MTH 235

Lec1 :

Separable differential equations

Def : [A first order differential equation for $y(t)$ is

$$y'(t) = f(t, y(t)) \qquad (1)$$

where f is given and $y' = \frac{dy}{dt}$]

Remarks : *. A differential eq. is an equation for a function and its derivatives

*. Eq (1) is called normal form of this equation.

Example :

(1) $y' - 2y = 0$ $\Rightarrow$ $y' = 2y$

(2) $y' = 2y + 3$ ✓

(3) $y' = y + t$ ✓

(4) $y' = \sin(t)\, y$ ✓

(5) $3ty\, y' - y^2 - ty + t^2 = 0$ $\Rightarrow$ $3ty \cdot y' = y^2 + ty - t^2 \Rightarrow y' = \frac{y^2 + ty - t^2}{3ty}$ normal form

Remarks : a). y : dependent variable $\left.\begin{matrix} t \\ x \end{matrix}\right\}$ independent variable

b) $y(t)$ is called the unknown function

c). ODE : ordinary differential equation $F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^ny}{dx^n}\right) = 0$

x – independent

y – dependent

d) PDE : partial differential equation $F\left(x_1, x_2, \cdots, x_n, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \cdots, \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \cdots\right) = 0$

where $x_1, x_2, \cdots, x_n$ are the independent variable.

Def: An initial value problem (IVP) is to solve a differential eq.

$$y'(t) = f(t, y(t))$$

which also satisfy an initial condition $y(t_0) = y_0$.

where t_0, y_0 are given numbers

Seperate equations

Def: A seperable differential equation is

$$h(y)\, y' = g(t)$$

where h, g are given function

Ex: $$y' = \frac{2t}{y+t^2 y} \qquad y(0) = 3.$$

solution: $$y' = \frac{2t}{y(1+t^2)} \Rightarrow y \cdot \frac{dy}{dt} = \left(\frac{2t}{1+t^2}\right) \Rightarrow y\,dy = \left(\frac{2t}{1+t^2}\right) dt$$

$$\int y\,dy = \int \frac{2t}{1+t^2}\,dt \Rightarrow \frac{1}{2}y^2 = \int \frac{2t}{1+t^2}\,dt \underset{d\varphi = 2t\,dt}{\overset{\varphi = 1+t^2}{=\!=}} \int \frac{d\varphi}{\varphi} = \ln|\varphi| = \ln|1+t^2| + C$$

$$\Rightarrow y = \pm\sqrt{2\ln(1+t^2) + C} \Rightarrow y(t) = \pm\sqrt{2\ln(1+t^2) + C}$$

$$y(0) = \pm\sqrt{2\ln(1) + C} = 3 \Rightarrow \pm\sqrt{C} = 3 \Rightarrow C = 9 \Rightarrow \boxed{y(t) = \pm\sqrt{2\ln(1+t^2) + 9}}$$

Theorem: All solution of $h(y)\, y' = g(t)$ are

$H(y) = G(t) + C$, where $H(y) = \int h(y)\, dy$ $G(t) = \int g(t)\, dt$ (or $y = H^{-1}(G(t)+c)$)

Proof: $\int h(y(t))\, y'(t)\, dt = \int g(t)\, dt$ Subsitude $y = y(t)$

$dy = y'(t)\, dt$

$$\underbrace{\int h(y)\, dy}_{H(y) + C_2} = \underbrace{\int g(t)\, dt}_{} \to G(t) + C_1$$

$\Rightarrow H(y) = G(t) + C$ (implicit form)

$\Rightarrow y = H^{-1}(G(t) + C)$ (explicit form

Previous example: $y \cdot y' = \frac{2t}{1+t^2}$ $h(y) = y \Rightarrow H(y) = \frac{1}{2}y^2$. $\Rightarrow \ldots$

$g(t) = \frac{2t}{1+t^2} \Rightarrow G(t) = \ln(1+t^2)$

Lec2: population model * exponential Growth - Decay

* the logistic equation.

① exponential growth - decay

$y' = ay$ < $a>0$ Exp Growth ; $a<0$ Exp Decay

with linear term:

$y' = ay + b$ a, b Constant

Notation: $r = |a|$ $b=0$ then. $(r>0)$

ex: $y' = 2y + 3$

Sol: $\frac{dy}{dt} = 2y+3 \Rightarrow \int \frac{1}{2y+3}\, dy = \int dt \Rightarrow \frac{1}{2}\ln|2y+3| = t + c \Rightarrow \ln|2y+3| = 2t + c$

$\Rightarrow |2y+3| = ce^{2t} \Rightarrow |y + \frac{3}{2}| = ce^{2t} \Rightarrow y = -\frac{3}{2} \underbrace{\pm c}_{\text{combine}} e^{2t} = -\frac{3}{2} + ce^{2t}$

Ex: $y' = ay + b$, $y(0) = y_0 \Rightarrow y(t) = \left(y_0 + \frac{b}{a}\right)e^{at} - \frac{b}{a}$

$a \neq 0$

Proof: $\frac{dy}{dx} = ay + b$ ($y \neq -\frac{b}{a}$) $\Rightarrow \int \frac{1}{ay+b}\,dy = \int dx \Rightarrow \frac{1}{a}\ln|ay+b| + C_1 = x + C_2$

$\Rightarrow \frac{1}{a}\ln|ay+b| = x + C_3 \Rightarrow \ln|ay+b| = ax + C_4 \Rightarrow ay + b = C_5 e^{ax} \Rightarrow y + \frac{b}{a} = C_6 e^{ax}$

$\Rightarrow y = C_7 e^{ax} - \frac{b}{a}$ For $y(0) = C_7 - \frac{b}{a} = y_0 \Rightarrow C_7 = y_0 + \frac{b}{a}$

$$\therefore \boxed{y(t) = \left(y_0 + \frac{b}{a}\right)e^{ax} - \frac{b}{a}}$$

- if $a > 0$, $b > 0 \Rightarrow y(t) \xrightarrow{t \to \infty} \infty$
- if $a > 0$ $b < 0 \Rightarrow \begin{cases} \text{if } y_0 > -\frac{b}{a} \Rightarrow p(t) \xrightarrow{t \to \infty} \infty \\ \text{if } y_0 < -\frac{b}{a} \Rightarrow p(t) \xrightarrow{t \to \infty} -\infty \end{cases}$

Ex: $P'(t) = rP(t)\left(1 - \frac{P(t)}{P_c}\right)$ where $P_c > 0$ is fixed // logistic equation

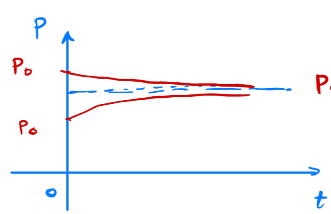
Sol: $\frac{dP}{dt} = rP - \frac{rP^2}{P_c} \Rightarrow \int \frac{1}{rP - \frac{rP^2}{P_c}}\,dP = \int dt \Rightarrow \int \frac{P_c}{rP_c \cdot P - rP^2}\,dP = \int dt$

$\Rightarrow \frac{P_c}{r}\int \frac{1}{P_c \cdot P - P^2}\,dP = t + C_1$ $\frac{P_c}{r} \cdot \int \frac{1}{P(P_c - P)}\,dP = \frac{P_c}{r} \cdot \frac{1}{P_c}\int \left(\frac{1}{P} + \frac{1}{P_c - P}\right)dP$

$\Rightarrow \frac{1}{r}\left[\int \frac{1}{P}\,dP + \int \frac{1}{P_c - P}\,dP\right] = \frac{1}{r}\left[\ln|P| - \ln|P_c - P|\right] = \frac{1}{r}\ln\left|\frac{P}{P_c - P}\right| \Rightarrow \ln\left|\frac{P}{P_c - P}\right| = rt + C_2$

$\Rightarrow \frac{P}{P_c - P} = C_3 e^{rt} \Rightarrow P = P_c C_3 e^{rt} - P C_3 e^{rt} \Rightarrow (1 + C_3 e^{rt}) \cdot P = P_c \cdot C_3 \cdot e^{rt}$

$\Rightarrow P(t) = \frac{P_c C_3 e^{rt}}{1 + C_3 e^{rt}} = \frac{P_c e^{rt}}{C_4 + e^{rt}}$

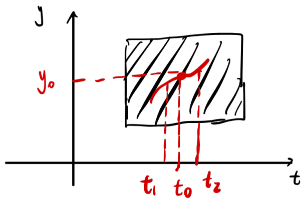


$\lim_{t \to \infty} P(t) = P_c$

Existance and uniqueness

Consider $y'(t) = f(y(t), t)$ $\quad y(t_0) = y_0$ $(*)$, assume $f(t,y)$ is Continuous in t.

and differentiate in y, on the rectangle (t,y) plane, Contain (t_0, y_0) in its interior



Theorem (Picard-lindelof):

There is $(t_1, t_2) \ni t_0$ Such that $(*)$, has a unique solution on (t_1, t_2) (This is Called a local solution)

Theorem: Let $\tilde{y}'(t) = f(t, \tilde{y}(t))$, $\tilde{y}(t_0) = \tilde{y}_0 \neq y_0$, then $\tilde{y}$ and y do not intersect.

(唯一性说明)

Rem: If $|f(t,y)| \leq M(1+|y|)$ for all t, then solution is global.

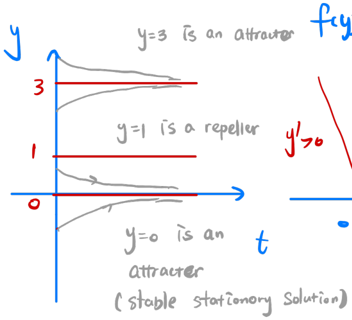
(解的延拓)

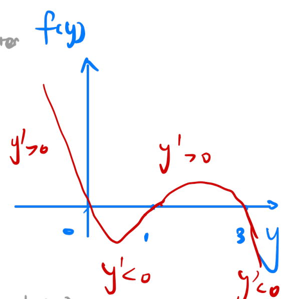
Autonomous equation

Def: $y' = f(y)$ is Called autonomous f is depend only on y

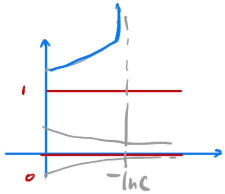
Ex: $y' = -y(1-y)(1-\frac{1}{3}y) = f(y)$ $\qquad f(y) = 0$ if $y=0$ or $y=1$ or $y=3$

$$\begin{cases} y(t) = 0 & \forall t \\ y(t) = 1 & \forall t \\ y(t) = 3 & \forall t \end{cases} \Rightarrow \text{are Stationary solutions}$$





Ex: $y' = y(y-1)$



$y(t) = \frac{1}{1-ce^t}$ $\quad (c \in \mathbb{R})$

$y(0) = \frac{1}{1-c} = y_0 \Rightarrow c = 1 - \frac{1}{y_0}$

Take $y_0 > 1$. Then $c > 0$ Then $1 - ce^t \neq 0 \Rightarrow t \in [0, -\ln c)$

Ex: Solve: $y' = \frac{3}{t}\cdot y + t^5$, $t > 0$

通解 $y(t) = v(t)\cdot y_h$ $\quad$ y_h 是 $y' = \frac{3}{t}y$ 的解

$$y_h' = \frac{3}{t} y_h \Rightarrow \frac{y_h'}{y_h} = \frac{3}{t} \Rightarrow \int \frac{y_h'}{y_h}dt = \int \frac{3}{t}dt \Rightarrow \int \frac{1}{y_h}dy_h = \int \frac{3}{t}dt$$

$$\Rightarrow \ln|y_h| = 3\ln|t| + C \Rightarrow y_h = \left(e^{\ln|t|}\right)^3 \cdot e^C \Rightarrow y_h = Ct^3 \Rightarrow y_h = t^3$$

$y(t) = v(t)\, t^3 \qquad y' = v't^3 + 3vt^2$

$$v't^3 + \cancel{3vt^2} = \frac{3}{t}(vt^3) + t^5 = \cancel{3vt^2} + t^5$$

$$\Rightarrow v' = t^2 \Rightarrow \frac{dv}{dt} = t^2 \Rightarrow v = \frac{1}{3}t^3 + c \Rightarrow y' = \left(\frac{1}{3}t^3 + c\right)t^3$$

$$\Rightarrow \boxed{y' = \frac{1}{3}t^6 + c_0 t^3}$$

$y' = a(t)y + b(t) \Rightarrow y = v(t)\, y_h$, where $y_h' = a(t)\, y_h$.

Ex2: $y' = P(t)\,y + q(t)\,y^n$ Bernoulli equation

and setting $v = \frac{1}{y^{n-1}}$

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以下是一个具体的实例，帮助你理解如何解决这种类型的方程。

方程实例

我们考虑以下 Bernoulli 方程：

$$y' = \frac{2}{t}y + t^2y^2$$

其中：

- $P(t) = \frac{2}{t}$,
- $Q(t) = t^2$,
- $n = 2$。

解法步骤

1. 写成标准形式

将方程改写为标准形式：

$$y' - \frac{2}{t}y = t^2y^2.$$

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2. 变量代换

我们使用代换 $z = y^{1-n} = y^{1-2} = \frac{1}{y}$，因此：

$$z = \frac{1}{y}, \quad y = \frac{1}{z}.$$

对 z 求导：

$$y' = -\frac{1}{z^2}z'.$$

将 y 和 y' 代入原方程：

$$-\frac{1}{z^2}z' - \frac{2}{t}\cdot\frac{1}{z} = t^2\cdot\frac{1}{z^2}.$$

整理后：

$$-\frac{1}{z^2}z' = \frac{2}{t}\cdot\frac{1}{z} + t^2\cdot\frac{1}{z^2}.$$

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3. 解线性方程

这是一个线性微分方程：

$$\boxed{z' + \frac{2}{t}z = -t^2.}$$

(a) 找积分因子：
积分因子 $\mu(t)$ 为：

$$\mu(t) = e^{\int \frac{2}{t}dt} = e^{2\ln t} = t^2.$$

(b) 乘以积分因子：
将整个方程乘以 t^2：

$$t^2z' + 2tz = -t^4.$$

左边是一个导数的展开：

$$\frac{d}{dt}(t^2z) = -t^4.$$

(c) 积分：
对两边积分：

$$t^2z = \int -t^4\,dt = -\frac{t^5}{5} + C.$$

解得：

$$z = -\frac{t^3}{5} + \frac{C}{t^2}.$$

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4. 还原 y

根据代换 $z = \frac{1}{y}$，可以得到：

$$\frac{1}{y} = -\frac{t^3}{5} + \frac{C}{t^2}.$$

因此：

$$y = \frac{1}{-\frac{t^3}{5} + \frac{C}{t^2}}.$$

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$$z' = -\frac{2}{t}z - t^2$$

$$z = v(t)\,z_h$$

$$z_h' = -\frac{2}{t}z_h$$

$$\int \frac{z_h'}{z_h}\,dt = \int -\frac{2}{t}\,dt$$

$$\ln(z_h) = -2\ln(t) + C$$

$$z_h = t^{-2}$$

$$z = \frac{v(t)}{t^2}$$

$$z' = \frac{v'\cdot t^2 - v\cdot 2t}{t^4} = \frac{v't - 2v}{t^3}$$

$$\frac{v't - 2v}{t^3} = -\frac{2}{t}\frac{v}{t^2} - t^2$$

$$(v't - \cancel{2v}) = -\cancel{2v} - t^5$$

$$v' = -t^4$$

$$v = -\frac{1}{5}t^5 + C$$

$$z = -\frac{1}{5}t^3 + \frac{C}{t^2}$$

$$y = \frac{1}{z} = \frac{1}{-\frac{t^3}{5} + \frac{C}{t^2}}$$

一阶线性微分方程通解

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线性微分方程的一般形式是：

$$y' + P(t)y = Q(t)$$

$$\frac{dy}{dt} = -P(t)y + Q(t)$$

通解的步骤：

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1. 找到积分因子

积分因子 $\mu(t)$ 是：

$\mu(t) = e^{\int P(t)\,dt}.$

$$y'\mu(t) + \mu(t)P(t)y = \mu(t)Q(t)$$

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2. 乘以积分因子

将方程两边都乘以积分因子 $\mu(t)$，目的是将左边化成一个可积的形式：

$\mu(t)y' + \mu(t)P(t)y = \mu(t)Q(t).$

左边可以写成：

$\frac{d}{dt}(\mu(t)y) = \mu(t)Q(t).$

$$\mu(t)\cdot y' + y\cdot \mu'(t)$$

Since $\mu(t) = e^{\int p(t)\,dt}$

$\mu'(t) = e^{\int p(t)\,dt} \cdot p(t)$

$= \mu(t)\,p(t)$

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3. 积分求解

对两边积分：

$$\mu(t)y = \int \mu(t)Q(t)\,dt + C,$$

其中 C 是积分常数。

解得：

$$y = \frac{1}{\mu(t)}\left(\int \mu(t)Q(t)\,dt + C\right).$$

这就是线性微分方程的**通解**。

举例：

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Scale invarint equation

一个微分方程是 **scale-invariant (尺度不变的)**，如果通过对变量进行缩放变换，方程形式保持不变。

假设方程是：

$F(t, y, y', y'', ...) = 0,$

如果我们对变量 t 和 y 进行以下缩放：

$t \to \lambda t, \quad y \to \lambda^k y,$

其中 $\lambda > 0$ 是一个缩放因子，且经过这种变换后，方程的形式不变，则称其为尺度不变方程。

解释

1. **缩放规则：**
   - 时间变量 t 被缩放为 λt；
   - 解函数 $y(t)$ 被缩放为 $\lambda^k y(t)$，其中 k 是一个标度指数，描述 y 随着 t 的缩放如何变化。
2. **形式不变性：**
如果在变量缩放后，方程中的所有项比例相等，则方程是尺度不变的。例如，在物理中，很多基本方程（如牛顿运动方程、热传导方程）满足某种形式的尺度不变性。

简单例子

1. 尺度不变的方程：

考虑方程：

$y' = \frac{y}{t}.$

对变量进行缩放 $t \to \lambda t$ 和 $y \to \lambda^k y$，此时：

$y' = \frac{dy}{dt} \to \frac{d(\lambda^k y)}{d(\lambda t)} = \frac{\lambda^k y'}{\lambda}.$

将缩放代入方程：

$\frac{\lambda^k y'}{\lambda} = \frac{\lambda^k y}{\lambda t}.$

两边的 λ^k 和 λ 抵消后，方程形式保持不[illegible] 因此，这是一个**尺度不变方程**。

1.5 Modeling Application.

Review logistic equation: $P'(t) = r\,P(t)\cdot\left(1-\frac{P(t)}{K}\right)$ $\boxed{r=3}$

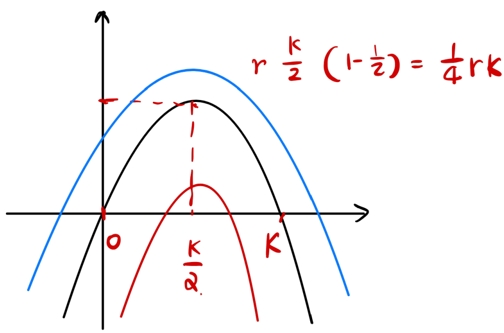
$r>0$ Const. growth rate. $\left[\frac{1}{\text{time}}\right]$

$K>0$ Const Carry Capacity [individuals]

$P' = rP\left(1-\frac{P}{K}\right)+M.$

- $M>0$ immigration rate. $= I = M.$
- $M<0$ emigration rate $= E = -M$
- hunting rate $= H = -M$

$f(P) = rP\left(1-\frac{P}{K}\right)$



$P' = rP\left(1-\frac{P}{K}\right)+M.$

① $M = I > 0.$

$$P_\pm = \frac{1}{2}\left(K \pm \sqrt{K^2+a^2}\right). \quad a^2 = \frac{4KI}{r}$$

② $M = -E < 0$, $E>0$

$$a^2 = \frac{4K|M|}{r}.$$

$$P_\pm = \frac{1}{2}\left(K \pm \sqrt{K^2-a^2}\right) \quad a^2 = \frac{4KE}{r}$$

Remark: $P_\pm = \frac{1}{2}\left(K \pm \sqrt{K^2-a^2}\right).$ $a^2 = \frac{4KE}{r}$ [emigration case]

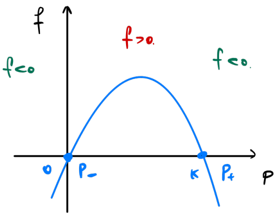
① $K^2-a^2>0$, $P_+ > P_- > 0 \Rightarrow 0 < E < \frac{rK}{4}.$

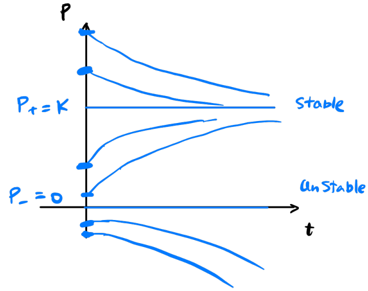
② $K^2-a^2=0$ $P_+ = P_- = \frac{K}{2}. \Rightarrow E = \frac{rK}{4}.$

③ $r^2-a^2<0$ No real P_+. P_- $\Rightarrow E > \frac{rK}{4}$

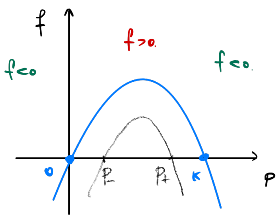
Remark: In emigration case. $f(P) = rP(1-\frac{P}{K}) - H$. ($H$: hunting ≥ 0)

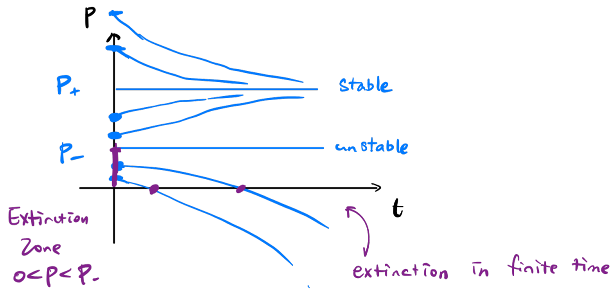
(1) $H = 0$



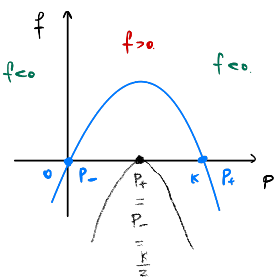


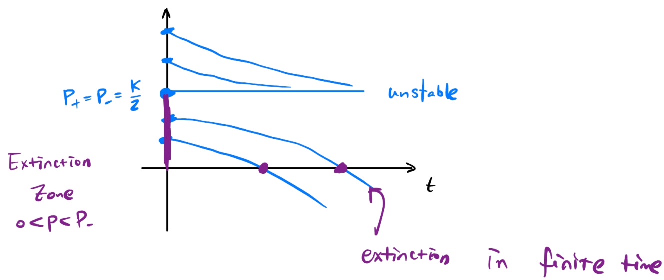
(2) $0 < H < \frac{rK}{4}$



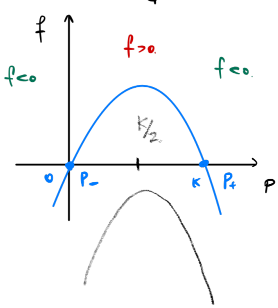


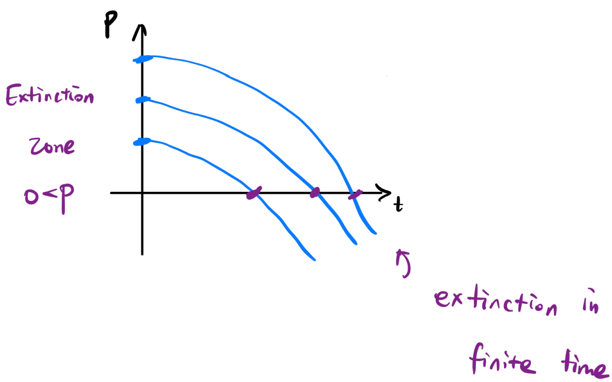
(3) $H = \frac{rK}{4}$





(4) $H > \frac{rK}{4}$





1.6 linear differential equation

$$y' = a(t)\, y + b(t)$$

$\ast$ linear variable coefficient - Homogeneous $(b=0)$

$$y' = a(t)\, y \qquad (b=0)$$

$$\int \frac{y'(t)\, dt}{y(t)} = \int a(t)\, dt \qquad A(t) = \int a(t)\, dt$$

$$\int \frac{1}{y(t)}\, dy(t) = \int a(t)\, dt = A(t)$$

$$\Rightarrow \ln(|y(t)|) + C_0 = A(t) \Rightarrow y = C_1 e^{A(t)}$$

$\ast$. Linear variable Coefficient - Non-Homogeneous

$$y' = a(t)\, y + b(t)$$

① $$\mu(t) = e^{\int -a(t)\, dt}$$

$$\Rightarrow y' \cdot \mu(t) = a(t) \cdot y \cdot \mu(t) + b(t)\, \mu(t)$$

$$\Rightarrow y' \cdot \mu(t) - a(t)\, y \cdot \mu(t) = b(t)\, \mu(t)$$

For $$\frac{d}{dt}(\mu(t) \cdot y) = y' \mu(t) + y \frac{d}{dt}(\mu(t))$$

$$= y' \cdot \mu(t) + y \frac{d}{dt}\left(e^{\int -a(t)\, dt}\right)$$

$$= y' \cdot \mu(t) + y\, \mu(t) \cdot [-a(t)]$$

$$\Rightarrow \frac{d}{dt}(\mu(t)\, y) = b(t)\, \mu(t)$$

$$\Rightarrow \mu(t)\, y = \int b(t)\, \mu(t)\, dt + C$$

$$\Rightarrow \boxed{y = \frac{1}{\mu(t)}\left[\int b(t)\, \mu(t)\, dt + C\right] \quad \text{where } \mu(t) = \int e^{-a(t)} dt}$$

$$\star \int e^{at} \cos(bt)\, dt = \frac{e^{at}(a\cos(bt) + b\sin(bt))}{a^2 + b^2}$$

$$\star \int e^{at} \sin(bt)\, dt = \frac{e^{at}(a\sin(bt) - b\cos(bt))}{a^2 + b^2}$$

1.11 Approximate Solutions

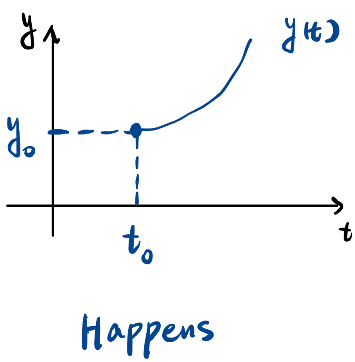
existence - uniqueness Theorem.

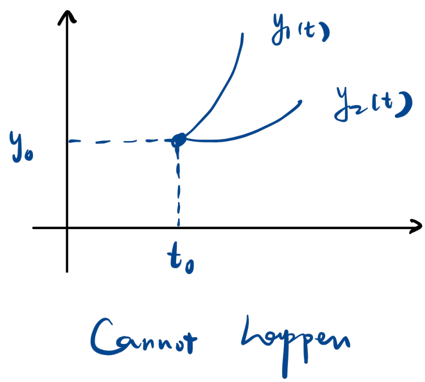
(IVP: $y'(t) = f(t, y(t))$ $\quad y(t_0) = y_0$ with f, $\partial_y f$ continuous

has a unique solution y, for t near t_0)

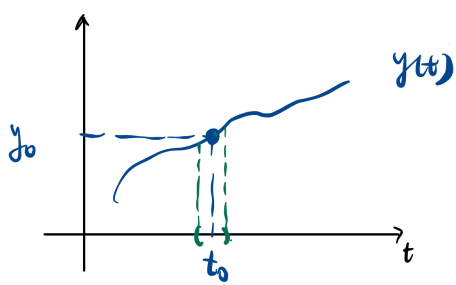
Remarks:

* No formula for the solution
* The initial condition determines only 1 solutions





* The "t near t_0" does not say how near



* The Euler Method

Idea 1) Consider the differential equation $y'(t) = f(t, y(t))$ $y(t_0) = y_0$

on a interval $[a, b]$

2) Introduce a partition of $[a, b]$ order $N > 0$

let $\Delta t = \dfrac{b-a}{N}$ time step

$$\{ t_0 = a, \quad t_1 = t_0 + \Delta t, \ \dots, \ t_n = t_0 + n\Delta t, \ \dots, \ t_N = t_0 + N\Delta t \}$$

$$= a + N \cdot \frac{b-a}{N} = b$$

3) Evaluate the equation of the partition.

$$y'(t_n) = f(t_n, y(t_n))$$

4) Approximate derivative by a finite rate of change in Δt

$$y'(t_n) \cong \frac{y(t_{n+1}) - y(t_n)}{\Delta t} \quad ; \quad t_{n+1} - t_n = \Delta t$$

Denote $y_{t_n} = y_n$

$$\Rightarrow \quad \frac{y_{n+1} - y_n}{\Delta t} = f(t_n, y_n), \quad y_0 = y(t_0)$$

$$\Rightarrow \quad y_{n+1} = f(t_n, y_n) \cdot \Delta t + y_n$$

eg: $y' = 2y + 3$ $\quad y(0) = 1$ $\quad$ on interval $[0, 3]$ using $\Delta t = 1$

<Sol>:

t_n	y_n
$t_0 = 0$	$y_0 = 1$
$t_1 = 1$	$y_1 = 6$
$t_2 = 2$	$y_2 = 21$
$t_3 = 3$	$y_3 = 66$

$t_0, \ t_1 = t_0 + \Delta t, \ \dots$

$t_0 = 0 \qquad 1 = \Delta t = \frac{3-0}{N} \Rightarrow N = 3$

$\{ t_0 = 0, \ t_1 = 1, \ t_2 = 2, \ t_3 = 3 \}$

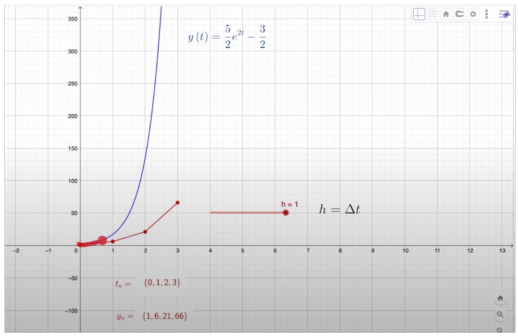
$$y_{n+1} = f(t_n, y_n)\,\Delta t + y_n$$

$$= (2y_n + 3)\,\Delta t + y_n = 3y_n + 3$$

$y_1 = 3y_0 + 3 = 6$

$y_2 = 3y_1 + 3 = 21$

$y_3 = 3y_2 + 3 = 66$



2.2. Second Order linear differential equations

A SOLDE is

$$y'' + a_1(t)\cdot y' + a_0(t)\cdot y = b(t) \qquad (1)$$

(1) is homogeneous iff $b=0$

(1) is Const ant Coeff iff a_1, a_0 are consts

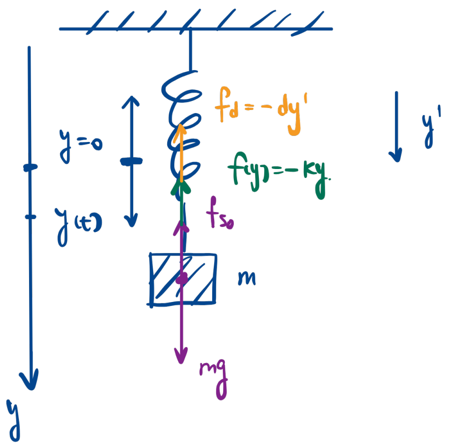
(1) is Variable Coeff iff either a_1 or a_0 is not a Constant

Example: Mass-Spring System

$f_s = -ky$

$f_d = -dy'$

$ma = f$



$$my'' = \underbrace{-ky}_{\text{Hooke's law}} \underbrace{- dy'}_{\text{Damping friction}}$$

$$my'' + dy' + ky = 0$$

Newton's Eq.

$$my'' + ky = 0 \Rightarrow y'(my'' + ky) = (0)\cdot y'$$

Chain Rule:

$$\Rightarrow my''\cdot y' + ky'y = 0$$

$$(y^2)' = 2y\cdot y'$$

$$\tfrac{1}{2}(y^2)' = y\cdot y'$$

$$\frac{m}{2}\left[(y')^2\right]' + \frac{k}{2}(y^2)' = 0$$

$$\left[(y')^2\right]' = 2y'\,y''$$

$$v(t) = y'(t)$$

$$\tfrac{1}{2}\left[(y')^2\right]' = y'y''$$

$$\Big(\underbrace{\tfrac{1}{2}mv^2 + \tfrac{1}{2}k\,y^2(t)}_{= E(t) = C}\Big)' = 0 \qquad E(t) = \frac{m}{2}v^2(t) + \frac{k}{2}y^2(t)$$

$$E'(t) = 0$$

$$y'(my'' + dy' + ky) = 0\ (y')$$

$$\Rightarrow my''y' + d(y')^2 + kyy' = 0 \qquad v = y'$$

$$\Rightarrow \tfrac{1}{2}m(v^2)' + \tfrac{1}{2}k(y^2)' = -dv^2$$

$$\Rightarrow \Big(\underbrace{\tfrac{1}{2}mv^2 + \tfrac{1}{2}ky^2}_{E(t)}\Big)' = -dv^2 \leq 0$$

$$E'(t) = -dv^2 \leq 0$$

$E(t)$: Not increasing

2.2 linear homogeneous

$$y'' + a_1(t)\, y' + a_0(t) \cdot y = b(t) \qquad \begin{aligned} y(t_0) &= y_0 \\ y'(t_0) &= y_1 \end{aligned}$$

if $a_1(t)$, $a_0(t)$, $b(t)$ continuous on interval (t_1, t_2), then exists a unique solution $y(t)$

Properties of Homogeneous Eq.

Notation: we write $y'' + a_1 y' + a_0 y = b$ as $L(y) = b$

where $L(y) = y'' + a_1 y' + a_0 y$

$$L(y_1 + y_2) = L(y_1) + L(y_2) \qquad L(cy) = c\,L(y)$$

* Superposition Property

let: $L(y) = y'' + a_1(t)\, y' + a_0(t) \cdot y$

Theorem: (If $y_1(t)$, $y_2(t)$ are sols of $L(y) = 0$

Then so is $y(t) = c_1 y_1(t) + c_2 y_2(t)$

for every constant of c_1, c_2)

2.3. Homogeneous Constant Coefficient

* Review. : general solution Theorem.

Theorem: If $y_1(t)$, $y_2(t)$ with $y_1(t) \neq c\,y_2(t)$ are sols of

$L(y_1) = 0$, $L(y_2) = 0$ with. $L(y) = y'' + a_1(t)\,y' + a_0(t)\cdot y$.

then every sol $y(t)$ of $L(y) = 0$ can be written as $y(t) = C_1 y_1(t) + C_2 y_2(t)$

Theorem: Given an SOLDE. $y'' + a_1 y' + a_0 y = 0$ let $r_\pm$ be the roots of

$$r^2 + a_1 r + a_0 = 0 \qquad D = a_1^2 - 4a_0$$

(A) $D > 0$. (two distinct root. r_1 and r_2) $y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

(B) $D = 0$ (one real repeated root r) $y(t) = (C_1 + C_2 t)e^{rt}$

(C) $D < 0$ (Complex conjugate roots $r = \alpha \pm \beta i$) $y(t) = e^{\alpha t}\left(C_1 \cos(\beta t) + C_2 \sin(\beta t)\right)$

Consider for this: $my'' + dy' + ky = 0 \Rightarrow y'' + \frac{d}{m}y' + \frac{k}{m}y = 0$

Define. $\alpha = \frac{d}{2m} \geq 0$ $\quad \omega_0 = \sqrt{\frac{k}{m}} > 0 \Rightarrow y'' + 2\alpha y' + \omega_0^2 y = 0$

$$\boxed{r_{+/-} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}. \qquad \alpha = \frac{d}{2m}. \qquad \omega_0 = \sqrt{\frac{k}{m}}}$$

①	$\alpha = 0$.	No Friction	Undamped Oscillation.
②	$0 < \alpha < \omega_0$	Small Friction	Damped Oscillation
③	$\alpha = \omega_0$	Critical Friction	Critical Decay
④	$\alpha > \omega_0$	Large Friction	Decay without oscillation

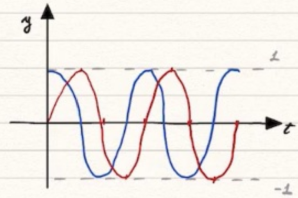
Summary Solutions of: $y'' + a_1 y' + a_0 y = 0$, $a_1, a_0 \in \mathbb{R}$.

Solve for $r_\pm$ sol. of $r^2 + a_1 r + a_0 = 0$

(4) $r_\pm = \pm \omega_0 i$, $\alpha = 0$

No Friction
Undamped.

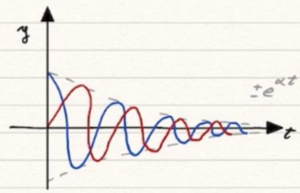
$y_+ = \cos(\omega_0 t)$, $y_- = \sin(\omega_0 t)$



(3) $r_\pm = \alpha \pm i\beta$, $\alpha < 0$.

Small Friction
Under Damped.

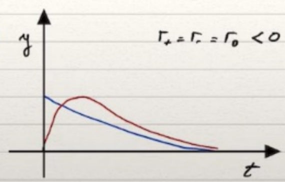
$y_+ = e^{\alpha t} \cos(\beta t)$, $y_- = e^{\alpha t} \sin(\beta t)$



(2) $r_+ = r_- = r_0$, real

Critical Friction
Critically Damped

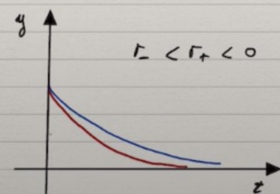
$y_+ = e^{r_0 t}$, $y_- = t\, e^{r_0 t}$



(1) $r_+ > r_-$, real:

Large Friction
Over Damped

$y_+ = e^{r_+ t}$, $y_- = e^{r_- t}$



Theorem: $y(t) = e^{\alpha t}\left(C_1 \cos(\omega t) + C_2 \sin(\omega t)\right)$

Can be written as $y(t) = A e^{\alpha t} \cos(\omega t - \phi)$

$A > 0$
$\omega > 0$
$\phi \in (-\pi, \pi]$

$$C_1 = A\cos\phi \qquad C_2 = A\sin\phi \implies \begin{cases} A = \sqrt{C_1^2 + C_2^2} \\ \tan\phi = \dfrac{C_2}{C_1} \end{cases}$$

2.5 Non-Homogeneous Equations

Theorem: If $L(y_1)=0$, $L(y_2)=0$ $y_1 \neq c\,y_2$ and $L(y_p)=f$ (p for "particular")

Then all sols of $L(y)=f$ are $y(t)=C_1 y_1(t)+C_2 y_2(t)+y_p(t)$

For particular guess (1) First guess: $ke^{at} \rightarrow ke^{at}$. $\quad kt^m \rightarrow k_m t^m + \dots + k_0$

$$\left.\begin{matrix} k\cos(bt) \\ k\sin(bt) \end{matrix}\right\} \longrightarrow k_1\cos(bt)+k_2\sin(bt)$$

(2) Possible Further guess
- If $L(y_{p_1})=0$ then $y_{p_2}=t\,y_{p_1}$
- If $L(y_{p_2})=0$ then $y_{p_3}=t^2 y_{p_1}$

2.6 Forced Oscillations

Find the solution of the IVP $my''+ky=F_0\cos(\omega t)$ $\quad y(0)=0 \quad y'(0)=0$

$\Rightarrow \quad y''+\frac{k}{m}y=\frac{F_0}{m}\cos(\omega t)$ $\quad$ we introduce $\omega_0=\sqrt{\frac{k}{m}}$, $f_0=\frac{F}{m}$

Therefore. solve: $\quad y''+\omega_0^2 y=f_0\cos(\omega t) \quad y(0)=0 \quad y'(0)=0$

(1) Solve the homogenous eq:

$$y''+\omega_0^2 y=0 \Rightarrow r^2+\omega_0^2=0 \Rightarrow r=\pm\omega_0 i \Rightarrow y(t)=C_1\cos(\omega_0 t)+C_2\sin(\omega_0 t)$$

(2) Find $y_p(t)$ particular sol. of non-hom. eq.

$L(y_p)=f_0\cos(\omega t)$ $\qquad$ Guess $y_p(t)=k_1\cos(\omega t)$

$$\Rightarrow L(y_p)=-k_1\omega^2\cos(\omega t)+\omega_0^2 k_1\cos(\omega t)=k_1\cos(\omega t)\cdot\underbrace{(-\omega^2+\omega_0^2)}_{\neq 0} \Rightarrow \omega\neq\omega_0 \Rightarrow k_1=\frac{f_0}{\omega_0^2-\omega^2}$$

$$\therefore \quad y_p(t)=\frac{f_0}{\omega_0^2-\omega^2}\cos(\omega t)$$

$$\Rightarrow y(t)=C_1\cos(\omega_0 t)+C_2\sin(\omega_0 t)+\frac{f_0}{\omega_0^2-\omega^2}\cos(\omega t)$$ put in $y(0)=0 \quad y'(0)=0$

$$\therefore \quad \boxed{y(t)=\frac{f_0}{\omega_0^2-\omega^2}\left(\cos(\omega t)-\cos(\omega_0 t)\right)}$$

当外力的频率 ω 等于系统的固有频率 ω_0 时，系统会发生**共振**，此时之前的方法不再适用，因为分母 $\omega_0^2 - \omega^2$ 变为零，导致解无定义。在这种情况下，我们需要采用不同的方法来求解。

共振情况的解法

1. **微分方程的形式：**
   考虑一个典型的强迫振动方程：
   $$y''(t) + \omega_0^2 y(t) = p\cos(\omega_0 t)$$
   这里 $\omega = \omega_0$。
2. **齐次方程的解：**
   首先，求解对应的齐次方程：
   $$y''(t) + \omega_0^2 y(t) = 0$$
   其通解为：
   $$y_c(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)$$
3. **特解的求解：**
   由于非齐次项 $p\cos(\omega_0 t)$ 与齐次解的形式相同，我们需要使用**待定系数法**的特殊形式。假设特解为：
   $$y_p(t) = t(A\cos(\omega_0 t) + B\sin(\omega_0 t))$$
   这里乘以 t 是为了避免与齐次解重复。
4. **求导并代入方程：**
   计算 $y_p(t)$ 的一阶和二阶导数：
   $$y_p'(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) + t(-A\omega_0\sin(\omega_0 t) + B\omega_0\cos(\omega_0 t))$$
   $$y_p''(t) = -2A\omega_0\sin(\omega_0 t) + 2B\omega_0\cos(\omega_0 t) - t\omega_0^2(A\cos(\omega_0 t) + B\sin(\omega_0 t))$$
   将 $y_p(t)$ 和 $y_p''(t)$ 代入原方程，并整理得到：
   $$-2A\omega_0\sin(\omega_0 t) + 2B\omega_0\cos(\omega_0 t) = p\cos(\omega_0 t)$$
   比较系数，得到：
   $$-2A\omega_0 = 0 \quad 和 \quad 2B\omega_0 = p$$
   解得：
   $$A = 0 \quad 和 \quad B = \frac{p}{2\omega_0}$$
5. **特解的形式：**
   因此，特解为：
   $$y_p(t) = \frac{p}{2\omega_0}t\sin(\omega_0 t)$$
6. **通解：**
   将齐次解和特解结合，得到通解：
   $$y(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t) + \frac{p}{2\omega_0}t\sin(\omega_0 t)$$

总结

当 $\omega = \omega_0$ 时，系统的响应包含一个随时间增长的项 $\frac{p}{2\omega_0}t\sin(\omega_0 t)$，这表明在共振情况下，系统的振幅会随时间线性增长，最终可能导致系统破坏。这种现象在工程和物理中非常重要，需要特别注意以避免共振带来的不利影响。

4.1 Laplace transformation

Def: The LT of the $f(t)$ on $D_f = [0, +\infty)$ is

$$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt \qquad (1)$$

defined for $s \in D_f$ where (1) converges

First Homework:

construct this table.

Basic Laplace Transform Table

$f(t)$	$F(s) = \mathcal{L}[f(t)]$	Domain
$f(t) = 1$	$F(s) = \frac{1}{s}$	$s > 0$
$f(t) = e^{at}$	$F(s) = \frac{1}{(s-a)}$	$s > a$
$f(t) = t^n$	$F(s) = \frac{n!}{s^{(n+1)}}$	$s > 0$
$f(t) = \sin(at)$	$F(s) = \frac{a}{s^2+a^2}$	$s > 0$
$f(t) = \cos(at)$	$F(s) = \frac{s}{s^2+a^2}$	$s > 0$
$f(t) = \sinh(at)$	$F(s) = \frac{a}{s^2-a^2}$	$s > \lvert a \rvert$
$f(t) = \cosh(at)$	$F(s) = \frac{s}{s^2-a^2}$	$s > \lvert a \rvert$
$f(t) = t^n e^{at}$	$F(s) = \frac{n!}{(s-a)^{(n+1)}}$	$s > \max\{a, 0\}$
$f(t) = e^{at}\sin(bt)$	$F(s) = \frac{b}{(s-a)^2+b^2}$	$s > \max\{a, 0\}$
$f(t) = e^{at}\cos(bt)$	$F(s) = \frac{(s-a)}{(s-a)^2+b^2}$	$s > \max\{a, 0\}$
$f(t) = e^{at}\sinh(bt)$	$F(s) = \frac{b}{(s-a)^2-b^2}$	$s > \max\{a, \lvert b \rvert\}$
$f(t) = e^{at}\cosh(bt)$	$F(s) = \frac{(s-a)}{(s-a)^2-b^2}$	$s > \max\{a, \lvert b \rvert\}$

$a = 0$

Property: $\mathcal{L}\{af + bg\} = a\mathcal{L}\{f\} + b\mathcal{L}\{g\}$

$\mathcal{L}\{cf\} = c \cdot \mathcal{L}\{f\}$

$\mathcal{L}[f'] = s\mathcal{L}[f] - f(0)$

$\mathcal{L}[f''] = s^2\mathcal{L}[f] - sf(0) - f'(0)$

Eg: $y'' - 6y' + 13y = e^{5t}$ $\quad y(0)=1 \quad y'(0)=2$

<Sol>: $\mathcal{L}[y'' - 6y' + 13y] = \mathcal{L}[e^{5t}]$

$$\Rightarrow L[y''] - 6L[y'] + 13L[y] = \frac{1}{s-5}$$

$$(s^2 L(y) - s y(0) - y'(0)) - 6(sL(y) - y(0)) + 13L(y) = \frac{1}{s-5}$$

$$\Rightarrow \mathcal{L}[y] = \frac{s-4}{s^2-6s+13} + \frac{1}{(s-5)(s^2-6s+13)}$$

$$s^2 - 6s + 13 = 0 \Rightarrow s_{\pm} = \frac{6 \pm \sqrt{36-4\times 13}}{2} = 3 \pm 2i. \quad \text{Complex root}$$

Work with real number. $s^2 - 6s + 13 = (s-a)^2 + b^2 = (s-3)^2 + 4$

$$\therefore H(s) = \frac{s-4}{(s-3)^2+4} + \frac{1}{(s-5)((s-3)^2+4)}$$

$$L[e^{at}\cos(bt)] = \frac{s-a}{(s-a)^2+b^2} \qquad \mathcal{L}[e^{at}\sin bt] = \frac{b}{(s-a)^2+b^2}$$

$$\frac{s-4}{(s-3)^2+4} = \frac{s-3}{(s-3)^2+2^2} - \frac{1}{(s-3)^2+2^2} = \frac{s-3}{(s-3)^2+2^2} - \frac{1}{2} \cdot \frac{2}{(s-3)^2+2^2}$$

$$= L[e^{3t}\cos(2t)] - \frac{1}{2}\mathcal{L}[e^{3t}\sin(2t)] \quad \star$$

$$\frac{1}{(s-5)((s-3)^2+4)} = \frac{A}{s-5} + \frac{Bs+C}{(s-3)^2+4} = A\,\mathcal{L}[e^{5t}] + \frac{B((s-3)+3)+C}{(s-3)^2+2^2}$$

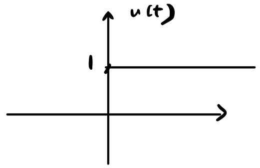
$$= A\,\mathcal{L}[e^{5t}] + \frac{B(s-3)}{(s-3)^2+2^2} + (3B+C) \cdot \frac{1}{2} \cdot \frac{2}{(s-3)^2+2^2}.$$

$$= A\,\mathcal{L}[e^{5t}] + B\,\mathcal{L}[e^{3t}\cos(2t)] + \left(\frac{3}{2}B + \frac{C}{2}\right)\mathcal{L}[e^{3t}\sin(2t)]$$

$\star$ -

4.3 Dicontinuous Sources

$u(t)$:



$$\mathcal{L}[u(t-c)] = \begin{cases} \dfrac{e^{-cs}}{s} & c \geq 0 \quad s>0 \\ \dfrac{1}{s} & c<0 \quad s>0 \end{cases}$$

$$\mathcal{L}[u(t-c)\cdot f(t-c)] = e^{-cs} \mathcal{L}[f(t)], \quad c \geq 0$$

$$\mathcal{L}[e^{ct} f(t)] = \mathcal{L}[f(t)]\Big|_{s-c} = F(s-c) \quad c \in \mathbb{R}$$

eg: $f(t) = \cos(2t)$ $\quad \mathcal{L}[f(t)] = \dfrac{s}{s^2+2^2}, \; s>0$

$$\mathcal{L}[u(t-3)\cos(2(t-3))] = e^{-3s}\mathcal{L}[\cos(2t)] = e^{-3s}\cdot\frac{s}{s^2+2^2}$$

$$\mathcal{L}[e^{5t}\cos(2t)] = \mathcal{L}[\cos(2t)]\Big|_{s-5} = \frac{s-5}{(s-5)^2+2^2}$$

eg: Find $g(t)$ $\quad \mathcal{L}[g(t)] = \dfrac{e^{-3s}}{s^2+5}$

<sol>: $\mathcal{L}[g(t)] = e^{-3s}\dfrac{1}{\sqrt{5}}\cdot\dfrac{\sqrt{5}}{s^2+(\sqrt{5})^2} = \dfrac{e^{-3s}}{\sqrt{5}}\mathcal{L}[\sin(\sqrt{5}t)] \Rightarrow g(t) = \dfrac{1}{\sqrt{5}} u(t-3)\sin(\sqrt{5}(t-3))$

Another 2 formula:

$$\mathcal{L}[\delta(t-a)] = e^{-as} \quad (a>0)$$

$$\mathcal{L}[\delta(t+a)] = 0 \quad (a>0)$$

$$\mathcal{L}[f(t)\,\delta(t-a)] = f(a)e^{-as} \quad (a>0)$$

Review of Solution formula:

$\vec{x}' = A\vec{x}$. with A 2x2, real, with eigenpairs

λ_+, $\vec{v}_+$ and λ_-, $\vec{v}_-$ are

(a) $\lambda_+ \neq \lambda_-$, real (A diagonalizable)

$$\vec{x}_+ = e^{\lambda_+ t}\vec{v}_+ , \quad \vec{x}_- = e^{\lambda_- t}\vec{v}_-$$

(b) $\lambda_\pm = \alpha \pm i\beta$ $\quad \vec{v}_\pm = \vec{a} \pm i\vec{b}$

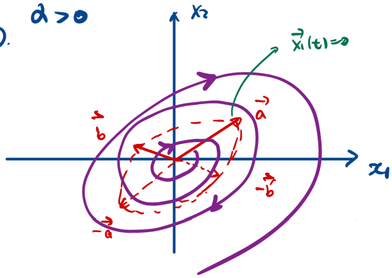
$$\vec{x}_1 = e^{\alpha t}\left(\cos(\beta t)\,\vec{a} - \sin(\beta t)\,\vec{b}\right)$$

$$\vec{x}_2 = e^{\alpha t}\left(\sin(\beta t)\,\vec{a} + \cos(\beta t)\,\vec{b}\right)$$

* Complex Eigenvalue.

$\vec{x_1}(t) = e^{\alpha t}\left(\vec{a}\cos(\beta t) - \vec{b}\sin(\beta t)\right)$ with $\begin{cases} \alpha > 0 & ① \\ \alpha < 0 & ② \\ \alpha = 0 & ③ \end{cases}$

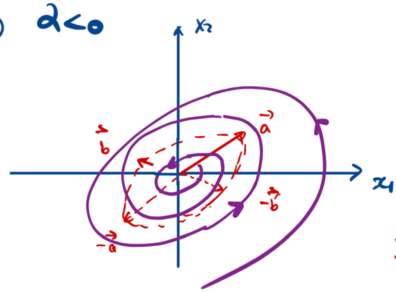
① $\alpha > 0$



$\vec{x}=0$ Source spiral

unstable

② $\alpha < 0$



$\vec{x}=0$ Sink spiral

stable

③ $\alpha = 0$

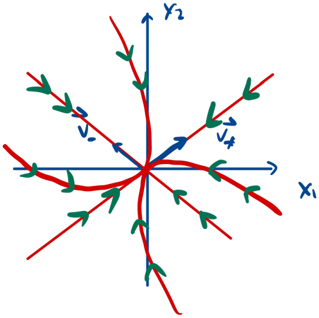
only ellipse

Center

顺/逆

代入数值去看

Another one: $\lambda_+ \neq \lambda_-$, Reals. $\vec{v}_+$ $\vec{v}_-$



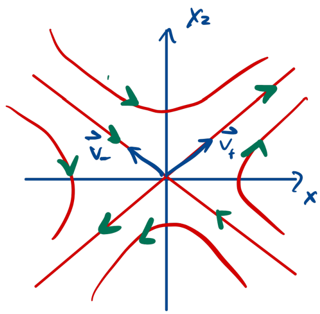
$\lambda_- < \lambda_+ < 0$

$\vec{x}=0$ Sink node.

stable

↓

Coexist



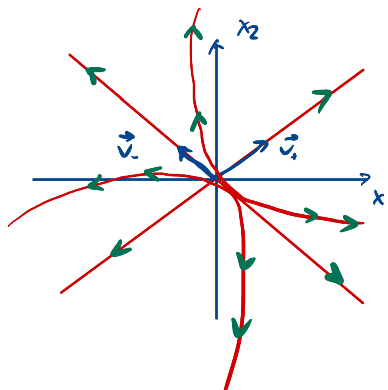
$\lambda_- < 0 < \lambda_+$

$\vec{x}=0$ Saddle node

unstable

↓

one species extinct



$0 < \lambda_- < \lambda_+$

$\vec{x}=0$ Source node

unstable

↓

both extinct