Math 6122: HW 5

Padmavathi Srinivasan

Due: Thursday, Feb 21st, start of class

In the following exercises, we will define and derive some properties of analogues of the 'norm' and 'trace' maps $\mathbb{C} \to \mathbb{R}$ (i.e., $a + ib \mapsto a^2 + b^2$ and $a + ib \mapsto 2a$) for an arbitrary finite extension of fields.

Let E/F be a finite extension of fields and let $\alpha \in E$. Let $M_\alpha : E \to E$ be the F-linear map induced by multiplication by α, i.e., the map $e \to \alpha e$. Let $f_\alpha = \det(xI - M_\alpha) \in F[x]$ be the characteristic polynomial of M_α (we have changed the usual sign for the characteristic polynomial to make f_α monic). Let $m_\alpha \in F[x]$ be the minimal polynomial of α.

1. If $E = F(\alpha)$, show that f_α equals the minimal polynomial m_α of α. (Hint: Choose a nice basis for $F(\alpha)/F$ and write down the matrix for M_α in this basis.)

2. Show that in general $f_\alpha = m_\alpha^{[E:F(\alpha)]}$. (Hint: Write down a matrix for M_α by using a nice 'product basis', that is by taking the product of the standard basis $\{1, \alpha, \alpha^2, \dots\}$ for $F(\alpha)/F$ and any basis for $E/F(\alpha)$.) [Remark: The polynomial m_α can also be identified with the minimal polynomial of the the linear transformation M_α.]

   The norm of α denoted $N_{E/F}(\alpha)$ is the determinant of M_α, and the trace of α denoted $\mathrm{Tr}_{E/F}(\alpha)$ is the trace of M_α.

3. Verify that $N_{E/F}(\alpha\beta) = N_{E/F}(\alpha)N_{E/F}(\beta)$ and $\mathrm{Tr}_{E/F}(\alpha + \beta) = \mathrm{Tr}_{E/F}(\alpha) + \mathrm{Tr}_{E/F}(\beta)$.

4. Let $\{\alpha_1 = \alpha, \alpha_2, \dots, \alpha_d\}$ be the roots of the minimal polynomial m_α (counted with multiplicity, so $d = \deg m_\alpha$) in a splitting field for α. Show that $\mathrm{Tr}_{E/F}(\alpha) = [E : F(\alpha)](\alpha_1 + \alpha_2 + \ldots + \alpha_d)$ and $N_{E/F}(\alpha) = (\alpha_1\alpha_2 \ldots \alpha_d)^{[E:F(\alpha)]}$.

5. Let let $E = F(\sqrt{D})$ for some $D \notin F^{\times 2}$ (i.e. a quadratic extension). Verify that $N_{E/F}(a+b\sqrt{D}) = a^2 - Db^2$ and $\mathrm{Tr}_{E/F}(a + b\sqrt{D}) = 2a$.

6. Let $E' \supset E \supset F$ be a tower of extensions such that E'/F is Galois. Let H be the subgroup of $\mathrm{Gal}(E'/F)$ corresponding to E, and let S be a set of coset representatives for G/H. Show that $f_\alpha(x) = \prod_{\sigma \in S}(x - \sigma\alpha)$. (Hint: First verify this for $E = F(\alpha)$ using the previous problem, and then prove this for general E using the relation between f_α and m_α.)

7. (Transitivity of trace) Use the previous problem to show that if $E' \supset E_1 \supset E_2 \supset F$ is a tower of extensions such that E'/F is Galois, then $\mathrm{Tr}_{E_1/F} = \mathrm{Tr}_{E_2/F} . \mathrm{Tr}_{E_1/E_2}$. [Remark: Transitivity of trace in fact holds in any tower of field extensions.]

   The trace map can be used to detect inseparability of extensions.

8. Show that if m_α is an inseparable polynomial, then $\mathrm{Tr}_{F(\alpha)/F} \equiv 0$.

9. Using the fact that every inseparable extension E/F can be factored as $E \supset E' \supset F$, where $E = E'(\alpha)$ for an inseparable element α (that is, the minimal polynomial of α over E' is inseparable) and the transitivity of trace, show that $\mathrm{Tr}_{E/F} \equiv 0$ for any inseparable extension E/F.

10. Using Problem 6 and Dedekind's lemma on linear independence of characters to show that $\mathrm{Tr}_{E'/F}$ does not identically vanish for a Galois extension E'/F (i.e., there exists $\alpha \in E'$ such that $\mathrm{Tr}_{E'/F}(\alpha) \neq 0$). Use transitivity of trace and the existence of Galois closures of separable extensions to conclude that $\mathrm{Tr}_{E/F}$ does not identically vanish for any separable extension.

Math 6122 - Homework 5

Caitlin Beecham (Discussed with Skye)

February 21, 2019

1

We let our basis for E/F be $\{\alpha^i | i \in \{0, \dots, n-1\}\}$. Then, the matrix M_α can be written as

$$M_\alpha = \begin{array}{cc} & \begin{matrix} 1 & \alpha & \alpha^2 & \alpha^3 & \dots & \alpha^{n-2} & \alpha^{n-1} \end{matrix} \\ \begin{matrix} 1 \\ \alpha \\ \alpha^2 \\ \alpha^3 \\ \vdots \\ \alpha^{n-2} \\ \alpha^{n-1} \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & 0 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & 0 & 0 & \dots & 0 & -a_2 \\ 0 & 0 & 1 & 0 & \dots & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & -a_{n-2} \\ 0 & 0 & 0 & 0 & \dots & 1 & -a_{n-1} \end{pmatrix} \end{array},$$

which gives us that

$$XI - M_\alpha = \begin{array}{cc} & \begin{matrix} 1 & \alpha & \alpha^2 & \alpha^3 & \dots & \alpha^{n-2} & \alpha^{n-1} \end{matrix} \\ \begin{matrix} 1 \\ \alpha \\ \alpha^2 \\ \alpha^3 \\ \vdots \\ \alpha^{n-2} \\ \alpha^{n-1} \end{matrix} & \begin{pmatrix} X & 0 & 0 & 0 & \dots & 0 & a_0 \\ -1 & X & 0 & 0 & \dots & 0 & a_1 \\ 0 & -1 & X & 0 & \dots & 0 & a_2 \\ 0 & 0 & -1 & X & \dots & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & X & a_{n-2} \\ 0 & 0 & 0 & 0 & \dots & -1 & X + a_{n-1} \end{pmatrix} \end{array}.$$

We can then use the cofactor expansion of the determinant along the last row to obtain that (in the case where n is odd)

$$\begin{aligned} det(XI - M_\alpha) &= (\sum_{i=0}^{n-2} (-1)^i a_i det(M_{n-1}^i)) + (X + a_{n-1}) det(M_{n-1}^{n-1}) \\ &= (\sum_{i=0}^{n-2} (-1)^i a_i X^i (-1)^{n-1-i}) + (X + a_{n-1}) X^{n-1} \\ &= (\sum_{i=0}^{n-2} (-1)^{n-1} a_i X^i) + X^n + a_{n-1} X^{n-1} \\ &= X^n + \sum_{i=0}^{n-1} a_i X^i = m_\alpha(X) \end{aligned}$$

or

$$\begin{aligned}
det(XI - M_\alpha) &= (\sum_{i=0}^{n-2} (-1)^{i+1} a_i det(M_{n-1}^i)) + (X + a_{n-1}) det(M_{n-1}^{n-1}) \\
&= (\sum_{i=0}^{n-2} (-1)^{i+1} a_i X^i (-1)^{n-1-i}) + (X + a_{n-1}) X^{n-1} \\
&= (\sum_{i=0}^{n-2} (-1)^n a_i X^i) + X^n + a_{n-1} X^{n-1} \\
&= X^n + \sum_{i=0}^{n-1} a_i X^i = m_\alpha(X)
\end{aligned}$$

if n is even (where M_j^i is the matrix obtained from M_α by deleting row i and column j where our indexing starts at 0) since all cofactor matrices are upper triangular which means that their determinants are the product of the elements on the diagonal. More precisely, $det(M_{n-1}^i) = \prod_{l=0}^{n-2} (M_{n-1}^i)_l^l = (\prod_{l<i} (M_\alpha)_l^l)(\prod_{l>i} (M_\alpha)_l^{l+1}) = (\prod_{l<i} X)(\prod_{l>i} -1) = X^i (-1)^{n-1-i}$.

2

Used the following source on computing determinants via blocks: https://math.stackexchange.com/questions/148532/general-expression-for-determinant-of-a-block-diagona https://arxiv.org/pdf/1112.4379.pdf

Now, say that $B := \{1 =: b_0, b_1, b_2, \ldots, b_r\}$ is a basis for $E/F(\alpha)$. Then, a basis for E/F is $\{\alpha^i b_j | i \in \{0, \ldots, n-1\}, j \in \{0, \ldots, r\}\}$. Then, we note that we can write the matrix induced by the linear transformation of E which is multiplication by α. Call it M_E. See attached figure. I'm sorry but this was waaay to hard to TeX–I tried.

3

We construct a basis using a tower of intermediate extensions. We note that $F \subseteq F(\alpha) \subseteq F(\alpha)(\beta) \subseteq E$. Let $m_\alpha(x) = \sum_{i=0}^n a_i \alpha^i$ be the minimal polynomial of α over F. Then, let $m_\beta(x) = \sum_{i=0}^r b_i x^i$ be the minimal polynomial of β over $F(\alpha)$. Then, let $1 =: c_0, \ldots, c_{s-1}$ be a basis for E over $F(\alpha)(\beta)$ (so here $[E : F(\alpha, \beta)] = s$). Then, using the basis $\{c_i \alpha^j \beta^k | i \in \{0, \ldots, s-1\}, j \in \{0, \ldots, n-1\}, k \in \{0, \ldots, r-1\}\}$ we note that we can write down a matrix which represents the linear transformation induced by multiplication by $\alpha\beta$ in this basis. (For future convenience, we impose an order on the basis given by $c_{i_1} \beta^{k_1} \alpha^{j_1} < c_{i_2} \beta^{k_2} \alpha^{j_2}$ if and only if $i_1 < i_2$ or ($i_1 = i_2$ and $k_1 < k_2$) or ($i_1 = i_2$ and $k_1 = k_2$ and $j_1 < j_2$). We say $c_{i_1} \beta^{k_1} \alpha^{j_1} = c_{i_2} \beta^{k_2} \alpha^{j_2}$ if and only if $i_1 = i_2$ and $j_1 = j_2$ and $k_1 = k_2$. This gives us an index for each basis element if we say that $c_0 \alpha^0 \beta^0 = c_0$ has index 0). Call such a matrix $M := M_{\alpha\beta}^E$. Then, we define the block C_j^i for $i, j \in \{0, \ldots, s-1\}$ of M as follows. Let $C_j^i := M_{[rnj:rn(j+1)-1]}^{[rni:rn(i+1)-1]}$ where $M_{[c:d]}^{[a:b]}$ denotes the submatrix of M using the ath through bth rows (inclusive) and cth through dth columns of M (inclusive). (Note: that these are indices of rows, NOT the corresponding basis elements and also note that row and column indexing starts at 0 according to my setup). Now, we note that C_j^i is the zero matrix for all $i \neq j$ which means that M is a block diagonal matrix. We now examine a non zero block, say C_1^1, noting that all of the diagonal blocks C_i^i are pairwise identical for all $i \in \{0, \ldots, s-1\}$. Now, what does C_1^1 look like? We can further break $C := C_1^1$ into blocks.

Define B^i_j for $i, j \in \{0, \dots, r-1\}$ by $B^i_j := C^{[ni:n(i+1)-1]}_{[nj:n(j+1)-1]}$. We then note that B^i_i is the zero matrix for all $i \in \{0, \dots, r-2\}$ and that $B^i_{i-1} = M_\alpha$ for all $i \in [r-1]$. That defines all blocks except those in the last column. Namely, we still have not determined B^i_{r-1} for all $i \in \{0, \dots, r-1\}$.

4

We note that M_E (which represents multiplication by α in E using the basis outlined in the figure in problem 2) is a block diagonal matrix in which each block is M_α. Thus, the trace of M_E is

$$Tr(M_E) = [E : F(\alpha)](-a_{d-1})$$

where a_{d-1} is a coefficient of the minimal polynomial of α over F given by $m_\alpha(x) = x^d + \sum_{i=0}^{d-1} a_i x^i$. We then note that if $\alpha_1 := \alpha, \alpha_2, \alpha_3, \dots, \alpha_d$ are the roots of m_α (not necessarily distinct), then we have that

$$\sum_{j=0}^{d} a_j x^j = \prod_{i=1}^{d} (x - \alpha_i)$$

Then, note that

$$\prod_{i=1}^{d} (x - \alpha_i) = \sum_{i=0}^{d} \Big((-1)^{d-i} \Big(\sum_{S \in \binom{[d]}{d-i}} \big(\prod_{s \in S} \alpha_s \big) \Big) (x^i) \Big),$$

which in particular means that

$$\begin{aligned} -a_{d-1} &= (-1)(-1)^{d-(d-1)} \Big(\sum_{S \in \binom{[d]}{1}} \big(\prod_{s \in S} \alpha_s \big) \Big) \\ &= (-1)^2 \Big(\sum_{i=1}^{d} \alpha_i \Big) \\ &= \sum_{i=1}^{d} \alpha_i. \end{aligned}$$

So, we get that $-a_{n-1} = \sum_{i=1}^{d} \alpha_d$ which means that

$$Tr(M_E) = [E : F(\alpha)](\sum_{i=1}^{d} \alpha_d)$$

and we are done.

Now it remains to show that $N_{E/F}(\alpha) = (\prod_{i=1}^{d} \alpha_i)^{[E:F(\alpha)]}$. We recall that the determinant of M_E can be expressed as

$$\det(M_E) = \prod_{i=1}^{[E:F(\alpha)]} \det(M_\alpha) = \prod_{i=1}^{[E:F(\alpha)]} N_{F(\alpha)/F}(\alpha).$$

We then recall that

$$M_\alpha = \begin{array}{c} \\ 1 \\ \alpha \\ \alpha^2 \\ \alpha^3 \\ \vdots \\ \alpha^{n-2} \\ \alpha^{n-1} \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & \alpha & \alpha^2 & \alpha^3 & \dots & \alpha^{n-2} & \alpha^{n-1} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & 0 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & 0 & 0 & \dots & 0 & -a_2 \\ 0 & 0 & 1 & 0 & \dots & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & -a_{n-2} \\ 0 & 0 & 0 & 0 & \dots & 1 & -a_{n-1} \end{pmatrix} \end{array},$$

and notice recall that $m_\alpha(x) = \det(XI - M_\alpha)$ which implies that $m_\alpha(0) = a_0 = \det(-M_\alpha) = (-1)^n \det(M_\alpha)$ (here I am using n and d interchangeably as the degree of m_α) which gives us that $N_{F(\alpha)/F}(\alpha) = \det(M_\alpha) \in \{a_0, -a_0\}$. Namely, if n is odd then $N_{F(\alpha)/F}(\alpha) = \det(M_\alpha) = -a_0$. Otherwise, $N_{F(\alpha)/F}(\alpha) = \det(M_\alpha) = a_0$. Now, we return to the formula,

$$\prod_{i=1}^{d}(x-\alpha_i) = \sum_{i=0}^{d}\Big((-1)^{d-i}\Big(\sum_{S\in\binom{[d]}{d-i}}\big(\prod_{s\in S}\alpha_s\big)\Big)(x^i)\Big),$$

which tells us that

$$\begin{aligned} a_0 &= (-1)^d\Big(\sum_{S\in\binom{[d]}{0}}\big(\prod_{s\in S}\alpha_s\big)\Big) \\ &= (-1)^d\Big(\prod_{i=1}^{d}\alpha_i\Big) \\ &= (-1)^d\prod_{i=1}^{d}\alpha_i. \end{aligned}$$

So, if n is odd then,

$$N_{F(\alpha)/F}(\alpha) = \det(M_\alpha) = -a_0 = (-1)(-1)^d\prod_{i=1}^{d}\alpha_i = (-1)^{n+1}\prod_{i=1}^{d}\alpha_i = \prod_{i=1}^{d}\alpha_i.$$

Similarly, if n is even, we get that (still using n and d interchangeably)

$$N_{F(\alpha)/F}(\alpha) = \det(M_\alpha) = a_0 = (-1)^d\prod_{i=1}^{d}\alpha_i = \prod_{i=1}^{d}\alpha_i.$$

Namely, we have that

$$\det(M_E) = \prod_{i=1}^{[E:F(\alpha)]}\det(M_\alpha) = \prod_{i=1}^{[E:F(\alpha)]}\prod_{i=1}^{d}\alpha_i = (\prod_{i=1}^{d}\alpha_i)^{[E:F(\alpha)]}.$$

5

We note that

$$M_{a+b\sqrt{D}} = \begin{array}{c} \\ 1 \\ \sqrt{D} \end{array} \begin{array}{c} \begin{array}{cc} 1 & \sqrt{D} \end{array} \\ \begin{pmatrix} a & bD \\ b & a \end{pmatrix} \end{array}$$

which tells us that $N_{E/F}(a+b\sqrt{D}) = a^2 - b^2 D$ and $Tr_{E/F}(a+b\sqrt{D})2a$.

6

First say $E = F(\alpha)$. Now, $H := \text{Gal}(E'/E) = \text{Gal}(E'/F(\alpha))$. When are two automorphisms σ_1 and σ_2 in the same coset of G/H? When $\sigma_1(\alpha) = \sigma_2(\alpha)$. Why? These automorphisms are in the same coset, more precisely $\sigma_1 \in \sigma_2 H$, exactly when $(\sigma_2)^{-1}(\sigma_1) \in H$, which means that $(\sigma_2)^{-1}\sigma_1(\alpha) = \alpha$ or $\sigma_2(\alpha) = \sigma_1(\alpha)$. Now, $\sigma(\alpha) \in \{\alpha_1 := \alpha, \alpha_2, \ldots, \alpha_k\}$ where $\{\alpha_1 := \alpha, \alpha_2, \ldots, \alpha_k\}$ are the (not necessarily distinct) roots of the minimal polynomial of α. Now, E'/F Galois means that any polynomial with a root in E' splits completely in E'. So, all roots $\{\alpha_1 := \alpha, \alpha_2, \ldots, \alpha_k\}$ are contained in E', which means that for all $i \in [k]$ there exists some $\sigma_i \in \text{Gal}(E'/F)$ such that $\sigma_i(\alpha) = \alpha_i$, each corresponding to a different coset of G/H (because we note that E'/F Galois implies E'/F separable which implies that the minimal polynomial of any element of E'/F is separable. Thus, all roots of the minimial polynomial of α actually are distinct meaning that $\alpha_i \neq \alpha_i$ for $j \neq i$). Thus, a valid set of coset representatives for G/H is such a set $S := \{\sigma_i | i \in [k]\}$. Thus, if we denote the minimal polynomial of α by $m_\alpha(x)$, we have $m_\alpha(x) = \prod_{i=1}^{k}(x - \alpha_i) = \prod_{\sigma_i \in S}(x - \sigma_i(\alpha))$. Finally, by noting that $f_\alpha(x) = m_\alpha(x)^{[E:F(\alpha)]}$ we see that in this case $[E : F(\alpha)] = 1$ which gives us $f_\alpha(x) = m_\alpha(x)$ in this case. Now, say that $E \supsetneq F(\alpha)$ and denote $H := \text{Gal}(E'/E)$ while $K := \text{Gal}(E'/F(\alpha))$. Now, once again we have that if $S := \{\sigma_i | i \in [k]\}$ where σ_i is a field automorphism fixing F such that $\sigma_i(\alpha) = \alpha_i$, then S is a set of coset representatives ofr G/K and $m_\alpha(x) = \prod_{\sigma_i \in S}(x - \sigma_i(\alpha))$. Finally, we note that if we have S a set of coset representatives for G/K and T a set of coset representatives for K/H (none of G/K or K/H are claimed to be groups), then $R := \{st | s \in S, t \in T\}$ is a set of coset representatives for G/H. Finally, one notes that $t(\alpha) = \alpha$ for all $t \in T$ since $t \in K := \text{Gal}(E'/F(\alpha))$. So, $\prod_{st \in R}(x - st(\alpha)) = \prod_{st \in R}(x - s(\alpha)) = \prod_{s \in S}(x - s(\alpha))^{|T|} = \prod_{s \in S}(x - s(\alpha))^{[E:F(\alpha)]} = (\prod_{s \in S}(x - s(\alpha)))^{[E:F(\alpha)]} = m_\alpha(x)^{[E:F(\alpha)]} = f_\alpha(x)$ and we are done.

7

Let $G := Gal(E'/F)$. Then, let $K := Gal(E'/E_1)$, $H := Gal(E'/E_2)$. Now, let S be a set of coset representatives for G/H and let T be a set of coset representatives for H/K. Then, $R := \{st | s \in S, t \in T\}$ is a set of coset representatives for G/K. Finally, note that $Tr_{E_1/F}(\alpha) = \sum_{q \in Q}(q(\alpha))$ where Q is a set of coset representatives for G/K. Recalling that R is such a set, we get that $Tr_{E_1/F}(\alpha) = \sum_{r \in R}(r(\alpha)) = \sum_{s \in S}\sum_{t \in T} st(\alpha) = \sum_{s \in S} s(\sum_{t \in T} t(\alpha)) = \sum_{s \in S} s(Tr_{E_1/E_2}(\alpha)) = Tr_{E_2/F}(Tr_{E_1/E_2}(\alpha))$ and we are done.

8

By definition m_α is irreducible. One can only have an inseparable irreducible polynomial in an infinite field of finite characteristic (or perhaps non-zero characteristic is a better way to say it). Now, for any field, its characteristic is either 0 or a prime number p. So, say F has characteristic p. We wish to show that p divides $Tr_{F(\alpha)/F}$. Well, first note that $Tr_{F(\alpha)/F} = \sum_{i=1}^{n} \alpha_i$ where $\{\alpha_i | i \in [n]\}$ are the not necessarily distinct roots of m_α. Now, we also note that as computed in problem 4 $\sum_{i=1}^{n} \alpha_i = \pm a_{n-1}$ where a_{n-1} is the coefficient of x^{n-1} in m_α. Next, one notes that m_α inseparable means that $\gcd(m_\alpha, m'_\alpha) \neq 1$. So, say $m_\alpha(x) = g(x)f(x)$ and $m'_\alpha(x) = g(x)h(x)$ where $g(x) =: \sum_{i=0}^{k} d_i x^i$, $f(x) =: \sum_{i=0}^{n-k} b_i x^i$, and $h(x) =: \sum_{i=0}^{n-k-1} c_i x^i$. Next, note that $m_\alpha(x) = \sum_{i=0}^{k}\sum_{j=0}^{n-k} d_i b_j x^{i+j}$ and $m'_\alpha(x) = \sum_{i=0}^{k}\sum_{j=0}^{n-k-1} d_i c_j x^{i+j}$. As noted before $Tr_{F(\alpha)/F} = \pm a_{n-1} = \pm(d_{k-1}b_{n-k} + d_k b_{n-k-1})$. However, m_α monic implies that $b_{n-k} = d_k = 1$. So, $Tr_{F(\alpha)/F} = \pm a_{n-1} = \pm(d_{k-1} + b_{n-k-1})$. Next, one notes that the coefficient of x^{n-2} in m'_α is $a'_{n-2} = (n -$

$1)a_{n-1} = \pm(n-1)(Tr_{F(\alpha)/F}) = \pm(n-1)(d_{k-1} + b_{n-k-1}) = (d_{k-1}c_{n-k-1} + d_k c_{n-k-2})$, but $a'_{n-1} = n = d_k c_{n-k-1} = c_{n-k-1}$ implies that $c_{n-k-1} = n$. So, $a'_{n-2} = (n-1)a_{n-1} = \pm(n-1)(Tr_{F(\alpha)/F}) = \pm(n-1)(d_{k-1} + b_{n-k-1}) = (nd_{k-1} + c_{n-k-2})$.

9

Say E/F is an inseparable extension with $E' \supset E \supset F$ and $E' = E(\alpha)$ for some $\alpha \in E'$ whose minimal polynomial over E is inseparable. Now, as shown in problem 7, $Tr_{E'/F} = Tr_{E/F}Tr_{E'/E}$. Now, since $Tr_{E'/E} = Tr_{E(\alpha)/E} \equiv 0$, we get that $Tr_{E'/F} = Tr_{E/F} * 0 \equiv 0$ and we are done.

10

We note that $\sigma : F^\times \to F^\times$ is a group homomorphism for any field automorphism σ (regardless of what is fixed by σ, I'm just saying it's an automorphism). Now $Tr_{E'/F}(\alpha) = \sum_{\sigma \in Gal(E'/F)} \sigma(\alpha)$. Say $Tr_{E'/F}(\alpha) = 0$ for all $\alpha \in E'$. Then, Dedekind's lemma says that $\sum_{\sigma \in Gal(E'/F)} a_i \sigma_i = \sum_{\sigma \in Gal(E'/F)} \sigma_i$ must satisfy $a_i = 0$ for all $i \in |Gal(E'/F)|$, a contradiction. Now, say we have a separable extension E/F. We know that there exists field $E' \supset E$ such that E'/F is Galois. Now, $Tr_{E'/F} = Tr_{E/F}Tr_{E'/E}$. Since E'/F is Galois, we just showed that $Tr_{E'/F}$ is not the zero function. Assume for contradiction that $Tr_{E/F}$ were the zero function. Then, one would have $Tr_{E'/F} = 0(Tr_{E'/E}) \equiv 0$, a contradiction and we are done.