Summary of "Realization of Groups with Pairing as Jacobians of Finite Graphs" by Gaudet et al

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Jacobian Group

Q: What is the Jacobian of a graph?
A: A finite abelian group constructable in 2 ways.

1. Use the Smith Normal Form of the Laplacian matrix or
2. Define an equivalence relation on certain "numberings" of the vertices of G.

Either construction gives the "same" group (up to isomorphism).

Jacobian Group using the Laplacian

Notation: $n := |V(G)|$.

- Compute the Laplacian $L = A - D$.
- Find its Smith Normal Form $M = SLT$ (with diagonal entries $m_1, \ldots, m_n$).
- Then, $Jac(G) =$ torsion part of the group $\prod_{i=1}^{n} \mathbb{Z}/m_i\mathbb{Z}$.

Jacobian Group using the Laplacian

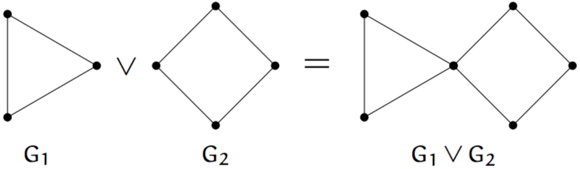
Q: What is meant by torsion part?
A: Subgroup of only the elements of finite order.

Theorem of Finitely Generated Abelian Groups:
Every finitely generated abelian group K has the form

$$K \cong \underbrace{\mathbb{Z}/p_1^{r_1}\mathbb{Z} \times \mathbb{Z}/p_2^{r_2}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_k^{r_k}\mathbb{Z}}_{\text{torsion part}} \times \mathbb{Z}^r$$

Wedge Sum of Two Graphs

Definition: The Wedge Sum of two graphs G_1, G_2 with chosen vertices v_1, v_2 is the quotient of the disjoint union of $G_1 \sqcup G_2$ by identifying v_1 and v_2 in the union.
Example:



Goal of this Paper

Goal: Prove that any odd order finite abelian group H occurs as the Jacobian of some graph G. Namely, for all finite abelian groups H of odd order, we wish to construct a graph G such that

$$Jac(G) = H.$$

Method: Note that any finite abelian group H is isomorphic to

$$H \cong \prod_i \mathbb{Z}/p_i^{r_i}\mathbb{Z},$$

for some finite list of primes $\{p_i\}$.

Plan

If for any odd prime p_i and positive integer r_i, we can construct a graph G_i with Jacobian group

$$Jac(G_i) = \mathbb{Z}/p_i^{r_i}\mathbb{Z}.$$

Then, if $H \cong \prod_i \mathbb{Z}/p_i^{r_i}\mathbb{Z}$, by letting the graph $G = \bigvee_i G_i$ be the wedge sum of these graphs, we get

$$\begin{aligned} Jac(G) &= Jac(\bigvee_i G_i) \\ &= \bigoplus_i Jac(G_i) \\ &= \bigoplus_i \mathbb{Z}/p_i^{r_i}\mathbb{Z} = H \end{aligned}$$

and we get the desired Jacobian group.

Plan

Step 1: we first show that for any odd prime p_i and positive integer r_i, we can construct a graph G_i with Jacobian group

$$Jac(G_i) = \mathbb{Z}/p_i^{r_i}\mathbb{Z}.$$

(So, here we are just constructing each cyclic factor as the Jacobian of some graph).
Then, if $H \cong \prod_i \mathbb{Z}/p_i^{r_i}\mathbb{Z}$, by letting the graph $G = \bigvee_i G_i$ be the wedge sum of these graphs, we get

$$\begin{aligned} Jac(G) &= Jac(\bigvee_i G_i) \\ &= \bigoplus_i Jac(G_i) \\ &= \bigoplus_i \mathbb{Z}/p_i^{r_i}\mathbb{Z} = H \end{aligned}$$

and we are done.

Plan Continued

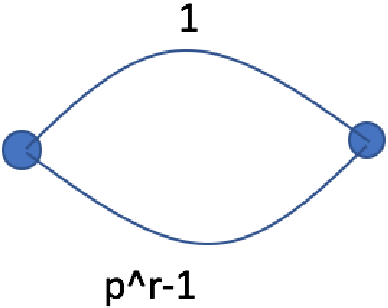
In our construction of a graph G such that $Jac(G) = \mathbb{Z}/p^r\mathbb{Z}$, we define a certain type of graph called a Banana Graph on a tuple $s = (s_1, \ldots, s_m)$ where if the s_i satisfy certain properties, we indeed get that $Jac(G) = Jac(B_s) = \mathbb{Z}/p^r\mathbb{Z}$. A lot of the very

tedious number theoretic arguments I will flesh out serve the sole purpose of constructing the $s = (s_1, s_2, \ldots, s_m)$ with the desired properties. At the end of the day, the desired Banana Graphs will be very simple. There are actually only three possible cases.

Plan Continued

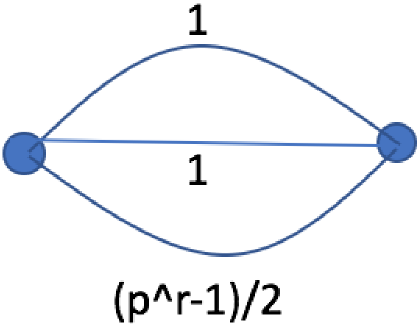
At the end of the day, the desired Banana Graphs will be one of the following.

Desired Banana Graphs



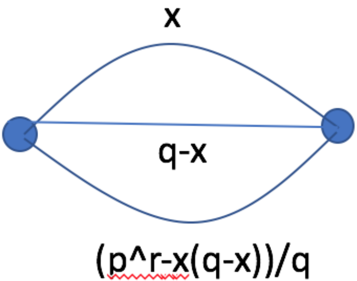
p prime

Desired Banana Graphs



p prime

Desired Banana Graphs



p prime

x < q a certain integer

Slight Problems

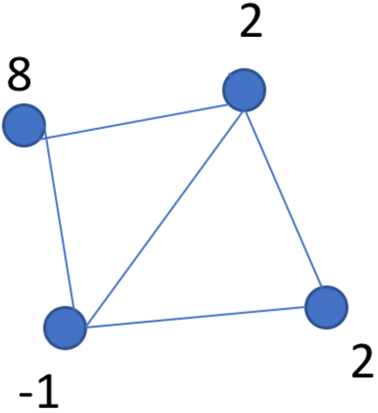
We can easily prove a relaxed statement of the above. Namely, we can show (relaxed version of Main Theorem) that there exists a finite set of primes $\mathcal{P} \subseteq \mathbb{Z}$ such that for any abelian group H whose order is not divisible by any prime $p \in \mathcal{P}$, we can construct a graph G with

$$Jac(G) \cong H.$$

The reason for the above weakening is that many lemmas I will discuss only work for "sufficiently large" p. However, if the Generalized Riemann Hypothesis holds, then we can say that this list of forbidden primes is just $\mathcal{P} = \{2\}$.

Jacobian Group using Vertex Numberings

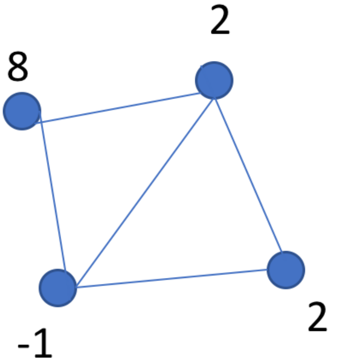
Def: A vertex numbering is a map $D : V(G) \to \mathbb{Z}$.



Jacobian Group using Vertex Numberings

Def: Degree of a vertex numbering D is

$$\sum_{v \in V} D(v).$$



Deg(D) =
8+2+2+(-1) = 11

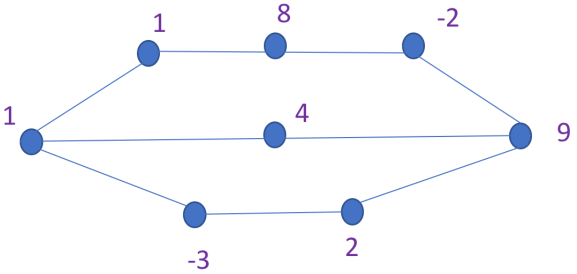
Jacobian Group using Vertex Numberings

Q: What is a principal vertex numbering?
A: A vertex numbering D arising from another vertex numbering f in a special way. How?

Definition of Principal Numbering D from f

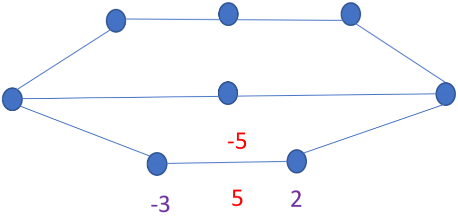
- Take a numbering $f : V(G) \to \mathbb{Z}$.
- Define a function $B : \overrightarrow{E}(G) \to Z$ on the edges.
- Define new numbering $D(v)$ using $B(e)$ on edges e adjacent to v.

Definition of Principal Numbering D from f



f(w) in
purple

Definition of Principal Numbering *D* from *f*

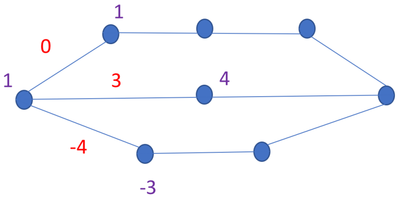


f(w) in
purple

B(xy) = f(y)-f(x)
B(yx) = f(x)-f(y)

Definition of Principal Numbering *D* from *f*

D(v) = 0+3+(-4)
= -1



f(w) in
purple

B(xy) = f(y)-f(x)
B(yx) = f(x)-f(y)

D(v) = sum of
B(e) for all
outgoing edges
e from x

Definition of Principal Numbering D from f

To summarize: The numbering D is principal if it arises from some f by the above process. Or equivalently put, the "image" of this process is the set of principal numberings.
Also, Degree 0 principal numberings are exactly those reachable from the 0 numbering through a sequence of chip-firing moves.
Example: [On board]

Jacobian Using Equivalence Relation

We can add and subtract numberings. It's just component wise (each component being a vertex) addition.

$$D_1 + D_2 := \bigoplus_{v \in V} D_1(v) + D_2(v).$$

Jacobian Using Equivalence Relation

Definition of equivalence relation, $\sim$
$D_1 \sim D_2$ if

$$D_1 - D_2 \text{ is a principal numbering } .$$

Example: [On board]

Jacobian Using Equivalence Relation

Now the Jacobian group is

$$\{\text{degree 0 numberings}\}/\sim.$$

(This should remind us of the Class Group a little!)

Jacobian Group and Number of Spanning Trees

Fun Fact: If G is a connected graph, then $|Jac(G)| = (\#$ of spanning trees of $G)$.

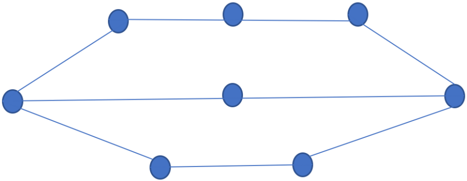
Jacobians of Banana Graphs

Defintion: A Banana Graph on the tuple $s = (s_1, \ldots, s_m)$ is constructed as follows:

- Take 2 vertices.
- Add m edges between them.
- Subdivide the ith edge $s_i - 1$ times.

Example: $B_{(4,2,3)}$.

B(4,3,2)



Jacobians of Banana Graphs

Q: How do we count the number of spanning trees in a Banana Graph?

A: We get a spanning tree by removing an edge from every path p_2 except one. Count the number of ways to do that.

$$\left(\underbrace{\sum_{i=1}^{m}}_{\text{choose path } p_i \text{ to keep}} \left(\underbrace{\prod_{j\in[m]\setminus\{i\}}}_{\text{choose edge from } p_j \text{ to remove}} s_j \right)\right)$$

Jacobians of Banana Graphs

Proposition 14: Fix a prime p, positive integer r. If we have positive integers $(s_1, \ldots, s_m)$ each coprime to p such that

$$\underbrace{\sum_{j \in [m]} \prod_{i \in [m] \setminus \{j\}} s_i}_{\#\text{ of spanning trees } = |Jac(G)|} \quad = \quad p^r,$$

then

$$Jac(B_s) \cong (\mathbb{Z}/p^r\mathbb{Z}, \langle \cdot, \cdot \rangle).$$

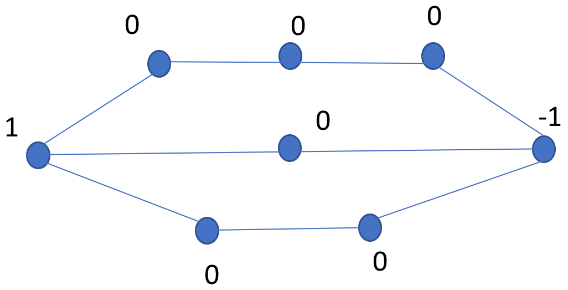
Jacobians of Banana Graphs

Hint: The real punchline is that $Jac(G)$ is **cyclic**. (We already knew the rest (e.g. abelian, finite of order p^r)).

So, we will show it's cyclic by finding a generator.

Jacobians of Banana Graphs

Q: What will our generator be?
A: Take $D = 1v + (-1)w + 0 + 0 + \cdots + 0 = v - w$.



D(w) in black

Jacobians of Banana Graphs

I claim $[D]$ is a generator of $Jac(G)$. It suffices to show its order in the group is $|Jac(G)| = p^r$.

By Lagrange's Theorem (order of an element divides order of the group) we know

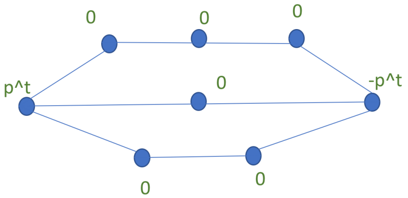
$$o([D]) = p^t$$

where $t \in \{0, 1, \ldots, r\}$. We will eventually show $t = r$.

Jacobians of Banana Graphs

For now, $o([D]) = p^t$ implies that $p^t[D] = e_{Jac(G)} = \{\text{class of principal numberings}\}$, which means p^tD is a principal labeling. So, it arises from some labeling $f : V(G) \to \mathbb{Z}$ and corresponding edge map $B : \overrightarrow{E}(G) \to \mathbb{Z}$ as before. Before any analysis, let's compute $p^tD(v)$. We just multiply each vertex number by p^t to get

$$\begin{aligned} p^tD &= p^t(1v + -1w + 0) \\ &= p^tv + -p^tw + 0. \end{aligned}$$



p^tD(w) in green

Jacobians of Banana Graphs

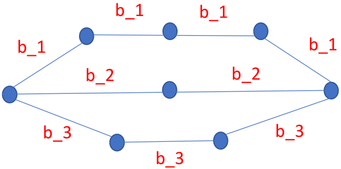
So, $p^t D$ principal means there exists $f : V(G) \to \mathbb{Z}$ such that using corresponding $B : \overrightarrow{E}(G) \to \mathbb{Z}$ defined by $B(xy) = f(x) - f(y)$ and then defining D' by

$$D'(u) = \sum_{\text{edges } \overrightarrow{uy} \in \delta^+(u)} B(\overrightarrow{uy})$$

one actually obtains $D' = p^t D$. (That was just the definition of principal).

Jacobians of Banana Graphs

Lemma: For any two edges $e, r \in p_i$ (both directed from v to w), one has $B(e) = B(r)$. So, on any path p_i from v to w, B is the constant function equal to b_i on all forward edges.

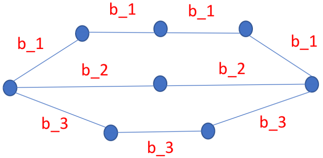


B(e) = b_i for all forward edges e on path i

Jacobians of Banana Graphs

Notation: The vertices in p_i are called $v, p_i^1, p_i^2, p_i^3 \ldots, p_i^{s_i-1}, w$. Call $b_i := B(p_i^1)$. Proof of Lemma (by induction): (Base case

trivial–Here's the inductive step). We know $B(p_i^{k-1}p_i^k) = b_i$ and we wish to show $B(p_i^k p_i^{k+1}) = b_i$.
Well, recall by definition of B, we have $B(xy) = f(x) - f(y)$. So, $b_i = B(p_i^{k-1}p_i^k) = f(p_i^{k-1}) - f(p_i^k)$ and $B(p_i^k p_i^{k+1}) = f(p_i^k) - f(p_i^{k+1})$.



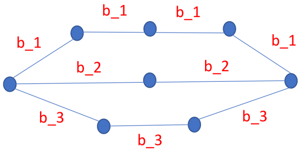
B(e) = b_i for all forward edges e on path i

Jacobians of Banana Graphs

Now, by definition of principality,

$$
\begin{aligned}
p^t D(p_i^k) &= \sum_{\text{edges } \overrightarrow{e} \in \delta^+(u)} B(e) \\
&= B(p_i^k p_i^{k-1}) + B(p_i^k p_i^{k+1}) \\
&= -B(p_i^{k-1} p_i^k) + B(p_i^k p_i^{k+1}) \\
&= -b_i + B(p_i^k p_i^{k+1}) \\
0 &= -b_i + B(p_i^k p_i^{k+1}),
\end{aligned}
$$

which gives $B(p_i^k p_i^{k+1}) = b_i$, proving the lemma. □



B(e) = b_i for all forward edges e on path i

Jacobians of Banana Graphs

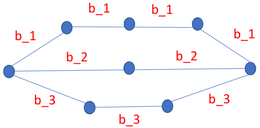
Now, we continue to prove the theorem that $[D]$ is a generator of $Jac(G)$.

Observe: For all $i \in [m]$, one has $f(v) - f(w) = b_i s_i$. Why?

We have a telescoping sum

$$\begin{aligned} f(v) - f(w) &= (f(v) - f(p_i^1)) + (f(p_i^1) - f(p_i^2)) \\ &\quad + (f(p_i^2) - f(p_i^3)) + \cdots + (f(p_i^{s_i-1}) - f(v)) \\ &= b_i + b_i + b_i + \cdots + b_i \\ &= b_i s_i. \end{aligned}$$

So, we can isolate $b_i = \frac{f(v)-f(w)}{s_i}$.



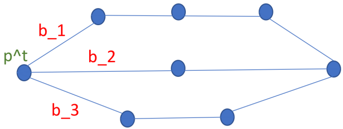
B(e) = b_i for all forward edges e on path i

Jacobians of Banana Graphs

$p^t D$ principal means that $p^t D$ came from the corresponding f, B so we have

$$\begin{aligned} p^t D(v) &= \sum_{e \in \delta^+(v)} B(e) \\ &= \sum_{i \in [m]} B(vp_i^1) \\ &= \sum_{i \in [m]} b_i \\ p^t &= \sum_{i \in [m]} b_i. \end{aligned}$$

p^tD(v) = p^t
=b_1+b_2+b_3



p^tD(w) in green

Jacobians of Banana Graphs

Continuing on we have

$$\begin{aligned}
p^t &= \sum_{i\in[m]} b_i \\
&= \sum_{i\in[m]} \frac{f(v)-f(w)}{s_i} \\
&= (f(v)-f(w)) \sum_{i\in[m]} \frac{1}{s_i} \\
&= (f(v)-f(w)) \sum_{i\in[m]} \frac{1}{s_i} \frac{\prod_{j\in[m]} s_j}{\prod_{j\in[m]} s_j} \\
&= \frac{(f(v)-f(w))}{\prod_{j\in[m]} s_j} \sum_{i\in[m]} \frac{\prod_{j\in[m]} s_j}{s_i}.
\end{aligned}$$

Jacobians of Banana Graphs

Continuing on we have

$$\begin{aligned} p^t &= \left(\frac{(f(v)-f(w))}{\prod_{j\in[m]} s_j}\right)\left(\sum_{i\in[m]} \frac{\prod_{j\in[m]} s_j}{s_i}\right) \\ &= \left(\frac{(f(v)-f(w))}{\prod_{j\in[m]} s_j}\right)\left(\underbrace{\sum_{i\in[m]} \prod_{j\in[m]\setminus\{i\}} s_j}_{\#\text{ of spanning trees}=|Jac(G)|=p^r}\right) \\ &= \frac{(f(v)-f(w))}{\prod_{j\in[m]} s_j} p^r. \end{aligned}$$

Jacobians of Banana Graphs

Now we have

$$\begin{aligned} p^t &= \frac{(f(v) - f(w))}{\prod_{j \in [m]} s_j} p^r \\ p^t \prod_{j \in [m]} s_j &= (f(v) - f(w)) p^r \\ \prod_{j \in [m]} s_j &= (f(v) - f(w))(p^{r-t}). \end{aligned}$$

Now, by assumption $t < r$ which means $p \mid RHS$ which implies $p \mid LHS$. So, $p \mid \prod_{j \in [m]} s_j$. Since p is prime, this means $p \mid s_k$ for some $k \in [m]$. However, that is a contradiction, since $gcd(s_j, p) = 1$ for all s_j. Thus, we have shown $t = r$, which means we found a generator and $Jac(G)$ is cyclic.

Quadratic Reciprocity

Definition: The Legendre Symbol of an integer a with respect to a prime p is defined as

$$(\frac{a}{p}) = \begin{cases} 1 & \text{if } n^2 \equiv a (\bmod\ p) \text{ for some } n \in \mathbb{Z} \\ -1 & \text{otherwise} \end{cases} .$$

This is also sometimes referred to as the "quadratic character" mod p. Then, the Law of Quadratic Reciprocity says that if p, q distinct primes, then

$$(\frac{p}{q})(\frac{q}{p}) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}},$$

which is useful because the RHS is easily computable, and if we know one of $(\frac{p}{q})$ or $(\frac{q}{p})$ (which recall are always ± 1) then we can easily can easily find the other without determining whether the number p is a square mod p.

Results on Quadratic Residues

Definition: A Dirichlet Character χ mod q is a function $\chi : \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}$ constructed from a group homomorphism (where both group operations are multiplication)

$$\phi : \mathbb{Z}/q\mathbb{Z}^{\times} \to \mathbb{C}^{\times}.$$

which we extend to take input in all of $\mathbb{Z}/q\mathbb{Z}$ by assigning $\chi(g) = 0$ for non-units $g \in \mathbb{Z}/q\mathbb{Z} \setminus (\mathbb{Z}/q\mathbb{Z})^{\times}$ (and just $\chi = \phi$ for the units).

Results on Quadratic Residues

Definition: The Principal Character mod n (n need not be prime) is the Dirichlet Character $\chi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}$ defined by

$$\chi(g) = \begin{cases} 1 & \text{if } gcd(g, n) > 1 \\ 0 & \text{otherwise} \end{cases} .$$

This is just one specific Dirichlet Character that exists for every natural number n. One also notes that the set of characters on $\mathbb{Z}/n\mathbb{Z}$ forms a group under pointwise multiplication

$$\chi_1\chi_2(g) := \chi_1(g)\chi_2(g),$$

and in this group the identity element is the Principal Character.

Results on Quadratic Residues

Definition: A Dirichlet Character on $\mathbb{Z}$ is a function $\chi : \mathbb{Z} \to \mathbb{C}$ which is the lift of a character $\chi_k : \mathbb{Z}/k\mathbb{Z} \to \mathbb{C}$ for some $k \in \mathbb{Z}$. (Lift means exactly what one would expect. It means that we extend χ_k to $\tilde{\chi_k}$ by assigning $\tilde{\chi_k}(x) = \chi_k(\bar{x})$ where $\bar{x}$ is the equivalence class of $x \mod k$).

Results on Quadratic Residues

Definition: A "Modulus" of two characters χ_1 mod 4 and χ_2 mod p is a number N such that $4, p \mid N$ and $\tilde{\chi_1}(n) = \tilde{\chi_2}(n)$ for all n with $gcd(n, N) = 1$.
Definition: The Conductor of a Character χ_N mod N is be the smallest positive divisor $c \mid N$ such that there exists a character χ_c mod c with $\tilde{\chi_N}(a) = \tilde{\chi_c}(a)$ for all $a \in \mathbb{Z}$ coprime to N. Definition:

The Conductor of a group of Characters is the LCM of the Conductors of each element.

Results on Quadratic Residues

Proposition 17: For any sufficiently large prime p, there exists a prime q such that

- q is a non-square mod p,
- $q \equiv 3 \bmod 4$ and
- $q < 2\sqrt{p}$.

Results on Quadratic Residues

Proof: Let χ_1 be the non-principal character mod 4. (One can show that there are only two valid characters mod 4 by recalling that the character must restrict to a group homomorphism over the units in $\mathbb{Z}/4\mathbb{Z}$, and we pick the non-principal one), which takes values

$$\begin{aligned}
\chi_1(0) &= 0\\
\chi_1(1) &= 1\\
\chi_1(2) &= 0\\
\chi_1(3) &= -1.
\end{aligned}$$

Let χ_2 be the quadratic character mod p which is just the Legendre Symbol $\chi_2(g) = (\frac{g}{p})$ which indicates whether g is a square mod p.

Clearly, the Conductor $f \in \mathbb{Z}$ of the group generated by χ_1, χ_2 must be $f \geq LCM(4, p)$ and the exponent divides 2. Also, in fact that is an equality. Then, we define a form $\chi = 1 + \chi_1\chi_2 - \chi_1 - \chi_2$ and the above allows us to apply a Theorem 1.4 cited in `http://pollack.uga.edu/generalsplit6.pdf`.

Results on Quadratic Residues

Theorem 1.4 guarantees the existence of a number q_2 such that

$$q_2 << (4p)^{\frac{1}{4}+\epsilon f \epsilon} << 2p^{\frac{1}{4}+2\epsilon} < 2\sqrt{p}$$

such that $\chi(q_2) \neq 0$.

Then, by construction of $\chi = 1 + \chi_1\chi_2 - \chi_1 - \chi_2$, one knows that $\chi_2(q_2) \neq 0$ means that $\chi_1(q_2) = \chi_2(q_2) = -1$.

So, $\chi_1(q_2) = -1$ which means that q_2 is not a square mod 4 which means $q_2 \equiv 3$ mod 4 and $\chi_2(q_2) = (\frac{q_2}{p}) = -1$ mod p means that q_2 is not a square mod p. $\Box$.

Galois Extensions

A field extension $K/\mathbb{Q}$ is Galois if it is the splitting field of a separable polynomial over $\mathbb{Q}$. Intuitively, a Galois Extension is one that is both normal and separable. Normal means that any polynomial with a root in K splits completely in K. So intuitively for any such polynomial K has "all the roots". Separable means that the minimal polynomial of any element in K is separable (means no repeated roots in the algebraic closure). In fact, one need only check the separability of the generators of the field $K/\mathbb{Q}$. Long story short, a Galois Extension is one with full automorphism group, meaning that $|Aut(K/\mathbb{Q})| = [K : \mathbb{Q}] = dim_{\mathbb{Q}}(K)$.
Why do normal and separable guarantee a full automorphism group? Well, any automorphism of $K/\mathbb{Q}$ sends the roots of an irreducible polynomial to other roots of that polynomial. Normal means we have all the roots available and separable means that none of them coincide. Examples to come.

Galois Extensions

NON-EXAMPLE: $K := \mathbb{Q}((2)^{\frac{1}{3}})$ is not a Galois field extension because it is not normal. Namely, it contains one root of $m(x) = x^3 - 2$ but not all of them. (One can see this by noting that $\mathbb{Q}((2)^{\frac{1}{3}}) \subseteq \mathbb{R}$, yet the roots are $(2)^{\frac{1}{3}} \in \mathbb{R}$ and $(2)^{\frac{1}{3}}\zeta_3, (2)^{\frac{1}{3}}\zeta_3^2 \in \mathbb{C} \setminus \mathbb{R}$). So, it does not have full automorphism group since any automorphism of $K/\mathbb{Q}$ must send roots to roots which means there is only one automorphism which sends $(2)^{\frac{1}{3}}$ to itself.

Galois Extensions

NON-EXAMPLE: $K := \mathbb{F}_p(\frac{t}{p}) \cong \mathbb{F}_p(t)[x]/(x^p - t)$ is not a Galois field extension because it is not separable. Namely, the minimal polynomial $(x^p - t)$ of the element $\frac{t}{p} \in K$ is not separable. Namely, it factors as $(x - \frac{t}{p})^p$. This means that $K/\mathbb{Q}$ does not have full automorphism group, though now for a different reason. It contains all the roots of $(x^p - t)$ but they all coincide which means that once again the the automorphism group $(\mathbb{F}_p(\frac{t}{p}))/(\mathbb{F}_p(t)) = \{id\}$ is trivial.
Fun fact: In most fields, a polynomial being irreducible implies that it is separable. (Fields in which this holds are called ”Perfect fields”). Some examples of perfect fields are:

- Fields of infinite characteristic and
- Finite fields.

However, as shown above, in infinite fields of finite characteristic it is possible for an irreducible polynomial to have repeated roots.

Galois Extensions

EXAMPLE: $K := \mathbb{Q}(\sqrt{2}) \cong \mathbb{Q}[x]/(x^2 - 2)$ is a Galois Extension over $\mathbb{Q}$ since it is normal and separable. (For both of these properties it suffices to check just the generators of this field extension, namely $\{\sqrt{2}\}$). Its minimal polynomial is $x^2 - \sqrt{2} = (x + \sqrt{2})(x - \sqrt{2})$ which is separable. Also, note that K is normal since $-\sqrt{2} \in K$. The extension $K/\mathbb{Q}$ has full automorphism group $Aut(K/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$. Note that $|Aut(K/\mathbb{Q})| = 2 = [K : \mathbb{Q}]$.

Galois Extensions

EXAMPLE: $K = \mathbb{Q}(\sqrt{-1}, \sqrt{\alpha})$ is a Galois Extension with Galois group

$$Gal(K/\mathbb{Q}) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

where any $\sigma_i \in Gal(K/\mathbb{Q})$ is completely determined by where it sends $\sqrt{-1}$ and $\sqrt{\alpha}$.

Galois Extensions

Namely, note that the requirement

$$\sigma : \sqrt{-1} \mapsto \pm\sqrt{-1} \text{ and}$$
$$\sigma : \sqrt{\alpha} \mapsto \pm\sqrt{\alpha}.$$

gives a natural intuitive connection between the Galois Group and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, which is why I chose to define it this way.

Galois Extensions

However, note that a more cannonical way is to define it as a "simple extension" $K = Q(\theta)$ (one formed by adjoining one element θ which we call "primitive"). How do we find such a primitive element θ?

Note that by the Galois Correspondence, every intermediate field K' (where $\mathbb{Q} \subseteq K' \subseteq K$) is the fixed field of some subgroup of the Galois Group $Aut(K/\mathbb{Q})$. Now, since there are only finitely many subgroups, there are only finitely many intermediate fields. Any element not in one of these intermediate fields will be a primitive element.

Galois Extensions

In our case, the intermediate fields are $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{\alpha})$, and $\mathbb{Q}(\sqrt{i}\sqrt{\alpha})$. These are fixed fields of the subgroups of $(K/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ which correspond to the subgroups $\langle(0,1)\rangle$,$\langle(1,0)\rangle$,$\langle(1,1)\rangle$ in $\mathbb{Z}/2\mathbb{Z}$).
Then, $\theta = \sqrt{-1} + \sqrt{\alpha}$ is a primitive element. Now, that we have a primitive element θ a basis for K as a $\mathbb{Q}-$vector space is $\{1, \theta, \theta^2, \theta^3\}$. (In general it is $\{1, \theta, \theta^2, \dots, \theta^{n-1}\}$ where $n = [K : \mathbb{Q}]$ is the degree of the extension).

Galois Extensions

Definition: The Discriminant of a Galois Field Extension $K/\mathbb{Q}$ (of degree $[K:\mathbb{Q}] =: n$) is

$$\Delta_{K/\mathbb{Q}} = (det(\sigma_i(\theta^j)))^2 \quad \text{where } i, j \in [n]$$

where θ is a primitive element of this extension (namely a number $\theta \in K$ such that $\mathbb{Q}(\theta) = K$) which means that $\{1, \theta, \theta^2, \dots, \theta^{n-1}\}$ is a basis as a vector space for K over $\mathbb{Q}$, and
$\{\sigma_i : i \in [n]\} = Gal(K/\mathbb{Q})$ is the set of field automorphisms of K which fix $\mathbb{Q}$ pointwise.
Note: one can even define the Discriminant for fields that are not Galois by looking at a Galois closure $L \supsetneq K \supsetneq \mathbb{Q}$. Then, one defines $G := Aut(L/\mathbb{Q})$ and $H := Aut(L/K)$. One can then look at coset representatives $\sigma \in gH$ for all cosets in G/H (although those cosets do not form a group in this case). One then uses these σ to calculate the discriminant.

Galois Extensions

EXAMPLE: For the field $K = \mathbb{Q}(\sqrt{-1}, \sqrt{\alpha}) = \mathbb{Q}(\sqrt{-1} + \sqrt{\alpha})$, the field discriminant is $\Delta_{K/Q} = (det(\sigma_i((\sqrt{-1} + \sqrt{\alpha})^j)))^2$ where $\theta = i + \sqrt{\alpha}$ is our primitive element.

Galois Extensions

Q: What's the point?
A: We will need to use the fact that a certain field extension is Galois to apply a useful theorem from another paper.

Results on Quadratic Residues

Proposition 18: For sufficiently large prime p and integer $r > 1$, there exist non-squares (modulo p) $q_1 = 1 \mod 4$ and $q_2 = 3$ mod 4, with $q_1, q_2 < 2\sqrt{p^r}$. Proof: As before let χ_1 be the

nontrivial character mod 4 and let χ_2 be the quadratic character (Legendre symbol $(\frac{\cdot}{p})$) mod p. We want $q_2 = 3$ mod 4 and q_2 not square mod p, which translate to $\chi_1(q_2) = -1$ and $\chi_2(q_2) = -1$.

Results on Quadratic Residues

To find such q_2, consider the field extension $K = \mathbb{Q}(\sqrt{-1}, \sqrt{\alpha})$ where $\alpha = (-1)^{\frac{p-1}{2}} p$. This extension is degree 4 (and recall: It's

Galois since its the splitting field of a separable polynomial, namely the product of the minimal polynomials of i and $\sqrt{\alpha}$ respectively). Its discriminant is $(4p)^2$, with field conductor $4p$. (The discriminant and field conductor are related: `https://en.wikipedia.org/wiki/Conductor-discriminant_formula`)

Results on Quadratic Residues

Now, the above properties, namely $K/\mathbb{Q}$ Galois, $[K:\mathbb{Q}] = 4$ and conductor of $K/\mathbb{Q}$ equal to $4p$, along with the fact that χ_1, χ_2 are quadratic characters (meaning they have order two in their respective character groups), one can apply Theorem 1.7 in `http://pollack.uga.edu/generalsplit6.pdf`. Theorem 1.7

gives us an upper bound on the desired prime

$$q_2 << 2p^{\frac{1}{2}+\epsilon} < 2\sqrt{p^r}.$$

Results on Quadratic Residues

To get the desired prime q_1 we apply Theorem 1.7 again, however now requiring that $q_1 = 1 \mod 4$ and q_1 not square mod p, which translate to $\chi_1(q_1) = 1$ and $\chi_2(q_1) = -1$, and we have found the desired q_1, q_2 and we are done. $\square$.

Statement of the Generalized Riemann Hypothesis

Say $\chi : \mathbb{Z} \to \mathbb{C}$ is a Dirichlet character. One then defines the Dirichlet L-function as

$$L(\chi, s) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

which takes in input in $\{s \in \mathbb{C} : Re(s) > 1\}$. By analytic continuity, this function can then be extended to take input in all of $\mathbb{C}$. Generalized Riemann Hypothesis: For every Dirichlet

Character χ and every complex number $s \in \mathbb{C}$ such that $L(\chi, s) = 0$ and $s \notin \mathbb{R}^-$ (we call such s ”non-trivial” zeros of $L(\chi, s)$), one has $Re(s) = \frac{1}{2}$.

Note: Plugging in $\chi(n) = 1$ gives $L(\chi, s)$ equal to the Riemann-Zeta function. This then gives the Standard Riemann Hypothesis (which you've likely heard stated as ”The real part of every non-trivial zero of the Riemann-Zeta function is $\frac{1}{2}$”).

Results on Quadratic Residues

Proposition 19 (CONDITIONAL ON THE GENERALIZED RIEMANN HYPOTHESIS): For any prime $p > 10^9$, there exists a prime number q such that

- $q = 3 \bmod 4$,
- q a non-square mod p and
- $q < 2\sqrt{p}$.

Note: This is EXACTLY our earlier proposition except now we have replaced ”for sufficiently large p” with ”for all $p > 10^{9}$”. The bound given here 10^9 is sufficiently small that we can handle all cases not handled by this theorem (namely primes $< 10^9$) by brute force computer search.

Results on Quadratic Residues

Proof: Let $\alpha = (-1)^{\frac{p-1}{2}} p$.
Consider $K = \mathbb{Q}(\sqrt{-1}, \sqrt{\alpha})$ an extension of degree 4.

Definition of Ring of Integers

Definition: The ring of integers O_k of a field extension $K/\mathbb{Q}$ is the set of $\{\beta \in K : m_{\beta/\mathbb{Q}}(x) \in \mathbb{Z}[x]\}$ whose minimal polynomials have only integer coefficients. They are meant to be a ring $O_K \subseteq K$ analogous to $\mathbb{Z} \subseteq \mathbb{Q}$.

Properties of O_K

Properties of the Ring of Integers O_k: Any ideal $I \subseteq O_K$ factors uniquely as a product of prime ideals

$$I = \prod_i P_i^{e_i}.$$

Note that this is essentially saying that O_K may not be a UFD because elements may not factor uniquely, it is still something close to a UFD in that IDEALS factor uniquely.

Properties of O_K

EXAMPLE: Take $K := \mathbb{Q}(\sqrt{-5})$. Then, one can check $O_K = \mathbb{Z}\sqrt{-5}$. (Computing O_K is actually a very non-trivial task. One first takes a $\mathbb{Q}$-basis $\mathcal{B}$ of K and scales the elements until they belong to O_K, which really involves clearing denominators to modify their minimal polynomials to get related ones with integer coefficients. Then, one knows that this new set $\mathcal{B}' \subseteq O_K$. It may not be a $\mathbb{Z}$-basis though as it may not span all of O_K. However, the span $span(\mathcal{B}')$ has rank equal to O_K, which means its index in O_K is finite. So we sandwich O_K as $span(\mathcal{B}') \subseteq O_K \subseteq \frac{1}{\Delta_{K/\mathbb{Q}}} span(\mathcal{B}')$ and looks at subgroups of $(\frac{1}{\Delta_{K/\mathbb{Q}}} span(\mathcal{B}'))/O_K$ which contain $span(\mathcal{B}')$ to find an actual $\mathbb{Z}$-basis).

Properties of O_K

EXAMPLE: Take $K := \mathbb{Q}(\sqrt{-5})$. Then, one can check $O_K = \mathbb{Z}\sqrt{-5}$. As noted, O_K may not be a UFD and here it is not since $6 = (1+\sqrt{-5})(1-\sqrt{-5}) = 2 * 3$ and the two factorizations do not differ by units, so they really are "distinct" so to speak. However, ideals factor uniquely into products of prime ideals in O_K. (Even more generally, this holds true in any "Dedekind Domain" which is an integral domain that is Noetherian, Integrally Closed and Height 1).

Q: What's the point?

A: We will rely heavily on the fact that ideals in O_K factor uniquely into prime ideals in order to guarantee the existence of a prime number with certain properties later. (Namely, it will turn out that this prime number is the norm of a prime ideal in O_K).

Definition of Artin Symbol

Definition: If $(\frac{P}{K/\mathbb{Q}})$ is a Galois field extension $K/\mathbb{Q}$, $P \subseteq O_K \subseteq L$ is a prime ideal (Unramified over p: which means if we have $pO_K = P^{e_0} \prod_i Q_i^{e_i}$ where Q_i are prime ideal factors of O_k, then $e_0 = 1$), and $(p) \subseteq \mathbb{Z}$ a prime ideal, then define the Artin Symbol of the prime ideal P as

$$\sigma = (\frac{P}{K/Q})$$

to be the unique element $\sigma \in Gal(K/Q)$ such that for every element $\beta \in K$,

$$\sigma(\beta) \equiv \beta^p \bmod P$$

or equivalently stated

$$\sigma(\beta) - \beta^p \in P.$$

Q: What's the point?
A: Once again, we will use the existence of such a σ in order to apply a cited theorem which guarantees the existence of a prime number with certain properties.

Results on Quadratic Residues

Now, in our particular case $[K:\mathbb{Q}] = 4$ and $\Delta = (4p)^2$. Then, Theorem 5.1 in `https://www.jstor.org/stable/2153734?seq=13#metadata_info_tab_contents` states Theorem 5.1:

ASSUMING THE GENERALIZED RIEMANN HYPOTHESIS, Let $K/\mathbb{Q}$ be a Galois extension with $K \neq \mathbb{Q}$. Let $\Delta = |\Delta_{K/Q}|$ and $n = [K:Q]$. Let $\sigma \in Gal(K/Q)$. Then, there is a prime ideal

$P \subseteq O_K$ with $(\frac{P}{K/Q}) = \sigma$ of residue degree 1 satisfying $N(P) = |O_K/P| \leq (4 log \Delta + 2.5n + 5)^2$. (Here's where we needed to use the Artin Symbol!)

Results on Quadratic Residues

In our case, we have $\Delta = (4p)^2$ and $4 = [K : \mathbb{Q}]$. Let $\sigma \in Gal(K/\mathbb{Q})$. Then, Theorem 5.1 applied to our problem, says there exists a prime ideal $P \subseteq O_K$ with $(\frac{P}{K/Q}) = \sigma$ of residue degree 1 satisfying

$$\begin{aligned} q := N(P) = |O_K/P| &\leq (4log((4p)^2) + 2.5 * 4 + 5)^2 \\ &= (4log((4p)^2) + 15)^2. \end{aligned}$$

and $(4log((4p)^2) + 15)^2 < 2\sqrt{p}$ provided $p > 10^9$ proving the result. $\square$.

Note: recall that the norm of a prime ideal $P \subseteq O_K$ is $q := N(P)$ a always a prime number.

Results on Quadratic Residues

Lemma 20: Let q be an odd prime and $k \in \mathbb{Z}$ such that

$$(\frac{k}{q}) = (\frac{-1}{q}).$$

Then, there exists $0 < c < q$ such that $c(q-c) \equiv k \mod q$.
Proof: Consider the set

$$R_q = \{\ell \in \mathbb{F}_q : (\frac{\ell}{q}) = (\frac{-1}{q})\}.$$

Now, consider the map $\phi : \mathbb{F}_q \to \mathbb{F}_q$ defined by

$$\phi(x) = -x^2.$$

One knows that the image $\phi(\mathbb{F}_q) \subseteq R_q$. Why? That is equivalent to saying that

$$\{(\frac{-x^2}{q}) = (\frac{-1}{q})\} \text{ for all } x \in \mathbb{F}_q.$$

Results on Quadratic Residues

Does that hold? Namely, does it hold that

$$(\frac{-x^2}{q}) = (\frac{-1}{q}) \text{ for all } x \in \mathbb{F}_q?$$

Note that if ($-1 = a^2 \bmod q$) is a square mod p, then so is $-x^2 = (-1)x^2 = (ax)^2 \bmod q$. (Recall something being a square is the same as its Legendre Symbol taking the value 1). Also, if $-1 \neq x^2 \bmod q$ for all $x \in \mathbb{F}_q$ is a non-square mod p, then so is $-x^2$ since the Legendre Symbol is multiplicative which means $(\frac{-x^2}{q}) = (\frac{-1}{q})(\frac{x^2}{q}) = -1 * 1 = -1$. So, indeed $(\frac{-x^2}{q}) = (\frac{-1}{q})$ for all $x \in \mathbb{F}_q$. This means $\phi(\mathbb{F}_q) \subseteq R_q$.

Results on Quadratic Residues

Does ϕ surject onto R_q? That happens if for all $a \in R_q$ there exists $x \in \mathbb{F}_q$ such that $-x^2 = a$ which happens exactly when $x^2 + a$ has a root in $\mathbb{F}_q$.

Now, since $x^2 + a$ has at most 2 roots in $\mathbb{F}_q$, this map is at most 2 to 1 and since $|R_q| = \frac{q-1}{2} < |\mathbb{F}_q|$ and since x is the root of at most 1 polynomial $x^2 + b$ (varied over b) meaning that the map defined as the "inverse" of the preimage map $a \mapsto \{x : x^2 + a = 0\}$ is in fact a function (meaning it is well defined since for every input x there is a unique a such that $x^2 + a = 0$). So, for all $k \in R_q$, there exists $c \in \mathbb{F}_q$ such that $\phi(c) = -c^2 = k$ and we have that $k = -c^2 = c(q - c) \bmod q$ as desired. $\square$.

Results on Quadratic Residues

Lemma 21: Let p be a sufficiently large prime p with $p \equiv 1 \bmod 4$ and let $r \in \mathbb{Z}^+$. Then, there exists a prime q with $(\frac{q}{p^r}) = -1$ (so q is NOT a square mod p^r) and a positive integer $c < q$ such that

$$\frac{p^r - c(q-c)}{q}$$

is a positive integer.

Proof: By Proposition 18, there exists a non-residue q with $(\frac{-1}{q}) = (\frac{p^r}{q})(\in \{\pm 1\})$ and $\frac{q^2}{4} < p^r$. By Lemma 20, there exists $c \in \mathbb{Z}^+$ such that $p^r = -c^2 = c(q-c) \bmod q$. So, we have $q \mid (p^r - c(q-c))$ and also $(p^r - c(q-c))$ is positive since $c < q$. $\square$.

Results on Quadratic Residues

Proposition 22: For any sufficiently large prime p and integer $r \in \mathbb{Z}^+$, there exists $s = (s_1, \ldots, s_m)$ such that $gcd(s_i, p) = 1$ for all i, $\prod_{j=1}^m s_j$ is a non-residue ($=$ not a square) mod p, and

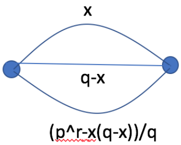
$$\sum_{i=1}^{m} \frac{\prod_{j=1}^m s_j}{s_i} = p^r$$

Proof: First case: $p \equiv 3 \bmod 4$. Choose $s = (s_1, s_2) = (1, p^r - 1)$ and right away we're done since s_1, s_2 coprime to p, their product is $s_1 s_2 = p^r - 1 \equiv -1 \bmod p$ which is not a square mod p. (Recall that $p = 1 \mod 4$ if and only if -1 is a square mod p!)

Results on Quadratic Residues

Second Case: $p \equiv 1 \mod 4$. Say we have x and q chosen (as in Lemma 21) such that $\frac{p^r - x(q-x)}{q}$ is a positive integer. Now, let

$$s_1 = x$$
$$s_2 = q - x$$
$$s_3 = \frac{p^r - x(q-x)}{q}$$



p prime
x < q a certain
integer

Results on Quadratic Residues

Both x and $q-x$ are smaller than p, which means they are both coprime to p. So, also their product $x(q-x)$ is coprime to p, which means that $p^r - x(q-x)$ is coprime to p (since otherwise if not coprime and p prime would mean $p \mid p^r - x(q-x)$ which means $p \mid x(q-x)$ a contradiction). Then, if $p^r - x(q-x)$ is coprime to p meaning $p^r - x(q-x)$ and p share no common factors, then dividing by q to get $\frac{p^r - x(q-x)}{q}$ can only remove some factors of $p^r - x(q-x)$ so the coprime property is preserved.

Results on Quadratic Residues

Second Case (continued): $p \equiv 1 \bmod 4$.
Now, is $\prod s_i = s_1 s_2 s_3$ a non-square mod p as desired? Well,

$$s_1 s_2 s_3 = x(q-x)\frac{p^r - x(q-x)}{q}.$$

Now, recall that we chose x and q to be as in Lemma 21 and Lemma 18, namely so that $(\frac{-1}{q}) = (\frac{p^r}{q})$. Now, denote $c := x(q-x)$. Then, $s_1 s_2 s_3 = c\frac{(p^r - c)}{q}$ we know $c\frac{(p^r-c)}{q} = \frac{cp^r - c^2}{q}$ is a square mod p if and only if $\frac{(-1)c^2}{q}$ is a square mod p.

Results on Quadratic Residues

Second Case (continued): $p \equiv 1 \mod 4$.

$$\frac{(-1)(x(q-x))^2}{q}$$

Now, since $p \equiv 1 \mod 4$, we have that $-1 \equiv y^2 \mod p$ is a square mod p^r. (Famous result by Fermat). Now the whole numerator is a square mod p. So that means that $s_1 s_2 s_3$ is a non-square mod p^r, which by definition of the Legendre symbol means $(\frac{s_1 s_2 s_3}{p^r})(= (\frac{q}{p^r})) = -1$. $\square$

Most Odd Groups Occur as Jacobians of Some Graph

That concludes the proof of the Relaxed Version of Main Theorem that there exists a finite set of primes $\mathcal{P} \subseteq \mathbb{Z}$ such that for any Abelian Group H whose order is not divisible by any prime $p \in \mathcal{P}$, we can construct a graph G with

$$Jac(G) \cong H.$$

Namely, for any given prime $p \notin \mathcal{P}$ and $r \in \mathbb{Z}+$, we have constructed the desired set $s = (s_1, \ldots, s_m)$ and corresponding Banana Graph G such that $Jac(G) = \mathbb{Z}/p^r\mathbb{Z}$. To strengthen the result to get the proof of the Main Theorem, we need one more result.

Even More Odd Groups Occur as Jacobians of Some Graph if the GRH Holds

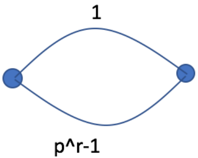
Proposition 23: Let p be an odd prime with $p \not\equiv 1 \mod 24$ and let $r > 1$ be an integer. Then, there exists $s = (s_1, \ldots, s_m)$ such that $\prod_i s_i$ is a non-square mod p and

$$\sum_{i=1}^{m} \frac{\prod_{j=1}^{m} s_j}{s_i} = p^r.$$

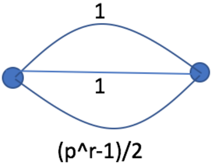
Even More Odd Groups Occur as Jacobians of Some Graph if the GRH Holds

Proof: We consider three cases.

1. If $p = 3 \mod 4$, we use $s = (s_1, s_2) = (1, p^r - 1)$.
2. If $p = 5 \mod 8$, we use $s = (s_1, s_2, s_3) = (1, 1, \frac{p^r - 1}{2})$. Now $p = 5 \mod 8$ means $p = 1 \mod 4$ which means that $-1 = p^r - 1 \mod p$ is a square mod p. Now, is $s_1 s_2 s_3$ a square? It is a square if and only if $\frac{1}{2} = 2^{-1}$ is a square mod p. Also, note 2^{-1} is a square mod p if and only if 2 is a square mod p. However, $p = 5 \mod 8$ means that 2 is not a square mod p.



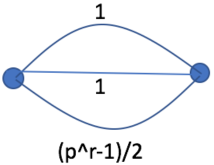
p prime



p prime

Even More Odd Groups Occur as Jacobians of Some Graph if the GRH Holds

3. $p = 2 \mod 3$. If $p = 3 \mod 4$ we are in the first case. Otherwise $p = 1 \mod 4$ and 2 is a non-square mod p. Once again we use $s = (s_1, s_2, s_3) = (1, 1, \frac{p^r - 1}{2})$. So, as stated in the hypothesis the only remaining case not covered by the above is $p = 1 \mod 24$. □



p prime

Even More Odd Groups Occur as Jacobians of Some Graph if the GRH Holds

Now, note that as stated earlier, conditional on the GRH, for all primes $p > 10^9$ (this bound is better than the "sufficiently large p" bound which doesn't use GRH) we have constructed the desired graph. Now, by the above lemma if $p \neq 1 \mod 24$ is an odd prime, we also have the desired graph. Now, the only remaining cases to handle are odd primes $q < 10^9$ with $q = 1 \mod 24$. We note that $= 1 \mod 24$ is a fairly restrictive condition, and these remaining cases are handled brute force by a computer.

Summary

We showed that there is some finite list of primes $\mathcal{P}$ such that any abelian group H of order not divisible by these primes occurs as the Jacobian of some graph.
Q: How did we show that?
A: By explicitly constructing such a graph.
Namely, we first showed that for any sufficiently large odd prime p_i and positive integer r_i, we can construct a graph G_i with Jacobian group

$$Jac(G_i) = \mathbb{Z}/p_i^{r_i}\mathbb{Z}.$$

Conditional on the GRH we have actually shown that the forbidden list of primes is now $\mathcal{P} = \{2\}$.

Summary

Then, if $H \cong \prod_i \mathbb{Z}/p_i^{r_i}\mathbb{Z}$, by letting the graph $G = \bigvee_i G_i$ be the wedge sum of these graphs, we get

$$\begin{aligned} Jac(G) &= Jac(\bigvee_i G_i) \\ &= \bigoplus_i Jac(G_i) \\ &= \bigoplus_i \mathbb{Z}/p_i^{r_i}\mathbb{Z} = H \end{aligned}$$

and we have constructed a graph whose Jacobian is the desired group H. Furthermore, if the GRH holds, we have that the list of forbidden primes is actually just $\mathcal{P} = \{2\}$, so in that case we can realize ANY odd order abelian group as the Jacobian of some graph.

Thanks for listening!