CS 6505 - Homework 4

Caitlin Beecham (Worked with Jennifer, Rosy, James)

2

By Savitch's theorem, NSPACE(n^2) $\subset$ SPACE(O(n^4)). Now, claim: $O(n^4) \leq o(n^5)$. Why? Well, say $g(n) = O(n^4)$. That means that there exists $c, m \in \mathbb{R}$ such that $g(n) \leq c * n^4$ for all $n \geq m$. Now, say $h(n) = o(n^5)$. That means that for all $c > 0$, there exists $m_c > 0$ such that $h(n) \leq c * n^5$. So, we know that $g(n) \leq c * n^4$. Is it true that $c * n^4 = o(n^5)$? (ARGUMENT *) Take arbitrary $k > 0$. We want to find $m_k > 0$ such that $cn^4 < kn^5$. That inequality will be satisfied if and only if $kn^5 - cn^4 > 0$. Take $m_k = \frac{c}{k}$. Then, $c(m_k)^4 = c(c^4/k^4) = c^5/k^4$. Then also $km_k^5 = k(c^5/k^5) = c^5/k^4$. So, $k(m_k)^5 \geq c(m_k)^4$. This also holds for $n \geq m_k$. So, $O(n^4) = o(n^5)$. This implies that SPACE($O(n^4)$) $\subset$ SPACE($o(n^5)$). Now, I claim $o(n^5) \leq n^5$ for sufficiently large n. Why? Well, $g(n) = o(n^5)$ means that for all $c > 0$, there exists $m > 0$ such that $g(n) \leq c * n^5$. Pick $c = 1$. There exists $m > 0$ such that $g(n) \leq n^5$ for all $n \geq m$. Thus, SPACE($o(n^5)$) $\subset$ SPACE(n^5). So, NSPACE(n^2) $\subset$ SPACE(n^5). Now, we wish to show that it is a strict subset, meaning there are Turing Machines in SPACE(n^5) which are not in NSPACE(n^2). By the Space Hierarchy Theorem, there exist languages decidable in SPACE($O(n^5)$) that are not decidable in SPACE($o(n^5)$). We mimic that proof to get the result that there exist languages decidable in SPACE(n^5) which are not decidable in SPACE($o(n^5)$).
Proof:
Construct the TM B as follows: Let L_B be language accepted by following Turing Machine B. On input x:

1. Mark $n^5 - log_2(|Q| * |\Gamma|^{(n^5)} * (n^5 + n))$ space on the tape.
2. Keep counter for number of steps taken by B.
3. If x $\neq \langle M \rangle, 1^n$, reject.
4. If M is not valid turing machine, reject.
5. Run M on x
   (a) If space used exceeds $n^5 - log_2(|Q| * |\Gamma|^{(n^5)} * (n^5 + n))$, reject. (STEP A)
   (b) If time used exceeds $|Q| * |\Gamma|^{(n^5)} * (n^5 + n)$, reject.
   (c) Else
      i. If M accepts, reject.
      ii. If M rejects, accept.

Now, by the way we have constructed this Turing Machine it will use at most n^5 space. (We allow it to use at most $n^5 - log_2(|Q| * |\Gamma|^{(n^5)} * (n^5 + n))$ space for the actual simulation of M and log($|Q| * |\Gamma|^{(n^5)} * (n^5 + n)$) space for the counter, which when added give s(n)). Now, assume we have a Turing Machine C which decides the same language as B in $O(n^4)$ space. So, we run B on $\langle C \rangle, 1^n$ for sufficiently large n. I claim for sufficiently large n that C uses less than $n^5 - log_2(|Q| * |\Gamma|^{(n^5)} * (n^5 + n))$ space. We note that $n^5 - log_2(K_1 * K_2^{(n^5)} * (n^5 + n)) = n^5 - log_2(K_1 * 2^{(log_2(K_2))(n^5)} * (n^5 + n))$ = $n^5 - (log_2(K_1) + (n^5) * log_2(K_2) + log_2(n^5 + n))$ and for sufficiently large n, $(log_2(K_1) + (n^5) * log_2(K_2) + log_2(n^5 + n)) <= \frac{1}{2}n^5$ since $(log_2(K_1) + (n^5) * log_2(K_2) + log_2(n^5 + n)) = o(n^5)$. The fact that $(log_2(K_1) + (n^5) * log_2(K_2) + log_2(n^5 + n)) <= \frac{1}{2}n^5$ for sufficiently large n implies that $\frac{1}{2}n^5 \leq n^5 - (log_2(K_1) + (n^5) * log_2(K_2) + log_2(n^5 + n))$ for such n. So, if C uses less than $\frac{1}{2}n^5$ space it will use less than $n^5 - (log_2(K_1) + (n^5) * log_2(K_2) + log_2(n^5 + n))$ space for large n. Now, $O(n^4) \leq \frac{1}{2}n^5$ by (ARGUMENT *) given earlier, which means $O(n^4)$ is less than $n^5 - (log_2(K_1) + (n^5) * log_2(K_2) + log_2(n^5 + n))$ for large n. This means that for large enough n, C will use less than the space allotted to it as it is simulated within B. B will do the opposite of what C does since it is not rejected for taking too much space (it is not rejected as part of (STEP A)). This is a contradiction. So the language L_B is decidable by B, but not decidable by any Turing Machine C in SPACE(O(n^4)). Since C is not in SPACE(O(n^4)) it is also not in NSPACE(n^2) (because NSPACE(n^2) $\subset$ SPACE(O(n^4)) as shown earlier in this problem. So, L_B is an example of a language which is in SPACE(n^5) but not in NSPACE(n^2).