Math 7014

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1 An application of the regularity lemma to Ramsey Theory

Definition: For all graphs H, let $R(H) := \min\{n \in N \| V(G)| \geq n$ implies $H \subseteq G$ or $H \subseteq \bar{G}\}$.
We note that such an R(H) exists since if $r = |V(H)|$, then $H \subseteq K_r$ and by Ramsey's Theorem we know that $R(r,r) := R(K_r, K_r)$ exists. (For those unfamiliar, $R(H_1, H_2) = \min\{n \in N \| V(G)| \geq n$ implies $H_1 \subseteq G$ or $H_2 \subseteq \bar{G}\}$).
Usually, $R(r,r)$ is exponential in r, but if we have H with bounded maximum degree, then we can show that $R(H)$ is linear in $|V(H)|$.
Theorem: (Chvatal, Rodl, Szemeredi, Trottor 1983)
For all $\Delta \geq 1$, there exists a constant $C = C(\Delta)$ such that $R(H) \leq C|V(H)|$ whenever $\Delta(H) \leq \Delta$.
Idea of the proof:

- Take G.
- Find an ϵ-regular partition of G.
- Construct the corresponding regularity graph, R, with no density threshold (just $d = 0$).
- Show that (since we chose the right parameters) such R has a K_m where m is at least a certain Ramsey Number which makes things work.
- With this K_m, we can get that any two coloring of it (which will arise from coloring an edge a certain color if the density of the corresponding pair was ≥ 0.5 or the other color if the density was < 0.5) has a monochromatic clique of size $\Delta + 1$.
- Basically what we're doing is constructing the desired C.

Proof:
Let $d = 0.5$, $\Delta \geq 1$, and $s \in N$ be given. Then, there exists $\epsilon_0 > 0$ such that the embedding lemma applies to ϵ-regular partitions with parameters $\epsilon \leq \epsilon_0$, $l \geq \frac{s}{\epsilon_0}$, and $d = 0.5$. Intuitively, we want to make such that n is large enough such that

we are guaranteed an ϵ-regular partition with those parameters. Namely, we want n large enough such that $K_m \subseteq R$ where $m \geq R(\Delta+1, \Delta+1)$. Let $m := R(\Delta+1, \Delta+1)$ and $\epsilon > 0$ such that $\epsilon \leq \epsilon_0$ and $2\epsilon < \frac{1}{m} - \frac{1}{m-1}$. We then apply Szemeredi's Regularity lemma to get a partition. Namely, we know that there exists $M = M(\epsilon, d)$ such that any graph G with $|V(G)| \geq M$ has an ϵ-regular partition $\{V_0, V_1, \ldots, V_k\}$ with exceptional set V_0 and $m \leq k \leq M$. We then throw away V_0 and look at $V_1, \ldots, V_k$. We choose $s := |V(H)|$. (The intuition here is that we might find H embedded completely within on V_i. It's unlikely but could happen). Then, $l = |V_1| = \cdots = |V_k|$. We see that $l = \frac{n-|V_0|}{k} \geq \frac{(1-\epsilon)n}{|M|}$. We want that $l \geq \frac{s}{\epsilon_0}$. So, we choose $n = \lceil \frac{SM}{(1-\epsilon_0)\epsilon_0} \rceil = \lceil \frac{M}{(1-\epsilon_0)\epsilon_0} |H| \rceil$. We set $C := \lceil \frac{M}{(1-\epsilon_0)\epsilon_0} \rceil$. We will see that a graph with at least n vertices has either H as a subgraph or its complement does.
Our initial regularity graph R, will correspond to all ϵ-regular pairs with any density. (The intuition here is that even if the density is small, the corresponding density in the complement will be large). Let R denote the regularity graph with parameters $\epsilon, l, d = 0$. We want to show that R contains a complete subgraph on m vertices. We note that since our partition is ϵ-regular, the number of pairs which are not ϵ-regular is less than ϵk^2, which means that $e(R) \geq (k \text{ choose } 2) - \epsilon k^2 = \frac{1}{2}k^2(1 - \frac{1}{k} - 2\epsilon) = \frac{1}{2}k^2\frac{m-2}{m-1}$. We want this quantity to be greater than $t_{m-1}k$ since that would imply that $K_m \subseteq R$. Let $R_1 :=$ the subgraph of R induced by the edges of R which correspond to pairs of density $\geq \frac{1}{2}$.
Let $R_2 :=$ the subgraph of R induced by edges of R corresponding to pairs of density $< \frac{1}{2}$. Note that $E(R) = E(R_1) \sqcup E(R_2)$, which means that $E(R_1), E(R_2)$ is a 2-edge coloring of R, which then induces a 2-edge coloring on the subgraph $K_m \subseteq R$. By Ramsey, we know that $K_{\Delta+1} \subseteq R_1$ or $K_{\Delta+1} \subseteq R_2$ (because we defined $m := R(\Delta+1, \Delta+1)$ and then choose n large enough that we got a $K_m \subseteq R$). If $K_{\Delta+1} \subseteq R_1$, then $H \subseteq R_1$ and $\phi(H) \subseteq \phi(R_1) \subseteq R^s$ (ϕ is the blow up map from R to R^s). Hence, the embedding lemma gives $H \subseteq G$. Otherwise if $K_{\Delta+1} \subseteq R_2$, then $H \subseteq R_2$ and $\phi(H) \subseteq \phi(R_2)$ which gives $H \subseteq \bar{G}$ and the proof is done). (Our C (which can be written in terms of H and Δ) works).

2 Bounds on traditional Ramsey Numbers

We will see that $2^{\frac{1}{2}r} \leq R(r,r) \leq 2^{2r}$. If I define $n := R(r,r)$ and then want to write r (the size of the largest clique or clique in the complement we are guaranteed) in terms of n, I get that $r \approx \log(n)$. We ask: what conditions can we impose on a class of graphs to get a larger r? What if we forbid certain subgraphs or induced subgraphs?
Erdos-Hajnal Conjecture: if forbid certain subgraphs ([these were not stated explicitly]) then can raise $r(n)$ to a polynomial function of n.
Definition: Given graphs H_1, H_2. Let $R(H_1, H_2) := \min\{n \in N | |V(G)| \geq n \text{ implies } H_1 \subseteq G \text{ or } H_2 \subseteq \bar{G}\}$. Once again, this number exists because the number $R(p,q) = R(K_p, K_q)$ exists as proved by Ramsey.

Example: What is $R(p, 2)$? We see that $R(p, 2) = p$, since if $|V(G)| = p$, then either G is a clique or it is missing some edge, which means that the complement has an edge (a K_2). Similarly, $R(2, q) = q$.
It can be shown that for $p, q \geq 3$, the following recursive formula holds. Namely, $R(p, q) \leq R(p-1, q)+R(p, q-1)$. Why? Consider a 2 edge coloring of K_n where $n := R(p-1, q) + R(p, q-1)$. Pick a vertex v. Look at the edges adjacent to it. It has at least $R(p-1, q)$ adjacent red edges or at least $R(p, q-1)$ adjacent blue edges. Look at the graph induced by the vertices adjacent to v via a red edge. This graph has at least $R(p-1, q)$ vertices, which means that it has a red K_{p-1} or a blue K_q. If it has a blue K_q, we are done. Otherwise it has a red K_{p-1}, which together with v, forms a red K_p and we are done. (The corresponding case for when v is adjacent to at least $K(p, q-1)$ blue edges is similar).
Corollary: $R(p, q) \leq \binom{p+q-2}{p-1}$. We get this bound from the recursive formular for $R(p, q)$ with base cases $R(2, q) = q$ and $R(p, 2) = p$.
Fact: $R(p, p) \leq \binom{2p-2}{p-1} \leq 2^{2p}$ (where the last inequality arises from Sterling's formula).
So, we have successfully constructed an upper bound on the diagonal Ramsey Numbers and next class we will construct a lower bound.