It is my hope that this document will convey my ability to read and understand non-trivial current research papers and also consolidate information coming from multiple sources into one cohesive narrative. This was originally a final project for my algebraic topology class, but the material it covers is more accurately a mix of, in descending order of importance, homological algebra, algebraic topology, and modular forms.

Overview of Algebraic Topology Final Project

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Part A

The idea is to motivate the significance of the computations of homology and cohomology of the groups $SL_2(\mathcal{O}_{-2})$ and $PSL_2(\mathcal{O}_{-2})$ and to detour as we do so into the relevant computations.

1. **Fermat's Last Theorem and an Analog of Serre's Modularity Conjecture for Imaginary Quadratic Fields**
   (a) Especially Brief Overview of Wiles' Proof and its Relation to Serre's Modularity Conjecture for Totally Real Fields
   (b) Analog of Serre's Modularity Conjecture for the Quadratic Number Field $\mathbb{Q}(i)$
      i. Involvement of the Cohomology $H^2(\Gamma, V)$ in this Conjecture
      ii. Computation of the Cohomology $H^2(\Gamma, V)$ via Instead Computing the Homology $H_0(\Gamma, St \otimes V)$ with Coefficients in the Steinberg Module
   (c) Detour into Related Computation of Cohomology $H^*(SL_2(\mathcal{O}_{-2}))$ and of the Homology $H_*(SL_2(\mathcal{O}_{-2}), St(2))$ with Coefficients in a Steinberg Module Including Explanation of Why these Groups are Loosely Analogous to Those Above
   (d) Outline of Tools Used in the Above Computation:
      i. Conceptual Relation between Homology/Cohomology of a Group and Homology/Cohomology of A Related Space which Is What One Actually Computes in Practice
      ii. Explanation of the Theory of Spectral Sequences and Shapiro's Lemma which Make the Above Decomposition Work
      iii. Explicit Results of the Above Computations
2. **Aiding the Understanding of Arithmetic of Bianchi Forms as Opposed to Modular Forms**
   (a) Remark that Understanding Torsion of Cohomology of Modular Groups (resp. Bianchi Groups) is Vital for Understanding the Arithmetic of Modular Forms (resp. Bianchi Forms)
   (b) Note that while the Necessary Cohomology of Modular Groups is Well-Understood and Readily Computed in the Literature Unfortunately Cohomology of Bianchi Groups is Not as Easily Computed or Readily Available
   (c) Note that $PSL_2(\mathcal{O}_{-2})$ is a Bianchi Group, which Motivates Our Explicit Computation of $H^*(PSL_2(\mathcal{O}_{-2}))$ in Order to Understand its Torsion
   (d) Remark that to Compute the Cohomology $H^*(PSL_2(\mathcal{O}_{-2}))$ it Suffices to Compute the Homology $H_*(PSL_2(\mathcal{O}_{-2}))$
   (e) Outline of Computation of $H_*(PSL_2(\mathcal{O}_{-2}))$ via Construction of Fundamental Domain and Analysis of the Action of the Group $PSL_2(\mathcal{O}_{-2})$ on this Fundamental Domain
      i. Explicit Construction of the Above Space and its Fundamental Domain for the Action of the Group which are Used to Compute Homology/Cohomology of the Group of Interest
      ii. Explanation of How to Compute Homology/Cohomology of this Group G by Decomposing the Fundamental Domain (resp. Group) into Pieces, which Are the 0,1, and 2 Cells of our Fundamental Domain (resp. Stabilizer Subgroups Γ_σ in G of those 0,1, and 2 Cells)

Part B

Also, as an **Appendix** I include a formal definition of homology/cohomology of a group via free resolutions as well as an intuitive explanation of cohomology of spaces.

Finally, in order to satisfy the computational aspect of the project, I myself complete two **Relevant Exercises** related to the material above. Furthermore, in tying together all material covered in the course, I provide relevant examples for each exercise, which motivate the statements to be proved.

1. Namely, the first relates relates the homology of a space to the homology of its covering group, which seems interesting given the depth with which we studied covering groups and covering spaces.
2. The second calls for computation of the homology $H_*(SL_2(\mathbb{Z}))$ which is closely related to our previous computations of $H_*(SL_2(\mathcal{O}_{-2}))$ since the ring of integers $\mathcal{O}_{-2}$ inside the field $\mathbb{Q}(\sqrt{-2})$ is meant to mimic the structure of the ring $\mathbb{Z}$ inside the field $\mathbb{Q}$. It also provides an interesting example of using an analog of a Mayer-Vietoris sequence to calculate the homology of a group rather than a space.

Algebraic Topology Final Project

1. Fermat's Last Theorem and an Analog of Serre's Modularity Conjecture for Imaginary Quadratic Fields

The reader is likely familiar with the statement of Fermat's Last Theorem which was proved by Andrew Wiles using the machinery of modular and elliptic curves and their representations. In particular the famed theorem states that the equation $a^n + b^n = c^n$ has no solutions in the integers for $n \geq 3$.

(a) Especially Brief Overview of Wiles' Proof and its Relation to Serre's Modularity Conjecture for Totally Real Fields

Wiles' proof has a few main steps. Namely, he assumes for contradiction that for some $n \geq 3$ there is such a solution (a, b, c) of the equation $a^n + b^n = c^n$ in the integers. By previous work of Frey, the existence of such a solution guarantees existence of an elliptic curve E which is not modular. However, in what constitutes the bulk of the proof, Wiles proves that all elliptic curves are modular. Namely, he first proves that to determine whether an elliptic curve is modular one need only determine whether an associated representation of the elliptic curve is modular, which is easier. Then, Wiles considers two associated representations $\rho(E, 3)$ and $\rho(E, 5)$ of the elliptic curve E and shows that they are modular, thus proving that E is modular, providing a contradiction and concluding the proof.

Serre's Modularity Conjecture makes an appearance in the literature of Mazur, which prompted Wiles' key insight known as the "3-5 switch" between consideration of associated representations $\rho(E, 3)$ and $\rho(E, 5)$ of the elliptic curve E. I will not go into depth on Mazur's use of the conjecture but instead state the key idea of the conjecture. Namely, Serre's Modularity Conjecture states in broad terms that every representation of the absolute Galois group $G_{\mathbb{Q}}$ over the totally real field $\mathbb{Q}$ is modular.

(b) Analog of Serre's Modularity Conjecture for the Quadratic Number Field $\mathbb{Q}(i)$

Rebecca Torrey formulates an Analog of Serre's Modularity Conjecture. The main difference is that Torrey allows for consideration of quadratic imaginary fields (such as $\mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{-2})$ in our case) instead of limiting consideration to totally real fields (like $\mathbb{Q}$). Torrey's analog instead asks whether every representation of the absolute Galois group $G_{\mathbb{Q}(i)}$ of the quadratic imaginary field $\mathbb{Q}(i)$ is modular.

i. Involvement of the Cohomology $H^2(\Gamma, V)$ in this Conjecture

The cohomology group $H^2(\Gamma, V)$ appears in Torrey's definition of a modular representation of the absolute Galois group $G_{\mathbb{Q}(i)}$, which is roughly speaking one that arises from a function $f \in H^2(\Gamma, V)$ where Γ is a "congruence subgroup" and V is a "Serre weight", which is not important for our purposes.

ii. Computation of the Cohomology $H^2(\Gamma, V)$ via Instead Computing the Homology $H_0(\Gamma, St \otimes V)$ with Coefficients in the Steinberg Module

Working via Borre-Serre duality, which might remind the reader of similarly used Poincare duality, we can relate homology and cohomology in order to translate our computation of cohomology into an easier computation of homology. Namely, Borre-Serre Duality dictates that to compute the cohomology $H^2(\Gamma, V)$ it suffices to compute the homology $H_0(\Gamma, St \otimes V)$ with coefficients in the "Steinberg Module" which appears again later.

(c) Detour into Related Computation of Cohomology $H^*(SL_2(\mathcal{O}_{-2}))$ and of the Homology $H_*(SL_2(\mathcal{O}_{-2}), St(2))$ with Coefficients in a Steinberg Module Including Explanation of Why these Groups are Loosely Analogous to Those Above

The analogy between our (Vogtmann's) computations of $H^*(SL_2(\mathcal{O}_{-2}))$ and $H_*(SL_2(\mathcal{O}_{-2}), St(2))$ to Torrey's computation of $H^2(\Gamma, V)$ and $H_0(\Gamma, St \otimes V)$ is quite vague, but nonetheless motivates consideration for homology and cohomology of groups $SL_2(\mathcal{O}_{-d})$ where d is a square-free positive integer. In particular, my very weak analogy draws connection between $SL_2(\mathcal{O}_{-2})$, $SL_2(\mathbb{Z})$ and congruence subgroups $\Gamma(n)$ in that $\mathcal{O}_{-2}$ mimics the structure of $\mathbb{Z}$ and that $SL_2(\mathbb{Z}) = \Gamma(1)$. So, Vogtmann's group of consideration $SL_2(\mathcal{O}_{-2})$ has similar structure to that of a "congruence subgroup" Γ, whose definition is not required for our purposes but is mentioned to motivate usefulness in the theory of modular forms. So a reader in Math 6441 should note that a congruence subgroup is some very fundamental definition in a recently popular subfield of number theory and leave it at that. Such observation merits the analogy between $H^*(SL_2(\mathcal{O}_{-2}), \mathbb{Z})$ and $H^2(\Gamma, V)$ where in our loose analogy we gloss over the fact that $\mathbb{Z}$ and V are quite different coefficient groups. In fact, such a discrepancy is amended by the fact that we compute the cohomology groups $H_*(SL_2(\mathcal{O}_{-2}), St(2))$ and $H_0(\Gamma, St \otimes V)$ each with coefficients in a "Steinberg Module".

(d) Outline of Tools Used in the Above Computation:

i. Conceptual Relation between Homology/Cohomology of a Group and Homology/Cohomology of A Related Space which Is What One Actually Computes in Practice

In order to understand the homology or cohomology of a group G we construct a space X on which G acts and cellularly and restrict our attention to a fundamental domain which, for each $n \geq 0$, contains one n-cell from each orbit of n cells. We then consider the stabilizer subgroups Γ_σ of each of these representatives σ which are usually simpler and smaller than G itself meaning we can compute their homology and cohomology groups directly using projective resolutions. Then, given the specific conditions of our problem, once we have these homology groups $H_*(\Gamma_\sigma, \mathbb{Z})$ or cohomology groups $H^*(\Gamma_\sigma, \mathbb{Z})$ we can feed these into a "spectral sequence" which eventually outputs the desired homology group $H_*(G, \mathbb{Z})$ or cohomology group $H^*(G, \mathbb{Z})$.

ii. Explanation of the Theory of Spectral Sequences and Shapiro's Lemma which Make the Above Decomposition Work

Specifically, the relevant Theorem 7.3 in "Cohomology of Groups" provides a "spectral sequence" whose "abutment", or roughly speaking limiting behavior, gives the homology groups $H_n(G, \mathbb{Z})$ and whose first page is given by homology groups $H_*(\Gamma_\sigma, \mathbb{Z})$ where σ is a cell.
In case the reader in unfamiliar with spectral sequences, I recall that a spectral sequence for a group G is a set $\{E^r_{p,q}\}$ indexed by natural numbers (in our case) $r, p, q \in \mathbb{N}_{\geq 0}$ where, for fixed r, the set $E^r := \{E^r_{p,q} : p, q \in \mathbb{N}_{\geq 0}\}$ is called the rth page of the sequence (which is an infinite matrix indexed by $(p, q) \in \mathbb{N}^2_{\geq 0}$). A key property about (most) spectral sequences is that as $r \to \infty$ one has that the pages E^r "stabilize" so to speak, meaning that for sufficiently large $R \in \mathbb{N}$ one has that $E^R = E^{R+1} = E^{R+2} = \ldots$ or that all subsequent pages are identical meaning equal element-wise. We denote this limit by $E^\infty := E^r$. As mentioned before, for useful spectral sequences one has that the page E^∞ somehow encodes the homology group $H_n(G, \mathbb{Z})$. In our case, we have that

$$H_n(G, \mathbb{Z}) = \bigoplus_{\{p,q \in \mathbb{N}_{\geq 0} : p+q=n\}} E^\infty_{p,q}. \tag{1}$$

Also, recall that for any spectral sequence, provided one has appropriate "differential maps", all pages E^r of the sequence can be determined from the first page E^1. By Theorem 7.3, the first page of our spectral sequence is

$$E^1_{p,q} = \bigoplus_{\sigma \in \Sigma_p} H_q(\Gamma_\sigma, \mathbb{Z}).$$

Thus, to summarize the approach: in order to compute $H_n(SL_2(\mathcal{O}), \mathbb{Z})$ one computes for each p-cell σ in the fundamental domain for the action of $SL_2(\mathcal{O}_{-2})$ the qth homology group $H_q(\Gamma_\sigma, \mathbb{Z})$ of Γ_σ.

Similarly one may follow the exact same process to compute cohomology via the spectral sequence

$$E_1^{pq} = \bigoplus_{\text{orbits of p cells}\sigma_p} H^q(\Gamma_{\sigma_p}, \mathbb{Z}) \implies H^{p+q}(SL_2(\mathcal{O}_{-2})).$$

iii. Explicit Results of the Above Computation Note that Vogtmann does not include a description for the fundamental domain for the action of $SL_2(\mathcal{O}_{-2})$ but rather includes a reference to Siegel's document "On Advanced Analytic Number Theory" which is what spurred the related work of Mendoza and Vogtmann. She does, however, include the cohomology groups for many subgroups of $SL_2(\mathcal{O}_{-2})$ which of course includes any possible stabilizer subgroups.

$$H^q(Q) = \begin{cases} \mathbb{Z} & q=0 \\ 0 & q=1(4) \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & q=2(4) \\ 0 & q=3(4) \\ \mathbb{Z}/8 & q=0(4), q>0 \end{cases}$$

$$H^q(D) = \begin{cases} \mathbb{Z} & q=0 \\ 0 & q=1(4) \\ \mathbb{Z}/4 & q=2(4) \\ 0 & q=3(4) \\ \mathbb{Z}/12 & q=0(4), q>0 \end{cases}$$

$$H^q(Te) = \begin{cases} \mathbb{Z} & q=0 \\ 0 & q=1(4) \\ \mathbb{Z}/3 & q=2(4) \\ 0 & q=3(4) \\ \mathbb{Z}/24 & q=0(4), q>0 \end{cases}$$

$$H^q(\mathbb{Z}/n) = \begin{cases} \mathbb{Z} & q=0 \\ 0 & q \text{ odd} \\ \mathbb{Z}/n & q \text{ even}, q>0 \end{cases}$$

These cohomology groups of the various stabilizer subgroups of the cells of our fundamental domain are then fed into the above spectral sequence giving the final result.

6.3 THEOREM. *The integral cohomology of $SL_2(\mathcal{O}_{-2})$ is given by*

$$H^q(SL_2(\mathcal{O}_{-2})) \begin{cases} \mathbb{Z} & q=0 \\ \mathbb{Z} & q=1 \\ \mathbb{Z}/3 & q\equiv 2(4) \\ \mathbb{Z}/2 \oplus \mathbb{Z}/6 & q\equiv 3(4) \\ \mathbb{Z}/2 \oplus \mathbb{Z}/24 & q\equiv 0(4), q>0 \\ \mathbb{Z}/12 & q\equiv 1(4), q>1 \end{cases}$$

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Also, for the reader who was particularly interested in the connection to the recent Analog of Serre's Modularity Conjecture I include the below result by Vogtmann as promised.

$$H_q(SL_2(\mathcal{O}_{-2}), St(2)_{-2}) = \begin{cases} 0 & q = 0 \\ \mathbb{Z} \oplus \mathbb{Z}/2 & q = 1 \\ \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/6 & q = 2 \\ \mathbb{Z}/3 & q \equiv 3(4) \\ \mathbb{Z}/12 & q \equiv 0(4), q > 0 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/24 & q \equiv 1(4), q > 1 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/6 & q \equiv 2(4), q > 2 \end{cases}$$

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2. **Aiding the Understanding of Arithmetic of Bianchi Forms as Opposed to Modular Forms**

   (a) Remark that Understanding Torsion of Cohomology of Modular Groups (resp. Bianchi Groups) is Vital for Understanding the Arithmetic of Modular Forms (resp. Bianchi Forms)

   We first give a definition of Modular Forms and Bianchi Modular Forms.

   A "Modular Form for the Modular Group $SL_2(\mathbb{Z})$" is by definition an analytic function f on $\mathcal{H}$ satisfying, among other properties, that

   $$f(\frac{az+b}{cz+d}) = (cz+d)^k f(z), \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \text{ and all } z \in \mathcal{H}.$$

   In contrast, a Bianchi Modular Form instead makes use of $SL_2(\mathcal{O}_{-d})$ where $\mathcal{O}_{-d}$ is a quadratic imaginary field.

   In particular, a "Bianchi Modular Form for the Modular Group $SL_2(\mathcal{O}_{-d})$" is by definition an analytic function f on $\mathcal{H}$ satisfying, among other properties, that

   $$f(\frac{az+b}{cz+d}) = (cz+d)^k f(z), \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathcal{O}_{-d}) \text{ and all } z \in \mathcal{H}.$$

   According to the article "On the Integral Cohomology of Bianchi Groups" by Sengun, the arithmetic structure of modular forms, which are is very well-understood. However, the structure of Bianchi modular forms is not. One reason is that a classical tool for understanding the arithmetic structure of modular forms is to examine the cohomology group of $PSL_2(\mathbb{Z})$ which implies that a similar tool for the Bianchi modular group would be to examine the cohomology groups of $PSL_2(\mathcal{O}_{-d})$ which were not as well-studied at the time of publication.

   (b) Note that while the Necessary Cohomology of Modular Groups is Well-Understood and Readily Computed in the Literature Unfortunately Cohomology of Bianchi Groups is Not as Easily Computed or Readily Available

   The issue is that while the cohomology groups of $PSL_2(\mathbb{Z})$ were long ago computed and readily available in the literature unfortunately the cohomology groups of $PSL_2(\mathcal{O}_{-d})$ were not until later, which is why Vogtmann's computations of those cohomology groups $H^*(PSL_2(\mathcal{O}_{-d}))$ may be useful.

   (c) Note that $PSL_2(\mathcal{O}_{-2})$ is a Bianchi Group, which Motivates Our Explicit Computation of $H^*(PSL_2(\mathcal{O}_{-2}))$ in Order to Understand its Torsion

   A Bianchi Group is by definition a projective linear group $PSL_2(\mathcal{O}_{-d})$ of an imaginary quadratic number field. In our case, we limit our consideration to the $d = 2$ cases for brevity, though Vogtmann includes computation for many other square-free, non-negative integers d.

   (d) Remark that to Compute the Cohomology $H^*(PSL_2(\mathcal{O}_{-2}), \mathbb{Z})$ it Suffices to Compute the Homology $H_*(PSL_2(\mathcal{O}_{-2}), \mathbb{Z})$

As per Hatcher pages 191-192, one can recover cohomology of a space using only its homology and coefficient group. The paper of Vogtmann computes $H_*(PSL_2(\mathcal{O}_{-2})) \cong H_*(F, \mathbb{Z})$. Then, by the argument given in Hatcher one can recover $H^*(F, \mathbb{Z}) \cong H^*(PSL_2(\mathcal{O}_{-2}))$.

(e) Outline of Computation of $H_*(PSL_2(\mathcal{O}_{-2}))$ via Construction of Fundamental Domain and Analysis of the Action of the Group $PSL_2(\mathcal{O}_{-2})$ on this Fundamental Domain

i. Explicit Construction of the Above Space and its Fundamental Domain for the Action of the Group which are Used to Compute Homology/Cohomology of the Group of Interest

In order to relate homology (resp. cohomology) of the group G with that of a space X we construct the space X as follows. We let X be certain subspace of the hyperbolic half plane $\hat{H}$ which is $(\mathcal{H}) \times (\mathbb{R}_{\geq} 0 \cup \infty)$ where $\mathcal{H} := \{z \in \mathbb{C} : Im(z) > 0\} \cup \{\infty\}$ is the extended complex upper half plane. Then, by considering elements $(z, \zeta) \in \hat{H}$ as vectors with two components we have a natural action of the group $G = PSL_2(\mathcal{O}_{-2})$ on $\hat{H}$ via standard matrix multiplication. Specifically,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} z \\ \zeta \end{pmatrix} := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ \zeta \end{pmatrix}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathcal{O}_{-2})$ and all $\begin{pmatrix} z \\ \zeta \end{pmatrix} \in \hat{H}$. Namely $X = I(k)$ where $I(k) \subseteq \hat{H}$ is defined via a notion of "Cusps". Namely, a "Cusp" of $\hat{H}$ is a point $\begin{pmatrix} z \\ \zeta \end{pmatrix} \in \hat{H}$ of the form $\begin{pmatrix} a \\ 0 \end{pmatrix}$ for some $a \in \mathcal{O}_{-2}$ or the point $\begin{pmatrix} a \\ 0 \end{pmatrix}$. Then, one considers a "distance function" from points of $\hat{H}$ to cusps λ of $\hat{H}$

$$d(\begin{pmatrix} z \\ \zeta \end{pmatrix}, \lambda) :=$$

defined by a formula not necessary for our purposes (but which does have interesting relationships to the class number of the ring of integers $\mathcal{O}_{-n}$ and can be found on page 577 of Schwermer and Vogtmann's paper for those interested).

Then, one lets

$$H(\lambda) = \{\begin{pmatrix} z \\ \zeta \end{pmatrix} \in \hat{H} : d(\begin{pmatrix} z \\ \zeta \end{pmatrix}, \lambda) \leq d(\begin{pmatrix} z \\ \zeta \end{pmatrix}, \lambda') \text{ for all cusps } \lambda' \neq \lambda\}$$

meaning the set of points who are closer to λ than to any other cusp. Then, let

$$I(k) = \{\begin{pmatrix} z \\ \zeta \end{pmatrix} \in \hat{H} : \begin{pmatrix} z \\ \zeta \end{pmatrix} \in H(\lambda) \cap H(\lambda') \text{ for some cusps } \lambda \neq \lambda'\}$$

meaning the set of points in at least two sets $H(\lambda), H(\lambda')$ for distinct cusps λ, λ'.
Then, it seems that $PSL_2((O_2))$ acts properly on $I(k)$. Furthermore, one obtains the fundamental domain $Y \cong I(k)/PSL_2(\mathcal{O}_{-2})$ which is isomorphic to the CW-complex F pictured below

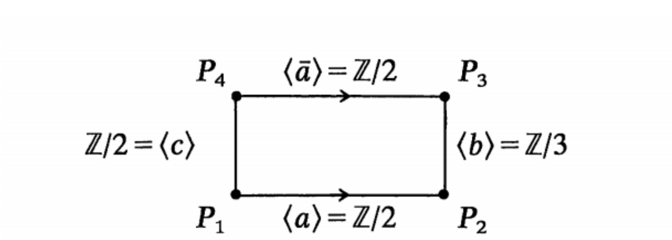

Figure 1: Pictorial representation of fundamental domain for the action of $PSL_2(\mathcal{O}_{-2})$ on $I(k)$

where the points P_1, P_2, P_3, P_4 are defined as
(Namely, the above rectangle is isomorphic but not equal to the fundamental domain Y since the actual fundamental domain is actually part of $I(k)$ which lies in the unit hemisphere with center $(0, 0) \in \mathbb{C} \times \mathbb{R}^+$. However, the above rectangle F is the projection of the fundamental domain Y onto the complex plane $\mathbb{C}$).

$$P_1 = \left(-\frac{\sqrt{2}}{2} i, \frac{\sqrt{2}}{2}\right), \quad P_2 = \left(\frac{1}{2} - \frac{\sqrt{2}}{2} i, \frac{1}{2}\right), \quad P_3 = \left(\frac{1}{2} + \frac{\sqrt{2}}{2} i, \frac{1}{2}\right), \quad P_4 = \left(\frac{\sqrt{2}}{2} i, \frac{\sqrt{2}}{2}\right).$$

ii. Explanation of How to Compute Homology/Cohomology of this Group G by Decomposing the Fundamental Domain (resp. Group) into Pieces, which are the 0,1, and 2 Cells of our Fundamental Domain (resp. Stabilizer Subgroups Γ_σ in G of those 0,1, and 2 Cells)

By a Theorem given in the textbook "Cohomology of Groups" by Brown we can compute the homology of the group $G = PSL_2(\mathcal{O}_{-2})$ via computation of the homology of smaller subgroups $\Gamma_\sigma \leq G$, which are the stablilizers of each cell $\sigma \subseteq F$. Namely, some of those stabilizer subgroups are already pictured in Figure 1. In particular if we consider the elements

$$a := \begin{pmatrix} 1 & \omega \\ \omega & -1 \end{pmatrix}, b := \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, c := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in PSL_2(\mathcal{O}_{-2}),$$

one has that the stabilizer subgroups of the zero cells are as follows,

$$\begin{aligned} \Gamma_1 &= \langle a, c \rangle \cong D_2 \\ \Gamma_2 &= \langle a, b \rangle \cong A_4 \\ \Gamma_3 &= \langle \bar{a}, b \rangle \cong A_4 \\ \Gamma_4 &= \langle \bar{a}, c \rangle \cong D_2 \end{aligned}$$

the stabilizer subgroups of the one cells are as follows,

$$\begin{aligned} \Gamma_{12} &= \langle a \rangle \cong \mathbb{Z}/2; \\ \Gamma_{23} &= \langle b \rangle \cong \mathbb{Z}/3; \\ \Gamma_{34} &= \langle \bar{a} \rangle \cong \mathbb{Z}/2; \\ \Gamma_{41} &= \langle c \rangle \cong \mathbb{Z}/2; \end{aligned}$$

and finally the stabilizer subgroup of the 2 cell is the multiplicative group $\{\pm 1\}$.
As before we have that

$$H_n(G, \mathbb{Z}) = \bigoplus_{\{p,q \in \mathbb{N}_{\geq 0} : p+q=n\}} E^\infty_{p,q}. \tag{2}$$

and again our first page is

$$E^1_{p,q} = \bigoplus_{\sigma \in \Sigma_p} H_q(\Gamma_\sigma, \mathbb{Z}).$$

Shown below are the homology groups of the various stabilizer subgroups of the 0,1, and 2 cells of the associated fundamental domain as computed by Vogtmann.

$$H_q(\mathbb{Z}/n) = \begin{cases} \mathbb{Z} & q=0 \\ \mathbb{Z}/n & q \text{ odd} \\ 0 & q \text{ even}, q>0 \end{cases}$$

$$H_q(D_2) = \begin{cases} \mathbb{Z} & q=0 \\ (\mathbb{Z}/2)^{(q+3)/2} & q \text{ odd} \\ (\mathbb{Z}/2)^{q/2} & q \text{ even}, q>0 \end{cases}$$

$$H_q(S_3) = \begin{cases} \mathbb{Z} & q=0 \\ \mathbb{Z}/2 & q \equiv 1\ (4) \\ 0 & q \equiv 2\ (4) \\ \mathbb{Z}/6 & q \equiv 3\ (4) \\ 0 & q \equiv 0\ (4),\ q>0. \end{cases}$$

$$H_q(A_4) = \begin{cases} \mathbb{Z} & q=0 \\ (\mathbb{Z}/2)^k \oplus \mathbb{Z}/3 & q=6k+1 \\ (\mathbb{Z}/2)^k \oplus \mathbb{Z}/2 & q=6k+2 \\ (\mathbb{Z}/2)^k \oplus \mathbb{Z}/6 & q=6k+3 \\ (\mathbb{Z}/2)^k & q=6k+4 \\ (\mathbb{Z}/2)^k \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/6 & q=6k+5 \\ (\mathbb{Z}/2)^k & q-6(k+1) \end{cases}$$

Now, since we have the homology groups $H_q(\Gamma_\sigma, \mathbb{Z})$, we have the first page $E^1_{p,q}$ by taking appropriate direct sums. Then, by standard computation of the subsequent pages of the spectral sequence we get that $E^\infty = E^2$ meaning the spectral sequence stabilizes quite early at page 2 (since differentials on later pages become zero quickly, which is a condition indicating stabilization). Thus, from E^1 one computes E^2, which by simply taking the direct sum of the entries along counter-diagonals on the second gives $H_n(PSL_2(\mathcal{O}_{-2}))$ as pictured below.

5.3 THEOREM. *The integral homology of $PSL_2(\mathcal{O}_{-2})$ is given by*

$$H_q(PSL_2(\mathcal{O}_{-2})) \cong \begin{cases} \mathbb{Z} & q=0 \\ \mathbb{Z} \oplus \mathbb{Z}/6 & q=1 \\ \mathbb{Z}/4 \oplus \mathbb{Z}/6 & q=2 \\ (\mathbb{Z}/2)^{2q/3} \oplus \mathbb{Z}/3 & q \equiv 0(3),\ q>0 \\ (\mathbb{Z}/2)^{2(q-1)/3} \oplus \mathbb{Z}/3 & q \equiv 1(3),\ q>1 \\ (\mathbb{Z}/2)^{2(q+1)/3} \oplus \mathbb{Z}/3 & q \equiv 2(3),\ q>2 \end{cases}$$

Figure 2: Integral homology of $PSL_2(\mathcal{O}_{-2})$ with $\mathbb{Z}$ coefficients

Appendix

- Intuitive Explanation of Cohomology of Spaces

  Recall the the homology group $H_*(X)$ of a space X is formed via a long exact sequence involving the chain groups $C_n(X)$. In order to define the "cohomology group" $H^*(X, M)$ "with coefficients in the module M" we utilize construction of dual groups and dual maps involving the usual chain groups $C_n(X)$ and boundary maps ∂ between them. Namely, for a fixed coefficient module M one considers the dual spaces $Hom(C_n(X), M)$ to $C_n(X)$ and dual maps δ to the usual boundary maps ∂. I note in passing that by definition of dual maps one has that the dual maps $\delta : Hom(C_n(X), M) \to Hom(C_{n+1}(X), M)$ go in the opposite direction of the usual maps $\partial : C_{n+1}(X) \to C_n(X)$. Still, just as before one can consider the quotients $H^n(X, M) := ker(\delta)/im(\delta)$ which we take as the definition of the nth cohomology group of the space X.

  One might wonder what motivates the need for both notions of the cohomology group and homology group since, as noted in Hatcher, provided one of the two groups and the associated coefficient group (or coefficient module) one can usually compute the other. However, the cohomology group encodes a bit more information than the homology group in the sense that it has a natural ring structure as a graded ring via the cup product.

- Formal Definition of Homology/Cohomology of a Group via Free Resolutions

  The reader may note that the actual definition of homology/cohomology of a group rather than a space has not yet been given. Indeed, I postponed such a definition for a reason. Note that we computed the homology/cohomology of groups $SL_2(\mathcal{O}_{-2})$ and $PSL_2(\mathcal{O}_{-2})$ by considering cellular complexes and the action of those groups those complexes. Indeed the core idea of Vogtmann's paper is that one can compute the homology/cohomology of a group by examining the action of the group on an associated space, which is a more familiar notion for Math 6441 students anyway. For that reason, I felt that giving the definition of homology/cohomology of a group, which is purely group-theoretic, would have distracted from the core idea that, as far as the Vogtmann paper is concerned, the homology $H_*(G, \mathbb{Z})$ of a group is determined exactly by how that group acts on a space.

  However, formally the homology/cohomology of a group is defined via "projective resolutions of the group" which are by definition exact sequences of projective modules over the group ring $\mathbb{Z}G$, written as

  $$P_* = \cdots \to P_i \to P_{i-1} \to \cdots \to P_1 \to P_0 \to M \to 0.$$

  Then, by definition

  $$H_*(G, M) = H_*(P_* \otimes_{\mathbb{Z}G} \mathbb{Z}).$$

  Note that as per "Cohomology of Groups" one has that

  $$M \otimes_{\mathbb{Z}G} \mathbb{Z} = M_G$$

  for any module M where

  $$M_G := \frac{M}{\langle gm - m : g \in G, m \in M\rangle}$$

  where, intuitively, quotienting by the specified ideal of M identifies each G-orbit to a single point in the quotient M_G meaning that an equivalent definition for homology of a group is

  $$H_*(G, M) = H_*((P_*)_G).$$

  where P_* is a projective resolution of M over $\mathbb{Z}G$.

  The reader should note that it is not easy in general to find a projective resolution of an arbitrary group, which is why geometric tools such as those used by Vogtmann are especially useful.

Related Exercises

1. As promised the first exercise relates the homology of a space to the homology of its covering group.

   Let Y be a path-connected space. If Y has a contractible, normal covering space X with covering group G, show that $H_*(Y) \cong H_*(G)$.

   Example motivating the utility of this result:
   Consider Y to be any elliptic curve $Y := E$ over $\mathbb{C}$ and note that since any elliptic curve over $\mathbb{C}$ is topologically isomorphic to $S^1 \times S^1$ (`https://wcnt.files.wordpress.com/2014/08/langlois-wcnt-2013.pdf`) we know that $\pi_1(E) = \mathbb{Z} \times \mathbb{Z}$. As noted in Hain's article, the universal cover X of E is $\mathbb{C}$. Recall that Proposition 1.39 of Hatcher implies that the covering group of any universal cover X of Y is normal, namely because the associated subgroup $H = p_*(X) \subseteq \pi_1(Y)$ is actually $H = \{e\}$ the trivial group, which is always normal, thus implying that the covering is normal. Thus, the above result applies to this case, meaning that we may compute the homology $H_*(Y)$ of the elliptic curve E by computing $H_*(G)$ which is easier since we know what G is. Namely, since X is a universal cover, we know per page 71 of Hatcher that $G(X) \cong \pi_1(Y) = \mathbb{Z} \times \mathbb{Z}$. So, the above result implies that $H_*(Y) = H_*(\mathbb{Z} \times \mathbb{Z}) = H_*(T^2)$ (since $T^2 = K[\mathbb{Z} \times \mathbb{Z}, 1]$), which finally implies that

   $$H_n(E) = \begin{cases} \mathbb{Z} \text{ if } n = 0 \\ \mathbb{Z}^2 \text{ if } n = 1 \\ \mathbb{Z} \text{ if } n = 2 \\ 0 \text{ if } n \geq 3 \end{cases}$$

which while obvious to someone fluent in the topic (perhaps since $Y \cong S^1 \times S^1$, is not to me since as a novice, previous to working through this example, I was not aware of that fact and at the point any elliptic curve seemed quite mysterious with an even more mysterious homology group).

Proof:
In our case, the ring in question is the group ring $\mathbb{Z}G$ and our module in question is $\mathbb{Z}$.
Note that by page 18 of "Cohomology of Groups" we know that the singular chain complex $C_*(X)$ provides a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Now, returning to the problem, since all free resolutions are projective we know that $C_*(X)$ provides a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Here one asks what exactly the associated module is. It seems it would be $C_0(X)$ and the associated resolution would be

$$\ldots C_3(X) \to C_2(X) \to C_1(X) \to C_0(X) \to \mathbb{Z} \to 0.$$

Note, that indeed the above resolution is exact since the fact that X is contractible means that $H_*(X) = H_*(pt) \cong 0$.

Then, by our second definition of $H_*(G, M)$ we see that

$$H_*(G, M) = H_*(C_*(X)_G).$$

Finally, the very key observation noted in "Cohomology of Groups" is that $C_*(X)_G \cong C_*(Y)$ whenever X is a "free G complex" and Y is the orbit complex X/G. Note that as per page 15 of "Cohomology of Groups" since X is a regular covering of the CW complex Y, we know that there exists a CW decomposition of X such that the covering group $G \cong \pi_1(Y)$ acts cellularly on X. Furthermore, much in line with the definition of a normal covering space we learned in class, the n-cells of X lying above a particular n-cell in Y lie in one single G-orbit meaning that X is indeed a free G-complex. Furthermore, note that the group of deck transformations of the base space Y is trivial which implies that Y is the orbit space X/G.

So, finally we see that $H_*(G) = H_*(C_*(X)_G) \cong H_*(C_*(Y)) = H_*(Y)$ as claimed.

2. As promised, the second calls for computation of the homology $H_*(SL_2(\mathbb{Z}))$ which is closely related to our previous computations of $H_*(SL_2(\mathcal{O}_{-2}))$ since the ring of integers $\mathcal{O}_{-2}$ inside the field $\mathbb{Q}(\sqrt{-2})$ is meant to mimic the structure of the ring $\mathbb{Z}$ inside the field $\mathbb{Q}$. It also provides an interesting example of the use of an analog of a Mayer-Vietoris sequence to compute homology of a group rather than a space.

It is a classical fact that $SL_2(\mathbb{Z}) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$. Calculate $H_*(SL_2(\mathbb{Z}))$.

As noted in "Cohomology of Groups" since $SL_2(\mathbb{Z}) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$ we have the following exact sequence

$$\cdots \to H_n(A) \to H_n(G_1) \oplus H_n(G_2) \to H_n(G) \to H_{n-1}(A) \cdots \to H_0(A) \to H_0(G_1) \oplus H_0(G_2) \to H_0(G) \to 0.$$

where $SL_2(\mathbb{Z}) = G = G_1 *_A G_2$ meaning that $G_1 = \mathbb{Z}_4, G_2 = \mathbb{Z}_6, A = \mathbb{Z}_2$. Thus, it suffices to compute $H_*(\mathbb{Z}_4), H_*(\mathbb{Z}_6)$, and $H_*(\mathbb{Z}_2)$.

We do so for each group $G_m = \mathbb{Z}_m = \langle t | t^m = 1\rangle$ using the following $\mathbb{Z}G_m$ resolution of G_m found on page 151 of "Homology of Discrete Groups".

$$P(G_m)_* = \cdots \to \mathbb{Z}G_m \to^{t-1} \mathbb{Z}G_m \to^{\Delta} \mathbb{Z}G_m \to^{t-1} \mathbb{Z}G_m \to^{\epsilon} \mathbb{Z} \to 0$$

where the map $\mathbb{Z}G \to^{t-1} \mathbb{Z}G$ denotes multiplication by $t-1$ in the group ring $\mathbb{Z}G$ and $\mathbb{Z}G \to^{\Delta} \mathbb{Z}G$ denotes multiplication by $1+t+t^2+\cdots+t^{m-1}$ in the group ring $\mathbb{Z}G$. Also, note that this sequence is exact since $im(t-1) = \mathbb{Z}G(t-1)$ and $\Delta((\sum_{k=1}^{m-1} a_k t^k)(t-1)) = (\sum_{k=1}^{m-1} a_k t^k)\big((t-1)(1+t+t^2+\cdots+t^{m-1})\big) = (\sum_{k=1}^{m-1} a_k t^k)\big(t^m - 1\big) = 0$ and similarly we have that $ker(t-1) = im(\Delta)$ once again because $(t-1)(1+t+t^2+\cdots+t^{m-1}) = 0$.

Then, one computes

$$P(G_m)_* \otimes_{\mathbb{Z}G_m} \mathbb{Z} = \cdots \to \mathbb{Z} \to^0 \mathbb{Z} \to^m \mathbb{Z} \to^0 \mathbb{Z} \to 0.$$

Now, I show that the maps $0, m$ are indeed the maps cannonically induced by the maps $t-1, \Delta$ respectively. In particular, note that the map $t-1 : \mathbb{Z}G_m \to \mathbb{Z}G_m$ induces the map

$$(t-1) \otimes_{\mathbb{Z}G_m} id : \mathbb{Z}G_m \otimes_{\mathbb{Z}G_m} \mathbb{Z} \to \mathbb{Z}G_m \otimes_{\mathbb{Z}G_m} \mathbb{Z}.$$

Then, we note that $\mathbb{Z}G_m \otimes_{\mathbb{Z}G_m} \mathbb{Z}$ is isomorphic to $\mathbb{Z}$ via the map $\tau : \mathbb{Z}G_m \otimes_{\mathbb{Z}G_m} \mathbb{Z} \to \mathbb{Z}$ defined by

$$(\sum_{k=0}^{m-1} a_k t^k) \otimes_{\mathbb{Z}G_m} x \mapsto^{\tau} \sum_{k=0}^{m-1} a_k (t^k \cdot x) = (\sum_{k=0}^{m-1} a_k) x$$

where $\cdot$ indicates the specified group action of G_m on $\mathbb{Z}$ which we here specify to be the trivial action $t^k \cdot x = x$ for all $k \in [0 : m-1]$ and all $x \in \mathbb{Z}$. Note that the above is clearly a homomorphism of modules. Furthermore, consider the associated map on the direct product $\phi : \mathbb{Z}G_m \times \mathbb{Z} \to \mathbb{Z}$ then

$$\begin{tikzcd}
\mathbb{Z}G_m \times \mathbb{Z} \arrow[r, "\pi"] \arrow[dr, "\phi"'] & \mathbb{Z}G_m \otimes_{\mathbb{Z}G_m} \mathbb{Z} \arrow[d, "\tau"] \\
& \mathbb{Z}
\end{tikzcd} .$$

Then, note that since $ker(\phi) = ker(\pi)$ where π is the canonnical projection map and since π is surjetive that implies that τ is injective. Of course, it is also surjective since $\tau(1 \otimes_{\mathbb{Z}G_m} 1) = 1$.

So, all together we see that $0 = \tau \circ (t-1) \circ \tau^{-1}$ and $m = \tau \circ \Delta \circ \tau^{-1}$ are the proper induced maps which makes sense intuitively since $(t-1)(\sum_{k=0}^{m-1} a_k t^k \otimes_{\mathbb{Z}G_m} x) = (\sum_{k=0}^{m-1} a_k t^k \otimes_{\mathbb{Z}G_m} (t-1) \cdot x) = (\sum_{k=0}^{m-1} a_k t^k \otimes_{\mathbb{Z}G_m} 0) = 0$ by linearity of the tensor product and the fact that G_m acts trivially on $\mathbb{Z}$. Similarly note that $(1+t+t^2+\cdots+t^{m-1})(\sum_{k=0}^{m-1} a_k t^k \otimes_{\mathbb{Z}G_m} x) = (\sum_{k=0}^{m-1} a_k t^k \otimes_{\mathbb{Z}G_m} (1+t+t^2+\cdots+t^{m-1}) \cdot x) = (\sum_{k=0}^{m-1} a_k t^k \otimes_{\mathbb{Z}G_m} mx) = m(\sum_{k=0}^{m-1} a_k t^k \otimes_{\mathbb{Z}G_m} x)$. So, we have justified that $P(G_m)_* \otimes_{\mathbb{Z}G_m} \mathbb{Z}$ is what we claimed. All that remains is to compute $H_*(G_m, \mathbb{Z}) = H_*(P(G_m)_* \otimes_{\mathbb{Z}G_m} \mathbb{Z})$. Indeed we see that

$$H_n(G_m, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}_m & \text{if } n \text{ odd} \\ 0 & \text{if } n \geq 2 \text{ even} . \end{cases}$$

So, we have finally computed $H_*(\mathbb{Z}_2), H_*(\mathbb{Z}_4)$, and $H_*(\mathbb{Z}_6)$ as desired. Now, we use the aforementioned Mayer-Vietoris sequence which gives

$$\cdots \to H_n(\mathbb{Z}_2) \to H_n(\mathbb{Z}_4) \oplus H_n(\mathbb{Z}_6) \to H_n(G) \to H_{n-1}(\mathbb{Z}_2) \cdots \to H_0(\mathbb{Z}_2) \to H_0(\mathbb{Z}_4) \oplus H_0(\mathbb{Z}_6) \to H_0(G) \to 0.$$

Now, we note that many of these groups are zero, namely

$$H_n(\mathbb{Z}_2) = H_n(\mathbb{Z}_4) = H_n(\mathbb{Z}_6) = 0$$

for $n \geq 2$ even meaning that the maps

$$H_n(G) \to H_{n-1}(\mathbb{Z}_2)$$

are the zero map for n odd and also that the maps

$$H_n(\mathbb{Z}_2) \to H_n(\mathbb{Z}_4) \oplus H_n(\mathbb{Z}_6)$$

and

$$H_n(\mathbb{Z}_4) \oplus H_n(\mathbb{Z}_6) \to H_n(G)$$

are the zero map for n even meaning that the map

$$H_n(G) \to^{\kappa} H_{n-1}(\mathbb{Z}_2)$$

is injective for n even and also that the map

$$H_n(\mathbb{Z}_4) \oplus H_n(\mathbb{Z}_6) \to^{\beta} H_n(G)$$

is surjective for n odd.

In particular, consider the small part of our exact sequence

$$H_n(\mathbb{Z}_2) \to^{\alpha} H_n(\mathbb{Z}_4) \oplus H_n(\mathbb{Z}_6) \to^{\beta} H_n(G) \to H_{n-1}(\mathbb{Z}_2) = 0$$

where n odd and I have labeled the maps α, β for convenience.

Now, note that $im(\beta) \cong H_n(\mathbb{Z}_4) \oplus H_n(\mathbb{Z}_6)/ker(\beta)$ but β onto and $ker(\beta) = im(\alpha)$ implies that

$$H_n(G) \cong (H_n(\mathbb{Z}_4) \oplus H_n(\mathbb{Z}_6))/im(\alpha)$$

Now, in order to determine the map α note that as page 92 of "Homology of Linear Groups" states that since α is the map induced by the map $\mathbb{Z}_2 \to \mathbb{Z}_4 \oplus \mathbb{Z}_6$ which sends $1 \mapsto (2,3)$ then the map α is the exact same map for n odd defined on $\mathbb{Z}_2 \cong H_n(\mathbb{Z}_2)$ to $\mathbb{Z}_4 \oplus \mathbb{Z}_6 \cong H_n(\mathbb{Z}_4) \oplus H_n(\mathbb{Z}_6)$ and thus $im(\alpha) = \{(2,3),(0,0)\}$ which means that for n odd we have

$$H_n(G) \cong (\mathbb{Z}_4 \times \mathbb{Z}_6)/\{(2,3),(0,0)\}.$$

Now, clearly the above group has order 12, but by inspection we see that the element $(1,1)$ is a generator meaning it is cyclic.

$$\begin{aligned}
(0,0) &\equiv (0,0)\\
(1,1) &\equiv (1,1)\\
(2,2) &\equiv (2,2)\\
(3,3) &\equiv (1,0)\\
(4,4) &\equiv (0,4)\\
(5,5) &\equiv (1,5)\\
(6,6) &\equiv (2,0)\\
(7,7) &\equiv (3,1)\\
(8,8) &\equiv (0,2)\\
(9,9) &\equiv (1,3)\\
(10,10) &\equiv (2,4) \equiv (0,1)\\
(11,11) &\equiv (3,5) \equiv (1,2)
\end{aligned}$$

So, we see that

$$H_n(G) \cong \mathbb{Z}_{12}$$

for n odd.

Additionally, we see that

$$H_n(G) \cong 0$$

for $n \geq 2$ even by noting that κ injective implies either $H_n(G) = 0$ or $H_n(G) = \mathbb{Z}_2$ for $n \geq 2$ even. However, if $H_n(G) = \mathbb{Z}_2$ then that implies κ surjective and hence $\alpha : H_{n-1}(\mathbb{Z}_2) \to H_{n-1}(\mathbb{Z}_4) \oplus H_{n-1}(\mathbb{Z}_6)$ is the zero map for n even, which is a contradiction since we know that α between odd homology groups is not the zero map (page 92).

Thus, we have shown that

$$H_n(SL_2(\mathbb{Z}), \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}_12 & \text{if } n \text{ odd} \\ 0 & \text{if } n \geq 2 \text{ even .} \end{cases}$$

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