Name ______________ Date __________ Partners ______________

Lab 11 - Free, Damped, and Forced Oscillations

OBJECTIVES

- To understand the free oscillations of a mass and spring.
- To understand how energy is shared between potential and kinetic energy.
- To understand the effects of damping on oscillatory motion.
- To understand how driving forces dominate oscillatory motion.
- To understand the effects of resonance in oscillatory motion.

OVERVIEW

You have already studied the motion of a mass moving on the end of a spring. We understand that the concept of mechanical energy applies and the energy is shared back and forth between the potential and kinetic energy. We know how to find the angular frequency of the mass motion if we know the spring constant. We will examine in this lab the mass-spring system again, but this time we will have two springs — each having one end fixed on either side of the mass. We will let the mass slide on an air track that has very little friction. We first will study the free oscillation of this system. Then we will use magnets to add some damping and study the motion as a function of the damping coefficient. Finally, we will hook up a motor that will oscillate the system at practically any frequency we choose. We will find that this motion leads to several interesting results including wild oscillations.

Harmonic motions are ubiquitous in physics and engineering - we often observe them in mechanical and electrical systems. The same general principles apply for atomic, molecular, and other oscillators, so once you understand harmonic motion in one guise you have the basis for understanding an immense range of phenomena.

INVESTIGATION 1: FREE OSCILLATIONS

Time:

An example of a simple harmonic oscillator is a mass m which moves on the x-axis and is attached to a spring with its equilibrium position at $x = 0$ (by definition). When the mass is moved from its equilibrium position, the restoring force of the spring tends to bring it back to the equilibrium position. The spring force is given by

$$F_{spring} = -kx \tag{11.1}$$

where k is the spring constant. The equation of motion for m becomes

$$m\frac{d^2x}{dt^2} = -kx \tag{11.2}$$

This is the equation for *simple harmonic motion*. Its solution, as one can easily verify, is given by:

$$x = A_F \sin(\omega_F t + \delta_F) \tag{11.3}$$

where

$$\omega_F = \sqrt{k/m} \tag{11.4}$$

NOTE: The subscript "*F*" on ω_F, etc. refers to the *natural* or *free* oscillation.

A_F and δ_F are constants of integration and are determined by the initial conditions. [For example, if the spring is maximally extended at $t = 0$, we find that A_F is the displacement from equilibrium and $\delta = \pi/2$.]

We can calculate the velocity by differentiating with respect to time:

$$v = \frac{dx}{dt} = \omega_F A_F \cos(\omega_F t + \delta_F) \tag{11.5}$$

The kinetic energy is then:

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}kA_F^2 \cos^2(\omega_F t + \delta_F) \tag{11.6}$$

The potential energy is given by integrating the force *on the spring* times the displacement:

$$PE = \int_0^x kx dx = \frac{1}{2}kx^2 = \frac{1}{2}kA_F^2 \sin^2(\omega_F t + \delta_F) \tag{11.7}$$

We see that in the absence of friction, the sum of the two energies is constant:

$$KE + PE = \frac{1}{2}kA_F^2 \tag{11.8}$$

Activity 1-1: Free Motion of the Two Spring System

To perform this laboratory you will need the following equipment:

- motion detector
- mechanical vibrator
- air track and glider
- two springs with approximately equal spring constants
- ceramic magnets
- masking tape

1. Set up the system on the air track as the diagram on Figure 11.1 indicates. We will be using a two-spring system with the springs connected on either side of the glider

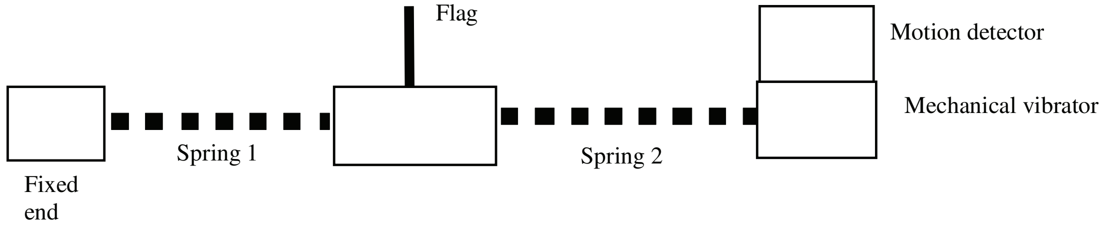

Figure 11.1:

2. Turn on the air supply for the air track. Make sure the air track is level. Check it by placing the glider on the track and see if it is motionless. Some adjustments may be necessary on the feet, but be careful, because it may not be possible to have the track level over its entire length.

3. Tape four ceramic magnets to the flag on top of the glider.

Never move items on the air track unless the air is flowing! You might scratch the surfaces and create considerable friction.

Both springs should be stretched in equilibrium. Move the mechanical vibrator so that the distance from end-to-end of the springs is about 75 cm. One spring is connected to a fixture on the end of the air track and the other spring should be connected to the mechanical vibrator. Once you settled on the length of the springs the mechanical vibrator should no longer be moved along the air track. Ask your TA if you have any questions.

1. Open the experimental file **L11.1-1 Two Spring System**.

2. Make sure the air is on in the air track. Verify the function generator is off. Let the glider remain at rest.

3. **Start** collecting the data and take data for 2-3 seconds with the glider at rest so you can obtain the equilibrium position. Then pull back the glider about 10-15 cm and let the glider oscillate until you have at least ten complete cycles. Then **stop** collecting data.

4. **Print** out this graph and include it with your report.

5. We can find the angular frequency by measuring the period T_F. Use the **Smart Tool** to find the time for the left most complete peak (write it down), count over several more peaks (hopefully at least 10) to the right and find the time for another peak (write it down).

   # cycles:__________

   First peak:__________ s Last peak: __________ s

6. Subtract the times for the two peaks and divide by the number of *complete cycles* to find the period. Read the sampling rate of the motion sensor, that is how many times per second the actual time is recorded.

Question 1-1: If the sampling rate is f_{sampling} what is the shortest time interval measured with your motion sensor? The sampling rate determines the smallest time division in your experiment just like a millimeter division on a regular ruler determines the shortest distance division for your length measurement. Keeping this analogy in mind, what would be the standard error σ_{time} in your time measurement? What is the relative error in your ability to measure time?

Relative error in time measurement: ________ %

7. Calculate the period T_F and determine f_F and ω_F. Knowing σ_{time} (the error in measurement of 1 second interval) calculate the corresponding errors for period, linear, and angular frequencies.

   Period T_F: ________ $\pm$ ________ s

   Frequency f_F: ________ $\pm$ ________ Hz

   Angular frequency ω_F: ________ $\pm$ ________ rad/s

> **NOTE:** The error you just calculated determines the precision of your experiment. Keep these values in mind for the duration of the experiment while making decisions if certain quantities can be neglected.

Question 1-2: Describe the motion that you observed in this activity. Does it look like it will continue for a long time?

INVESTIGATION 2: DAMPED OSCILLATORY MOTION

Time:

Equation (11.2) in the previous section describes a periodic motion that will last forever. This is true if the only force acting on the mass is the restoring force . Most motions in nature do not have such simple "free" oscillations. It is more likely there will be some kind of friction or resistance to damp out the free motion. In this investigation we will start the free oscillations like we did in the previous experiment, but we will add some damping. Automobile springs, for example, are damped (by shock absorbers) to reduce oscillations caused by rough road surfaces.

We'll be using magnetically induced eddy current damping which has the simple form:

$$F_v = -bv \tag{11.9}$$

This form also occurs between oiled surfaces or in liquids and gases (for low speeds).

The new equation of motion becomes:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \tag{11.10}$$

A simple sinusoid will not satisfy this equation. In fact, the form of the solution is strongly dependent upon the value of b.

If $b > 2\sqrt{mk} = 2m\omega_F$, the system is "over damped" and the mass will not oscillate. The solution will be a sum of two decaying exponentials.

If $b = 2m\omega_F$, the system is "critically damped" and, again, no oscillations occur. The solution is a simple decaying exponential. Shock absorbers in cars are so constructed that the damping is nearly critical. One does not increase the damping beyond critical because the ride would feel too hard.

For sufficiently small damping ($b < 2m\omega_F$), the solution is given by:

$$x = A_D e^{-t/\tau} \sin(\omega_D t + \delta_D) \tag{11.11}$$

where the decay constant (time for the amplitude to drop to $1/e$ of its initial value) is given by:

$$\tau \equiv 2m/b \tag{11.12}$$

and the angular frequency is given by:

$$\omega_D = \sqrt{\omega_F^2 - 1\big/\tau^2} \tag{11.13}$$

The subscript ("D", for "damping") helps to distinguish this angular frequency from the natural or free angular frequency, ω_F. Note that frequency of the damped oscillator, ω_D, is smaller than the natural frequency. For small damping one may neglect the shift.

A_D and δ_D are constants of integration and are again determined by the initial conditions. An example of Equation (11.11), with $\delta_D = \pi/2$, is shown in Figure11.2:

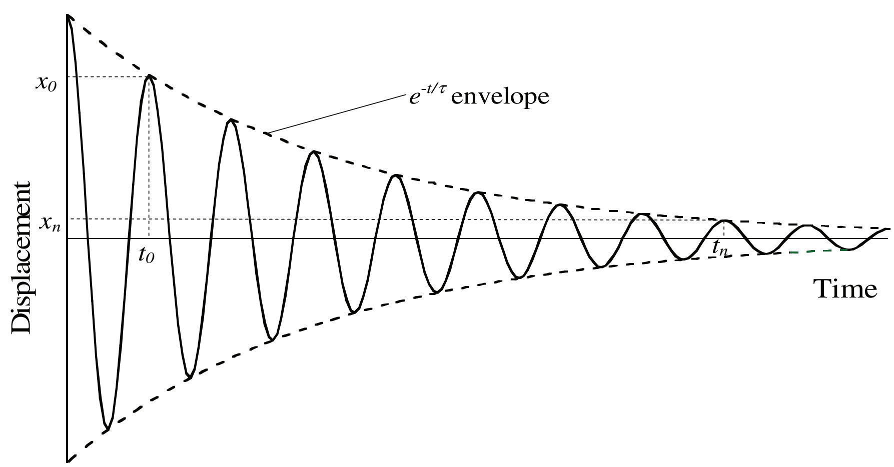

Figure 11.2:

Activity 2-1: Description of Damped Harmonic Motion

We can add damping by attaching one or more strong ceramic magnets to the side of the moving glider. In your study of electromagnetism, you will learn that a moving magnetic field sets up a second magnetic field to oppose the effects of the original magnetic field. This is explained by Lenz's Law. The magnets we attach to the moving glider will cause magnetic fields to be created in the aluminum air track that will oppose the motion of the glider. The effect will be one of damping. The opposing magnetic fields are caused by induced currents, called *eddy currents*, and they will eventually dissipate in the aluminum track due to resistive losses. The gliders themselves are made of non-magnetic material (aluminum) that allows the magnetic field to pass through (iron gliders would not work).

NOTE: Do not place the magnets near the bottom of the glider. The magnets' interaction with steel support rods would perturb the motion.

1. Place the four ceramic magnets symmetrically on each side of the glider, at top of the "skirt". Use a small piece of tape to keep the magnets in place.

2. Open the experiment file called **L11.2-1 Damped Oscillations.**

3. Make sure the air is on for the air track, the glider is at equilibrium and not moving on the air track.

4. **Start the computer**. Let the glider be at rest for a couple of seconds, so you can obtain the equilibrium position. Then pull back the glider about 10 – 15 cm and release it.

5. Let the glider oscillate through at least ten cycles before stopping the computer.

   Question 2-1: Does the motion look like Figure 11.2? Discuss.

6. **Print** out the graphs.

Activity 2-2: Determination of Damping Time

We can use Equation (11.11) and our data to determine the damping time τ. Refer to Figure 11.2. We measure the amplitude x_0 at some time t_0 corresponding to a peak. We then measure the amplitude x_N at t_N (the peak N periods later). [Remember that the period is given by $T = 1/f = 2\pi/\omega$.] From Equation (11.11) we can see that the ratio of the amplitudes will be given by:

$$x_0/x_N = e^{(t_N - t_0)/\tau} \tag{11.14}$$

From this we can get the decay time:

$$\tau = (t_N - t_0)\,/\,\ln(x_0/x_N) \tag{11.15}$$

1. Look at the data you took in the previous activity. Use the **Smart Tool** and find the equilibrium position of the glider. Then place the cursor on top of the first complete peak. Note both the peak height and time, and subtract the equilibrium position from the peak height to find the amplitude. Click on another peak that is about a factor of two smaller than the first peak. Determine again the amplitude and time.

Equilibrium position ________ m

	Peak 0		Peak N	
Peak Height:		m		m
Amplitude:		m		m
Time:		s		s

2. Use Equation (11.15) to find the decay time.

τ: ________ s

Activity 2-3: Determination of Damped Angular Frequency

Question 2-2: Use Equation (11.13) to calculate the theoretical value of the angular frequency of the damped oscillations. Show your work.

ω_D^{theo}: ________ rad/s

3. Determine the angular frequency for damping ω_D by measuring the time for ten cycles.

Time for ten cycles: ________ s, Period T_D: ________ s;

Frequency f_D: ________ Hz, ω_D: ________ rad/s

Question 2-3: Discuss the agreement between the measured and predicted damped angular frequencies.

Question 2-4: Are we justified in ignoring the difference in frequency between the damped and undamped oscillations? Discuss keeping Equation (11.13) and the precision of your experiment in mind.

INVESTIGATION 3: FORCED OSCILLATORY MOTION

Time:

In addition to the restoring and damping forces, one may have a force which keeps the oscillation going. This is called a driving force. In many cases, this force will be sinusoidal in time:

$$F_{driving} = F_0 \sin \omega t \tag{11.16}$$

The equation of motion becomes

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=F_0\sin\omega t \tag{11.17}$$

This equation differs from Equation (11.10) by the term on the right, which makes it *inhomogeneous.* The theory of linear differential equations tells us that *any* solution of the inhomogeneous equation added to any solution of the homogeneous equation will be the general solution.

We call the solution to the homogeneous equation the *transient solution* since for all values of b, the solution damps out to zero for times large relative to the damping time τ.

We shall see that the solution of the inhomogeneous equation does not vanish and so we call it the *steady state* solution and denote it by x_{ss}. It is this motion that we will now consider.

The driving term forces the general solution to be oscillatory. In addition, there will likely be a phase difference between the driving term and the response.

Without proof, we will state that a solution to the inhomogeneous equation can be written as:

$$x_{ss}=A(\omega)\sin(\omega t+\delta) \tag{11.18}$$

Plugging Equation (11.18) into Equation (11.17), we see that the amplitude of the steady state motion is given by:

$$A(\omega)=\frac{F_0/m}{\sqrt{\left(\omega_F^2-\omega^2\right)^2+(2\omega/\tau)^2}} \tag{11.19}$$

and the phase shift is given by

$$\delta=-\tan^{-1}\left(\frac{2\omega/\tau}{\omega_F^2-\omega^2}\right) \tag{11.20}$$

where w_F is the free oscillation frequency. To discuss Equation (11.18), we look at its variation with the driving frequency ω. First consider the limit as the driving frequency goes to zero. There is no phase shift (the displacement is in phase with the driving force) and the amplitude is constant:

$$A_0\equiv A(0)=F_0\Big/m\omega_F^2=F_0/k \tag{11.21}$$

This is just what you should expect if you exert a constant force F_0 on a spring with spring constant k.

We can write the amplitude in terms of the zero frequency extension:

$$A(\omega)=A_0\omega_F^2\Big/\sqrt{(\omega_F^2-\omega^2)^2+(2\omega/\tau)^2} \tag{11.22}$$

For low driving frequencies, the phase shift goes to zero. The displacement will vary as $\sin\omega t$ and be in phase with the driving force. The amplitude is close to the zero frequency limit:

$$A(\omega<\omega_F)\approx A_0 \tag{11.23}$$

For very high driving frequencies, the phase shift goes to 180°. The displacement will again vary as $\sin(\omega t)$ but it will be 180° out of phase with the driving force. The amplitude drops off rapidly with increasing frequency:

$$A(\omega > \omega_F) \approx A_0(\omega_F/\omega)^2 \tag{11.24}$$

At intermediate frequencies, the amplitude reaches a maximum where the denominator of Equation (11.19) reaches a minimum:

$$\omega_R = \sqrt{\omega_F^2 - \frac{2}{\tau^2}} \tag{11.25}$$

The system is said to be at *resonance* and we denote this condition with the subscript "R". Note that ω_R is lower than ω_D.

At resonance (actually at ω_F), the displacement is 90° out of phase with the driving force. The amplitude is given by:

$$A_R \equiv A(\omega = \omega_F) = A_0 \cdot (\omega_F \tau/2) \tag{11.26}$$

From Equation (11.26) we see that the amplitude is proportional to the damping time and would go to infinity if there was no damping. In many mechanical systems one must build in damping, otherwise resonance could destroy the system. [You may have experienced this effect if you've ever ridden in a car with bad shock absorbers!]

The *resonant amplification* (also known as "Quality Factor") is defined to be the ratio of the amplitude at resonance to the amplitude in the limit of zero frequency:

$$Q \equiv A_R/A_0 = \omega_F \tau/2 \tag{11.27}$$

$A(\omega)$ is shown for various values of Q in Figure 11.3 below.

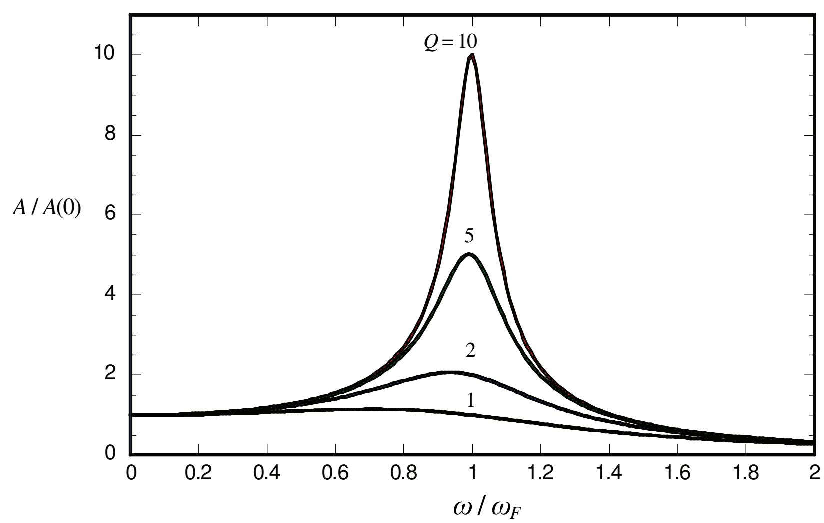

Figure 11.3:

We can find another useful interpretation of Q by considering the motion of this damped oscillator with no driving force. Instead imagine simply pulling the mass away from its equilibrium position and releasing it from rest. As we have seen, the mass will oscillate with continuously decreasing amplitude. Recall that in a time τ the amplitude will decay to $1/e$ of its original value. The period of oscillation is given by $T = 2\pi/\omega_F$, hence:

$$Q = \pi \frac{\tau}{T_F} \tag{11.28}$$

This says that Q is just π times the number of cycles of the oscillation required for the amplitude to decay to $1/e$ of its original value.

Activity 3-1: Observing Driven Oscillations

We finally are ready to study what happens to simple harmonic motion when we apply a force to keep the oscillation moving.

Question 3-1: From your measurements so far, using Equation (11.27), what Q do you expect for this system? Show your work.

The only extra equipment you will need is the

- Digital Function Generator – Amplifier

NOTE: The generator displays the frequency, f, not the angular frequency, ω. Remember that $\omega = 2\pi f$.

The equipment is the same as for Investigation 2, but you will need to utilize the mechanical vibrator to drive the spring. Keep the four magnets taped to the side of the glider or the resonant motion will be too large. We need to have the damping.

1. Unlock the oscillation driver. Turn on the power for the Function Generator driver (switch on back). Adjust the amplitude knob from zero to about halfway. **Ask your TA for help if you need it.** The variable speed motor oscillates the end of one of the springs at the specified frequency.
2. Set the generator to the very low frequency of 0.1 Hz. Make use of the Range buttons of the generator to set the most significant figures on the frequency display. This frequency is low enough that we can use it to measure A_0, the amplitude for zero frequency.

3. Make sure the air for the track is on. The vibrator should be moving left and right very slowly.

4. Every time you turn the motor on, it sets the glider into a superposition of two types of motion. One is the damped oscillations (first instance of the motor moving is similar to your setting the glider into motion) we studied earlier, and the persistent driven oscillations that we are about to study.

   Question 3-2: Will both of these two types of motion contribute to the amplitude of the glider oscillation over time? Explain.

   Question 3-3: Think about the definition of τ and by how much the damped amplitude of the system reduces after τ seconds. How long (in units of of τ) should you wait before the amplitude of the damped oscillations is 10% of the original amplitude? 1% of the original amplitude?

Question 3-4: Assume a typical precision of measuring the amplitude in our experiment to be 1%. Use your measured values of τ, to determine how long you need to wait for the transient part of the motion to die out, i.e., fall below your ability (think experimental error) to detect them?

5. Open the file **L11.3-1 Driven Oscillations**. Make sure that the generator is connected to the driver and that the voltage probe is connected to the generator. [The red leads go to the red jacks and the black to black.]

6. **Start the computer** and take data for at least three cycles of the motion. Verify that the motion of the *driver* is in phase with the driving voltage. Ask your TA for help if you have problems.

7. Once you have a good set of data, **stop the computer**.

   Question 3-5: Describe the motion of the glider, especially the phase relation between the driver and the motion of the glider.

8. Find the peak-to-peak values of the displacement for the oscillatory motion. Note that this is twice the amplitude! This measurement can be done easily by fitting a sine function to the appropriate region of the data. Ask your TA if you are not familiar with this technique.

$2A_0$ ______________ [A_0 is your "zero frequency" amplitude.]

Question 3-6: We claim that for our purposes, we can take $\omega_R \approx \omega_F$. Use your measured values of ω_F and τ and discuss the validity of this claim keeping Equation (11.25) and the precision of your experiment in mind.

Question 3-7: For what driving frequency do you expect to obtain maximum amplitude and what will peak-to-peak amplitude distance do you expect (Hint: use Equations (11.25) and (11.26))? Show your work.

$f_R^{\text{predicted}}$: ________ Hz $2A_R^{\text{predicted}}$: ________ m

9. Set the driving frequency slightly below (about few hundredth of one Hz) of your predicted resonant frequency $f_R^{\text{predicted}}$. Wait until the transients die out and then **start the computer.** Slowly increase the driving frequency in small increments until you find the maximum amplitude. In case of such a large amplitude you can use smart tool to measure the amplitude quite reliably. Write down what resonant frequency and amplitude did you find?

$f_R^{\text{experimental}}$: ________ Hz $2A_R^{\text{experimental}}$: ________ m

10. **Print** a copy of the graph showing your work in finding the resonance frequency and the amplitude.

Question 3-8: How well do your two resonant frequencies agree? Would you expect them to be the same? Explain.

11. Set the driver frequency to twice the resonant frequency. Wait for the transients to decay and then **start the computer** and observe the motion. Take data for at least three cycles and then **stop the computer.**

    Question 3-9: Describe the motion of the glider, especially the relative phase between the driver and the glider. Does the phase difference match the value discussed in the paragraph before the Equation (11.24) ?

Question 3-10: Using Equation (11.27) and your experimental values of $A_R^{\text{experimental}}$ and A_0 calculate Q from your data. Discuss agreement with your prediction made in Question 3-1.

Please clean up your lab area