CS 6505 - Homework 7

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2

We show that $max_{p\in P}min_{e\in p}c_e \leq min_{c\in C}max_{e\in c}c_e$ and $max_{p\in P}min_{e\in p}c_e \geq min_{c\in C}max_{e\in c}c_e$ by contradiction. First, assume $max_{p\in P}min_{e\in p}c_e > min_{c\in C}max_{e\in c}c_e$ for some graph G. Let G be some minimal graph with that property (one such that removing any edge or vertex results in a graph in which $max_{p\in P}min_{e\in p}c_e \leq min_{c\in C}max_{e\in c}c_e$). Now, say we remove an edge, f, from G. This edge, f, may be contained in some s-t cuts c. It holds that the new value $max_{e\in c\setminus\{f\}}c_e \leq max_{e\in c}c_e$ for all $c \neq \{f\}$ (since if the edge f was an edge of maximum capacity the max capacity edge within this cut may decrease. However, it will certainly not increase). Now, one also must notice that f itself may have been an s-t cut. If (case 1) f did not attain the optimal value of $min_{c\in C}max_{e\in c}c_e$, then we have that $min_{c'\in C'}max_{e\in c'}c_e \leq min_{c\in C}max_{e\in c}c_e$ (where C' is the set of s-t cuts in $G - \{f\}$ and c' are the cuts in C') because each value $max_{e\in c'}c_e \leq max_{e\in c}c_e$ and the minimum $min_{c\in C}max_{e\in c}c_e$ was obtained by some other edge still in the graph after removal of f. However, if f attained the optimal value of $min_{c\in C}max_{e\in c}c_e$ and $\{f\} \in C$ (meaning it is itself an s-t cut), then (case 2) the value $min_{c\in C}max_{e\in c}c_e$ may increase. What could happen to the value $max_{p\in P}min_{e\in p}c_e$ after the removal of f? For all p containing f, the value $min_{e\in p}c_e$ may increase (if f attained the unique minimum c_e) or stay the same. If there exists a path q with only one edge, f, and f attains the optimal value (case 3) $max_{p\in P}min_{e\in p}c_e$, then the value $max_{p'\in P'}min_{e\in p'}c_e \leq max_{p\in P}min_{e\in p}c_e$ may decrease. If not, ie if f itself is not such a path with f attaining the optimal value, (case 4), the value $max_{p'\in P'}min_{e\in p'}c_e \geq max_{p\in P}min_{e\in p}c_e$ is non-decreasing. So, say we have case 1 and case 4. Then, we have that $max_{p'\in P'}min_{e\in p'}c_e \geq max_{p\in P}min_{e\in p}c_e > min_{c\in C}max_{e\in c}c_e \geq min_{c'\in C'}max_{e\in c'}c_e \geq min_{c'\in C'}max_{e\in c'}c_e$, which implies that $G' = G - \{f\}$ is still a counterexample, which implies that our original counterexample G wasn't minimal, a contradiction. Now, say we have case 1 and case 3. We have that $max_{p\in P}min_{e\in p}c_e > min_{c\in C}max_{e\in c}c_e \geq min_{c'\in C'}max_{e\in c'}c_e$. Either, $max_{p'\in P'}min_{e\in p'}c_e > min_{c'\in C'}max_{e\in c'}c_e$, in which case, we once again get a contradiction to minimality or $max_{p'\in P'}min_{e\in p'}c_e \leq min_{c'\in C'}max_{e\in c'}c_e$. Then, we know by the fact that we are in case 1, f was not the unique minimum among the maximum capacities in each cut. So, there is some cut whose maximum capacity edge is less than or equal to c_f. Take such a cut that minimizes the max capacity edge. We also know by the fact that we are in case 3 that it is the case that $\{f\}$ itself is an s-t path and the value $c_f = max_{p\in P}min_{e\in p}c_e$ with f being the unique edge attaining that maximum value. In this case, mimicking what we did in class, take this cut $E(S,\bar{S})$ which minimizes the max capacity edge where S is the component containing s and $\bar{S}$ is the component containing t. Contract $\bar{S}$ to a point $\bar{t}$ to get $\tilde{G}$. In this graph, some s-t cuts in the original graph may decrease in size (ie lose some edges), which means that $min_{\tilde{c}\in\tilde{C}}max_{e\in\tilde{c}}c_e \leq min_{c\in C}max_{e\in c}c_e$ unless there existed some cut consisting of a single edge which no longer exists in $\tilde{G}$ which was the unique edge attaining $min_{c\in C}max_{e\in c}c_e$. However, that cannot happen as we chose our cut to minimize the max capacity edge. So, we have that $min_{\tilde{c}\in\tilde{C}}max_{e\in\tilde{c}}c_e \leq min_{c\in C}max_{e\in c}c_e$. Also, f was the unique edge attaining $c_f = max_{p\in P}min_{e\in p}c_e$ and f still exists in $\tilde{G}$, so $c_f = max_{p\in P}min_{e\in p}c_e = max_{\tilde{p}\in\tilde{P}}min_{e\in\tilde{p}}c_e$. By the fact that $\tilde{G}$ is a subgraph strictly smaller than G, it is no longer a counterexample. So, $c_f = max_{p\in P}min_{e\in p}c_e = max_{\tilde{p}\in\tilde{P}}min_{e\in\tilde{p}}c_e \leq min_{\tilde{c}\in\tilde{C}}max_{e\in\tilde{c}}c_e$ and $min_{\tilde{c}\in\tilde{C}}max_{e\in\tilde{c}}c_e \leq min_{c\in C}max_{e\in c}c_e$ implies that $c_f = max_{p\in P}min_{e\in p}c_e \leq min_{c\in C}max_{e\in c}c_e$ so that G wasn't a counterexample to start with, a contradiction. Now, say we have case 2 and case 4. Since we are in case 2, we know that f attained the optimal value of $c_f = min_{c\in C}max_{e\in c}c_e$ with $\{f\}$ an s-t cut. Contract this cut. Call it $F(S,\bar{S})$ where S is the component containing s and $\bar{S}$ is the component containing t. Say G' is the contracted graph. It is strictly smaller so it is no longer a counter example, which means, $max_{p\in P \text{ in G'}}min_{e\in p \text{ in G'}}c_e \leq min_{c\in C \text{ in G'}}max_{e\in c \text{ in G'}}c_e$. Also, the fact we are in case 2 implies $min_{c\in C \text{ in G'}}max_{e\in c \text{ in G'}}c_e = min_{c\in C}max_{e\in c}c_e = c_f$. Case 4 implies that $max_{p\in P}min_{e\in p}c_e \leq max_{p\in P \text{ in G'}}min_{e\in p \text{ in G'}}c_e \leq min_{c\in C \text{ in G'}}max_{e\in c \text{ in G'}}c_e = min_{c\in C}max_{e\in c}c_e = c_f$. So, our original G wasn't a counterexample, a contradiction. Say we are in case 2 and case 3. This is the case where f attains the optimal value $c_f = min_{c\in C}max_{e\in c}c_e$ and it also obtained the optimal value $c_f = max_{p\in P}min_{e\in p}c_e$. In this case, since $c_f = c_f$ we get equality of $min_{c\in C}max_{e\in c}c_e = max_{p\in P}min_{e\in p}c_e$ in the original graph, a contradiction. So, we have shown that $max_{p\in P}min_{e\in p}c_e \leq min_{c\in C}max_{e\in c}c_e$. The reverse inequality, $max_{p\in P}min_{e\in p}c_e \geq min_{c\in C}max_{e\in c}c_e$, holds for the following reason. The LHS is the largest possible flow that can be sent along an augmenting path from s to t. We show that this value must be at least the RHS. If not, along an augmenting path containing the edge achieving the optimal value of the right hand side there is some smaller bottleneck, eg an edge w with $c_w < RHS$. Consider all edges r with $c_r < RHS$. By the optimality of the RHS, these edges do not form an s-t cut (since otherwise we would have an s-t cut whose max capacity edge has capacity less than the RHS, a contradiction). Delete these edges. Since G is still connected, there is a path, q, from s to t. Its edges all have capacity $c_f \geq RHS$ (since we deleted all edges whose capacities were less). Thus $min_{e\in q}c_e \geq RHS$ and thus $max_{p\in P}min_{e\in p}c_e \geq RHS$. Now, we are done.