u^2 - v^2 \in [0,1]$ $y = 2uv \in [0,2]$

$x = 1 - v^2$ $y = 2v \Rightarrow x = 1 - \left(\frac{y}{2}\right)^2$

S_3: $0 \le u \le 1$ $v = 1$

$x = u^2 - v^2 \in [-1, 0]$ $y = 2uv \in [0,2]$

$x = u^2 - 1$ $y = 2u \Rightarrow x = \left(\frac{y}{2}\right)^2 - 1$

S_4: $u = 0$ $0 \le v \le 1$

$x = u^2 - v^2 \in [-1, 0]$ $y = 2uv \in [0,2]$

$x = -v^2$ $y = 0$

$\boxed{\text{Jacobian}}$

Def^n: If T is a transformation $x = g(u,v)$ $y = h(u,v)$, then Jacobian is defined to be

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Change of variable

$\boxed{\text{Theorem}}$ Suppose T is a continuous transformation whose Jacobian is non-zero and that maps a region S in the uv plane to a region R in xy plane. Futhermore, assume f is continuous on R and that R and S are either type 1 or type2 region. Suppose that T is one to one expect possibly along the boundary of S, then

$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v),\, y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

① Polar coordinate Jacobian

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| = \begin{pmatrix} \frac{\partial x}{\partial r}, & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r}, & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r\cos^2\theta + r\sin^2\theta$$

$$= r$$

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(x(r,\theta),\, y(r,\theta))\,|r|\,dr\,d\theta$$

② Cylindrical coordinate Jacobian

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \quad J = \det\begin{pmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \cos\theta\,(-1)^{1+1}\left(r\cos\theta - 0\right) + (-r\sin\theta)\,(-1)^{1+2}\left(\sin\theta - 0\right) + 0$$

$$= r\cos^2\theta + r\sin^2\theta = r$$

$$\therefore\ dx\,dy\,dz = |r|\,dr\,d\theta\,dz$$

③ Spherical Coordinate Jacobian

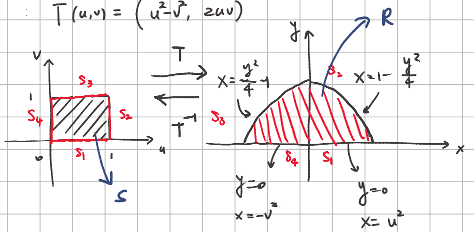
$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho\cos\phi$

$$J = \det\begin{pmatrix} x_\rho & x_\phi & x_\theta \\ y_\rho & y_\phi & y_\theta \\ z_\rho & z_\phi & z_\theta \end{pmatrix} = \begin{pmatrix} \sin\phi\cos\theta & \rho\cos\theta\cos\phi & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\theta\cos\phi & \rho\sin\phi\cos\theta \\ \cos\theta & -\rho\sin\phi & 0 \end{pmatrix} = \rho^2\sin\phi$$

$\therefore\ dx\,dy\,dz = \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$

$\boxed{\text{Example}}$ $\iint_R y\,dA \qquad x = u^2 - v^2 \quad y = 2uv$

$T(u,v) = (u^2-v^2,\ 2uv)$



$$\iint_R y\,dA = \iint_S 2uv\,|J|\,du\,dv$$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4(u^2+v^2)$$

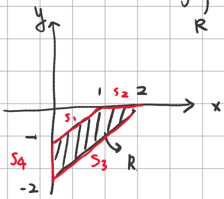
$$\therefore \int_0^1\int_0^1 2uv\,(4u^2+4v^2)\,du\,dv = \int_0^1\int_0^1 8uv(u^2+v^2)\,du\,dv$$

For $\displaystyle\int_0^1 8uv(u^2+v^2)\,du \ \underset{d\varphi = 2u\,du}{\overset{\varphi = u^2}{=\!=}}\ \int_0^1 4v(\varphi+v^2)\,d\varphi$

$$= \int_0^1 4v\varphi + 4v^3\,d\varphi = 2v\varphi^2 + 4v^3\varphi\Big|_0^1 = 2v + 4v^3$$

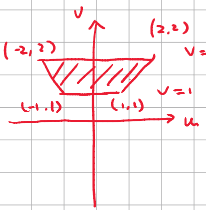
$$\therefore \int_0^1 2v + 4v^3\,dv = v^2 + v^3\Big|_0^1 = \underline{\underline{2}}$$

$\boxed{\text{Example}}$ $\displaystyle\iint_R e^{(x+y)/(x-y)}\,dxdy$ Region as below



$$u = x+y \quad\Rightarrow\quad x = \tfrac{1}{2}(u+v) \quad\Rightarrow T(u,v) = \left(\tfrac{1}{2}(u+v),\ \tfrac{1}{2}(u-v)\right)$$
$$v = x-y \qquad\quad y = \tfrac{1}{2}(u-v)$$

$$\therefore J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$



S_1: $y = x-1 \Rightarrow \frac{1}{2}(u-v) = \frac{1}{2}(u+v) - 1 \Rightarrow u-v = u+v-2$
$\Rightarrow v = 1$

S_2: $y = 0 \Rightarrow u = v$.

S_3: $y = x-2 \Rightarrow \frac{1}{2}(u-v) = \frac{1}{2}(u+v) - 2 \Rightarrow u-v = u+v-4$
$\Rightarrow v = 2$

S_4: $x = 0 \Rightarrow u = -v$

$$\therefore \iint_R e^{(x+y)/(x-y)}\,dx\,dy = \int_1^2\int_{-1}^{1} e^{\frac{u}{v}} \left|-\frac{1}{2}\right| du\,dv = \frac{3}{4}(e - e^{-1}).$$

☆ Jacobian are absolute value

15.4 Double integral in Polar Coordinate

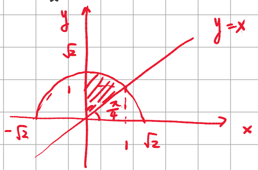
$\boxed{\text{Theorem}}$ If f is continuous on Polar rectangle

$R = \{ (r,\theta) \mid a \le r \le b,\ \alpha \le \theta \le \beta \}$ then $\displaystyle\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(x(r,\theta), y(r,\theta))\, r\,dr\,d\theta$

$\boxed{\text{Example}}$

Convert $\displaystyle\int_0^1 \int_x^{\sqrt{2-x^2}} x+2y\ dy\,dx$ into polar coordinate

$$\int_{x=0}^{x=1} \int_{y=x}^{y=\sqrt{2-x^2}}$$



$0 \le r \le \sqrt{2}$

$\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

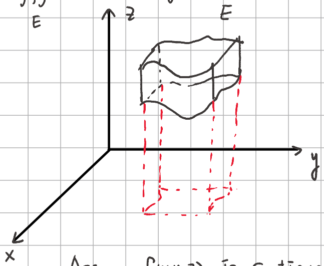
$$\therefore \int_0^1 \int_x^{\sqrt{2-x^2}} x+2y\ dy\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\sqrt{2}} (r\cos\theta + 2r\sin\theta)\, r\,dr\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{3}r^3\cos\theta + \frac{2}{3}r^3\sin\theta \Big|_0^{\sqrt{2}} d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{2\sqrt{2}}{3}(\cos\theta + 2\sin\theta)\,d\theta = \frac{2\sqrt{2}}{3}(\sin\theta - 2\cos\theta)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{2\sqrt{2}}{3}\left[1 - \left(\frac{\sqrt{2}}{2} - 2\cdot\frac{\sqrt{2}}{2}\right)\right] = \frac{2\sqrt{2}}{3}\left(1 + \frac{\sqrt{2}}{2}\right)$$

15.7 Triple integral

$$\iiint_E f(x,y,z)\,dz\,dy\,dx$$



Area and volumn

$\boxed{\text{Theorem}}$: Volumn of a closed bounded region E in space given by

$$\iiint_E 1\,dV$$

$\boxed{\text{Theorem}}$ (Fubini's theorem)

Assume $f(x,y,z)$ is continuous : $E = [a,b]\times[c,d]\times[r,s]$

then $\displaystyle\iiint_E f(x,y,z)\,dV = \int_r^s\int_c^d\int_a^b f(x,y,z)\,dx\,dy\,dz$. furthermore we can envaluate the integral

in any order $\displaystyle\int_c^d\int_r^s\int_a^b f(x,y,z)\,dx\,dz\,dy$

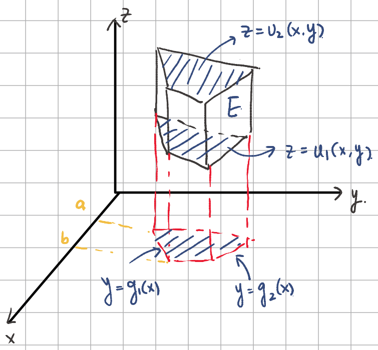
$\boxed{\text{Example}}$

Evaluate $\displaystyle\iiint_E xyz+1\,dV$ where $E=[1,5]\times[2,3]\times[-2,0]$

<Solution> : $\displaystyle\int_{-2}^{0}\int_{2}^{3}\int_{1}^{5}(xyz+1)\,dx\,dy\,dz = \int_{-2}^{0}\int_{2}^{3} \underline{yz(\tfrac{1}{2}x^2)+x\Big|_1^5}\,dy\,dz = \int_{-2}^{0}\int_{2}^{3} 12yz+4\,dy\,dz$

$\hookrightarrow yz\cdot\frac{25-1}{2}+4 = 12yz+4$

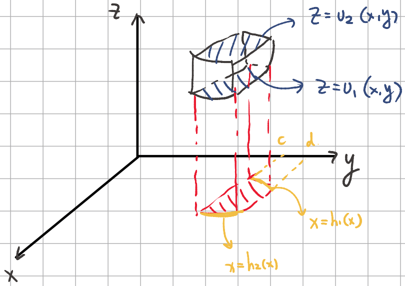
$\displaystyle = \int_{-2}^{0} 12z(\tfrac{1}{2}y^2)+4y\Big|_2^3\,dz = \int_{-2}^{0}\left(12z\cdot\frac{9-4}{2}+4\right)dz = \int_{-2}^{0} 30z+4\,dz = 15z^2+4z\Big|_{-2}^{0} = -52$



Type I : Projection of E is onto xy plane (dy dx)

$$E=\{(x,y,z)\mid a\le x\le b,\ g_1(x)\le y\le g_2(x),\ u_1(x,y)\le z\le u_2(x,y)\}$$

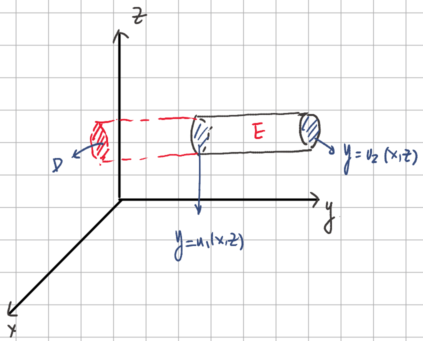
$$\iiint_E f(x,y,z)\,dV = \int_a^b\int_{g_1(x)}^{g_2(x)}\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$



Type II (dxdy)

$$E=\{(x,y,z)\mid c\le y\le d,\ h_1(x)\le y\le h_2(x),\ u_1(x,y)\le z\le u_2(x,y)\}$$

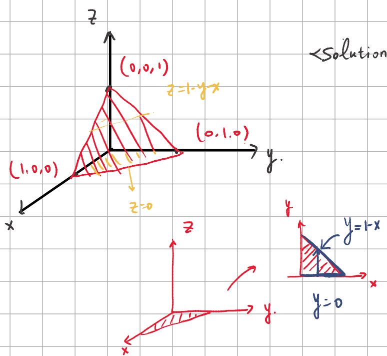
$$\iiint_E f(x,y,z)\,dV=\int_c^d\int_{h_1(x)}^{h_2(x)}\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dx\,dy$$



Type III: projection on xz plane

$$E=\{(x,y,z)\mid (x,y)\in D\ \ u_1(x,z)\le y\le u_2(x,z)\}$$

$\boxed{\text{Example}}$ Evaluate $\iiint_D x\,dV$ where D is the region in the first octant bounded by $x+y+z=1$



<Solution>:

$$\int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\int_{z=0}^{z=1-x-y} x\,dz\,dy\,dx$$

$$\int_{y=0}^{y=1-x} x(1-x-y)\,dy = x(1-x)^2-\frac{x}{2}(1-x)^2=\frac{x}{2}(1-x)^2$$

$$\int_0^1 \frac{x}{2}(1-x)^2dx=\int_0^1\frac{x}{2}(1-2x+x^2)\,dx=\int_0^1 \frac{x}{2}-x^2+\frac{x^3}{2}\,dx$$

$$=\frac{1}{4}x^2-\frac{1}{3}x^3+\frac{1}{8}x^4\Big|_0^1=\frac{1}{4}-\frac{1}{3}+\frac{1}{8}=\frac{3}{8}-\frac{1}{3}=\frac{1}{24}$$

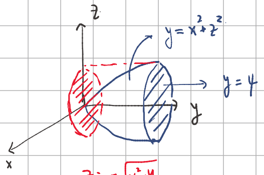
along z axies hit first plane, end plane

along y axies hit first line, end line

along x axies ...

Example

Evaluate $\iiint_E \sqrt{x^2+z^2}\,dV$ where E is bounded by $y=x^2+z^2$ & $y=4$



$$\iint_D \int_{y=x^2+z^2}^{y=4} \sqrt{x^2+z^2}\,dy\,dA$$

$$\begin{cases} x = r\cos\theta \\ z = r\sin\theta \end{cases}$$

$$= \iint_D \sqrt{x^2+z^2}\,(4-x^2-z^2)\,dA$$

$dx\,dz = |r|\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^2 r^2(4-r^2)\,dr\,d\theta = \frac{128\pi}{15}$$

15.7

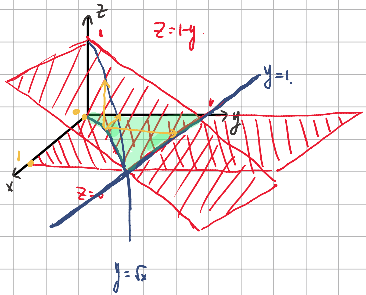
Example $\int_0^1\int_{\sqrt{x}}^1\int_0^{1-y} f(x,y,z)\,dz\,dy\,dx$ Rewrite this in $dz\,dx\,dy$. $dy\,dz\,dx$

$$\int_0^1\int_{\sqrt{x}}^1\int_0^{1-y}$$

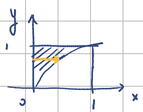
$z=0 \to z=1-y$

$y=\sqrt{x} \to y=1$

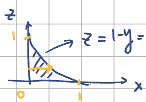
$x=0 \to x=1$



$$\int_0^1\int_0^{y^2}\int_0^{1-y}$$ $dz\,dx\,dy$

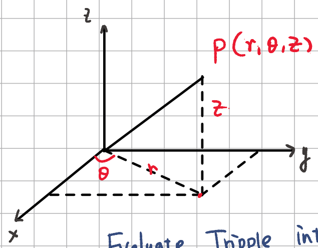


$$\int_0^1\int_0^{(1-z)^2}\int_{\sqrt{x}}^{1-z}$$ $dy\,dx\,dz$



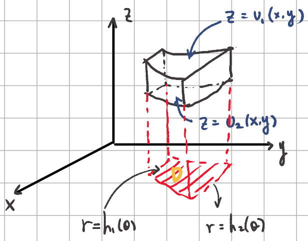
$z=1-y=1-\sqrt{x} \Rightarrow x=(1-z)^2$

15.8 Triple integral in Cylindrical Coordinate



$\tan\theta = \frac{y}{x}$ $\quad r^2 = x^2 + y^2$

Evaluate Tripple integral

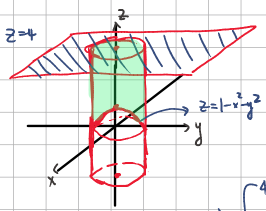


$E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \leq z \leq u_2(x,y)\}$

$D = \{(r,\theta) \mid \alpha \leq \theta \leq \beta,\ h_1(\alpha) \leq r \leq h_2(\beta)\}$

$$\iiint_E f\, dV = \int_\alpha^\beta \int_{h_1(\alpha)}^{h_2(\beta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f \quad r\, dz\, dr\, d\theta$$

$\boxed{\text{Example}}$ Evalute $\iiint_E k\sqrt{x^2+y^2}\, dV$ where E lies within cylinder $x^2+y^2=1$ between the plane $z=4$ & above paraboloid $z=1-x^2-y^2$



$$\iiint_E k\sqrt{x^2+y^2}\, dV = k\int_0^{2\pi}\int_0^1\int_{z=1-x^2-y^2}^{z=4} r \cdot r\, dz\, dr\, d\theta$$

$z = 1-x^2-y^2 = 1-r^2$

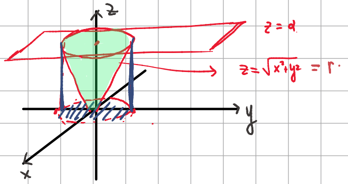
$\therefore \int_{1-r^2}^{4} r^2\, dz = r^2(3+r^2) \qquad \int_0^1 3r^2 + r^4\, dr = r^3 + \frac{1}{5}r^5 \Big|_0^1 = \frac{6}{5}$

$\therefore \int_0^{2\pi} \frac{6}{5}\, d\theta = \frac{12\pi}{5} \qquad \therefore \iiint_E k\sqrt{x^2+y^2}\, dV = \frac{12\pi k}{5}$

$\boxed{\text{Example}}$ Evaluate $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx$

$$\int_{x=-2}^{x=2}\int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}}\int_{z=\sqrt{x^2+y^2}}^{z=2}$$

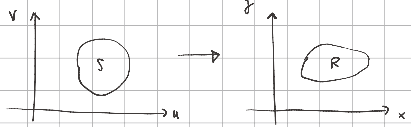
$$\iiint_E (x^2+y^2)\,dz\,dy\,dx = \int_0^{2\pi}\int_0^{2}\int_{z=r}^{z=2} r^2\cdot r\,dz\,dr\,d\theta$$



$$\int_{z=r}^{z=2} r^3\,dz = r^3(2-r) = 2r^3-r^4$$

$$\int_0^2 2r^3-r^4\,dr = \frac{1}{2}r^4-\frac{1}{5}r^5 = 8-\frac{32}{5} = \frac{8}{5} \quad \therefore \frac{16}{5}\pi$$

Q1 Transformation



$\begin{cases} x = 6u+4v^3 \\ y = 7u+v^3 \end{cases}$ what is the value of C so that $\iint_R 1\,dx\,dy = \iint_S cv^2\,du\,dv$

Q2 $\displaystyle\int_0^4\int_{2x}^{4} e^{\frac{2x}{y}}\,dy\,dx$ Evaluate by reversing the order of integral

$\boxed{\text{MCQ}}$ Find average value $f(x,y) = x^2+y$ over $R=[0,4]\times[2,5]$

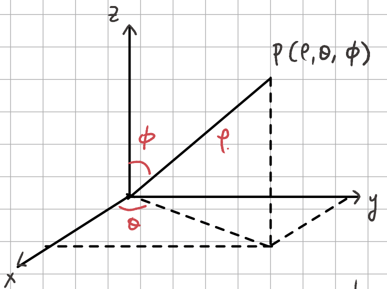
$R=[1,5]\times[1,4]$ Approximate $\iint_R xy^2\,dA$ $\Delta x=2$ $\Delta y=1$ and (x_k, y_k) lower right

$f(x,y)$ is continuous then $\displaystyle\int_0^2\int_0^3 f(x,y)\,dy\,dx = \int_0^3\int_0^2 f(x,y)\,dy\,dx$ $\boxed{\text{T/F}}$

15.9 Triple Integral in Sperical Coordinate.

$\boxed{\text{Spherical Coordinate}}$ (ρ, θ, ϕ)

$x = \rho\sin\phi\cos\theta \quad y = \rho\sin\phi\sin\theta \quad z = \rho\cos\phi \quad \rho^2 = x^2 + y^2 + z^2$



$$\iiint_E f(x,y,z)\,dV$$

$$= \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

where $E = \{(\rho,\theta,\phi) \mid a \le \rho \le b,\ \alpha \le \theta \le \beta,\ c \le \phi \le d\}$

$\boxed{\text{Example}}$ Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\,dV$ where $B = \{(x,y,z) \mid x^2+y^2+z^2 \le 1\}$

<solution>

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$$

$x^2+y^2+z^2 = \rho^2 \Rightarrow (x^2+y^2+z^2)^{3/2} = (\rho^2)^{3/2} = \rho^3$

$$\int_0^\pi \int_0^{2\pi} \int_0^1 e^{\rho^3}\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

method d

$$\int_0^\pi \int_0^{2\pi} \int_0^1 \rho^5\sin\phi\,d\rho\,d\theta\,d\phi = \left[\int_0^1 \rho^2 e^{\rho^3}d\rho\right]\left[\int_0^{2\pi} 1\,d\theta\right]\left[\int_0^\pi \sin\phi\,d\phi\right] = \frac{4\pi}{3}(e-1)$$

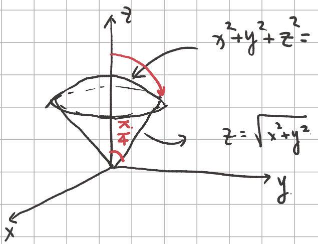
16.5 Curls and Divergence

$\boxed{\text{Def}^n}$ If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ curl $\vec{F} = \nabla \times \vec{F}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$\boxed{\text{Example}}$ Find volume of the solid that has above the Cone $z=\sqrt{x^2+y^2}$

& below the sphere $x^2+y^2+z^2=z$



<Solution>:
$$\begin{cases} x=\rho\sin\phi\cos\theta \\ y=\rho\sin\phi\sin\theta \\ z=\rho\cos\phi \end{cases}$$

$$x^2+y^2+z^2=z \Rightarrow \rho^2=\rho\cos\phi \Rightarrow \rho=\cos\phi$$

$$z=\sqrt{x^2+y^2}=\sqrt{\rho^2\sin^2\phi\,(1)}=\rho\sin\phi$$

$$\Rightarrow \rho\cos\phi=\rho\sin\phi \Rightarrow \tan\phi=1 \Rightarrow \phi=\frac{\pi}{4}$$

$$E=\left\{(\rho,\theta,\phi)\,\middle|\,0\le\theta\le2\pi,\ 0\le\phi\le\frac{\pi}{4},\ 0\le\rho\le\cos\phi\right\}$$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_{\rho=0}^{\rho=\cos\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\frac{\pi}{8}$$

$\boxed{\text{Example}}$ $\vec{F}=\langle xz,\ xyz,\ -y^2\rangle$

$$\nabla\times\vec{F}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix}=(-2y-xy)\vec{i}-(0-x)\vec{j}+(yz-0)\vec{k}=\langle -2y-xy,\ x,\ yz\rangle$$

$\boxed{\text{Theorem}}$ If $\vec{F}$ is a vector field defined on the $\mathbb{R}^3$ whose component f is continuous partial derivative, and Curl $\vec{F}=\nabla\times\vec{F}=0 \Rightarrow \vec{F}$ vector filed Conservative

- We say that a vector field $\vec{F}$ is Irrotational at a point (x,y,z) If curl $\vec{F}=0$ at (x,y,z)

无旋的

$\boxed{\text{Def}^n}$ Divergence ($\operatorname{div} \vec{F}$)

If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a vector field on $\mathbb{R}^3$ and $\frac{\partial P}{\partial x}, \frac{\partial Q}{\partial y}, \frac{\partial R}{\partial z}$ exists

then divergence of $\vec{F} \Rightarrow \operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

$$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot \vec{F}$$

$\boxed{\text{Example}}$ $\vec{F} = \langle xz, xyz, -y^2 \rangle$

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = z + xz + 0 = z(1+x)$$

$\boxed{\text{Theorem}}$

* If $\vec{F} = \langle P, Q, R \rangle$ is a vector field on $\mathbb{R}^3$ and P, Q, R have continuous second partial derivative, then $\operatorname{div}(\operatorname{curl} \vec{F}) = 0$

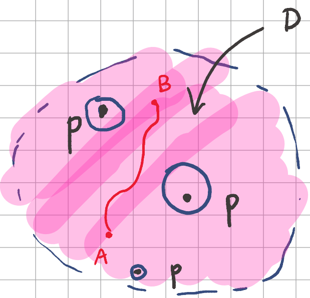
* We say that a vector field $\vec{F}$ is <u>incompressible</u> at a point (x, y, z) if $\operatorname{div}(\vec{F}) = 0$ at (x, y, z)

不可压缩的

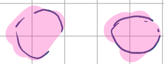
gradient	curl	divergence
$\nabla(f) = \langle f_x, f_y, f_z \rangle$	$\operatorname{curl}(\vec{F}) = \nabla \times \vec{F}$	$\operatorname{div}(\vec{F}) = \nabla \cdot \vec{F}$
input a scalar	input a vector	input a vector
output a vector	output a vector	output a scalar

$\boxed{\text{Def}^n}$

(a) D is open, desk that can center p that lies entirely in D

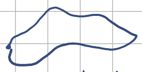


(b) D is connected



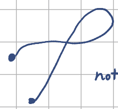
not connected

(c) closed curve & simple curve

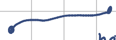


closed curve

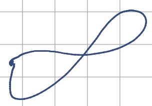
simple



not simple & closed



not closed curve



closed but not simple

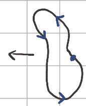
(e) A simple-connected region (No doughnuts or holes in D)

16.4 Green's Theorem

$\boxed{\text{Def}^n}$

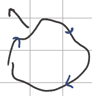
(a) A simple closed curve C has positive orientation if its parametrization traverses the curve exactly once in a counter clockwise (逆时针) direction

positive



(b) A simple closed curve C has negative orientation if its parametrization traverses the curve exactly once in a clockwise (顺时针) direction

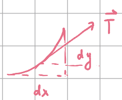
negative



拓展推导: $\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$ $\quad \oint_C \vec{F}\cdot\vec{n}\,ds = \iint_D \nabla\cdot\vec{F}\,dA$

左侧 Prof: $\vec{F} = \langle P, Q\rangle$ $\quad \vec{n}$ 是曲线单位法向量 $\quad ds$ 是曲线上弧长元素

$ds = \sqrt{(dx)^2 + (dy)^2}$



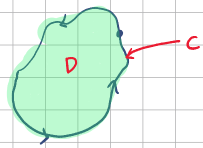
$\Rightarrow \vec{F}\cdot\vec{T}\,ds = \langle P, Q\rangle \langle dx, dy\rangle = P\,dx + Q\,dy$

$\vec{T}\cdot ds = \langle dx, dy\rangle$

$d\vec{r} = \vec{T}\cdot ds$

$\boxed{\text{Theorem 4.2}}$ Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If $\vec{F} = \langle P, Q \rangle$ have continuous partial derivative on an open region that contains D then

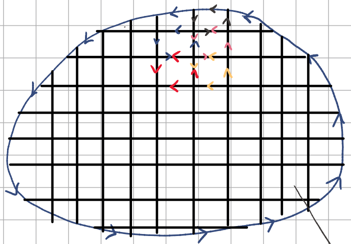
$$\int_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA$$



or

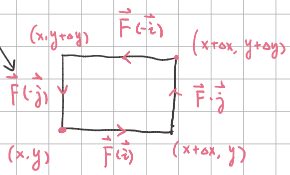
$$\int_C \vec{F} \cdot \underbrace{\vec{T}\,ds}_{d\vec{r}} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Proof:



Inside the curve will all cancel out

idea $\sum$(circulation of points on curve) = $\sum$(circulation of points inside curve)



Top: $\vec{F}(x, y+\Delta y) \cdot (-\vec{i})\,\Delta x = -P(x, y+\Delta y)\Delta x$

Bottom: $\vec{F}(x, y) \cdot (\vec{i})\,\Delta x = P(x, y) \cdot \Delta x$

Right: $\vec{F}(x+\Delta x, y)\,(\vec{j}) \cdot \Delta y = Q(x+\Delta x, y)\,\Delta y$

Left: $\vec{F}(x, y) \cdot (-\vec{j}) \cdot \Delta y = -Q(x, y)\,\Delta y$

$\vec{F} \cdot \vec{i} = P(x, y)$

$\vec{F} \cdot \vec{j} = Q(x, y)$

For $\vec{F} = \langle P, Q \rangle$

$\vec{i} = \langle 1, 0 \rangle \quad \vec{j} = \langle 0, 1 \rangle$

$\boxed{\vec{F} \cdot \vec{i} \cdot \Delta x = \vec{F} \cdot \vec{T} \cdot ds}$

Circulation of $\square$ = Top + Bottom + Right + Left

$$= -P(x, y+\Delta y)\,\Delta x + P(x, y)\,\Delta x + Q(x+\Delta x, y)\,\Delta y + -Q(x, y)\,\Delta y$$

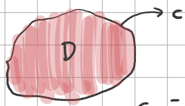
$$= \frac{\left(-P(x, y+\Delta y) + P(x, y)\right)}{\Delta y}\,\Delta y\,\Delta x + \frac{Q(x+\Delta x, y) - Q(x, y)}{\Delta x}\,\Delta x\,\Delta y$$

$$= (-P_y + Q_x)\,\Delta x\,\Delta y = (Q_x - P_y)\,dx\,dy = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

C is parametrized by $\vec{r}(t) = \langle x(t), y(t) \rangle$ $\vec{F} = \langle P, Q \rangle$ then.

$$\int_C \vec{F}\, d\vec{r} = \int_C \langle P, Q \rangle \langle dx, dy \rangle\, dt = \int_C P\,dx + Q\,dy \;\; dt$$

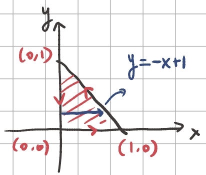
Then we use the Green's Theorem:



$$\int_{C = \partial D} P\,dx + Q\,dy \;\; dt = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

C is boundary of D

$\boxed{\text{Example}}$ Evaluate $\int_C x^4\,dx + xy\,dy$ where C is a triangular curve consisting line segment from (0,0) to (1,0) from (1,0) to (0,1) from (0,1) to (0,0)



$\int_C P\,dx + Q\,dy$ where $P = x^4$ $Q = xy$

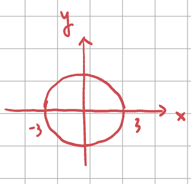
according Green Theorem: $\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$

$$= \iint_D (y - 0)\, dx\,dy = \int_{y=0}^{y=1} \int_{x=0}^{x=1-y} y \; dx\,dy$$

$$= \int_0^1 y(1-y)\,dy = \int_0^1 y - y^2\,dy = \frac{1}{2}y^2 - \frac{1}{3}y^3 \Big|_0^1 = \frac{1}{6}$$

$\boxed{\text{Example}}$

Evaluate $\oint_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^2+1})\,dy$, where $C: x^2 + y^2 = 9$

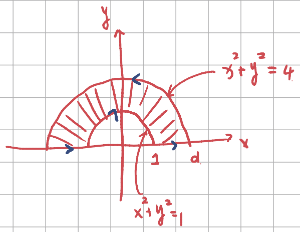


$P = 3y - e^{\sin x}$ $\quad Q = 7x + \sqrt{y^2+1}$

$\frac{\partial P}{\partial y} = 3$ $\quad \frac{\partial Q}{\partial x} = 7$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = 4 \iint_D dx\,dy = 4 \int_0^{2\pi} \int_0^3 r\,dr\,d\theta = 36\pi$$

$\boxed{\text{Example}}$ Evaluate $\oint_c y^2 dx + 3xy\,dy$ c is shown as below



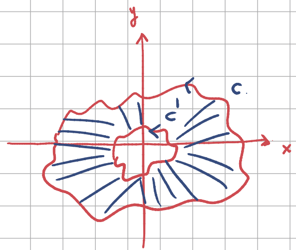
$P = y^2 \quad Q = 3xy$

$\dfrac{\partial P}{\partial y} = 2y \quad \dfrac{\partial Q}{\partial x} = 3y \quad \Rightarrow \iint_D (3y-y)\,dA$

$$= \int_{\theta=0}^{\theta=\pi}\int_{r=1}^{r=2} r^2 \sin\theta\, dr\, d\theta = \int_0^{\pi} \sin\theta\, \frac{7}{3}\, d\theta$$

$$= -\frac{7}{3}\cos\theta\Big|_0^{\pi} = \frac{14}{3}$$

$\boxed{\text{Example}}$ If $\vec{F}(x,y) = \left\langle \dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right\rangle$ show that $\int_c \vec{F}\cdot d\vec{r} = 2\pi$ for every positive oriented simple closed path that enclosed the origin



- at $x=0$ $y=0$ $\vec{F}$ does not exist
- So create curve C & c'
- So we newly created curve such $c \cup c'$ such that it gives me region D

Then $\displaystyle\int_c Pdx+Qdy + \int_{-c'} Pdx+Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D \left(\frac{y^2-x^2}{(x^2+y^2)^2} - \frac{(y^2-x^2)}{(x^2+y^2)^2}\right) dA = 0$

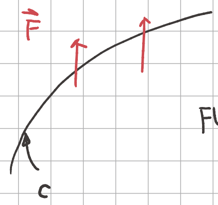
$$\Rightarrow \int_c Pdx+Qdy = -\int_{-c'} Pdx+Qdy = \int_{c'} Pdx+Qdy$$

$$\int_c \vec{F}\,d\vec{r} = \int_{c'} \vec{F}\,d\vec{r} \qquad \vec{r}(t) = \langle a\cos t, a\sin t\rangle \quad t\in[0, 2\pi]$$

$$\therefore \int_c \vec{F}\,d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t))\, r'(t)\, dt = \int_0^{2\pi} dt = 2\pi$$

16.4 (b) Green Theorem

$\boxed{\text{Flux}}$



Flux over the Curve $C: \vec{r}(t) \quad a \le t \le b$

$$\text{Flux} \rightarrow \oint_C \vec{F} \cdot \vec{n} \cdot ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{n} \cdot |r'(t)| \, dt$$

$\vec{n}$ ↓ unit normal vector

$$\text{where} \quad \vec{n} = \pm \left\langle \frac{y'(t)}{|r'(t)|}, -\frac{x'(t)}{|r'(t)|} \right\rangle \quad \begin{cases} \text{upward} \Rightarrow \vec{n} = \langle \_\_, +\_\_ \rangle \\ \text{downward} \Rightarrow \vec{n} = \langle \_\_, -\_\_ \rangle \end{cases}$$

$\boxed{\text{Example}}$ Consider $\vec{F} = \langle x+y, x^2+y^2 \rangle$ What is the downward flux across the line segment from $(0,0)$ to $(-6,0)$?

<Solution>: Parameter equation of $C = \vec{r}(t)$ $\quad \vec{r}(t) = \langle -6t, 0 \rangle \quad \vec{r}'(t) = \langle -6, 0 \rangle$

$\Rightarrow x(t) = -6t \quad y(t) = 0$

$$\therefore \Rightarrow \vec{n} = \pm \left\langle \frac{y'(t)}{|r'(t)|}, \frac{-x'(t)}{|r'(t)|} \right\rangle = \pm \left\langle 0, \frac{6}{6} \right\rangle = \pm \langle 0, 1 \rangle \Rightarrow \text{downward is } \langle 0, -1 \rangle$$

$$\oint \vec{F} \cdot \vec{n} \, ds = \int_0^1 \langle -6t, 36t^2 \rangle \cdot \langle 0, -1 \rangle \cdot |6| \, dt = \int_0^1 -6 \times 36 t^2 \, dt = -6 \times 36 \times \frac{1}{3} t^3 \Big|_0^1 = -72$$

$\boxed{\text{Example}}$

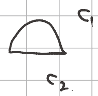
$\vec{F} = \langle x+y, x^2+y^2 \rangle$ downward flux across line segment $(7,0)$ to $(0,-7)$

<solution>: $\vec{r}(t) = \langle 7-7t, -7t \rangle \quad \vec{r}'(t) = \langle -7, -7 \rangle \quad \vec{n} = \pm \left\langle \frac{-7}{7\sqrt{2}}, \frac{+7}{7\sqrt{2}} \right\rangle = \pm \left\langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$

Because we need downward $\vec{n} = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$

Theorem

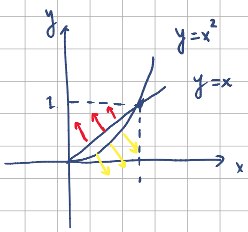
Flux is additive. Let's say $C = C_1 \cup C_2$



Then we can have flux: $\oint_C \vec{F}\cdot\vec{n}\,ds = \oint_{C_1} \vec{F}\cdot\vec{n}\,ds + \oint_{C_2} \vec{F}\cdot\vec{n}\,ds$

Example

Find the outward flux, of the vector field $\vec{F} = \langle xy, y^2 \rangle$ across the region defined by $y = x^2$ and $y = x$ for $0 \le x \le 1$



① upward normal vector

② downward normal vector

① $y = x^2$ $\quad \vec{r}(t) = \langle t, t^2 \rangle \quad \vec{r}'(t) = \langle 1, 2t \rangle$

$\vec{n} = \pm \left\langle \frac{2t}{\sqrt{1+4t^2}}, \frac{-1}{\sqrt{1+4t^2}} \right\rangle$ for downward $\Rightarrow \vec{n} = \left\langle \frac{+2t}{\sqrt{1+4t^2}}, \frac{-1}{\sqrt{1+4t^2}} \right\rangle$

$$\oint_{C_1} \vec{F}\cdot\vec{n}\,ds = \int_0^1 \vec{F}(r(t))\cdot\vec{n}\cdot|\vec{r}'(t)|\,dt = \int_0^1 \langle t^3, t^4 \rangle \cdot \left\langle \frac{+2t}{\sqrt{1+4t^2}}, \frac{-1}{\sqrt{1+4t^2}} \right\rangle \sqrt{1+4t^2}\,dt$$

$$= \int_0^1 \frac{t^4}{\sqrt{1+4t^2}} \cdot \sqrt{1+4t^2}\,dt = \frac{1}{5}t^5 \Big|_0^1 = \frac{1}{5}$$

② $y = x$ $\quad \vec{r}(t) = \langle t, t \rangle \quad \vec{r}'(t) = \langle 1, 1 \rangle \quad \vec{n} = \pm \left\langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right\rangle$

for upward. $\vec{n} = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$

$$\oint_{C_2} \vec{F}\cdot\vec{n}\,d\vec{s} = \int_0^1 \langle t^2, t^2 \rangle \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \sqrt{2}\cdot dt = 0$$

$$\Rightarrow \oint_C \vec{F}\cdot\vec{n}\,d\vec{s} = \oint_{C_1} \vec{F}\cdot\vec{n}\,ds + \oint_{C_2} \vec{F}\cdot\vec{n}\,ds = \frac{1}{5}$$

Vector form of Green's Theorem:

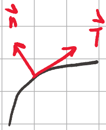
Let $\vec{F} = P\vec{i} + Q\vec{j} \Rightarrow \oint_C \vec{F}\cdot d\vec{r} = \oint P\,dx + Q\,dy = \iint_D \operatorname{curl}(\vec{F})\cdot\vec{k}\,dA$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

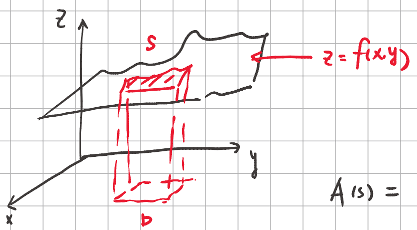
Calculate flux using green theorem

$\vec{F} = P\vec{i} + Q\vec{j} \qquad C = \vec{r}(t) = (x(t), y(t))$

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_D \operatorname{div}\vec{F}\,dA = \iint_D \nabla\cdot\vec{F}\,dA$$



15.6 / 16.6 Surface Area



The area of surface with equation

$z = f(x,y) \quad (x,y)\in D$, where f_x f_y continuous

$$A(S) = \iint_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\;dA$$

Suppose $z = f(x,y)$. $\quad A(S) = \iint_D \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\;dA$

$\boxed{6}$ **Definition** If a smooth parametric surface S is given by the equation

$$\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k} \qquad (u,v)\in D$$

and S is covered just once as (u,v) ranges throughout the parameter domain D, then the **surface area** of S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\,dA$$

where $\quad \mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \qquad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$

16.7 Surface Integral

Parametric Surface $\vec{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v) \rangle$

$$\vec{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \qquad \vec{r}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

$$\iint_S f(x,y,z)\, dS = \iint_D f(\vec{r}(u,v))\, |\vec{r}_u \times \vec{r}_v|\, dA$$

$\boxed{\text{Example}}$

Compute surface integral $\iint_S x^2 ds$ where S is unit sphere $\rho = 1$

<Solution>: $x = \sin\phi\cos\theta \quad y = \sin\phi\sin\theta \quad z = \cos\phi$

$0 \le \phi \le \pi \quad 0 \le \theta \le 2\pi \quad \vec{r}(\theta,\phi) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle$

$|\vec{r}_\theta \times \vec{r}_\phi| = \sin\phi \qquad \iint_S x^2 ds = \int_0^{\pi}\int_0^{2\pi} (\sin\phi\cos\theta)^2 \sin\phi\, d\phi\, d\theta = \frac{4\pi}{3}$

Any surface $z = g(x,y)$ $\quad x = x,\ y = y,\ z = g(x,y)$

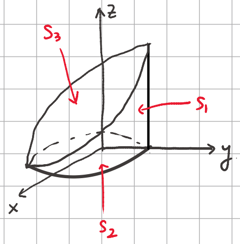
$$\iint_S f(x,y,z)\, dS = \iint_D f(x,y,g(x,y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\ dA$$

$\boxed{\text{Example}}$ $\iint_S y\, ds$ where S is surface $z = x + y^2$. $0 \le x \le 1,\ 0 \le y \le 2$.

<Solution>: $\frac{\partial z}{\partial x} = 1 \qquad \frac{\partial z}{\partial y} = 2y$

$$\iint_S y\, ds = \int_0^1\int_0^2 y\sqrt{y^2+2}\ dy\, dx = \frac{13\sqrt{2}}{3}$$

Example



$S = S_1 \cup S_2 \cup \cdots \cup S_n$

$$\iint_S f(x,y,z)\,dS = \iint_{S_1} f(x,y,z)\,dS + \cdots + \iint_{S_n} f(x,y,z)\,dS$$

$\iint_S z\,dS$ where $S_1 \Rightarrow$ cylinder $x^2+y^2=1$

$S_2 \Rightarrow$ disk $x^2+y^2 \le 1$

$S_3 \Rightarrow$ part of plane $z = 1+x$

$\boxed{S_1}$ $x^2+y^2=1$ $x=\cos\theta$, $y=\sin\theta$ $z=z$

$0 \le \theta \le 2\pi$ $0 \le z \le 1+\cos\theta$

$$\Rightarrow \iint_S z\,dS = \int_0^{2\pi}\int_0^{1+\cos\theta} z\,|\vec{r}_\theta \times \vec{r}_z|\,dz\,d\theta = \frac{3\pi}{2}$$

$\vec{r}(\theta,z) = \langle \cos\theta, \sin\theta, z \rangle$ $|\vec{r}_\theta \times \vec{r}_z| = 1$

$\boxed{S_2}$ $x^2+y^2 \le 1$ $x = r\cos\theta$ $y = r\sin\theta$ $z=0$ $\Rightarrow \iint_S z\,dA = \iint_D 0\,dA = 0$

$0 \le \theta \le 2\pi$ $0 \le r \le 1$

$\boxed{S_3}$ $z = 1+x \Rightarrow \frac{\partial z}{\partial x} = 1$ $\frac{\partial z}{\partial y} = 0$ $f(x,y,z) = z$

$$\iint_S z\,dS = \iint (1+x)\sqrt{1+1+0}\,dA = \int_0^{2\pi}\int_0^1 (1+r\cos\theta)\,r\sqrt{2}\,dr\,d\theta = \sqrt{2}\pi$$

Therefore: all answer is $\sqrt{2}\pi + \frac{3\pi}{2}$

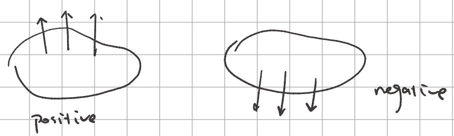
Cylinder : oriented

Mobius strip : non-oriented.

two natural orientation

Upward

downward



$\boxed{\text{Def}^n}$ If f is a continuous field defined on an oriented surface S

with unit normal vector $\vec{n}$, then the surface integral of $\vec{F}$ over S

$$\iint_S \vec{F}\cdot d\vec{s} = \iint_S \vec{F}\cdot\vec{n}\cdot ds$$

Flux of F across S

If S has parametrization $\vec{r}(u,v)$ $\quad \iint_S \vec{F}\, d\vec{s} = \iint_D \vec{F}\cdot(\vec{r_u}\times\vec{r_v})\, dA$ ①

$$\vec{n} = \frac{-\frac{\partial g}{\partial x}\vec{i} - \frac{\partial g}{\partial y}\vec{j} + \vec{k}}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}} = \frac{\vec{r_u}\times\vec{r_v}}{|\vec{r_u}\times\vec{r_v}|}$$

Proof ①: $\displaystyle \iint_S \vec{F}\, d\vec{s} = \iint_S \vec{F}\cdot\vec{n}\, ds = \iint_S \vec{F}\cdot\frac{\vec{r_u}\times\vec{r_v}}{|\vec{r_u}\times\vec{r_v}|}\, ds = \iint_D \vec{F}\cdot\frac{\vec{r_u}\times\vec{r_v}}{|\vec{r_u}\times\vec{r_v}|}\cdot|\vec{r_u}\times\vec{r_v}|\, ds$

$$= \iint_D \vec{F}\cdot(\vec{r_u}\times\vec{r_v})\, ds$$

$\boxed{z = g(x,y)}$

$$\mathbf{F}\cdot(\mathbf{r}_x\times\mathbf{r}_y) = (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k})\cdot\left(-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}\right)$$

Thus Formula 9 becomes

10 $$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$$

For upward flux

For downward flux we need to add (−)

16.8 Stoke's Theorem

$\boxed{\text{Stoke's Theorem}}$

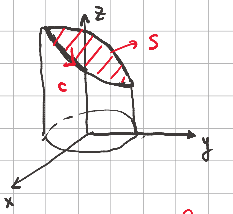
Let S be oriented piecewise smooth surface, that is bounded by a simple, closed, piecewise smooth bounding curve C with positive orientation

Let $\vec{F}$ be a vector field whose component have continuous partial derivative on open region in $\mathbb{R}^3$ that contain S, then :

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}(\vec{F})\cdot d\vec{S} = \iint_S \nabla\times\vec{F}\cdot d\vec{S} \qquad C = \partial S$$

Since $\int_C \vec{F}\,d\vec{r} = \int_C \vec{F}\cdot\vec{T}\,ds \qquad \iint_S \nabla\times\vec{F}\cdot d\vec{S} = \iint_S \nabla\times\vec{F}\cdot\vec{n}\,dS$

$\boxed{\text{Example}}$ Evaluate $\int_C \vec{F}\,d\vec{r}$ where $\vec{F} = \langle -y^2, x, z^2\rangle$ and C is the curve of the intersection of the plane $y+z=2$ & cylinder $x^2+y^2=1$



We know $\iint_S \vec{F}\,d\vec{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$

$\vec{F} = \langle -y^2, x, z^2\rangle \qquad \text{curl}(\vec{F}) = \langle 0, 0, 1+2y\rangle$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}(\vec{F})\cdot d\vec{S} = \iint_S \langle 0,0,1+2y\rangle\cdot d\vec{S} = \iint_S (1+2y)\,dA$$

$g(x,y) = z = 2-y$

$$= \iint_S \left(-0\,\frac{\partial g}{\partial x} - 0\,\frac{\partial g}{\partial y} + 1+2y\right) dA$$

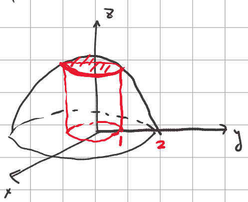
$$= \int_0^{2\pi}\int_0^1 (1+2r\sin\theta)\, r\,dr\,d\theta = \pi$$

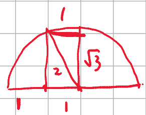
$\boxed{\text{Example 2}}$ Use Stoke's Theorem to complete $\iint_S \text{curl}(\vec{F})\cdot d\vec{S}$ where $\vec{F} = \langle xz, yz, xy \rangle$

S - part of the sphere $x^2+y^2+z^2=4$ that is inside the cylinder $x^2+y^2=1$ above xy-plane

$\vec{r}'(t) = \langle -\sin t, \cos t, 0 \rangle dt$

<Solution>





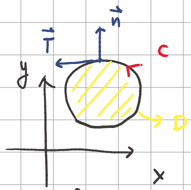
C: $\vec{r}(t) = \langle \cos t, \sin t, \sqrt{3} \rangle$

$\vec{F} = \langle \sqrt{3}\cos t, \sqrt{3}\sin t, \sin t\cos t \rangle$

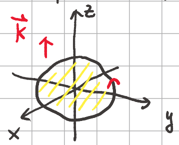
$$\iint_S \text{curl}(\vec{F})\, d\vec{S} = \int_C \vec{F}\cdot d\vec{r}$$

$$= \int_0^{2\pi} \langle \sqrt{3}\cos t, \sqrt{3}\sin t, \sin t\cos t \rangle \langle -\sin t, \cos t, 0 \rangle dt = 0$$

Green Theorem: $\int_C \vec{F}\cdot d\vec{r} = \iint_D (\text{curl } \vec{F})\cdot \vec{k}\, dA$



(C is closed curve that bounds D in xy plane)



Some Transformation about Green Theorem:

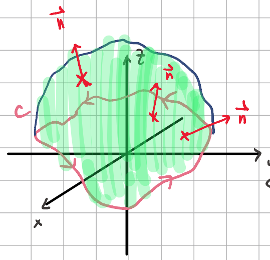
(A) $\int_C \vec{F}\, d\vec{r} = \int_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\, dA = \int_C \vec{F}\cdot\vec{T}\, ds$

(B) $\int_C \vec{F}\cdot\vec{n}\, ds = \int_C \vec{F}\cdot d\vec{s} = \int_a^b \vec{F}(\vec{r}(t))\, \vec{n}\cdot|\vec{r}'(t)|\, dt = \iint_D \nabla \vec{F}\, dA$

$$= \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA = \iint_D \text{div } \vec{F}\, dA$$

In this case: $\vec{n} = \left\langle \frac{y'(t)}{|r'(t)|}, -\frac{x'(t)}{|r'(t)|} \right\rangle$ downward

Stokes Theorem:



$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\text{curl } \vec{F}) \cdot \vec{n} \, dS$$

Divergence Theorem:

E $\longrightarrow$ Simple solid region

S $\longrightarrow$ boundary surface of E

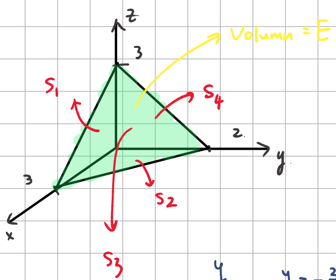
positive (outward) orientation

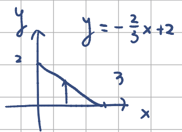
$$\iint_S \vec{F}\cdot\vec{n} \, dS = \iiint_E \text{div } \vec{F} \, dV$$

Ex: E is a Solid region in the first octant that lies beneath the plane $2x+3y+2z=6$

Let S be the boundary of E (S consists of four Triangles) If $\vec{F} = \langle x^2, y^2, z^2 \rangle$

use divergence Theorem to write





$$\iint_S \vec{F}\cdot\vec{n} \, dS = \iiint \text{div}(\vec{F}) \, dV$$

$$= \iiint 2x+2y+2z \, dV$$

$$= \int_0^3 \int_0^{-\frac{2}{3}x+2} \int_0^{3-x-\frac{3}{2}y} (2x+2y+2z) \, dz \, dy \, dx$$

Flux

$$\iint_S \vec{F}\, d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\, dS = \iint_D \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\, dA$$

If S has parametrazation $\vec{r}(u,v)$

$$\vec{n} = \frac{-\frac{\partial g}{\partial x}\vec{i} - \frac{\partial g}{\partial y}\vec{j} + \vec{k}}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

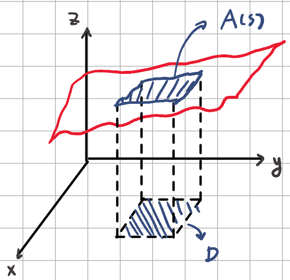
$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$$

For upward

$$\vec{F}\cdot(\vec{r}_x \times \vec{r}_y) = (P\vec{i} + Q\vec{j} + R\vec{k})\cdot\left(-\frac{\partial g}{\partial x}\vec{i} - \frac{\partial g}{\partial y}\vec{j} + \vec{k}\right)$$

$$z = g(x,y)$$

Area Surface Summarize



① $z = g(x,y)$ $\quad (x,y) \in D$

$$A(S) = \iint_D \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\; dA$$

② $\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}$, $(u,v) \in D$

$$A(S) = \iint_D |\vec{r}_u \times \vec{r}_v|\, dA$$

Surface Integral

① $\vec{r}(u,v) = X(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}$

$$\iint_S f(x,y,z)\, dS = \iint_D f(\vec{r}(u,v))\, |\vec{r}_u \times \vec{r}_v|\, dA$$

② $z = g(x,y)$

$$\iint_S f(x,y,z)\, dS = \iint_D f(x,y,g(x,y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\quad dA$$