Translating Graph Operations to Polytope Operations

Caitlin Beecham

7/20/19 - Present

Motivation: Say I have a graph $G = (V, E)$, then I can make several polytopes from it. For instance, I can make polytopes in $R^{|V|}$ such as

- Independent Set Polytope
- Vertex Cover Polytope
- Min Vertex Cover Polytope.
- Independent Set Polytope for Graphic Matroid.

I can also construct polytopes in $R^{|E|}$ such as

- Hamiltonian Cycle Polytope
- Matching Polytope
- Perfect Matching Polytope

My question is: What do different graph operations like vertex deletion, edge deletion, edge contraction, vertex cuts, edge cuts, or associating a set of vertices to a single vertex translate into as maps on the polytopes (of whichever type e.g. Min Vertex Cover Polytope or PM polytope) before and after the operation?

Note: I can consider an "edge" type operations effect on both the polytopes in $R^{|V|}$ AND $R^{|E|}$.

I can ask: Are there maps from $R^{|V|}$ to $R|V| - 1$ which map the Independent set polytope of G to the independent set polytope of $G - \{v_0\}$? Are these linear maps? Are they "convex"?

I can also ask the same of edge deletion and edge contraction. (I think these will be the most fruitful).

NOTE: Once I have answered some of these questions I can then translate certain graph properties to polytope properties. For instance, I know that 3 connected graphs can all be contracted edge by edge down to K_4. What does that mean for certain polytopes on 3-connected graphs?

Also: What set of polytopes on a graph serve to determine the graph? Like, if I know the PM polytope of a graph, do I know the graph itself? Or do I need say the PM Polytope and also the Min Vertex Cover Polytope? Think about this.

So the idea is that if I want to characterize "polytopes of 3-connected graphs" I need to ask myself what is "the polytope of a 3-connected graph"? What polytope or pair of polytopes uniquely determines a graph? Could it be the Independent Set Polytope alone? Well, yeah I think perhaps. Note: for these questions I may need to ask, whether I need to know the actual matrix representation or just the polytope itself. For instance, I am thinking that if in order to uniquely determine a graph from two of its polytopes, say one in $R^{|V|}$ and another in $R^{|E|}$, I may need some way of relating these like knowing which rows in the polytope in $R^{|V|}$ (say the Independent Set Polytope which means the rows index the edges) correspond to which columns of the matrix representation of the polytope in $R^{|E|}$ (coordinates of $R^{|E|}$). It will be interesting to see when in this research one needs to know some certain cannonical representation of a polyhedron (say with the rows in some labeled order) versus just the solution set itself (regardless of representation).

Ok: one answer. If I just take the convex hull of independent sets of a labeled graph including incidence vectors of independent sets of size 1, it will always be the same–just the $|V|-$dimensional cube. Perhaps, I can consider only independent sets of size ≥ 2.

Note: It would be beneficial if lots of this research asks that one only know the actual polyhedron rather than a certain representation because for instance in the graph reconstruction and graph isomorphism problems we want to forget about the labels on the edges and vertices.

For instance: For graph reconstruction, it would be great if we figured out what vertex deletion and edge contraction meant for these polytopes and if these characterization did not depend on the representation of the polyhedron but only the polyhedron itself.

Thinking about Graph Isomorphism: What does permuting the labels of the vertices do to say the Independent Set Polytope? Or any of the other polytopes?

Also: Thinking about Graph Reconstruction, say I have a polytope in $Q \subseteq R^N$ on the set of N vertices of some graph. I can ask for which Independent Set Polytopes $P \subseteq R^{N+1}$ could one contract an edge and end up with Q?

Once I figure out which linear maps represent these graph operations, I can ask: what do the kernels of these maps tell us?

REALLY Important: Once I know what the linear maps for these graph operations are, I can ask: What if I make a "restricted polytope" and look at the image under these linear maps of JUST this restricted polytope then what do I get?

Like I could say the "Restricted Independent Set" Polytope of a graph $G = (V, E)$ with N vertices in R^N is one using only $N - 1$ vertices BUT STILL LIVING IN R^N. So, for instance we require that the Nth coord of x be 0 (I could also delete the inequalities arising from the edges incident to v_N, but I believe simply requiring that $x_N = 0$ has the same effect. Yes, for the independent set polytope it does).

I can then look at the map which corresponds to vertex deletion, edge deletion, or edge contraction and apply these maps to each of the restricted polytopes and examine their respective ranges. Note: one advantage of using this in Graph Reconstruction is that say I know ahead of time (obviously in Graph Reconstruction we do not) what our entire set of edges is (which gives us the inequalities of the independent set polytope which depend on the specific edges in the graph) I can then iteratively "delete vertices" by applying my linear map of deletion or by simply requiring that coordinate of my solution to the independent set polytope be 0. NOTE that these two alternate operations correspond to the exact distinction between knowing the original graph or not.

When we apply the map from $R^{|V|}$ down to $R^{|V|-1}$ this is akin to not knowing the original graph. However, if one simply requires the nth coordinate of $x \in R^{|V|}$ of the Independent Set Polytope be 0 (by adding an equality constraint), this is akin to still knowing the original graph (I mean we still have the matrix of Independent Set Polytope inequalities which I believe uniquely determines the graph).

If I add the $x_n = 0$ constraint to the independent set polytope,this means the dimension of the polytope WILL go down which is good and consistent with the fact that it also does when we use our linear map from $R^{|V|}$ to $R^{|V|-1}$ to represent this vertex deletion. So, reminder to myself: this polytope with the added equality constraint will now NOT be full dimensional.

I can ask: in general in some of these operations, the ones in which I remain in the same space, so $R^{|V|}$ or $R^{|E|}$ my polytope will shrink. What does the part of the polytope cut off tell us? What does the kernel of this map tell us?

Is the kernel of my vertex deletion map $\phi_1 : R^{|V|} \to R^{|V|-1}$ related in any meaningful way to the kernel of the vertex deletion map $\phi_2 : R^{|V|} \to R^{|V|-1}$?

I can also think simply about the dimension of the kernels of these maps rather than what the kernels are themselves.

Is there some way to quantify vertex or edge connectivity this way? Say as the dimension of the kernel of some map? That may work, so I should think about this.

I ask: If some series of these maps results in the zero polytope (meaning just the origin), what does that mean? If it results in an empty polytope (whose inequalities are infeasible), what does that mean? If they result in a non-empty polytope, which is not just a single point but still has 0 volume in whatever space I am in, what does that mean?

NOTE: Whether the polytope I am considering has zero volume in its current storing space may be a way for me to track whether this is actually a part of the graph or the whole graph. For instance, if all my maps are from $R^{|V|} \to R^{|V|}$ but like in vertex deletion of the Independent Set Polytope I simply require the coordinate of that vertex to be 0, then the dimension can track the number of vertices in my current graph. So, maybe I start with N vertices in R^N and maybe my Independent Set Polytope is full-dimensional, so of dimension N (note that I am not sure a priori it is full-dimensional; it may not be depending on the graph; I have to think about it), and when I delete a vertex it is of dimension N and so on and so forth.

It might be useful to ask: If I delete a vertex (or delete an edge or contract an edge) and the dimension of the polytope (of whatever type, Ind Set, Matching, etc) does not decrease, what does that mean? If it does decrease what does that mean?

Is there some connection between vertices for which dimension decreases or not and cut vertices say? Or maybe if there are no vertices then the vertices for which dimension of our polytope decreases after deletion of them this means that these vertices belong to SOME minimum vertex cut. Find out!

Question: Given a polytope (Ind Set, Matching, etc) of a graph, how do I know if the graph is connected?

Question: What interesting polytopes can I make from directed graphs?

TODO: Think about flow polytopes. I can also see what operations on flow polytopes look like and what all this stuff looks like about properties on directed graphs.

I can also flip all this around and say I'm just given a linear map, when does this linear map correspond to some graph operation? I mean the answer is probably almost never.

Question: How can I change coordinates (like permuting vertices)? Or even something really strange like the following: So, obviously say I am in $R^{|V|} = R^N$. I can use $v_i \in V$ (well, their indicator vectors) as my basis, OR I could change basis, to something like $(0.5, 1), (-1, 0.3)$ of R^2 (instead of $(1, 0), (0, 1)$). What if I then considered these new basis vectors and rewrote my linear program in terms of them? So somehow I am thinking of these new basis vectors as an alternate set of vertices of some LP relaxation of whatever my original IP problem was (like Min Vert Cover, Max Matching, etc). I can in fact, vary over all possible bases and see what the new representations of my polytope look like. (Note that my polytope itself is the same through all this. I am simply writing it using different bases which give different coordinate systems. Oh, no that doesn't really make sense. I mean: just think about the extreme points, so before the extreme point in the orig coordinate system was maybe $(0.5, 1)$ which was NOT integral, but now in my new system it is the first basis vector which means it is $(0, 1)$ which IS now integral).

Question: Can I write all this stuff compactly using change of base matrices and stuff?

Question: For which basis vectors does my polytope become integral? This actually translates pretty easily into trying to use the extreme points as basis vectors. However, this cannot be done easily as the number of extreme points for general polytopes can be WAY more than the dimension of the ambient space. For instance, take an octagon in R^2. I cannot use all the extreme points as basis vectors since there are 8 of them.

TODO: It would be nice to think about what it means when integrality is preserved under some map or when its not. What I am trying to do here is have some kind of "relaxation" of permuting the vertices. So, permuting the vertices is kind of like a change of basis where we simply have a change of basis matrix which is a permutation matrix. A relaxation of the idea of permuting vertices could be a different change of basis matrix which is not a permutation matrix. Then, maybe in the way one solves an LP relaxation to solve an IP, one can solve this relaxation to find the right permutation.

Like, more precisely. Say I am given two labeled graphs. And I want to find which permutation of the vertices of the 2nd graph gives us a graph closest to the first (using some reasonable metric on the distance between labeled graphs). (Here I am assuming they have the same number of vertices, but maybe I do not need to assume that). I might be able to instead of varying over all permutations rather vary over all changes of basis and see which gives the closest. This may be easier as this is a continuous process perhaps which means if things are "convex" this could actually work really well.

Now, I have to think does the change of basis actually permute the vertices? Let me work out an example.

So, consider the independent set polytope on P_3 given by:

$$\begin{aligned} x_1 + x_2 &\leq 1 \\ x_2 + x_3 &\leq 1 \\ 0 \leq x_i &\leq 1 \forall i \in [3] \end{aligned}$$

Say I change basis and write this polytope differently. What do I mean by that precisely? Say I decide to use the basis $x_1' = (0, 0, 1), x_2' = (0, 1, 1), x_3' = (1, 1, 1)$?

Then, I can write x_1, x_2, x_3 in terms of this new basis. Namely,

$$\begin{aligned}
x_1 &= x'_3 - x'_2 \\
x_2 &= x'_2 - x'_1 \\
x_3 &= x'_1
\end{aligned}$$

So, we can write our old polytope using this new basis as

$$\begin{aligned}
(x'_3 - x'_2) + (x'_2 - x'_1) &\leq 1 \\
(x'_2 - x'_1) + (x'_1) &\leq 1 \\
0 \leq (x'_3 - x'_2) &\leq 1 \\
0 \leq (x'_2 - x'_1) &\leq 1 \\
0 \leq x'_1 &\leq 1
\end{aligned}$$

or, cleaned up, we get

$$\begin{aligned}
-x'_1 + x'_3 &\leq 1 \\
x'_2 &\leq 1 \\
0 \leq -x'_2 + x'_3 &\leq 1 \\
0 \leq -x'_1 + x'_2 &\leq 1 \\
0 \leq x'_1 &\leq 1
\end{aligned}$$

Let's use another basis which is just a permutation.

Say $\hat{x}_1 = x_2, \hat{x}_2 = x_3, \hat{x}_3 = x_1$.

Then, if we write our original polytope using this new basis we get from the

ORIG

$P = \{x \in R^3 :$

$$\begin{aligned}
x_1 + x_2 &\leq 1 \\
x_2 + x_3 &\leq 1 \\
0 \leq x_i &\leq 1 \forall i \in [3]
\end{aligned}$$

$\}$

to the NEW $P' = \{x \in R^3 :$

$$\begin{aligned}
(\hat{x}_3) + (\hat{x}_1) &\leq 1 \\
(\hat{x}_1) + (\hat{x}_2) &\leq 1 \\
0 \leq \hat{x}_i &\leq 1 \forall i \in [3]
\end{aligned}$$

$\}$

NOTE that this polytope $P \neq P'$. I believe this happens if and only if our change of basis is NOT a valid graph automorphism. (Of course throughout all this I am really ASSUMING that the Independent Set Polytope of a Graph uniquely determins the graph, which intuitively I think that it should).

For instance, note that the change of basis $\tilde{x}_1 = x_3, \tilde{x}_2 = x_2, \tilde{x}_3 = x_1$ IS a valid graph automorphism of P_3 and I believe their independent set polytopes are the same.

Here we write the orig using our new basis. Here is the orig: $P = \{x \in R^3 :$

$$\begin{aligned} x_1 + x_2 &\le 1 \\ x_2 + x_3 &\le 1 \\ 0 \le x_i &\le 1 \forall i \in [3] \end{aligned}$$

$\}$

which we then write as $\tilde{P} = \{x \in R^3 :$

$$\begin{aligned} \tilde{x}_3 + \tilde{x}_2 &\le 1 \\ \tilde{x}_2 + \tilde{x}_1 &\le 1 \\ 0 \le \tilde{x}_i &\le 1 \forall i \in [3] \end{aligned}$$

$\}$

which has the EXACT same matrix rep as P. Both are $Ax \le b$ with

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and

$$b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and that is a valid graph automorphism.

Lemma 1: The Independent Set Polytope of a graph (labeled?, unlabeled?) uniquely determines the graph. (I believe I mean labeled independent set polytope deterines labeled graph).

Proof of Lemma 1:

Lemma 2 (Well, Conjecture): Say $G = (V, E)$ has Independent Set Polytope $P \subseteq R^{|V|}$ represented by $Ax \le b, 0 \le x_i \le 1$ for all $i \in V$ where each row A^e of A corresponds to an edge $e \in E(G)$ and is of the form $\left(A^e_j = 1 \text{ if } j \in e \text{ and } A^e_j = 0 \text{ otherwise} \right)$ so that $A \in R^{|E|}_{|V|}$ and the vector $b = 1 \in R^{|E|}$. Then, let $\pi \in \{0,1\}^{|V|}_{|V|}$ be a permutation matrix (namely one such that $\sum_{i \in [|V|]} A^i_k = 1$ for all $k \in [|V|]$ and $\sum_{i \in [|V|]} A^k_i = 1$ for all $k \in [|V|]$). Then, assuming Lemma 1, π is a valid graph automorphism if and only if

$P' := \{x \in R^{|V|}$:

$$(A\pi)x \le b$$
$$0 \le x_i \le 1$$

$\} = P$, the original polyhedron.

Put in terms of matrices, this is true if and only if $A\pi = A$. (Note: This looks different form the usual $PAP^T = A$ we're used to seeing because our $A \in R^{|E|}_{|V|}$ is an INCIDENCE matrix whereas the usual $PAP^T = A$ is talking about the ADJACENCY matrix).

PROOF:

Now, let's look at a relaxation, where π is ANY change of basis matrix. So $\pi \in R^{|V|}_{|V|}$ such that $det(\pi) \neq 0$. We are interested in whether $\pi P = P$, where

$$\pi P := \{x \in R^{|V|} :$$
$$A\pi x \le b$$
$$0 \le \pi x_i \le 1$$
$$\}$$

where x_i is the standard ith basis vector.

Say one defines a metric, $m : \{(G_1, G_2) : |V(G_2)| = |V(G_2)|\} \to \mathbb{R}$, on labeled graphs G_1, G_2 such that $m(G_1, G_2) = 0$ if and only if G_1, G_2 are isomorphic as labeled graphs, which one then translates into a metric on the respective independent set polytopes (NOTE this specifically means that the vertices are labeled HOWEVER this does NOT mean that the order of the edges (which are ineqs of the polytope) matters so we ONLY care about the polytope itself, NOT its representation. Though, we do care about the specific embedding into R^N. I mean that's the whole point we're permuting the basis vectors which is what's changing our embedding into R^N in that we're changing which vertex corresponds to which coordinate). TODO: Come up with this metric $M : (P, P') \to \mathbb{R}$ where $P, P' \subseteq R^{|V|}$ are polytopes.

Then, one would like to solve the following minimization problem which is

$$min_{\pi \in (R^{|V|}_{|V|})^\times} \{M(P, \pi P)\}$$

NOTE: In the above formula, I am assuming that the change of basis matrices are exactly those with non-zero determinant. Well, actually I just wrote invertible real matrices. Note, that if I instead look at Z^n later, I will need that my change of basis matrices are invertible over Z^n, not just R^n.

I then want to ask whether there is a $\pi \in$

$$min_{\pi \in (R^{|V|}_{|V|})^\times} \{M(P, \pi P)\}$$

such that π is a permutation matrix. If not, I would like to pick some "nearest" permutation matrix.

I could do this several ways. For instance I could use randomized rounding. (Analogous to, say I want to randomly round $5 + (3/4)$ to an integer. I round it to 6 with probability 1/4 and I round it to 5 with probability 3/4) I should be able to come up with something similar.

Note: To think about this minimization problem, I should think about what happens to polytopes when I first transform the input using a change of basis. In fact, I could see whether I can find some useful visualization tool on Wolfram Alpha.

Note: I belive it is true that the Indepedent Set Polytope of any Graph is an Integral Polytope which is the convex hull of the indicator vectors of its independent sets. TODO: check this. TODO: Examine for which π for which πP is integral, even if it is NOT equal to P. I believe this correspond to actual permutations of the vertices (most likely, it would be by very small chance they do not).

Here's a connection to probability:
Given a labeled graph G_0. I can look at its orbit under the set of all permutations of its vertices. For each labeled graph G_i in its orbit I can consider the number of permutations $n_i := \#\{\sigma \in S_{|V|}$ such that $\sigma(G_0) = G_i\}$. Then, taking a random permutation $\sigma \in S_{|V|}$ of the labeled graph G_0, one obtains $\sigma(G_0) = G_i$ with probability $\frac{n_i}{n!}$. This is a discrete probability. One could also look at a continuous probability.

Say among all change of basis matrices of determinant 1, one could look at the probability of getting out a certain polytope. I mean, they should all have probability almost 0, but maybe they are non-zero. I doubt it. I mean they may all be 0, though it is possible.

Let $I = \{\text{polytopes } P \subseteq R^N : convhull(P \cap Z^N) = P\}$ denote the set of integral polyhedra $P \subseteq R^N$.

One could ask, among the set $Q = \{\pi(P) : \pi \in R^{|v|}_{|V|} \text{ with } det(\pi) = 1\} \cap I$ of all integral polyhedra resulting from our changes of basis π, and all π in the set $S := \{\pi : \pi(P) \in Q\}$. One asks, for fixed $Q_0 \in Q$, say one chooses $\pi \in S$ uniformly at random, what is the probability that $\pi(P) = Q_0$.

1 Linear Algebra Related Ideas

1. I can look at the indicator vectors of sets of independent sets of G of size ≥ 2, consider them vectors in $R^{|V|}$ then look at their span over R. I can also look at their span over C. I can also look at their span over a field extension. What roots would I be adding? There is a polynomial associated with a graph called its characteristic polynomial. See page 87 of Handbook of Graph Theory. This seems to be a good idea! There is existing work on this! On decks and characteristic polynomials and reconstruction problem. Still open.

2. NOTE: it seems to be a good idea to connect linear algebra and polyhedral theory. They are closely related. Polyhedra are like cutting off affine subspaces which are translations of linear subspaces.

3. `https://mathoverflow.net/questions/24978/spectral-graph-theory-interpretability-of-eig`

2 Just graph theory ideas

Can I come up with a meaningful generalization of a biparite graph for hypergraphs (k uniform)? like maybe we have k parts and each edge has one vertex in each part.

Can I come up with an algorithm like Blossoms algorithm for hypergraphs? augmenting path? see 7/24/19 page A, need to be careful

characterize mathcing size for hypergraphs

3 Useful Sources: DO NOT REMOVE I REFERENCE THIS SECTION A LOT AS NOT A HYPERREF

1. This resources is great because they make some kind of polytope from a graph where the face poset corresponds to some poset of tubes (which satisfy among other properties disjoint or one contained in the other for every pair of tubes (like laminar family from Swati's class)). REALLY cool because this paper describes what happens when one changes the underlying graph–it describes exactly what happens to the polytope when one (1) contracts an edge or (2) deletes an edge. TODO: read or skim relevant parts of this and understand exactly what the point of the "tubing" process is here. Is it general enough to use on lots of classes or problems or is its scope very limited? `https://arxiv.org/pdf/1005.2551.pdf`

2. `http://www.sfu.ca/~mdevos/notes/misc/perf_graph.pdf`
   TFAE: (1) G perfect, (2) Ind set polytope (denoted P(G)) of G is integral (3) $\overline{G}$ is perfect. Also, a one way implication: ($P(G)$ integral) implies that ($P(G \setminus X)$ integral for all $X \subseteq V$).

3. `https://riliu.math.ncsu.edu/724/notesse5.html`.
   I often translate polytopes to graphs–this translates the other way.

4. `https://arxiv.org/pdf/1203.5175.pdf`
   Seems fun to read. Connects polytopes to proper edge colorings of a graph somehow. Could be useful, but not sure.

5. `https://arxiv.org/pdf/1107.0632.pdf`
   These guys may have implemented the idea I came up with. Graph Isomorphism polytopes are apparently a thing. I will play around on my own to understand since that will stick in my memory longer than reading the paper or perhaps play around then read. I assume they have not SOLVED GI however, since I would have heard that.

6. `https://arxiv.org/abs/math/0106093` "Abstract: We show that the problem to decide whether two (convex) polytopes, given by their vertex-facet incidences, are combinatorially isomorphic is graph isomorphism complete, even for simple or simplicial polytopes. On the other hand, we give a polynomial time algorithm for the combinatorial polytope isomorphism problem in bounded dimensions. Furthermore, we derive that the problems to decide whether two polytopes, given either by vertex

or by facet descriptions, are projectively or affinely isomorphic are graph isomorphism hard.
The original version of the paper (June 2001, 11 pages) had the title "On the Complexity of Isomorphism Problems Related to Polytopes". The main difference between the current and the former version is a new polynomial time algorithm for polytope isomorphism in bounded dimension that does not rely on Luks polynomial time algorithm for checking two graphs of bounded valence for isomorphism. Furthermore, the treatment of geometric isomorphism problems was extended."

7. `https://link.springer.com/article/10.1007/BF02240204`
TLDR: Graphs are ds-isomorphic (doubly stochastic isomorphic) if there's some kind of doubly stochastic matrix whose conjugate of one adjacency matrix is the other. (Like generalizations of permutation matrices–instead of having one 1 in each row/column, we have that the row and columns sums are both 1). "Main question:For which graphsG the polytope ofds-automorphisms ofG equals the convex hull of the automorphisms ofG?" "Abstract: Two graphs G and G having adjacency matrices A andB are called ds-isomorphic iff there is a doubly stochastic matrix X satisfying XA=BX. Ds-isomorphism is a relaxation of the classical isomorphism relation. In section 2 a complete set of invariants with respect to ds-isomorphism is given. In the case where A=B (ds-automorphism) the main question is: For which graphs G the polytope of ds-automorphisms of G equals the convex hull of the automorphisms of G? In section 3 a positive answer to this question is given for the cases where G is a tree or where G is a cycle. The corresponding theorems are analoga to the well known theorem of Birkhoff on doubly stochastic matrices."

8. `https://pdfs.semanticscholar.org/7c7d/a63dd2406fadb2c4eee6a1398845359b4380.pdf` TLDR: (Note: the following is very closely paraphrased from the abstract. Do not include anywhere without rewriting or using quotation marks). There is a ring structure called "coherent algebra" arising from the adjacency matrices of graphs which is closed under conjugate transposition and element-wise multiplication of matrices, also containing the identity matrix and that which consists entirely of 1's (identity for the element-wise multiplication operation). Coherent algebras are useful for testing graph isomorphism and also for finding the automorphism parition of a graph (which has to do with stable colorings). Weisfeiler and Leman gave an algorithm which finds the coherent algebra of an adjaceny matrix of a graph. For a while, no runtime complexity bound was known. This paper implements the algorithm called STABCOL and analyzes time complexity. They give detailed description, program listing, and instructions on how to use this algorithm.

9. "On Vertices and Facets of Combinatorial 2-Level Polytopes". Theorem (5) about stable set/clique trade-off in Section 3 looks useful. `https://infoscience.epfl.ch/record/222832/files/On%20vertices%20and%20facets%20of%20combinatorial%202-level%20polytopes.pdf`

10. Definition of the Permutahedron "is an (n1)-dimensional polytope whose faces are in bijection with the strict weak orderings on n letters. In particular, the n! vertices

of Pn correspond to all permutations of n letters" https://arxiv.org/pdf/1005.2551.pdf

4 Implemented Algorithms I Can Use

-
- https://pdfs.semanticscholar.org/7c7d/a63dd2406fadb2c4eee6a1398845359b4380.pdf TLDR: Finds Coherent Algebra of Adjacency Matrix–has to do with automorphism paritition and stable colorings. (Note: the following is very closely paraphrased from the abstract. Do not include anywhere without rewriting or using quotation marks). There is a ring structure called "coherent algebra" arising from the adjacency matrices of graphs which is closed under conjugate transposition and element-wise multiplication of matrices, also containing the identity matrix and that which consists entirely of 1's (identity for the element-wise multiplication operation). Coherent algebras are useful for testing graph isomorphism and also for finding the automorphism parition of a graph (which has to do with stable colorings). Weisfeiler and Leman gave an algorithm which finds the coherent algebra of an adjaceny matrix of a graph. For a while, no runtime complexity bound was known. This paper implements the algorithm called STABCOL and analyzes time complexity. They give detailed description, program listing, and instructions on how to use this algorithm.
- Here is a list of ways to use Matlab to characterize graphs using linear algebra. This stuff is really advanced, not trivial user guide or anything. There are actual proofs here. It might be useful. https://arxiv.org/pdf/1812.04379.pdf

5 Notes on Source 3 From Useful Source Section Above

https://riliu.math.ncsu.edu/724/notesse5.html.

- The 1-skeleton of a polytope is a graph. So, if polytope pointed (and bounded I suppose), make graph where verts are the verts of polytope and edges are the 1d faces.
- Balinski's Theorem: if P is a d dimensional Polytope, then its 1-skeleton graph G(P) is d-connected. Proof is included in the source.
- https://arxiv.org/pdf/1005.2551.pdf
- If d is 3 then, G(P) is planar.
- Also, if we take the polar (polytope of P) called P^* then $G(P^*)$ planar and the (sorry, a) dual planar graph of G(P). Recall, planar duals depend on the specific plane embedding.
- Some kind of converse to above: A graph with ≥ 4 edges is the 1-skeleton graph of a dim-3 polytope if and only if it is ((1) simple, (2) planar, (3) 3-connected). So, the

above is almost kind of an if and only if except we need to add simple. Perhaps the forward direction (P 3-dim implies planar G(P)) does not ALWAYS result in a simple graph. So simple is sufficient but not necessary for $G(P)$ to a be a 1-skeleton of a 3-dim polytope. Can come back and look at details. I was just guessing here.

- There is something here called a ΔY transformation which is some kind of edge contracting, subgraph replacement (replacing a triangle with a structure that looks like a Y) here. Seems cool. Note sure if on a 1-skeleton graph. Can come back and read.
- All for now, NOTE: if I click next at bottom of the page there are a series of pages which seem as useful and niche, yet accesible as this. COME LOOK LATER.

TODOS:

- Insert definition of "isomorphic as labeled graphs".
- Look over all notation and decide if I should add things like $I(n)$ if I depends on n but I didn't make note of it.
- TODO: read or skim relevant parts of this and understand exactly what the point of the "tubing" process is here. Is it general enough to use on lots of classes or problems or is its scope very limited? `https://arxiv.org/pdf/1005.2551.pdf`
- Pick a couple sections of papers to read (but be very purposeful about which I choose– I want some cohesive direction of where I am heading this semester. Today my goal is to start figuring out what that might be). I want to make some incremental progress on an open problem by the end of the semester. It does not need to be much, but I want at least one new small result in one of these subfields I really like so that Professor Thomas might be willing to give me another RA. I should also present relevant sections in the graph theory working group to demonstrate the fact that I have been using my time well.
- Understand Tutte Polynomials. `https://en.wikipedia.org/wiki/Tutte_polynomial`. I like how this has to do with connectivity and matroids. Once I understand this (apparently it encodes connectivity of a graph). I can look at polytopes associated with whatever the relevant matroid is.
- Order this book from the library: Godsil, Chris; Royle, Gordon (2004), Algebraic Graph Theory, Springer, ISBN 978-0-387-95220-8. Chapter 15.
- Look in the Handbook of Graph Theory for accessible open problems that might relate to these topics I like. Note: I don't love the field theory/polynomial stuff THAT much. I am just interested in looking at vector spaces over these fields (where I adjoin roots of these polynomials). I think it could be interesting to ask myself what roots of these polynomials mean.
- Read that math stack exchange answer on what eigenvalues of the adjacency matrix of a graph mean and at least understand some of the answers which seem interesting.

(It seems to me that in the past whatever has deeply interested me even if it didn't seem useful at the time would have been useful to learn at the time it interested me).

- Order this book from the library: Bollobs, Bla (1998), Modern Graph Theory, Springer, ISBN 978-0-387-98491-9. Chapter 10.
- Order this book from the library: Biggs, Norman (1993), Algebraic Graph Theory (2nd ed.), Cambridge University Press, ISBN 0-521-45897-8. Chapter 13.

Questions:
What does the intersection of two matching polytopes mean graph-wise? Union of two polytopes? When one takes one of these polytopes and deletes edges until a graph is disconn