# CS 6505 - Homework 8

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## 1

We use induction on the number of vertices in A that satisfying that condition implies there exists a matching saturating A. Base case: If A has one vertex, a, and this condition is satisfied, it has at least one neighbor. Picking an edge between a and that neighbor completes a matching saturating A. Inductive step: Say we have a graph with bipartition A, B with $|A| = k$ which satisfies that condition (Hall's condition). We wish to show it has a matching saturating A. Pick a vertex $a \in A$. It has some neighbor $b \in B$ via the edge $e = ab$. Delete a and b from G to get G'. Now, the $|A'| = k - 1$ which implies that as long as Hall's condition is satisfied in G', we can apply the inductive hypothesis. There are two cases. Case 1: Something slightly stronger than Hall's Condition was satisfied in G. Namely, For all $S \subseteq A$, $|N_G(S)| \geq |S| + 1$. This means that then $|N_{G'}(S)| \geq |N_G(S)| - 1 \geq |S| + 1 - 1 = |S|$ and we have Hall's Condition in G' which means we can apply the inductive hypothesis and get a matching saturating A in G. So, in case 1 we're done. Now, consider case 2, the negation of case 1. Case 2: This means that there exists some subset $S \subsetneq A$ with $|N_G(S)| = |S|$. In this case, we break G into 2 graphs. Let the first graph, $G_1 = S \cup N_G(S)$ and let $G_2 = (A \setminus S) \cup (B \setminus N_G(S))$. To start, $G_1$ still satisfies Hall's Condition, which means that $G_1$ has a perfect matching. Now, I claim that $G_2$ also satisfies Hall's Condition. We show this by contradiction. Assume not. Assume there exists some subset $T \subseteq A \setminus S$ such that $|N_{G_2}(T)| < |T|$. Now, consider $T \cup S$ and calculate $|N_G(T \cup S)| = |N_G(T)| + |N_G(S)| - |N_G(S) \cap N_G(T)| \leq |N_G(T)| + |N_G(S)| < |T| + |S| = |S \cup T|$ (since $|N_{G_2}(T)| < |T|$ and $|N_G(S)| = |S|$) which is a contradiction since that would mean Hall's Condition wasn't satisfied in the original graph. So, that can't happen and that means Hall's Condition is satisfied in $G_2$. Then, we can apply the inductive hypothesis to get a matching of $G_2$ which saturates $A \setminus S$. Combining this matching of $G_2$ with the aforementioned matching of $G_1$, we get a matching of $G$ which saturates A.

Now, for the reverse implication, we need to show that if there exists such a matching then Halls condition is satisfied. If there exists such a matching of size —A—, then it must saturate every vertex of A (since the only edges that exist are between A and B). Now, assume Hall's Condition is not satisfied, namely that for some $X \subseteq A$, $|N(X)| < |X|$. Well, since every vertex of A is matched in particular, every vertex $x \in X$ is matched. These vertices must be matched to some neighbor, all of which are contained in $N(X)$ and also no two $x_1, x_2 \in X$ are matched to the same $y \in B$, which implies that for each $x \in X$, there is one $y \in N(X)$ which gives that $|N(X)| \geq |X|$, a contradiction.