CS 6505 - Homework 10

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4

The volume of the ellipsoid within the box has axis lengths equal to the lengths of the box sides. Thus, by the equation for the volume of an n-dimensional ellipsoid, the volume of the smaller ellipsoid is $\frac{2}{n}\frac{\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})}\prod_{i=1}^{n}(\frac{a_i}{2})$. Next, we note that for the minimum volume ellipsoid containing the box the corners of the box will lie on the boundary of such an ellipsoid. Thus, to find the axis lengths for the new ellipsoid, we plug in one of the corners of the box, x, and solve the following equation for the new axis lengths $\tilde{a_i}$:

$$x^T \begin{bmatrix} (\tilde{a_1})^2 & 0 & 0 & \dots & 0 \\ 0 & (\tilde{a_2})^2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & (\tilde{a_n})^2 \end{bmatrix} x = 1$$

Plugging in a corner $x = (\frac{a_1}{2}, \frac{a_2}{2}, \dots, \frac{a_n}{2})$, we get

$$\begin{bmatrix} \frac{a_1}{2} & \frac{a_2}{2} & \dots & \frac{a_n}{2} \end{bmatrix} \begin{bmatrix} (\tilde{a_1})^2 & 0 & 0 & \dots & 0 \\ 0 & (\tilde{a_2})^2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & (\tilde{a_n})^2 \end{bmatrix} \begin{bmatrix} \frac{a_1}{2} \\ \frac{a_2}{2} \\ \vdots \\ \frac{a_n}{2} \end{bmatrix} = \sum_{i=1}^{n} \frac{1}{4} a_i^2 \tilde{a_i}^2 = 1$$

Next, we note that every axis will scale by the same factor, so we have that $\tilde{a_i} = ka_i$ for all i. We substitute this expression in to get

$$\sum_{i=1}^{n} \frac{k^2}{4} a_i^4 = k^2 \sum_{i=1}^{n} \frac{1}{4} a_i^4 = 1 \tag{1}$$

Solving for k, we get

$$k = \sqrt{\frac{4}{\sum_{i=1}^{n} a_i^4}}. \tag{2}$$

So, our new axis lengths are

$$\tilde{a_i} = \sqrt{\frac{4}{\sum_{i=1}^{n} a_i^4}} a_i. \tag{3}$$

Plugging these values into the equation for volume of an ellipsoid we get that the volume of the larger ellipsoid is

$$\frac{2}{n}\frac{\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})}\prod_{i=1}^{n}\left((\frac{a_i}{2}) \sqrt{\frac{4}{\sum_{j=1}^{n} a_j^4}} \right) \tag{4}$$