Demonstration of Ability to Explain Real Analysis Concepts to Undergraduate Students

Princeton Real Analysis Qualifying Exam Questions

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1 Measure Theory

Question 1.1. *What is a measurable function? Is the composition of two measurable functions measurable?*

It is a function $f : E \to \mathbb{R}$ such that the pre-image of any open set is measurable, or written precisely one such that $f^{-1}(U) \subseteq E$ is measurable whenever $U \subseteq \mathbb{R}$ is open. No, not necessarily. One would need the outer function to be continuous. Namely, if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are measurable, we ask whether $g \circ f : \mathbb{R} \to \mathbb{R}$ is measurable, which amounts to asking whether $f^{-1} \circ g^{-1}(U)$ is measurable for all $U \subseteq \mathbb{R}$. The answer is no. What we do know is that $g^{-1}(U)$ is measurable, but that is where our argument falls apart because f measurable does not ensure that pre-images of measurable sets are measurable. So, we cannot say anything about $f^{-1}(g^{-1}(U))$.

Question 1.2. *What is the measure of a set on the real line? When is a set measurable? Name some measurable sets. What are the operations that you can do to measurable sets to get measurable sets?*

We first define exterior measure, which is a very general notion of volume of a set.

For an interested college freshman who might ask why we need a more general notion of volume, the answer is that our method of calculating volume of an interval $[1/2, 3/5]$ or a rectangle $[1, 2] \times [e, 3.5] \subseteq \mathbb{R}^2$ which have volume $3/4 - 1/2$ in $\mathbb{R}^1$ and $(2-1)(3.5-e)$ respectively relies on the notion of breaking the region into solid blocks. Even when computing the areas of slightly more complex regions, like ellipses or areas under a curve in the plane, we are relying on the fact that each portion of such a set solidly covers the area it occupies. However, certain sets of interest do not. Namely, the set $(\mathbb{R} \setminus \mathbb{Q}) \cap [0, 1]$ of irrationals in $[0, 1]$ is dense but so is its complement $\mathbb{Q} \cap [0, 1]$ of rationals in $[0, 1]$. As a result, $(\mathbb{R} \setminus \mathbb{Q}) \cap [0, 1]$ contains no interval, which means it is unclear how to calculate its volume using the very basic notion of volume described above. Using measure theory, however, we can compute the “Lesbegue measure” of this set to be 1 and the measure of the rationals $\mathbb{Q} \cap [0, 1]$ in $[0, 1]$ to be 0, which means in some sense that the interval $[0, 1]$ is almost entirely made up of irrational numbers, an idea one could not glean without measure theory.

To provide one more intuitive and conceptually digestible definition of a measurable set before jumping into the specifics, a measurable set of $\mathbb{R}^n$ is a set $U \subseteq \mathbb{R}^n$ that can be written as the union of a finite set of intervals (boxes) plus or minus sets of very small "exterior measure". So, measurable sets are sets that can be approximated arbitrarily well by a finite number of boxes. To be fair, the definition of measure means that sets that look very different from a finite union of boxes can be measurable since subtracting a set of small exterior measure from a finite union of boxes can wildly alter the set's appearance. For example, the sets described above, $(\mathbb{R} \setminus \mathbb{Q}) \cap [0, 1]$ and $\mathbb{Q} \cap [0, 1]$, which were described above as motivation for defining measure of a set, are measurable. The first differs from the interval (box) $[0, 1]$ by a set of small exterior measure $\mathbb{Q} \cap [0, 1]$, which has measure 0, and the second differs from the empty set, technically finite union of boxes, by a set of small measure, namely itself $\mathbb{Q} \cap [0, 1]$.

The punchline is that measure theory allows us to compute the measure, a volume of sorts, of any "measurable set" in $\mathbb{R}^n$. Pretty much any set someone can think of is "measurable", which is what makes the theory so useful. However, as a small point, we note that some sets are not measurable and thus still elude our approach to compute a "volume" of sorts. These sets are very contrived–the most famous one is the Vitali set, constructed as a set of representatives of the cosets of $\mathbb{Q}$ in $\mathbb{R}$. It should be clear, even at first glance, that such a set is very strange since the number of these cosets is uncountable and since we need to invoke the Axiom of Choice just to be able to construct this set in the first place. So, the point is that most sets in $\mathbb{R}^n$ are measurable, and as such we are able to extend the ideas from calculus like differentiation and integration to lots of weird sets and functions. For instance, we can come up with a meaningful definition of the integral $\int_0^1 \chi_{x \in \mathbb{Q}} d\mu$ using our new Lebesgue measure μ. This is an improvement over using Riemann integration because the function $\chi_{x \in \mathbb{Q}}$ is not Riemann integrable due to the fact that its lower Riemann integral and its upper Riemann integral, which are 0 and 1 respectively, do not agree. Note that the theory of Lebesgue integration defines the Lebesgue unambiguously as $\int_0^1 \chi_{x \in \mathbb{Q}} d\mu = 0$ to be zero. So, we begin to get a sense of the improvements measure theory affords the study of calculus.

Question 1.3. *Are all subsets of* $\mathbb{R}$ *measurable? Give me a non-measurable subset.*

No. The Vitali set is not measurable. One can show that the Vitali set is not measurable since the set of differences of points in the Vitali set does not contain an interval around the origin, which shows that the Vitali set is not measurable since any measurable set satisfies that property.

The Vitali set is constructed as follows. Consider the cosets $\mathbb{R}/\mathbb{Q}$ of the set of rational numbers in the set of real numbers. Using the Axiom of Choice, select one representative from each of the uncountably many cosets and call this set of representatives the Vitali set.

Question 1.4. *What is the measure of a countable set? Can an uncountable set have zero measure?*

Q: What is the measure of a countable set?
A: It is 0.
Q: Can an uncountable set have zero measure?
A: Yes, the unit interval, $[0, 1] \times \{0\}$, has zero measure in $\mathbb{R}^2$.

2 Riemann Integration

Question 2.1. *Why is Lebesgue integration so much better than Riemann integration?*

Put simply, it allows us to integrate functions over sets that we wouldn't be able to. It allows integration of trickier functions and also allows integration over trickier domains. Riemann integration relies on dividing the domain over which we integrate into intervals which we might not be able to do for strange sets. Since the Lebesgue integral of a function $f : \mathbb{R}^n \to \mathbb{R}$ is defined very straightforwardly as the the "measure under the function". Namely, for a non-negative, measurable function $f : \mathbb{R}^n \to \mathbb{R}$ the Lebesgue integral is defined as $\int_E f d\mu := |\{(x, y) \in \mathbb{R}^{n+1} : x \in E, 0 \leq y \leq f(x)\}|_{\mathbb{R}^{n+1}}$. That allows us to integrate over strange sets like $E = \mathbb{Q}$ or $E = (\mathbb{R} \setminus \mathbb{Q}) \cap [1, e]$.

For the student who might already have some exposure, he or she may have heard that Lebesgue integration is accomplished by dividing up the y axis (range) rather than the x axis (domain) as we do for Riemann integration. That is true. Namely, the above definition I mentioned can be obtained as a sum the measures of finer and finer slices ("horizontally along the the y axis"). In particular, note that if we denote $M_f := |\{(x, y) \in \mathbb{R}^{n+1} : x \in E, 0 \leq y \leq f(x)\}|_{\mathbb{R}^{n+1}}$ then clearly $M_f := \lim_{m\to\infty} \sum_{n\in\mathbb{N}} \sum_{k\in[m]} M^{n,n+1}_{f,k,m}$ where $M^{n,n+1}_{f,k,m} := |\{(x, f(x)) \in \mathbb{R}^{n+1} : x \in E, n+\frac{k-1}{m} \leq f(x) \leq n+\frac{k}{m}, 0 \leq y \leq n+\frac{k-1}{m}\}|_{\mathbb{R}^{n+1}}$. Here, by the way, is where the definition of a measurable function comes into play. Namely, f measurable means by one of many equivalent definitions that $\{x \in E : n \leq f(x) \leq n+1\}$ is measurable and one notes that in order for the set $\{(x, f(x)) \in \mathbb{R}^{n+1} : x \in E, n+\frac{k-1}{m} \leq f(x) \leq n+\frac{k}{m}, 0 \leq y \leq n+\frac{k-1}{m}\}$ to be measurable for appropriate fixed n, m, k one needs $\{x \in \mathbb{R}^n : x \in E, n+\frac{k-1}{m} \leq f(x) \leq n+\frac{k}{m}\}$ to be measurable (since the former is the direct product of the latter with an interval).

Question 2.2. *What is a necessary and sufficient condition for a function to be Riemann integrable?*

One necessary and sufficient condition is that its lower and upper Riemann integrals exist and are equal. Another is that the set of discontinuities of f have measure zero.

Question 2.3. *Is it possible that the characteristic function of an open set, E, is not Riemann integrable?*

In order for such a function to fail to be Riemann integrable, the boundary of the set E would need to have positive measure since the set of discontinuities of its characteristic function is exactly its boundary.

3 Different Kinds of Convergence

Question 3.1. *A sequence of continuous functions converges pointwise. What is a condition you can impose to make the limit continuous?*

You can impose the requirement that f_n converges uniformly to f. For fixed $x \in \mathbb{R}$, uniform convergence provides $N(\epsilon/3) \in \mathbb{N}$ such that $|f - f_n|(x) < \epsilon/3$ for all $n \geq N(\epsilon/3)$ and all

$x \in \mathbb{R}$. Note that for any $n \in \mathbb{N}$ and $x, x' \in \mathbb{R}$ we have that $|f(x) - f(x')| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x')| + |f_n(x') - f(x')|$. Also, uniform convergence implies that for all $n \geq N(\epsilon/3)$, we have that $|f(x) - f_n(x)| < \epsilon/3$ and $|f(x') - f_n(x')| < \epsilon/3$ for all $x, x' \in \mathbb{R}$. Also, continuity of f_n for each $n \in \mathbb{N}$ implies that one has $\delta_{n,x}(\epsilon/3) > 0$ such that $|f_n(x) - f_n(x')| < \epsilon/3$ whenever $|x - x'| < \delta_{n,x}(\epsilon/3)$. So, for arbitrary $n \geq N(\epsilon/3)$ we have that $|f(x) - f_n(x)| + |f_n(x) - f_n(x')| + |f_n(x') - f(x')| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$ whenever $|x - x'| < \delta_{n,x}(\epsilon/3)$. The specific $n \geq N$ does not matter; we only need one. We see that using any $n \geq N(\epsilon/3)$ in the above equation gives that $|f(x) - f(x')| < \epsilon$ whenever $|x - x'| < \delta_{n,x}(\epsilon/3)$, proving continuity.

Question 3.2. *State/prove the Arzelà–Ascoli theorem.*

The Arzelà–Ascoli Theorem states that for any sequence $\{f_n : \mathbb{R} \to \mathbb{R} : n \in \mathbb{N}\}$ of equicontinuous and uniformly bounded functions has a subsequence that converges compactly.

Note that $\{f_n : \mathbb{R} \to \mathbb{R} : n \in \mathbb{N}\}$ equicontinuous means by definition that for each $x \in \mathbb{R}$ and each $\epsilon > 0$ there exists $\delta > 0$ such that $|x - x'| < \delta$ implies that $|f_n(x) - f_n(x')| < \epsilon$ for all $n \in \mathbb{N}$. Also $\{f_n : \mathbb{R} \to \mathbb{R} : n \in \mathbb{N}\}$ uniformly bounded means by definition that there exists $M \in \mathbb{R}_{\geq 0}$ such that $|f_n(x)| \leq M$ for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$. So, to put it simply and very roughly in terms an engineering undergraduate might appreciate, the notions of equicontinuity and uniform boundedness ensure that a given parameter "works" for every *function* in the family. (These notions differ from uniform continuity which ensures one parameter "works" for every *point* in the domain). Finally, note that $f_n \to f$ compactly means, by definition, that $f_n \to f$ uniformly on every compact subset $K \subseteq \mathbb{R}$. For an undergraduate who might not know, $f_n \to f$ uniformly on $K \subseteq \mathbb{R}$ means by definition that for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ for all $x \in \mathbb{R}$ and all $n \geq N$. So, the key point is that one N works for every *point* $x \in \mathbb{R}$ in the domain simultaneously.

4 Sequences and Series

Question 4.1. *When does the sum of $1/n^p$ (for fixed real p) converge? What does this imply if we replace p by a complex number?*

It converges when $p > 1$. If we replace p by a complex number it converges on the right half plane. More formally to determine whether we have convergence for $p \in \mathbb{C}$ we note that

$$n^p = n^{x+iy} = e^{(x+iy)ln(n)} = e^{xln(n)}e^{iyln(n)}$$

and thus

$$|n^p| = |e^{xln(n)}| = |(e^{ln(n)})^x| = |n^x| = |n|^x$$

meaning that the above sum converges absolutely when $\Re(p) > 1$.

Question 4.2. *Consider the series*

$$1 - 1/2 + 1/3 - 1/4 + \ldots$$

What does this converge to? Can you show why this converges? Can you rearrange the series to converge to something else? What's a general statement about the convergence of alternating series?

Q: What does this converge to? Can you show why this converges?
A: It converges to $ln(2)$. Namely, note that $ln(1+x) = \int \frac{1}{1+x}dx = \int \left(\sum_{n\geq 0}(-1)^n x^n\right)dx = \sum_{n\geq 0}(-1)^n(\int x^n dx) = \sum_{n\geq 0}(-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ for $-1 < x \leq 1$. So, plugging in $x = 1$ we see that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = ln(1+1) = ln(2)$.

Q: Can you rearrange the series to converge to something else?
A: Yes. Since the series converges conditionally but does not converge absolutely, we can rearrange it to converge to any real number. Intuitively, the reason for that is that the fact that the series converges conditionally means that the tails $T_n = S - S_n$, where S_n is the nth partial sum, go to zero. However, the fact that the series doesn't converge absolutely means that in each tail T_n there are infinitely many positive terms and infinitely many negative terms and, what's more, the sum of the positive terms in the nth tail is infinite and the sum of the negative terms in the nth tail is infinite. So, in each tail the sum of the positive terms and the sum of the negative terms are both infinite but the absolute value of the individual terms goes to zero as $n \to \infty$.

Intuition: The fact that we have an infinite amount of positive and negative value to add means that we can reach far enough out to get to any real number. The fact that the terms get arbitrarily small in absolute value ensures that once we reach a number and choose terms in a smart way we can stay at that value and converge to it.

Now, to explain the above process with the right balance of "precision" and simplicity for a group of engineering students in required calculus course TA'd by graduate students: we note that in order to reorder the series to converge to some real $a > 0$ we do as follows. Label our series as $a_n := (-1)^{n+1}\frac{1}{n}$ for $n \geq 1$ and thus $S_n := \sum_{k\in[1:n]} a_n$. We reorder the series to converge to a by going through the terms in order adding each positive term we encounter until our new partial sum has surpassed a, then returning to the start of the original list, adding the negative terms we passed over until we have dipped below a, and then we return to the start of the list of unadded terms and scan until we reach the next unadded positive term and add those until we surpass a again, whence we return to the start of the list and scan until we reach the next unadded negative term and add those until we dip below a, and so on. This will produce a series converging to a. Namely, the fact that we have an infinitely amount of positive value and negative value to add at each step in our process ensures that we can carry out this process indefinitely, namely since it ensures once we dip below we always have enough positive value to add to get back above a and vice versa, and the fact that $|a_n| \to 0$ as $n \to \infty$ ensures that our series converges to a.

Q: What's a general statement about the convergence of alternating series? A: An alternating series converges if and only if its terms decrease in absolute value. The intuitive reason for this is that we can group terms in pairs to see that the series converges.

Question 4.3. *Evaluate the limit as n goes to infinity of*

$$\frac{1}{n} + \frac{1}{n+1} + \ldots + \frac{1}{2n}.$$

It is $ln(2)$. Namely the above sum is a Riemann sum whose limit is a Riemann integral. (For the interested undergraduate: The easy way to recognize a sum as a Riemann sum approximating an integral is to note that (1) the number of terms changes as n changes (as opposed to just the value of each term) and that (2) there is some clear way to factor $\frac{1}{n}$ out of each term). Note that $\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} = \frac{1}{n}\frac{1}{1+0} + \frac{1}{n}\frac{1}{1+\frac{1}{n}} + \frac{1}{n}\frac{1}{1+\frac{2}{n}} + \ldots + \frac{1}{n}\frac{1}{1+1} = \sum_{k\in[0:n]} \frac{1}{n}\frac{1}{1+\frac{k}{n}}$ and thus $\lim_{n\to\infty}(\frac{1}{n} + \frac{1}{n+1} + \ldots + \frac{1}{2n}) = \int_0^1 \frac{1}{1+x}dx = ln(1+x)\Big|_{x=0}^{1} = ln(2)$.