Assignment 3

ISYE 7686 - Advanced Combinatorial Optimization

April 13, 2019

Due Date: April 12, 2019.

In preparing solutions for this assignment, please abide by the Georgia Tech academic honor code. We will not tolerate copying of answers. Discussions are encouraged but the final answer must be based upon your understanding of the question. Also write the names of the people you discussed the questions with.

Q1. Let $G = (V, E)$ be a graph. Let $\mathcal{I}$ be the collection of those subsets Y of E so that Y has at most one circuit. Show that $(E, \mathcal{I})$ is a matroid.

We need to show (1) $\emptyset \in I$, (2) $I \in \mathcal{I}$ and $J \subseteq I$ implies that $J \in \mathcal{I}$, and (3) $I, J \in \mathcal{I}$ and $|I| < |J|$ implies that there exists $j \in J \setminus I$ such that $I \cup \{j\} \in \mathcal{I}$. Clearly, $\emptyset \in I$ since the set of no edges does not have a cycle. Also, $I \in \mathcal{I}$ means that I has at most one cycle, and then $J \subseteq I$ means that J also has at most one cycle since J is obtained from I by some deletion of edges and deleting edges cannot create a cycle. Finally, say $I, J \in \mathcal{I}$ and $|I| < |J|$. So, I and J each contain at most one cycle. Now, assume for contradiction that for all $j \in J \setminus I$, $I \cup \{j\}$ has at least 2 cycles.

To start, this must mean that I had a cycle to begin with. Otherwise, if it did not, then note that adding one edge may create at most one cycle. Why? Assume adding an edge e to a graph I (some abuse of notation here since I is a set of edges) with no cycles created two cycles C_1, C_2. Then, $e \in C_1$ and $e \in C_2$. However, then note that $C_1 \setminus e$ and $C_2 \setminus e$ are both paths (not necessarily disjoint) with the same endpoints u, v (where e is the edge uv). Denote $C_1 \setminus e =: P_1 =: uw_1w_2 \dots w_kv$ and $C_2 \setminus e =: P_2 =: vr_1r_2 \dots r_lu$. Now, either P_1 and P_2 are internally disjoint, in which case, $P_1 \cup P_2$ is a cycle in the original graph, a contradiction, or they are not internally disjoint. Then, let i be the minimum index $i \in [k]$ such that $w_i \in P_2$ and let j be the maximum index $j \in [l]$ such that $r_j \in P_1$. If $i > 1$ or $j < l$, then $uw_1w_2 \dots w_i(= r_j)r_{j+1} \dots r_lu$ is a cycle in the original graph, a contradiction. Otherwise, if $i = 1$ and $j = l$, instead let i' be the maximum index $i' \in [k]$ such that $w_{i'} \in P_2$ and let j' be the minimum index $j' \in [l]$ such that $r_{j'} \in P_1$, then if $j' > 1$ or $i' < k$, one has that $vr_1r_2 \dots r_{j'}(= w_{i'})w_{i'+1} \dots w_kv$ is a cycle in the original graph, a contradiction. So, we have deduced that $i = 1$, $j = l$, $i' = k$, $j' = 1$. Now, let if such an i'' exists let i'' be the minimum index for which $w_{i''} \in P_2$ and $w_{i''+1} \notin P_2$ and i''' be the minimum index with $i''' > i''$ and $w_{i'''} \in P_2$. By what we have deduced, $i'' > 1$ and also $i''' \leq k$. Now, $w_{i''}, w_{i'''} \in P_2$ so there exist $j'', j''' \in [l]$ such that $w_{i''} = r_{j''}$ and $w_{i'''} = r_{j'''}$ then $w_{i''}w_{i''+1} \dots w_{i'''}(= r_{j'''})r_{j'''+1} \dots r_{j''}(= w_{i''})$ is a cycle in the original graph, a contradiction.

Now, if such an i'' does not exist then that means $w_1, w_2, \dots, w_k \subseteq P_2$. However, $C_1 \neq C_2$ means that $P_1 \neq P_2$, so then, there exists $j'' \in [l]$ such that $r_{j''} \notin P_1$. Let j' be the maximum index less than j'' such that $r_{j'} \in P_1$. Also, let j''' be the minimum index greater than j'' such that $r_{j'''} \in P_1$. Then, there exists $i^* \in [k]$ such that $r_{j'''} = w_{i^*}$ and $r_{j'} = w_{i^*+1}$. Then, $(w_{i^*+1} =) r_{j'} r_{j'+1} \dots r_{j'''-1} r_{j'''} (= w_{i^*}) w_{i^*+1}$ is a cycle in the original graph. So, I contains a cycle.

Now, in particular, this means that I contains exactly one cycle. We can now relate its number of edges and vertices. Denote $E(I) := I$ and $V(I) = V(G[I])$. Also, let $C(I)$ denote the number of connected components of $G[I]$. Then, $E(I) = V(I) - c(I) + 1$. Why? The fact that I contains exactly one cycle means that there exist some edge e (pick any e on that cycle in I) such that $I - e$ is a forest. Note that for any forest F, one has $|E(F)| = |V(F)| - C(F)$ since $E(F) = \cup_{i=1}^{C(F)} E(F_i)$ where F_i is the ith connected component of F. Then, $|E(F)| = \sum_{i=1}^{C(F)} |E(F_i)| = \sum_{i=1}^{C(F)} (|V(F_i)| - 1) = (\sum_{i=1}^{C(F)} |V(F_i)|) - C(F) = |V(F)| - C(F)$. So, $|I| - 1 = |I \setminus \{e\}| = |V(I \setminus \{e\})| - C(I \setminus \{e\}) = |V(I)| - C(I)$ (noting that $V(I) = V(I \setminus \{e\})$ since there exist a path between the endpoints of e, called u_1 and u_2 in I, since e belonged to a cycle in I and also noting that $C(I \setminus \{e\}) = C(I)$ since deleting an edge from a cycle cannot disconnect any component). Finally, we see $|I| - 1 = |V(I)| - C(I)$ or $|I| = |V(I)| - C(I) + 1$.

Now, note that by assumption for all $j \in J \setminus I$, $I \cup \{j\} \notin \mathcal{I}$, which means $I \cup \{j\}$ has at least 2 cycles. In particular, this means that for every edge $j \in J \setminus I$, j has both ends in the same component of I. Otherwise, we could add it to I without creating any new cycles. So, now if one looks at the components $\{C_k : k \in [C(I)]\}$ of I. The fact that every edge $j \in J \setminus I$ (and also every edge $j \in J \cap I \subseteq I$) has both ends in the same component C_k means that for all edges $u_1 u_2 := j \in J$, one has that $u_1, u_2 \in C_l$ for some $l \in [C(i)]$. Now, that means that $|E(J)| \leq (\sum_{k=1}^{C(I)} |V(C_k)| - 1) + 1 = |E(I)|$, a contradiction to $|E(I)| < |E(J)|$, and we are done.

Q2. Let $G = (V, E)$ be a hypergraph, that is each $e \in E$ is a hyperedge, in other words $e \subseteq V$. We say that $X \subseteq E$ a forest-representable if one can choose for each $e \in X$ two nodes in e such that the chosen pairs when viewed as edges form a forest on V. Prove that $(E, \mathcal{I})$ is a matroid where $\mathcal{I} = \{X \subseteq E \mid X \text{ is forest-representable}\}$. This is called the hypergraphic matroid.

Clearly, the empty set is in our matroid. Also, the subset property clearly holds since, if one takes $X \in \mathcal{I}$, then that means that there exist pairs $v_1^i, v_2^i \in H_i$ for all $i \in [|X|]$ such that the edges formed by these pairs is a forest. If one takes $Y \subseteq X$, then for all $H_i \in Y$ use the same pairs v_1^i, v_2^i.

Now, we wish to show that the growth property holds. Say one has independent sets $X, Y \in \mathcal{I}$ with $|X| < |Y|$ and that means that there exist corresponding forests for X, Y. Choose these forests F_1, F_2 (among all possible choices that work) such that $|F_1 \cap F_2|$ is maximized. Now, recall that forests are matroids, so (as long as) $|F_1| < |F_2|$ there exists an edge (not hyperedge) $e (= v_1 v_2) \in F_2 \setminus F_1$ such that $F_1 \cup \{e\}$ is a forest. Now, so long as there exists some hyperedge $H \in Y \setminus X$ such that $v_1, v_2 \in H$, we are done, as we then know that $X \cup \{H\} \in \mathcal{I}$. Otherwise, for all $H \in Y \setminus X$ one has that $v_1 \notin H$ or $v_2 \notin H$.

Q3. Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid and $r : 2^S \to \mathbb{Z}$ be its rank function. Show that

(i) $r(\emptyset) = 0$ and monotone submodular, i.e. r satisfies:

(a) (monotonicity) $r(X) \leq r(Y)$ for $X \subseteq Y \subseteq S$,

(b) (submodularity) $r(X) + r(Y) \geq r(X \cup Y) + r(X \cap Y)$ for all $X, Y \subseteq S$.

We wish to show that $r(X) \leq r(Y)$ whenever $X \subseteq Y \subseteq S$. Recall,

$$r(X) = \max_{I \in \mathcal{I}, I \subseteq X} |I|.$$

So, there exists some independent subset $J \subseteq X$ of size $r(X)$. Since $X \subseteq Y$, one also has $J \subseteq Y$ and still $J \in \mathcal{I}$, so clearly $r(Y) \geq |J| = r(X)$, which proves monotonicity.

Now, we wish to prove submodularity. Here I used the reference "Introduction to Graph Theory" by Douglas B West (specifically page 356). We wish to compare the quantities $r(X), r(Y), r(X \cap Y)$, and $r(X \cup Y)$. First, say the rank of $X \cap Y$ being $r(X \cap Y)$ means that there exists some independent subset $I \subseteq X \cap Y$ with $|I| = r(X \cap Y)$ and no other independent set $I' \subseteq X \cap Y$ satisfies $|I'| > |I|$. So, simply put, $|I|$ is maximum (within $X \cap Y$). Now, consider the set $X \cup Y$. Either $r(X \cup Y) = r(X \cap Y) = |I|$ or $r(X \cup Y) > r(X \cap Y) = |I|$. (Here we are using monotonicity). If $r(X \cup Y) = r(X \cap Y) = |I|$ we do nothing. Otherwise, $r(X \cup Y) > |I|$ implies that there are independent sets of size greater than $|I|$ and the growth property tells us that we can grow I to an independent set $J \supseteq I$ with $|J| = r(X \cup Y)$ and $J \subseteq X \cup Y$. Now, we note that $|J| = |J \cap X| + |J \cap Y| - |I|$. Why? It suffices to show that $|I| = |(J \cap X) \cap (J \cap Y)|$. To do that, we note that clearly $(J \cap X) \cap (J \cap Y) = J \cap (X \cap Y)$ is independent since it is contained in J. So, $|J \cap (X \cap Y)| \leq r(X \cap Y) = |I|$. However, since $I \subseteq J$ and also $I \subseteq X \cap Y$, one has $I \subseteq J \cap (X \cap Y)$ which means that $|I| \leq |J \cap (X \cap Y)|$. So, we have $|I| = |(J \cap X) \cap (J \cap Y)|$.

Next, one notes that $|(J \cap X) \cup (J \cap Y)| = |(J \cap X)| + |(J \cap Y)| - |(J \cap X) \cap (J \cap Y)| = |(J \cap X)| + |(J \cap Y)| - |I|$. Furthermore one notes that $(J \cap X) \cup (J \cap Y) = J$ since $J \subseteq X \cup Y$. So, finally, putting this all together one has $|J| = |(J \cap X)| + |(J \cap Y)| - |I|$ or equivalently put

$$\begin{aligned} |I| + |J| &= |(J \cap X)| + |(J \cap Y)| \\ r(X \cap Y) + r(x \cup Y) &= |(J \cap X)| + |(J \cap Y)| \\ &\leq r(X) + r(Y) \end{aligned}$$

since $(J \cap X)$ and $(J \cap Y)$ are independent sets contained in X and Y respectively, and we are done.

(ii) $r(X \cup \{e\}) \leq r(X) + 1$ for all $X \subseteq S$ and $e \in S$.

There is some independent set $I \subseteq X \cup \{e\}$ with $|I| = r(X \cup \{e\}$. Now, either $e \in I$ or $e \notin I$. If $e \notin I$, then $I \subseteq X$ is also and independent set contained in X, so $|I| \leq r(X) \leq r(X \cup \{e\}) = |I|$, which means that $r(X) = r(X \cup e)$. Now, otherwise $e \in I$. In this case, $I \setminus \{e\}$ is an independent set in X, which means that $|I \setminus \{e\}| = |I| - 1 \leq r(X)$ or $|I| \leq r(X) + 1$ and $|I| = r(X \cup \{e\})$, so we are done.

Also show the converse, i.e. if $r : 2^S \to \mathbb{Z}$ is a submodular function satisfying above properties, then it is a rank function of some matroid.

We construct such a matroid. Namely, let $\mathcal{I} = \{I \in 2^S : r(I) = |I|\}$. We wish to show that $M = (S, \mathcal{I})$ is a matroid. Well, $r(\emptyset) = 0$ implies that $\emptyset \in \mathcal{I}$.

We wish to show that $S \subseteq T$ and $T \in \mathcal{I}$ implies that $S \in \mathcal{I}$. Well, $T \in \mathcal{I}$ implies that $r(T) = |T|$. Then, $S \subseteq T$ (and both finite) implies that there exist $\{t_1, t_2, \ldots, t_k\} \subseteq T \backslash S$ such that $S = T \backslash \{t_1, \ldots, t_k\}$. Call $S_0 := S$ and $S_{k+1} := T$, then define recursively $S_j := S_{j+1} \backslash \{t_j\}$.

Well, property (ii) implies that $r(X) \leq |X|$ for all $X \in 2^S$. Why? We use induction on the size of $|X|$. Base case: $r() = 0 \leq |\emptyset|$. Define $X_0 := \emptyset$, $X_i := \cup_{j \in [i]} x_j$ for $i \in [n]$. (Here, $X = \cup_{k \in [n]} x_k =: X_n$). Inductive step: we know that $r(X_j) \leq |X_j|$ for all $j \leq i$. We wish to show that $r(X_{i+1}) \leq |X_{i+1}|$. Note, $|X_{i+1}| = |X_i| + 1$. Property (ii) tells us that $r(X_{i+1}) \leq r(X_i) + 1 \leq |X_i| + 1 = |X_{i+1}|$ and we are done.

Now, we can show inductively that $r(S_j) = |S_j|$. Base case: $r(S_{k+1}) = r(T) = |T| = |S_{k+1}|$. Inductive step: Now assume $r(S_i) = |S_i|$ for all $i \geq j$, we wish to show that $r(S_{j-1}) = |S_{j-1}|$. Well, $S_{j-1} = S_j \setminus \{t_{j-1}\}$. Property (ii) guarantees $r(S_j) \leq r(S_{j-1}) + 1$. Now, by the inductive hypothesis, $r(S_j) = |S_j|$, which implies $|S_j| \leq r(S_{j-1})+1$ or $r(S_{j-1}) \geq |S_j|-1$. Now does it also hold that $r(S_{j-1}) \leq |S_j|-1$? Well, by my above lemma $r(S_{j-1}) \leq |S_{j-1}| = |S_j|-1$, so yes, and we are done. We have shown $r(S_a) = |S_a|$ for all $a \in \{0, \ldots, k-1\}$ which means that $S_a \in \mathcal{I}$ for all a, which concludes our proof of the subset property.

Now, we wish to show that the growth property holds. So, take $I, J \in \mathcal{I}$ with $|I| < |J|$. We wish to show that there exists $j \in J \setminus I$ such that $I \cup \{j\} \in \mathcal{I}$. Assume not. Assume for all $j \in J \setminus I$, one has that $I \cup \{j\} \notin \mathcal{I}$, which by our defined matroid happens exactly when $r(I \cup \{j\}) < |I \cup \{j\}| = |I| + 1$. Or better put when $r(I \cup \{j\}) = |I|$.

Definition: Now, say $J \setminus I =: \{j_1, \ldots, j_k\}$. Then, define $J_m := \cup_{i=1}^m j_i$ where $m \in [k]$.

Lemma: $r(I \cup J_m) \leq r(I \cup J_{m-1})$ for all $m \in [k]$.

Proof: Take $A'' := I \cup J_{m-1}$ and $B'' := I \cup \{j_m\}$. Then, submodularity says

$$\begin{aligned} r(A'') + r(B'') &\geq r(A'' \cup B'') + r(A'' \cap B'') \\ r(I \cup J_{m-1}) + r(I \cup \{j_m\}) &\geq r(I \cup J_m)) + r(I) \\ r(I \cup J_{m-1}) + |I| &\geq r(I \cup J_m)) + |I| \\ r(I \cup J_{m-1}) &\geq r(I \cup J_m)) \end{aligned}$$

Now, using the above lemma and noting that $r(I \cup J_1) = r(I \cup \{j_1\}) = |I|$ gives $r(I \cup J) = r(I \cup J_k) \leq r(I \cup J_{k-1}) \leq r(I \cup J_{k-2}) \leq \cdots \leq r(I \cup J_1) = |I| < |J|$ which provides a contradiction, as $J \subseteq I \cup J$ and $r(J) = J$ (since independence means rank equal to size for our matroid). Then, monotonicity implies that $r(I \cup J) \geq r(J) = |J|$. So, we have shown that the growth property holds. Thus, we have defined a valid matroid using this rank function.

Q4. A function $f : 2^S \to \mathcal{R}$ is supermodular if $f(A) + f(B) \leq f(A \cup B) + f(A \cap B)$ for all $A, B \subseteq S$. Let $A_1, \ldots A_n$ be random events and $S = \{A_1, \ldots, A_n\}$ and let for any $X = \{A_{i_1}, \ldots, A_{i_k}\} \subseteq S$, we let $f(X) = \mathbb{P}(A_{i_1}, \ldots, A_{i_k})$ be the probability that all events in X occur. Show f is a supermodular function.

First, note that the fact that these are random independent events means that whenever $A \cap B = \emptyset$, one has that $Pr(A \text{ and } B) = Pr(A)Pr(B)$. So, note

$$\begin{aligned}
Pr(A) &= Pr((A \cap B) \text{ or } (A \setminus B)) = Pr(A \cap B) + Pr(A \setminus B) \\
Pr(B) &= Pr((A \cap B) \text{ or } (B \setminus A)) = Pr(A \cap B) + Pr(B \setminus A) \\
Pr(A \cup B) &= Pr((A \setminus B) \text{ or } (B \setminus A) \text{ or } (A \cap B)) = Pr((A \setminus B)) + Pr(B \setminus A) + Pr(A \cap B).
\end{aligned}$$

(Note: that above in all 3 cases the union bound is actually an equality since $A \cap B$, $A \setminus B$ and $B \setminus A$ are mutually exclusive events).

Now,

$$\begin{aligned}
Pr(A) + Pr(B) &= (Pr(A \cap B) + Pr(A \setminus B)) + (Pr(A \cap B) + Pr(B \setminus A)) \\
&= Pr(A \cap B) + (Pr(A \setminus B) + Pr(A \cap B) + Pr(B \setminus A)) \\
&= Pr(A \cap B) + (Pr(A \cup B)),
\end{aligned}$$

and we are done.

Q5. Given a graph $G = (V, E)$, let $\mathcal{I} = \{F \subseteq E \mid |E(S) \cap F| \leq 2|S| - 3 \text{ for all } S \subseteq V \text{ with } |S| > 1\}$. We want to show that $(E, \mathcal{I})$ defines a matroid.

(a) For $F \subseteq E$, let

$$f_F(S) = \begin{cases} 2|S| - 3 - |E(S) \cap F| & \text{if } |S| > 1 \\ 0 & \text{if } |S| \leq 1. \end{cases} \tag{1}$$

Observe that $F \in \mathcal{I}$ is equivalent to f_F being nonnegative. Observe that the function f_F is *not* submodular. For $F \in \mathcal{I}$, call a set $S \neq \emptyset$ tight if $f_F(S) = 0$. Show that if A and B are tight and $|A \cap B| > 1$ then $A \cup B$ is tight (and so is $A \cap B$ but you won't need that).

Well, first say some set S is tight. First define the vertex set $S' := S \cap V(F)$. Now, F a forest implies that $E(S) \cap F = E(S') \cap F$ also a forest, which further implies that $|E(S) \cap F| = |E(S') \cap F| \leq |S'| - 1 \leq |S| - 1$. So, if S is tight, we have that $2|S| - 3 = |E(S) \cap F| \leq |S| - 1$ which implies that $|S| \leq 2$. Now, $|A \cap B| > 1$ implies that $|A \cap B| \geq 2$ which implies $|A|, |B| \geq 2$. Now, A, B tight implies that

$$\begin{aligned}
2|A| - 3 - |E(A) \cap F| = 0 \\
2|B| - 3 - |E(B) \cap F| = 0
\end{aligned}$$

and as just proven we know $|A|, |B| \leq 2$. So, in particular, $|A| = |B| = 2$ and since $2 \leq |A \cap B| \leq |A|, |B| = 2$, we know that $|A \cap B| = 2$ and since $A \cap B \subseteq A$ with $|A \cap B| = |A|$, we have that $A \cap B = A$, and similarly $A \cap B = B$. So, $A = B$, which means that $A \cup B = A$, which is tight by assumption.

(b) Prove that $(E, \mathcal{I})$ satisfy the matroid axioms.

Clearly $\emptyset \in \mathcal{I}$. Now, if $F \in \mathcal{I}$ and $E \subseteq F$, is $E \in \mathcal{I}$? Yes. Observe that for all $S \subseteq V$ with $|S| > 1$, one has that $E(S) \cap E \subseteq E(S) \cap F$ which implies that $|E(S) \cap E| \leq |E(S) \cap F| \leq 2|S| - 3$. So, $E \in \mathcal{I}$. Now, is the growth property satisfied? Namely, say we have $E, F \in \mathcal{I}$ and $|E| < |F|$. We wish to show that there exists $f \in F \setminus E$ such that $E \cup \{f\} \in \mathcal{I}$. Assume not, then for all $f \in F \setminus E$, $E \cup \{f\} \notin \mathcal{I}$, which means that there exists $S_f \subseteq V$ with $|S_f| \geq 2$ such that $2|S_f| - 3 - |E(S_f) \cap (E \cup \{f\})| < 0$. Equivalently written

$$2|S_f| - 3 < |E(S_f) \cap (E \cup \{f\})| \leq |E(S_f) \cap E| + 1.$$

Also, since $E \in \mathcal{I}$ one knows

$$2|S_f| - 3 \geq |E(S_f) \cap E|,$$

which means one cannot have $|E(S_f) \cap (E \cup \{f\})| = |E(S_f) \cap E|$ since then that would imply $2|S_f| - 3 < |E(S_f) \cap (E \cup \{f\})| = |E(S_f) \cap E| \leq 2|S_f| - 3$, which gives $2|S_f| - 3 < 2|S_f| - 3$, a contradiction. So, $|E(S_f) \cap (E \cup \{f\})| = |E(S_f) \cap E| + 1$. Now, we have

$$2|S_f| - 3 < |E(S_f) \cap (E \cup \{f\})| = |E(S_f) \cap E| + 1 \leq 2|S_f| - 2.$$

Now $2|S_f| - 3 < |E(S_f) \cap (E \cup \{f\})|$ means $2|S_f| - 3 \leq |E(S_f) \cap (E \cup \{f\})| - 1 = |E(S_f) \cap E| \leq 2|S_f| - 3$. So, S_f is tight with respect to E, which means that $|S_f| \leq 2$ (by our earlier argument). Also, since $|S_f| \geq 2$, we know that exactly $|S_f| = 2$. So, $2|S_f| - 3 = 1 < |E(S_f) \cap (E \cup \{f\})| = |E(S_f) \cap E| + 1$. So, $1 < |E(S_f) \cap (E \cup \{f\})|$ means $1 \leq |E(S_f) \cap (E \cup \{f\})| - 1 = |E(S_f) \cap E| = 2|S_f| - 3 = 1$. So, $|E(S_f) \cap E| = 1$ and $|E(S_f) \cap (E \cup \{f\})| = 2$. However, this is a contradiction, as $|S_f| = 2$ implies that $E(S_f) \leq 1$ and also, $2 = |E(S_f) \cap (E \cup \{f\})| \leq |E(S_f)| \leq 1$. So we have $2 \leq 1$, a contradiction. We've shown that the growth property holds.

Q6. Let $G = (V, E)$ be a loopless undirected graph. Let A be the matrix obtained from the $V \times E$ incidence matrix of G by replacing in each column exactly one of the two $1's$ by -1 (i.e. orient the edges in an arbitrary manner).

(i) Show that a subset Y of E is a forest if and only if the columns of A with index in Y are linearly independent.

We first show that if the columns are linearly dependent, then it is not a forest. If the columns are linearly dependent then there exists some set of constants $c_y \in \mathbb{R}$ which are not all zero such that

$$\sum_{y \in Y} c_y A_y = 0.$$

In particular, this means that for every row $v \in V$, one has

$$\sum_{y \in Y} c_y A_y^v = 0,$$

which matches the definition of flow conservation. In particular, one can think of the vector $(c_y)_{y \in Y}$ as a vector of flows on each edge. If one has a circulation with some nonzero flow value on an edge, then that graph contains a cycle. Thus, the underlying graph (induced by the columns of A) contains a cycle and is not a forest.

Now, we wish to show that if the columns indexed by $Y \subseteq E$ are independent, then they induce a forest. Assume not. Then, they contain a cycle (in the underlying undirected graph). In such a case, say the edges of the cycle are $e_1, \ldots, e_k$. Label $e_i = v_i v_{i+1}$ for all $i \in [k-1]$ and then $e_k = v_k v_1$. Construct the following vector $(c_y)_{y \in Y}$ as follows. If $y \notin e_1, \ldots, e_k$, set $c_y = 0$. Otherwise, $y = e_i$ for some $i \in [k]$. Then, if $A_{e_i}^{v_i} = 1$ and $A_{e_i}^{v_{i+1}} = -1$, set $c(y) = 1$. Otherwise, $A_{e_i}^{v_i} = -1$ and $A_{e_i}^{v_{i+1}} = 1$, and we set $c(y) = -1$. Then, one has

$$\sum_{y \in Y} c_y A_y = 0,$$

which means that the columns indexed by Y are linearly dependent. (By construction the vector $(c_y)_{y \in Y}$ was not the zero vector).

(ii) Derive that any graphic matroid is a linear matroid.

In particular, take the matrix representation defined above using the field $GF(3)$. (So, one has 0,1, and 2 which is $-1 \bmod 3$). As shown in part (i), the linear matroid corresponding to this matrix coincides with the graphic matroid.