

Math 6441 - Homework 5

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1. Show that if \tilde{X} is an n fold covering space of the CW complex X , then $\chi(\tilde{X}) = n\chi(X)$. (You need to say something about the CW structure on \tilde{X} . For this problem you just need to describe the structure, but don't need to prove the structure is correct (though you should think through this).

Note that if \tilde{X} is an n -fold covering space of CW-complex X , then if l_k is the number of k cells in X then nl_k is the number of k cells in \tilde{X} . Thus, we have that

$$\chi(\tilde{X}) = \sum_{k=0}^{\infty} (-1)^k nl_k = n \left(\sum_{k=0}^{\infty} (-1)^k l_k \right) = n\chi(X).$$

2. Show that a surface of genus g can be a covering space of a surface of genus h only if $g = n(h - 1) + 1$ for some integer n . (Here all surfaces are assumed to be oriented).

Example 2.36 in Hatcher proves that

$$H_n(M_r) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}^{2r} & \text{if } n = 1 \\ \mathbb{Z} & \text{if } n = 2 \\ 0 & \text{if } n \geq 3 \end{cases}$$

Thus, we see that

$$\text{rank}(H_n(M_r)) = \begin{cases} 1 & \text{if } n = 0 \\ 2r & \text{if } n = 1 \\ 1 & \text{if } n = 2 \\ 0 & \text{if } n \geq 3 \end{cases}$$

So, we have that

$$\chi(M_g) = \sum_{i=0}^2 (-1)^i \text{rank}(H_i(X)) = 2 - 2g$$

and that

$$\chi(M_h) = \sum_{i=0}^2 (-1)^i \text{rank}(H_i(X)) = 2 - 2h.$$

Now, if M_g is a covering space of M_h , then it is an n -fold covering space for some $n \geq 1$. By Exercise 5, we know that $\chi(M_g) = n\chi(M_h)$ or that $2 - 2g = n(2 - 2h)$ or that $1 - g = n(1 - h)$ or that $g = 1 - n + nh = n(h - 1) + 1$ as claimed.

3. Given a sequence of abelian groups $G_0 = \mathbb{Z}, G_2, \dots, G_n$ show that there is a CW complex X such that $H_i(X) \cong G_i$.
 - We let $X = \bigvee_{k=0}^n Y_k$ where Y_k is constructed so that $H_k(Y_k) \cong G_k$ and $H_j(Y_k) = 0$ for all $j \in \mathbb{N}$ with $j \neq k$. The wedge can be performed at any points $\{y_k \in Y_k : k \in [0 : n]\}$.
 - Then, since for any $y_k \in Y_k$ we will show that (y_k, Y_k) is a good pair we may invoke Corollary 2.25 which says that $\tilde{H}_j(\bigvee_{k=0}^n Y_k) \cong \bigoplus_{k=0}^n \tilde{H}_j(Y_k)$ for all $j \in \mathbb{N}$. Then, since for any space Z and for $j \geq 1$ one has that $\tilde{H}_j(Z) \cong H_j(Z)$ we have that $H_j(\bigvee_{k=0}^n Y_k) \cong \bigoplus_{k=0}^n H_j(Y_k)$ for all $j \geq 1$.
 - By our future construction of Y_k as in bullet point 1, one has for each $j \geq 1$ that the direct sum $\bigoplus_{k=0}^n H_j(Y_k)$ has only one non-zero term (the j th term) and in particular is exactly $\bigoplus_{k=0}^n H_j(Y_k) \cong H_j(Y_j)$. Thus, for $j \geq 1$ one will have that $H_j(\bigvee_{k=0}^n Y_k) \cong G_j$, again by our chosen construction as in bullet point 1. We will postpone the proof that $H_0(\bigvee_{k=0}^n Y_k) \cong \mathbb{Z}$ until the end of our construction of Y_k .
 - We now explain how to construct Y_k . In particular, there are two cases to consider. Either G_k is finitely generated or it is not.
 - If G_k is finite generated, then $G_k \cong \mathbb{Z}^{a_k} \oplus \left(\bigoplus_{i=0}^{m_k} \mathbb{Z}/b_i^k \mathbb{Z} \right)$ where $a_k \in \mathbb{N}_{\geq 0}, m_k \in \mathbb{N}_{\geq -1}$ (with the convention that if $m_k = -1$ then the sum $\bigoplus_{i=0}^{m_k} \mathbb{Z}/b_i^k \mathbb{Z}$ is empty) and $b_i^k \in \mathbb{N}_{\geq 1}$. In this case, we let $Y_k = (\bigvee_{i=0}^{a_k} S^k) \vee (\bigvee_{i=0}^{m_k} D^{k+1} \cup_{f_i^k} S^k)$ where $f_i^k : \partial D^{k+1} \rightarrow S^k$ is an attaching map of degree b_i^k . To construct this map explicitly we note that ∂D^{k+1} is canonically isomorphic to S^k via $\psi_k : \partial D^{k+1} \rightarrow S^k$.

- As in Example 2.31, one defines $q_i^k : S^k \rightarrow \bigvee_{l=1}^{b_i^k} S^k$ as $\phi_k \circ \pi_k$ where π_k is the quotient map $\pi_k : S^k \rightarrow S^k / \sim$ by $z \mapsto [z]$ for all $z \in S^k$ where \sim is defined as follows and $\phi_k : S^k / \sim \rightarrow \bigvee_{i=1}^{b_i^k} S^k$ is the canonical isomorphism. Namely, consider b_i^k pairwise disjoint non-empty open balls $B_1^k, B_2^k, \dots, B_{b_i^k}^k \subseteq S^k$ and define $z_1 \sim z_2$ if and only if $z_1, z_2 \in S^k \setminus (\bigcup_{j=1}^{b_i^k} B_j^k)$.
- Then, one defines $p_i^k : \bigvee_{l=1}^{b_i^k} S^k \rightarrow S^k$ as in Example 2.31 as the “identification of the summands to a single sphere”.
- Finally one defines $f_i^k = p_i^k \circ q_i^k \circ \psi_k$ and one notes that f_i^k has degree b_i^k for all $i \in [0 : m_k]$.
- If G_k is not finitely generated then consider an infinite set of generators $\{g_a\}_{a \in \alpha}$ of G and note that $G_k \cong \langle g_a : a \in \alpha \rangle^{ab} / N$ where we use additive notation for the abelianization meaning $G_k \cong \langle g_a : a \in \alpha \rangle_+ / N$ where the subscript $+$ just means the operation is addition. $N = \langle y_b : b \in \beta \rangle_+ = \langle \sum_{a \in \alpha} k_a^b x_a : b \in \beta \rangle_+$ (so really $y_b : b \in \beta$ is just the set of relations that hold in G). Now, we let X be the result of the following construction. Namely, we start with $X^n := \bigvee_{a \in \alpha} S_a^n$ and construct X by attaching $n+1$ cells e_b^{n+1} using maps $f_b : S^n (\cong \partial e_b^{n+1}) \rightarrow X^n$ such that $\deg(\text{proj}_a \circ f_b) = k_a^b$. Then the value of the cellular boundary map on the $n+1$ cell e_b^{n+1} will be just $\partial_{n+1}^{CW} = \sum_k \text{index}_{(X^n \rightarrow X^n / X^{n-1}) \circ (\text{inclusion}(\partial e_b^{n+1}) \rightarrow X^n)} x_a$ (where x_a correspond to the n cells) So, we see that since $d_{n+1}(y_b) = \sum_a k_a^b x_a$ is just inclusion into F then we get that $H_n(X) \cong G$ and $H_k(X) = 0$ for $k \neq n$.

4. A cochain $\phi \in C^1(X; G)$ can be thought of as a function from paths in X to G . Show that if $\delta\phi = 0$ then

- $\phi(\gamma * \eta) = \phi(\gamma) + \phi(\eta)$
- if γ is homotopic to η rel endpoints, then $\phi(\gamma) = \phi(\eta)$
- ϕ is a coboundary if and only if $\phi(f)$ only depends on the endpoints of f , for all f
- Show that the discussion above gives a homomorphism $H^1(X; G) \rightarrow \text{Hom}(\pi_1(X); G)$ (Note: the universal coefficients theorem says that this map is an isomorphism if X is path connected).

Indeed an element $\phi \in C^1(X; G)$ is just $\phi \in \text{Hom}(C_1(X), G)$. Indeed, since $C_1(X) = \{\text{paths in } X\}$. Note that $\delta = \partial_2^*$ meaning that $\delta(\phi)(D) = \phi(\partial(D))$ and thus, we must have that $\phi(\partial(D)) = 0$ for all $D \in C_2(X)$.

Throughout this problem we will need the following functions:

$$\begin{aligned}
 p : \mathbb{R}^3 &\rightarrow \mathbb{R}^2 & p(x, y, z) &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
 q : \mathbb{R}^2 &\rightarrow \mathbb{R}^2 & q(x, y, z) &= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\
 r : \mathbb{R}^2 &\rightarrow \mathbb{R}^2 & r(x, y) &= (x, 2y - |x|) \\
 s : \mathbb{R}^2 &\rightarrow \mathbb{R}^2 & s(x, y) &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\
 f : \mathbb{R}^2 &\rightarrow \mathbb{R}^2 & f(x, y) &= (y - x)^+(0, 1) + (y - x)^-(1, 0)
 \end{aligned}$$

Note that all of the above are continuous, either because they are linear or because their component functions are continuous functions of x and y . (In particular note that $f(a) = a^+ := \max(a, 0)$ and $f(a) = a^- := -\min(a, 0)$ are continuous functions of $a \in \mathbb{R}$).

(a) Let

$$g : \{0 \times [0, 1]\} \cup \{[1, 0] \times 0\} \rightarrow X$$

be defined by

$$\begin{aligned}
 g(0, a) &= \gamma(1 - a) \\
 g(b, 0) &= \eta(b).
 \end{aligned}$$

Then, we have that

$$g(f(p(\Delta^2)))$$

is a 2-simplex in X .

Now, note that since $\delta = \partial^*$ we have that $\delta(\phi D) = \phi(\partial D)$ for all 2-simplices $D \subseteq X$. So, by hypothesis $\phi(\partial D) = 0$ for all 2-simplices meaning that in particular the statement holds for the 2-simplex $g(f(p(\Delta^2)))$.

So, finally we obtain $\partial(g(f(p(\Delta^2)))) = g(f(p([\hat{e}_0, e_1, e_2]))) - g(f(p([e_0, \hat{e}_1, e_2]))) + g(f(p([e_0, e_1, \hat{e}_2]))) = \eta - (\gamma * \eta) + \gamma$. Now, since $\phi(\partial D) = \phi(\eta) - \phi(\gamma * \eta) + \phi(\gamma) = 0$ we see that $\phi(\eta) + \phi(\gamma) = \phi(\gamma * \eta)$.

- (b) • Now, note that $s(r(q(p(\Delta^n)))) = [0, 1] \times [0, 1]$ and note that $\gamma \sim \eta$ relative to endpoints means that there exists a continuous function

$$F : \mathbb{R}^2 \rightarrow X$$

such that $F(0, t) = \gamma(t)$ and $F(1, t) = \eta(t)$ and so that $F(s, 0)$ and $F(s, 1)$ are constant for $s \in [0, 1]$.

- Now, note that if one partitions γ into $\gamma = \alpha * \beta$ (in the obvious canonical way) then we see that by our above construction

$$\begin{aligned} s(r(q(p([\hat{e}_0, e_1, e_2]))) &= \alpha \\ s(r(q(p([e_0, \hat{e}_1, e_2]))) &= \eta \\ s(r(q(p([e_0, e_1, \hat{e}_2]))) &= \beta \end{aligned}$$

and since

$$D := s(r(q(p(\Delta^n))))$$

is a 2-simplex in X satisfying $\partial D = \alpha - (-\eta) + \beta = 0$ and since by hypothesis $\phi(\partial D) = \phi(\alpha) - \phi(\eta) + \phi(\beta) = 0$ we see that $\phi(\eta) = \phi(\alpha) + \phi(\beta)$ and by part (a) we get that $\phi(\eta) = \phi(\alpha * \beta) = \phi(\gamma)$.

- (c) By definition a coboundary ϕ is an element $\phi \in \text{im}(\delta_0)$. Then that implies that $\phi = \delta_0(h)$ for some $h \in \text{Hom}(C_0(X), G)$. Indeed, then $\phi(v) = (\delta_0 \circ h)(v) = h(\partial_1(v))$. Now, note that $\partial_1(v)$ depends only on the endpoints of v for all paths v proving necessity. Now, to prove sufficiency suppose that $\phi(v)$ depends only on the endpoints of v ,

- (d) We simply define $\psi : H^1(X, G) \rightarrow \text{Hom}(\pi_1(X), G)$ by

$$\psi([\phi])([\gamma]) = \phi(\gamma)$$

for all loops $\gamma \subseteq X$ and all $[\phi] \in H^1(X, G)$. Note that this map is well defined for two reasons. Firstly because part (b) insures that $\phi(a) = \phi(b)$ for all loops $a, b \in X$ with $[a]_{\pi_1(X)} = [b]_{\pi_1(X)}$ and secondly because $\psi([\phi]) = 0$ for all $\phi \in \text{im}(\delta_0)$. Namely note that by part (c) for such ϕ we have that $\phi(\gamma)$ depends only on the endpoints of γ and thus for a loop γ we have that $\phi(\gamma) = \phi(\{\gamma(0)\}) = \phi(\{\gamma(0)\} * (-\{\gamma(0)\})) = \phi(\{\gamma(0)\}) - \phi(\{\gamma(0)\}) = 0 \in G$. Finally, note that $\psi([\phi])$ is a homomorphism for all $[\phi] \in H^1(X, G)$ since $\psi([\phi])([a][b]) = \psi([\phi])([a * b]) = \phi(a * b) = \phi(a) + \phi(b)$.

5. In Hatcher's book, the cohomology of $\mathbb{R}P^n$ is computed. Use this computation (you do not have to reproduce it) to show that there is no map $\mathbb{R}P^n \rightarrow \mathbb{R}P^m$ inducing a nontrivial map $H^1(\mathbb{R}P^m; \mathbb{Z}/2\mathbb{Z}) \rightarrow H^1(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$ if $n > m$.

Note that as Hatcher correctly states (as a total ring) $H^*(\mathbb{R}P^n, \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$ meaning that each individual coordinate (corresponding to the k th powers of α for $0 \leq k \leq m$) are each \mathbb{Z}_2 .

So, if there were such a non-trivial map, it would send $f : H^*(\mathbb{R}P^m, \mathbb{Z}_2) (\cong \mathbb{Z}_2[\alpha]/(\alpha^{m+1})) \rightarrow H^*(\mathbb{R}P^n, \mathbb{Z}_2) (\cong \mathbb{Z}_2[\beta]/(\beta^{n+1}))$ it would send $\alpha \mapsto f^* \beta$. Now, since f is a homomorphism and $\alpha^{m+1} = 0$ we would need that $\beta^{m+1} = 0$ in $\mathbb{Z}_2[\beta]/(\beta^{n+1})$. However, that does not hold since $m+1 < n+1 = \min(k \in \mathbb{Z} : \beta^k = 0)$.

6. Show that any map $S^4 \rightarrow S^2 \times S^2$ must induce the zero map on H^4 .

First note that $H^4(S^2 \times S^2) \cong H_4(S^2 \times S^2) \cong H^2(S^2) \otimes_{\mathbb{Z}} H^2(S^2) \cong \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z}$. Take a generator $\alpha \in H^2(S^2)$ and then note that $\alpha \cup \alpha$ is a generator of $H^4(S^2 \times S^2)$ and Prop 3.10 in Hatcher says that $f^*(\alpha \cup \alpha) = f^*(\alpha) \cup f^*(\alpha) \in H^4(S^4)$. However, if $f^*(\alpha) \cup f^*(\alpha) \in H^4(S^4)$ that implies that $f^*(\alpha) \in H^2(S^4)$ but $H_2(S^4) = 0$ which implies that $f^*(\alpha) \cup f^*(\alpha) = 0$ and thus $f^* = 0$.