Asha Bharadwaj, Caitlin Dutta, John Higgins, Alexis Smith
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We consider a dynamic model of loan repayment. $T_i \in \{1,2,3,4\}$ denotes the observed loan duration, and Y_{it} denotes an indicator variable equal to one if the loan is pre-paid at the end of period t. T_i is related to Y_{it} in the following manner:

$$T_i = \begin{cases} 1 & \text{if } Y_{i0} = 0 \\ 2 & \text{if } Y_{i0} = 0 \text{ and } Y_{i1} = 1 \\ 3 & \text{if } Y_{i0} = Y_{i1} = 0 \text{ and } Y_{i2} = 1 \\ 4 & \text{if } Y_{i0} = Y_{i1} = Y_{i2} = 0 \end{cases}$$

We assume that, at each period t, a loan is repaid only if

$$\alpha_t + X_i\beta + Z_{it}\gamma + \varepsilon_{it} < 0$$

Rearranging this, we find that the loan will be repaid only If

$$\varepsilon_{it} < -\alpha_t - X_i\beta - Z_{it}\gamma$$

This means that a loan will *not* be repaid in period t if $\varepsilon_{it} \geq -\alpha_t - X_i\beta - Z_{it}\gamma$.

We assume that X_i is a time-invariant vector of borrow characteristics, Z_{it} is a vector of time varying characteristics (such as the borrower's FICO score), and $\varepsilon_{it} = \rho\,\varepsilon_{it-1} + \eta_{it}$ if $t > 1$ and $\eta_{it} \sim N(0,1)$. We assume that $\varepsilon_{it} \sim N(0, \sigma_0^2)$, with $\sigma_0^2 = \frac{1}{(1-\rho)^2}$.

Given this setup, we must derive the likelihood associated with loan duration T_i. I believe that the likelihood given in the problem set is not quite correct, so I have attempted to derive it based on the above assumptions.

We first wish to find the likelihood of observing $T_i = 1$ given X_i, Z_i, and θ. $T_i = 1$

corresponds to the loan being paid off at $t = 0$. This means that

$$\begin{aligned} P(T_i = 1 \mid X_i, Z_i, \theta) &= P(\alpha_0 + X_i\beta + Z_{i0}\gamma + \varepsilon_{i0} < 0) \\ &= P(\varepsilon_{i0} < -\alpha_0 + -X_i\beta - Z_{i0}\gamma) \\ &= \Phi\left(\frac{-\alpha_0 + -X_i\beta - Z_{i0}\gamma}{\sigma_0}\right) \end{aligned}$$

since $\varepsilon_{i0} \sim N(0, \sigma_0^2)$ implies that $\frac{\varepsilon_{i0}}{\sigma_0} \sim N(0,1)$. Hence, we have that

$$P(T_i = 1 X_i, Z_i, \theta) = \Phi\left(\frac{-\alpha_0 + -X_i\beta - Z_{i0}\gamma}{\sigma_0}\right)$$

We now must find $P(T_i = 2 \mid X_i, Z_i, \theta)$. $T_i = 2$ corresponds to the loan not being repaid in $t = 0$ and being repaid in $t = 1$. Thus, we have that

$$\begin{aligned} P(T_i = 2 \mid X_i, Z_i, \theta) &= P(\alpha_0 + X_i\beta + Z_{i0}\gamma + \varepsilon_{i0} \geq 0, \alpha_1 + X_i\beta + Z_{i1}\gamma + \varepsilon_{i1} < 0) \\ &= P(\varepsilon_{i0} \geq -\alpha_0 - X_i\beta - Z_{i0}\gamma, \varepsilon_{i1} < -\alpha_1 - X_i\beta - Z_{i1}\gamma) \\ &= P(\varepsilon_{i0} \geq -\alpha_0 - X_i\beta - Z_{i0}\gamma, \rho\,\varepsilon_{i0} + \eta_{i1} < -\alpha_1 - X_i\beta - Z_{i1}\gamma) \\ &= P(\varepsilon_{i0} \geq -\alpha_0 - X_i\beta - Z_{i0}\gamma, \eta_{i1} < -\alpha_1 - X_i\beta - Z_{i1}\gamma - \rho\,\varepsilon_{i0}) \\ &= \int_{-\alpha_0 - X_i\beta - Z_{i0}\gamma}^{\infty} \Phi(-\alpha_1 - X_i\beta - Z_{i1}\gamma - \rho\,\varepsilon_{i0}) \frac{\phi(\varepsilon_{i0}/\sigma_0)}{\sigma_0}\, d\,\varepsilon_{i0} \end{aligned}$$

We now consider $P(T_i = 3 \mid X_i, Z_i, \theta)$. This corresponds to the loan not being repaid in $t = 0$ or $t = 1$ and being repaid in $t = 2$. For compactness, we define $b_0 = -\alpha_0 - X_i\beta - Z_{i0}\gamma$,

$b_1 = -\alpha_1 - X_i\beta - Z_{i1}\gamma$, and $b_2 = -\alpha_2 - X_i\beta - Z_{i2}\gamma$. We deduce the following:

$$\begin{aligned}
P(T_i = 3 \mid X_i, Z_i, \theta) &= P(\alpha_0 + X_i\beta + Z_{i0}\gamma + \varepsilon_{i0} \geq 0, \alpha_1 + X_i\beta + Z_{i1}\gamma + \varepsilon_{i1} \geq 0, \\
&\quad \& \ \alpha_2 + X_i\beta + Z_{i2}\gamma + \varepsilon_{i2} < 0) \\
&= P(\varepsilon_{i0} \geq b_0, \varepsilon_{i1} \geq b_1, \varepsilon_{i2} < b_2) \\
&= P(\varepsilon_{i0} \geq b_0, \varepsilon_{i1} \geq b_1, \eta_{i2} < b_2 - \rho\,\varepsilon_{i1}) \\
&= \int_{b_0}^{\infty} \int_{b_1}^{\infty} \Phi(b_2 - \rho\,\varepsilon_{i1})\phi(\varepsilon_{i1} - \rho\,\varepsilon_{i0}) \frac{\phi(\varepsilon_{i0}/\sigma_0)}{\sigma_0}\, d\,\varepsilon_{i1}\, d\,\varepsilon_{i0}
\end{aligned}$$

Finally, we find $P(T_i = 4 \mid X_i, Z_i, \theta)$. This corresponds to the loan not being paid off in $t = 0, 1$, or 2. The setup is very similar as the previous one, but with a flipped inequality for ε_{i2}:

$$\begin{aligned}
P(T_i = 4 \mid X_i, Z_i, \theta) &= P(\alpha_0 + X_i\beta + Z_{i0}\gamma + \varepsilon_{i0} \geq 0, \alpha_1 + X_i\beta + Z_{i1}\gamma + \varepsilon_{i1} \geq 0, \\
&\quad \& \ \alpha_2 + X_i\beta + Z_{i2}\gamma + \varepsilon_{i2} \geq 0) \\
&= P(\varepsilon_{i0} \geq b_0, \varepsilon_{i1} \geq b_1, \varepsilon_{i2} \geq b_2) \\
&= P(\varepsilon_{i0} \geq b_0, \varepsilon_{i1} \geq b_1, \eta_{i2} \geq b_2 - \rho\,\varepsilon_{i1}) \\
&= \int_{b_0}^{\infty} \int_{b_1}^{\infty} [1 - \Phi(b_2 - \rho\,\varepsilon_{i1})]\phi(\varepsilon_{i1} - \rho\,\varepsilon_{i0}) \frac{\phi(\varepsilon_{i0}/\sigma_0)}{\sigma_0}\, d\,\varepsilon_{i1}\, d\,\varepsilon_{i0}
\end{aligned}$$

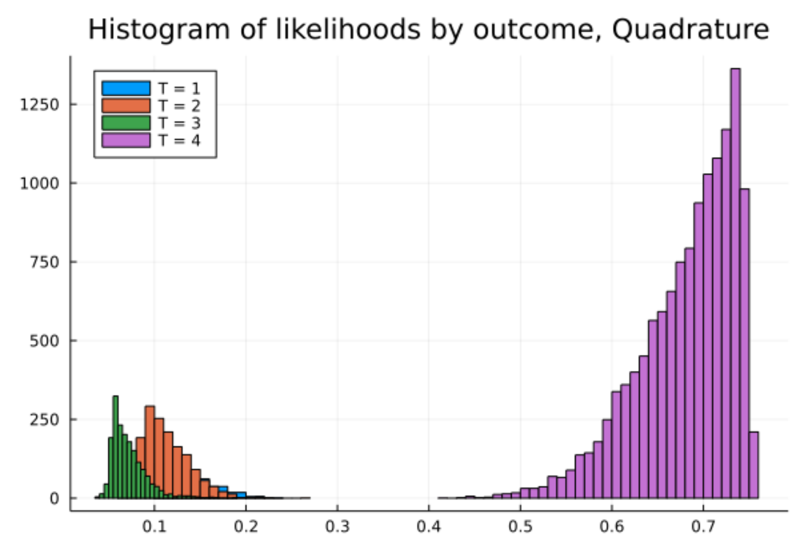
Combining these and using the above definitions of b_0, b_1, and b_2, the likelihood is given by the following:

$$P(T_i \mid X_i, Z_i, \theta) = \begin{cases} \Phi(b_0/\sigma_0) & \text{if } T_i = 1 \\ \int_{b_0}^{\infty} \Phi(b_1 - \rho\,\varepsilon_{i0}) \frac{\phi(\varepsilon_{i0}/\sigma_0)}{\sigma_0}\, d\,\varepsilon_{i0} & \text{if } T_i = 2 \\ \int_{b_0}^{\infty}\int_{b_1}^{\infty} \Phi(b_2 - \rho\,\varepsilon_{i1})\phi(\varepsilon_{i1} - \rho\,\varepsilon_{i0}) \frac{\phi(\varepsilon_{i0}/\sigma_0)}{\sigma_0}\, d\,\varepsilon_{i1}\, d\,\varepsilon_{i0} & \text{if } T_i = 3 \\ \int_{b_0}^{\infty}\int_{b_1}^{\infty} [1 - \Phi(b_2 - \rho\,\varepsilon_{i1})]\phi(\varepsilon_{i1} - \rho\,\varepsilon_{i0}) \frac{\phi(\varepsilon_{i0}/\sigma_0)}{\sigma_0}\, d\,\varepsilon_{i1}\, d\,\varepsilon_{i0} & \text{if } T_i = 4 \end{cases}$$

$$\begin{aligned} \text{with } b_0 &= -\alpha_0 - X_i\beta - Z_{i0}\gamma \\ b_1 &= -\alpha_1 - X_i\beta - Z_{i1}\gamma \\ b_2 &= -\alpha_2 - X_i\beta - Z_{i2}\gamma \end{aligned}$$

Problem 1.

Solution: We wrote a routine which evaluates the log-likelihood function using Gaussian quadrature. We used nodes and weights with precision of 20. We find that the log-likelihood of the initial parameter vector with $\alpha_0 = 0$, $\alpha_1 = -1$, $\alpha_2 = -1$, β_0, $\gamma = 0.3$, and $\rho = 0.5$ is equal to -13598.877. We plot the histogram of likelihoods for each outcome below:



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Problem 2.

Solution: We now provide a routine which evaluates the log-likelihood using the GHK method. We create random samples from target distributions by first using Halton sequences to generate uniform draws from the interval $[0, 1]$ and then transforming these draws using the quantile function of the specified distribution.

The approach of the GHK method is as follows. We want to express the probability of some joint event as the product of independent conditional probabilities. We use the following fact about conditional probability extensively:

$$P(A \cap B) = P(A)P(B \mid A)$$

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As previously, the likelihoods of each outcome can be written in the following form:

$$P(T_i \mid X_i, Z_i, \theta) = \begin{cases} \Phi(b_0/\sigma_0) & \text{if } T_i = 1 \\ P(\varepsilon_{i0} \geq b_0, \varepsilon_{i1} < b_1) & \text{if } T_i = 2 \\ P(\varepsilon_{i0} \geq b_0, \varepsilon_{i1} \geq b_1, \varepsilon_{i2} < b_2) & \text{if } T_i = 3 \\ P(\varepsilon_{i0} \geq b_0, \varepsilon_{i1} \geq b_1, \varepsilon_{i2} \geq b_2) & \text{if } T_i = 4 \end{cases}$$

$$\begin{aligned} \text{with } b_0 &= -\alpha_0 - X_i\beta - Z_{i0}\gamma \\ b_1 &= -\alpha_1 - X_i\beta - Z_{i1}\gamma \\ b_2 &= -\alpha_2 - X_i\beta - Z_{i2}\gamma \end{aligned}$$

We will rewrite each of these likelihoods as the product of independent (conditional) probabilities. The likelihood for $T_i = 2$ can be rewritten as the following:

$$\begin{aligned} P(T_i = 2 \mid X_i, Z_i, \theta) &= P(\varepsilon_{i0} \geq b_0, \varepsilon_{i1} < b_1\gamma) \\ &= P(\varepsilon_{i0} \geq b_0)P(\varepsilon_{i1} < b_1 \mid \varepsilon_{i0} \geq b_0) \\ &= [1 - \Phi(b_0/\sigma)]P(\eta_{i1} < b_1 - \rho\,\varepsilon_{i0} \mid \varepsilon_{i0} \geq b_0) \\ &= [1 - \Phi(b_0/\sigma)]\Phi(b_1 - \rho\,\varepsilon_{i0} \mid \varepsilon_{i0} \geq b_0) \end{aligned}$$

To compute the above likelihood, we must sample ε_{i0} from its (truncated) distribution and find the sample average of the likelihood evaluated at each sampled ε_{i0}. Since we need $\varepsilon_{i0} \geq b_0$, ε_{i0} will be drawn from a truncated normal distribution with mean 0, variance σ_0^2, and support $[b_0, \infty)$. Since $\varepsilon_{i0} = \sigma\eta_{i0}$ where $\eta_{i0} \sim N(0, 1)$, we can draw ε_{i0} from a truncated normal distribution with mean 0, variance 1, and support $[b_0/\sigma, \infty)$ and scale by σ to get ε_{i0}.

We draw η_{i0} using the following procedure. For any $q \in [0, 1]$ (ours will be randomly generated using Halton sequences), we let

$$q = \frac{\Phi(\eta_{i0}) - \Phi(b_0/\sigma)}{1 - \Phi(b_0/\sigma)}$$

Then, inverting this and solving for η_{i0}, we obtain

$$\eta_{i0} = \Phi^{-1}[\Phi(b_0/\sigma) + q(1 - \Phi(b_0/\sigma))]$$

Finally, we scale by σ to get $\varepsilon_{i0} = \sigma\eta_{i0}$. This gives us a draw from the desired truncated distribution for η_{i0}. Let $\{q_k\}_{k=1}^{100}$ be drawn uniformly from $[0, 1]$ and $\{\eta_{i0}^k\}_{k=1}^{100}$ be the corresponding draws using the above procedure. The simulated likelihood of $P(T_i = 2 \mid X_i, Z_i, \theta)$ is the following:

$$\hat{P}(T_i = 2 \mid X_i, Z_i, \theta) = \frac{1}{100}\sum_{i=1}^{n}[1 - \Phi(b_0/\sigma)]\Phi(b_1 - \rho\,\varepsilon_{i0}^k)$$

We do a similar procedure for $T_i = 3$. Rewriting the likelihood as the product of conditional probabilities, we obtain:

$$\begin{aligned} P(T_i = 3 \mid X_i, Z_i, \theta) &= P(\varepsilon_{i0} \geq b_0, \varepsilon_{i1} \geq b_1, \varepsilon_{i2} < b_2) \\ &= [1 - \Phi(b_0/\sigma)] P(\eta_{i1} \geq b_1 - \rho\,\varepsilon_{i0} \mid \varepsilon_{i0} \geq b_0) \\ &\times P(\eta_{i2} < b_2 - \rho\,\varepsilon_{i1} \mid \varepsilon_{i1} \geq b_1 - \rho\,\varepsilon_{i0}, \varepsilon_{i0} \geq b_0) \\ &= [1 - \Phi(b_0/\sigma)][1 - \Phi(b_1 - \rho\,\varepsilon_{i0})]\Phi(b_2 - \rho\,\varepsilon_{i1}) \end{aligned}$$

where $\varepsilon_{i0} \geq b_0$ and $\varepsilon_{i1} \geq b_1$ in the last line.

Here, we must draw two samples recursively. First, we draw ε_{i0} as outlined in the previous section. Then, we must draw ε_{i1} such that $\varepsilon_{i1} \geq b_1$. Note that since $\varepsilon_{i1} = \rho\,\varepsilon_{i0} + \eta_{i1}$, this is equivalent to drawing $\eta_{i1} \geq b_1 - \rho\,\varepsilon_{i0}$. Thus, we first draw η_{i1} from a truncated $N(0, 1)$ distribution with support $[b_1 - \rho\,\varepsilon_{i0}, \infty)$. Using our Halton sequence, we generate a random $q \in [0, 1]$ and use this to find the corresponding η_{i1}:

$$q = \frac{\Phi(\eta_{i1}) - \Phi(b_1 - \rho\,\varepsilon_{i0})}{1 - \Phi(b_1 - \rho\,\varepsilon_{i0})} \iff \eta_{i1} = \Phi^{-1}[\Phi(b_1 - \rho\,\varepsilon_{i0}) + q(1 - \Phi(b_1 - \rho\,\varepsilon_{i0}))]$$

Then, we apply the following transformation to obtain ε_{i1}:

$$\varepsilon_{i1} = \rho\,\varepsilon_{i0} + \eta_{i1}$$

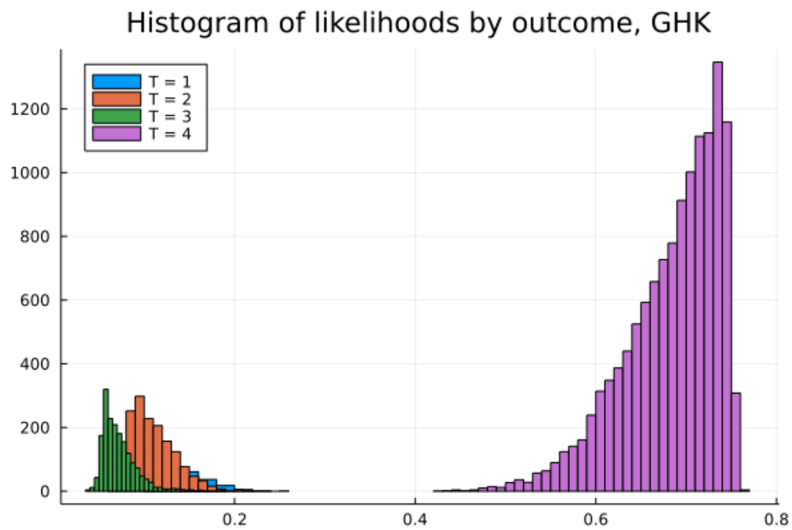
To find the simulated likelihood, we proceed as before. We generate two uniform random samples $\{q_k^1, q_k^2\}_{k=1}^{100}$ using Halton sequences and find the corresponding draws $\{\varepsilon_{i0}^k, \varepsilon_{i1}^k\}_{k=1}^{100}$ using the procedure outlined above. Then, the simulated likelihood is given by

$$\hat{P}(T_i = 3 \mid X_i, Z_i, \theta) = \frac{1}{100}\sum_{k=1}^{100}[1 - \Phi(b_0/\sigma)][1 - \Phi(b_1 - \rho\,\varepsilon_{i0}^k)]\Phi(b_2 - \rho\,\varepsilon_{i1}^k)$$

The sampling procedure for $T_i = 4$ is identical, and the simulated likelihood is given by

$$\hat{P}(T_i = 4 \mid X_i, Z_i, \theta) = [1 - \Phi(b_0/\sigma)][1 - \Phi(b_1 - \rho\,\varepsilon_{i0}^k)][1 - \Phi(b_2 - \rho\,\varepsilon_{i1}^k)]$$

Using this procedure, we find that the log-likelihood of the initial parameter guess is -13579.539, which is very close to the log-likelihood obtained in the first part. The histogram of likelihoods by outcome is included below. It matches very closely with that of quadrature:



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