Advanced Graph Theory

Lecture notes scribed by Caitlin Beecham

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1 An application of the regularity lemma to Ramsey Theory

Definition: For all graphs H, let $R(H) := \min\{n \in \mathbb{N} : |V(G)| \geq n \text{ implies } H \subseteq G \text{ or } H \subseteq \bar{G}\}$.

We note that such an R(H) exists since if $r = |V(H)|$, then $H \subseteq K_r$ and by Ramsey's Theorem we know that $R(r, r) := R(K_r, K_r)$ exists. (For those unfamiliar, $R(H_1, H_2) = \min\{n \in \mathbb{N} | |V(G)| \geq n \text{ implies } H_1 \subseteq G \text{ or } H_2 \subseteq \bar{G}\}$).

Usually, $R(r, r)$ is exponential in r, but if we have H with bounded maximum degree, then we can show that $R(H)$ is linear in $|V(H)|$.

Theorem: (Chvatal, Rodl, Szemeredi, and Trottor 1983)

For all $\Delta \geq 1$, there exists a constant $C = C(\Delta)$ such that $R(H) \leq C|V(H)|$ whenever $\Delta(H) \leq \Delta$.

Idea of the proof:

- Take G.

- Find an ϵ-regular partition of G.

- Construct the corresponding regularity graph, R, with no density threshold (just $d = 0$).

- Show that (since we chose the right parameters) such R has a K_m where m is at least $R(\Delta + 1)$.

- With this K_m, we can get that any two coloring of it (which will arise from coloring an edge a certain color if the density of the corresponding pair was ≥ 0.5 or the other color if the density was < 0.5) has a monochromatic clique of size $\Delta + 1$.

- Basically what we're doing is finding the desired C.

Proof:

Let $d = 0.5$ and $\Delta \geq 1$ be given. Then, there exists $\epsilon_0 > 0$ such that the embedding lemma applies to ϵ-regular partitions with parameters $\epsilon \leq \epsilon_0$, $l \geq \frac{s}{\epsilon_0}$, and $d = 0.5$. Intuitively, we want to make n large enough such that any graph with n vertices admits an ϵ-regular partition with those parameters. Namely, we want n large enough such that $K_m \subseteq R$ where $m \geq R(\Delta + 1, \Delta + 1)$.

Let $m := R(\Delta + 1, \Delta + 1)$ and $\epsilon > 0$ such that $\epsilon \leq \epsilon_0$ and $2\epsilon < \frac{1}{m} - \frac{1}{m-1}$. We then apply Szemeredi's Regularity lemma to get a partition. Namely, we know that there exists $M = M(\epsilon, d)$ such that any graph G with $|V(G)| \geq m$ has an ϵ-regular partition $\{V_0, V_1, \ldots, V_k\}$ with exceptional set V_0 and $m \leq k \leq M$. We then throw away V_0 and look at $V_1, \ldots, V_k$. We choose $s := |V(H)|$. (The intuition here is that we might find H embedded completely within V_i. It's unlikely but could happen). Then, $l = |V_1| = \cdots = |V_k|$. We see that $l = \frac{n - |V_0|}{k} \geq \frac{(1-\epsilon)n}{|M|}$. We want that $l \geq \frac{s}{\epsilon_0}$. So, we choose $n = \lceil \frac{M}{(1-\epsilon_0)\epsilon_0} = \lceil \frac{M}{(1-\epsilon_0)\epsilon_0} \rceil |H|$. We set $C := \lceil \frac{M}{(1-\epsilon_0)\epsilon_0} \rceil$. We will see that a graph with at least n vertices has either H as a subgraph or its complement does.

Our initial regularity graph R, will correspond to all ϵ-regular pairs (with any density). The intuition here is that even if a pair has small density, the corresponding pair in the complement will have large density.

Let R denote the regularity graph with parameters $\epsilon, l, d = 0$. We want to show that R contains a complete subgraph on m vertices. We note that since our partition is ϵ-regular, the number of pairs which are not ϵ-regular is less than ϵk^2, which means that $e(R) \geq \binom{k}{2} - \epsilon k^2 = \frac{1}{2}k^2(1 - \frac{1}{k} - 2\epsilon) > \frac{1}{2}k^2 \frac{m-2}{m-1}$. Thus, by Turan's Theorem $K_m \subseteq R$.

Let $R_1 :=$ the subgraph of R induced by the edges of R which correspond to regular pairs of density $\geq \frac{1}{2}$. Let $R_2 :=$ the subgraph of R induced by edges of R corresponding to pairs of density $< \frac{1}{2}$. Note that $E(R) = E(R_1) \sqcup E(R_2)$, which means that $E(R_1), E(R_2)$ is a 2-edge coloring of R, which then induces a 2-edge coloring on the subgraph $K_m \subseteq R$. By Ramsey's Theorem, we know that $K_{\Delta+1} \subseteq R_1$ or $K_{\Delta+1} \subseteq R_2$ (because we defined $m := R(\Delta + 1, \Delta + 1)$ and $k \geq m$). If $K_{\Delta+1} \subseteq R_1$, then $H \subseteq R_1$ and $\phi(H) \subseteq \phi(R_1) \subseteq R^s$ (ϕ is the blow up map from R to R^s). Hence, the embedding lemma gives $H \subseteq G$. Otherwise if $K_{\Delta+1} \subseteq R_2$, then $H \subseteq R_2$ and $\phi(H) \subseteq \phi(R_2)$ which gives $H \subseteq \bar{G}$ and the proof is done. Note that $n = C|H|$ when $C = \lceil \frac{M}{(1-\epsilon_0)\epsilon_0} \rceil$ depends on Δ only.

2   Bounds on traditional Ramsey Numbers

We will see that $2^{\frac{1}{2}r} \leq R(r,r) \leq 2^{2r}$. If we let $n := R(r,r)$ and then want to write r (the size of the largest clique or clique in the complement we are guaranteed) in terms of n, we get that $r \approx \log(n)$. We ask: what conditions can we impose on a class of graphs to get a larger r? What if we forbid certain subgraphs or induced subgraphs?
Erdös and Hajnal conjectured that if we forbid certain subgraphs ([these were not stated explicitly]) then we can raise $r(n)$ to a polynomial function of n.

Definition: Given graphs H_1, H_2, let $R(H_1, H_2) := \min\{n \in \mathbb{N} : |V(G)| \geq n$ implies $H_1 \subseteq G$ or $H_2 \subseteq \bar{G}\}$. Once again, this number exists because the number $R(p,q) = R(K_p, K_q)$ exists as proved by Ramsey.

Example: What is $R(p,2)$? We see that $R(p,2) = p$, since if $|V(G)| = p$, then either G is a clique or it is missing some edge, which means that the complement has an edge (a K_2). Similarly, $R(2,q) = q$.

It can be shown that for $p, q \geq 3$, the following recursive formula holds. Namely, $R(p,q) \leq R(p-1,q) + R(p,q-1)$. Why? Consider a 2 edge coloring of K_n where $n := R(p-1,q) + R(p,q-1)$. Pick a vertex v. Look at the edges adjacent to it. It has at least $R(p-1,q)$ adjacent red edges or at least $R(p,q-1)$ adjacent blue edges. Look at the graph induced by the vertices adjacent to v via a red edge. This graph has at least $R(p-1,q)$ vertices, which means that it has a red K_{p-1} or a blue K_q. If it has a blue K_q, we are done. Otherwise it has a red K_{p-1}, which together with v, forms a red K_p and we are done. (The corresponding case for when v is adjacent to at least $K(p,q-1)$ blue edges is similar).

Corollary: $R(p,q) \leq \binom{p+q-2}{p-1}$. We get this bound from the recursive formula for $R(p,q)$ with base cases $R(2,q) = q$ and $R(p,2) = p$.

Fact: $R(p,p) \leq \binom{2p-2}{p-1} \leq 2^{2p}$ (where the last inequality arises from Sterling's formula).

So, we have successfully constructed an upper bound on the diagonal Ramsey Numbers and next class we will construct a lower bound.

Definition: $R(p,g) :=$ the least integer n such that for any graph G on at least n vertices at least one of the following holds: $K_p \subseteq G$ or $K_q \subseteq \bar{G}$.

We already showed an upper bound on the diagonal Ramsey number $R(p,p)$. We now give a lower bound.

Claim: $2^{\frac{p}{2}} < R(p,p)$ for all integers $p \geq 4$.
Proof: The proof is based on a probabilistic argument. Set $n := 2^{\frac{p}{2}}$. Let $G(n, \frac{1}{2})$ be a randomly constructed graph on n vertices where any edge exists with probability $\frac{1}{2}$. We use the notation $V(G(n, \frac{1}{2})) =: \{1, 2, \ldots, n\}$. What is $Pr(G(n, \frac{1}{2}) \supseteq K_p)$? Well,

$$
\begin{aligned}
Pr(G(n, \frac{1}{2}) \supseteq K_p) &= Pr(\bigvee_{S \in \binom{[n]}{p}} G[S] \cong K_p) \\
&\leq \sum_{S \in \binom{[n]}{p}} Pr(G[S] \cong K_p) \\
&= \sum_{S \in \binom{[n]}{p}} \binom{n}{p}(\frac{1}{2})^{\binom{p}{2}} \\
&< \frac{n^p}{p!} 2^{-\binom{p}{2}} \\
&\leq \frac{(2^{\frac{p}{2}})^p}{p!} 2^{-\binom{p}{2}} \\
&\leq \frac{2^{\frac{p^2}{2}}}{2^{p+1}} 2^{\frac{-(p^2-p)}{2}} \leq \frac{1}{2}
\end{aligned}
$$

Now, because the probability of an edge is the same as the probability of a non-edge we also have that $Pr(\alpha(G(n, \frac{1}{2})) \geq p) < \frac{1}{2}$. So, with probability greater than zero, $\omega(G(n, \frac{1}{2})) < p$ and $\alpha(G(n, \frac{1}{2})) < p$, which means that in the frequentist sense of probability, there is at least one graph on n vertices with neither a K_p as a subgraph or an independent set of size p. Thus, $R(p,p) > 2^{\frac{p}{2}}$.

Notation: The homogeneous number of a graph, denoted $hom(G) = max\{\omega(G), \alpha(G)\}$. The upper bound on the Ramsey number $R(p,p) < 2^{2p}$ gives us the following lower bound on the homogeneous number $hom(G) \geq \frac{\log_2(n)}{2}$. Also, the lower bound on the Ramsey number $2^{\frac{n}{2}} < R(p,p)$ gives us the fact that there exist graphs with $hom(G) \leq 2\log_2(n)$. Such graphs are called "Ramsey graphs" or, by some, "Ramsey-like graphs".

Comment: Erdös and Hajnal studied Ramsey-like graphs in the 70's and they observed that the Ramsey-like graphs should have any small subgraph as an induced subgraph, which motivates the following definition.

Definition: given a positive integer ℓ, a graph G is said to be ℓ-universal if for all graphs H on ℓ vertices, G contains H as an induced subgraph.

Intuition: If look at increasingly large Ramsey graphs, they should be l-universal with increasing l.

Idea: If we exclude some subgraphs from a graph, then it cannot be l-universal.

Theorem [Erdos, Hajnal 1989]:

Let $\ell \in \mathbb{N}$ and $0 < c < \frac{1}{\ell}$. Then, there exists $n_0 = n_0(\ell, c)$ such that if G is a graph on at least n_0 vertices and $k < e^{\frac{c\sqrt{\log(n)}}{2}}$, then G is ℓ-universal or $hom(G) \geq \lfloor e^{\frac{c\sqrt{\log(n)}}{2}} \rfloor$.

Conjecture [Erdos Hajnal formally asked in 1989, question first appeared in 1977 paper]:

For all graphs H, there exists $\epsilon = \epsilon(H)$ such that for all large n and all H-free graphs G on n vertices, $hom(G) \geq n$.

Note: there are equivalent statements of this conjecture: one stated in the language of tournaments and another that has to do with whether a graph on n vertices contains a perfect subgraph with n^c vertices (where c is some constant which depends on certain parameters). As stated in its original form, mathematicians have more understanding of what happens for small H. However, when stated in terms of tournaments, it is easier to understand some cases where H can be large.

Definition: A graph G is "perfect" if for all induced subgraphs $L \subseteq G$ $\chi(L) = \omega(L)$.

Fact: For perfect graphs G, $hom(G) \geq |V(G)|^{\frac{1}{2}}$. Why? G perfect implies that in particular $\chi(G) = \omega(G)$. Take some coloring of G on a minimum number of colors. The color classes form a partition of G into $\chi(G)$ independent sets, each with size at most $\alpha(G)$. Thus, $|V(G)| \leq \chi(G)\alpha(G) = \omega(G)\alpha(G)$. Namely, $n \leq \omega(G)\alpha(G)$ which means that $max\{\alpha(G), \omega(G)\} \geq |V(G)|^{\frac{1}{2}}$.

Idea: Perfect graphs are well understood and when we solve this problem we might want an intermediate step. A good intermediate step here is to look for a perfect graph of a certain size, which will then contain an independent set or complete graph of a certain size. In particular, some researchers have narrowed their focus to subclasses of perfect graphs which can be constructed from 1 vertex graphs through some series of fixed operations. These graphs seem easier to work with.

Observation: If $H = K_p$, then we know that $R(p,q) \leq \binom{p+q-2}{p-1} \leq q^p$. (If we think of p as fixed, then the right hand side is a polynomial in p). If we think of $n = q^p$, then this expression translates to a lower bound on $hom(G)$. Namely, we get that for all H-free graphs G on n (large) vertices, $hom(G) \geq q \geq n^{\frac{1}{p}}$. Hence, the Erdös Hajnal Conjecture holds when H is complete or an independent set.

Idea: We have a graph H on ℓ vertices. If G is not ℓ-universal (which means that it is H-free for some H), we want to show that $hom(G)$ is close to k.

So, we want to embed H into G in a similar way to the blowup lemma used in the regularity lemma. Intuitively, since we want H as a subgraph, we want to be sure that whenever $uv \in E(R)$ (where R is our regularity graph) that we have high enough density in the corresponding pair so that the blowup vertices are neighbors. However, this is not enough. In this instance, we also want the subgraph to be induced. So, we need neighbors to correspond to neighbors but also non neighbors to correspond to non neighbors. In other words, "non-edges can't become edges". So, we break G into ℓ disjoint sets of vertices each with size $|V_i| \geq \lfloor \frac{n}{l} \rfloor$. So the idea is that if we have H as a subgraph of R, we want to blowup to G to get induced copies of H in G.

1. Choose u_1 from V_1 such that $|N(u_1) \cap V_j| \geq C|V_j|$ if $v_1 v_j \in E(H)$ or such that $|V_j \setminus (N(u_1) \cap V_j)| \geq C|V_j|$. If we can find such a u_1, we proceed to pick a u_2. If not, we will see that something else happens that ends up being helpful in a different way.