These Were My Personal Notes to Prepare for My First Research Meeting with Professor Xingxing Yu (I mentioned my old research focus was Graph Theory)

2019

1 Relevant Theorems and Ideas

NOTE: See menu to the left for scans of my notes and scans of relevant pages from the Handbook of Graph Theory. I have no idea if these will be useful or not. I gathered everything I found that is even marginally related to graphs of odd girth 5 or X binded graphs. I figure I can list all these and then pick out the couple things that might be useful later. Scans of these can be found in the menu on the left. Also, below are some links to anything that came up when I searched about graphs of odd girth 5.

- `https://www.researchgate.net/publication/271447827_On_the_odd_girth_and_the_circular_chromatic_number_of_generalized_Petersen_graphs`
- `http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.889.7132&rep=rep1&type=pdf`
- `https://www.math.uni-hamburg.de/home/schacht/2015/3n4k.pdf`
- `https://mathoverflow.net/questions/193716/what-is-the-smallest-4-chromatic-graph-of-girth`
- `https://arxiv.org/pdf/1610.03685.pdf`
- `https://pdfs.semanticscholar.org/5505/b5591a312852a42a267051a68b5c94dfb236.pdf`
- `ttps://www.sciencedirect.com/science/article/pii/S0095895602000278`
- `http://www.math.uchicago.edu/~may/VIGRE/VIGRE2007/REUPapers/FINALAPP/Ullery.pdf`

Ideas related to theorems listed here:
Note: I have not fleshed out these ideas. I am just brainstorming and writing whatever comes to mind. I will figure out which of these connections are valid later.

1. To start, in general there are two ways we could go about this: either try to prove it directly for the desired class of graphs or look at a minimal counter example. I should try both lines of thought.

2. Idea: Get 3 coloring by induction on two (or more) parts of graph see if I can make them agree on some common part to get a 3 coloring on whole graph.

3. This is very strange. When I take the contrapositive of the result given here `https://pdfs.semanticscholar.org/5505/b5591a312852a42a267051a68b5c94dfb236.pdf`I get that if the odd girth of G is $\geq$ r, then G has $< (\frac{r-3}{2})^2$ vertices (pretty sure I checked that's right the first is supposed to be $\geq$ and the other is $<$ (strict inequality)) which would mean that if r is 5, then G has $< \frac{5-3}{2}^2 = 1$ vertices. I am very confused. I have not read the proof of the result given in this paper, but I brought it in and we can all take a look and make sure this holds for small enough n. Well, I'm just realizing now that maybe the author did not consider the case $n = 1$, which means that the contrapositive of the statement hasn't been shown for $r = 5$.

4. Idea: Look at a block decomposition, either of a fork-free, triangle free graph G OR of a minimal counterexample (in terms of vertices and edges OR just vertices) depending on how we do this. Then, we know that pairwise, blocks overlap in a single vertex. So, the intuition here is that maybe it is easy to color each block and then make them agree on that overlapping vertex. Also, 3 coloring blocks is trivial I think since we just 2 color the starting cycle, then for each lobe of the ear 2 color the vertices not already colored (possibly using a different 2

colors though). Now, making these blocks agree on common vertices should be reasonably easy except for when we have one vertex in many blocks (like a vertex that is a star with adjacent blocks all being K_2's). We note that in many of these cases which would be difficult, the common vertex, v, has degree at least 4 which makes forks likely. The only cases in which a common vertex v in at least 2 distinct blocks has degree less than or equal to 3 is when (1) v belongs to one block which is not K_2 and another block which is K_2 or (2) v belongs to 1,2, or 3 blocks which are all K_2. Maybe we can try to color these high degree vertices first and then progress along the "tree of blocks" in a smart way to color each block successively. I could even apply induction on the number of blocks by deleting a block which is a "leaf" in the "tree of blocks". Also, note that it may come in handy to use the fact that a block is a MAXIMALLY 2-connected subgraph, not just any 2-connected subgraph. Can one always delete a lobe consisting of more than one edge? What if we start with the right base case? Also, perhaps can look at all possible lobes to delete and if u and v ever lie in different components of $G_{1,3}$ or $G_{1,2}$ respectively, we are done. This is unlikely to work though, since these colorings we get by deleting each of these paths are not related in any way. Note, my approach to create a uv-cut in $G_{1,3}$ could be strengthened if not working to create a uv-cut in one of $G_{1,2}$ or $G_{1,3}$ and maybe it's possible that if one doesn't work, then the other must. In cases where my approach doesn't work, try using discharging method to show it had to have had an induced fork or triangle or vertex of degree 2.

5. Along these lines: what if we decompose into blocks and handle blocks that have a C_5 and blocks that don't have a C_5 separately? I belive the blocks that don't have a C_5 are already handled by this dissertation. Also, I wonder: can a block have more than one odd cycle? Or instead maybe when we remove a cycle from each block it's bipartite. Is there a way one can remove vertices from such a block (with a C_5) to get a bipartite graph, which is then 2-colorable, then add those vertices back in and modify to get a 3 coloring. Note: I don't think that last part is likely to work, but I'm just writing down every idea that comes to mind.

6. Definition: G is k-vertex-critical if G is k-colorable and deletion of any vertex results in a graph that is $(k-1)$-colorable (though deleting an edge may still keep it k-colorable but NOT $(k-1)$-colorable). Definition: G is k-critical if G is k-colorable and deletion of any vertex or edge results in a graph that is $(k-1)$-colorable. So, k-critical IMPLIES k-vertex-critical but not vice versa.

7. The following theorem of Kainen (found here: `http://www.trentu.ca/academic/math/bz/gcolor.pdf`) says that if G is a k-critical graph and X, Y is a partition of $V(G)$, then there are at least kâĹŠ1 edges between X and Y . So, if G is 4-color-critical (by the definition used here this means that deleting any vertex OR EVEN just an edge results in a graph that is 3-colorable) then every vertex has degree at least 3.

8. What if, similar to something done in this dissertation, I split $V(G)$ into V_4, and $V_{\geq 5}$? (Note that, by the theorem of Kainen there are no vertices of degree less than 4). Then, well there is some work to do to apply the kind of results I wanted to use... but basically I wanted to look at some r-regular subgraph of G, so that I could possibly apply result found here: `http://www.math.uchicago.edu/~may/VIGRE/VIGRE2007/REUPapers/FINALAPP/Ullery.pdf`

9. Fun and useful fact: saying girth $= 4$ is the same as odd girth $= 5$ in a triangle free graph :) So, I could just apply these results to a subgraph of G which is regular and has girth 4. Oh! I should look up results on graphs with girth 4.

10. Ok, so generally I have this idea to looks at block decompositions. I could look at blocks of

$V(G)$ or blocks of $V_{\geq 5}$ or even blocks of V_4 and see how those relate. (Note: the reason I want to look at vertices of degree 4 was to find some regular subgraph, though now I see that I should look for a 4-regular subgraph even if it's not induced. So namely, it could include vertices that have degree ≥ 5 in G, but which in this non-induced subgraph have degree 4). Also, I wonder: can I have a vertex of degree 5 in a graph which is a minimal counterexample?

11. This article seems like it could be really useful: `https://arxiv.org/pdf/1610.03685.pdf`.

12. I recall that the size of largest independent set is related to the chromatic number. Namely, $\chi(G)\alpha(G) \geq |V(G)|$. So, since `http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.889.7132&rep=rep1&type=pdf` says that $\alpha(G) \geq \frac{4n-m-1}{7}$, that implies... oh wait, nope, this is the wrong direction of inequality.

13. This resource `https://www.math.uni-hamburg.de/home/schacht/2015/3n4k.pdf` says that either G has a vertex of degree $\leq \frac{2}{5}n$ or G is bipartite which means 2 colorable. So, this could narrow down some cases. It also introduces the notion of "homomorphism to the 5-cycle" which is also mentioned here: `https://www.researchgate.net/publication/271447827_On_the_odd_girth_and_the_circular_chromatic_number_of_generalized_Petersen_graphs`.

14. Theorem 6.4.3 in the following resource says that the join of an r-color-critical graph and an s-color-critical graph is an $(r + s)$-color-critical graph. `http://www.trentu.ca/academic/math/bz/gcolor.pdf`

15. Can I somehow use Theorem 6.4.6 given here? (`http://www.trentu.ca/academic/math/bz/gcolor.pdf`). So, say for instance that I have two minimal counterexamples (not necessarily distinct) and I look at their Hajos Construction. (Definition: Let G and H be two graphs. Let uu' be an edge in G and vv' be an edge in H. The graph $G\triangle H$ called the "Hajos Construction" is obtained by identifying u and v, deleting the edges uu' and vv and adding an edge $u'v'$). So their Hajos Construction would also be 4-color critical, but is it still fork-free and triangle free? I bet so. If so, I bet I could prove that relatively easily. However, I am now realizing that there seems to be little point in combining two minimal counterexaples unless the Hajos Construction ends up having the same number of vertices. Oh, actually it could be useful in that case. If I use the Hajos Construction and I get a graph with the same number of vertices then it would also be minimum. Oh wait, I am looking at minimal counterexamples. Maybe I should look at minimum counterexamples in terms of (1) number of vertices, (2) number of edges, or (3) number of vertices plus number of edges. In that case, this might give me a way of constructing different minimum counterexamples from current ones. Also, even if I'm still considering minimal counterexamples, I could ask the question of whether the Hajos Construction of two minimal counterexamples is still a minimal counterexample. I have no idea.

16. What if I delete a 5-cycle and apply induction? Or delete a block and apply induction on the number of blocks?

17. Random note: the line graphs of the triangle K_3 and the claw $K_{1,3}$ are isomorphic even though the graphs themselves aren't. I'm just writing this here since the Vizing bound comes from line graphs and there are some results about claw free graphs.

18. Another thing on my to-do list: don't USE Theorem 6.4.11 in `http://www.trentu.ca/academic/math/bz/gcolor.pdf` but rather look over the proof of Theorem 6.4.11 because I think it has some useful techniques. I think I could modify it to get a result using the fact that our graph is actually triangle free (by working through the proof backwards in a way).

19. Also, here's this question about smallest 4-chromatic graphs of girth 5: `https://mathoverflow.net/questions/193716/what-is-the-smallest-4-chromatic-graph-of-girth-5`. Note, that these graphs may still contain forks so some of them may not be in the class we are considering. Also, some of the graphs we are considering may have girth 4 so they are not of this type. There isn't containment either way.

20. Results on cubic 2-connected graphs: `https://www.sciencedirect.com/science/article/pii/S0012365X0300058X` `https://www.combinatorics.org/ojs/index.php/eljc/article/view/v15i1n38`

21. Uniquely 3-colorable graphs:

    `https://www.combinatorics.org/ojs/index.php/eljc/article/view/v15i1n38`

    `https://en.wikipedia.org/wiki/Uniquely_colorable_graph`

    Link A: `http://mathworld.wolfram.com/Uniquelyk-ColorableGraph.html`

    `https://www.sciencedirect.com/science/article/pii/S0095895678800100`

    (THERE ARE MISTAKES IN THIS: which are pointed out in Link A from Wolfram Alpha. They claim certain graphs to be uniquely 3-colorable which are NOT. However, there still may be useful results in this paper. `https://www.sciencedirect.com/science/article/pii/S0021980069800864`)

    `https://www.researchgate.net/publication/220077664_Uniquely_Colorable_Graphs`

    `https://www.arcjournals.org/pdfs/ijsimr/v2-i10/8.pdf`
    Theorem 2 in this seems useful:
    `https://www.sciencedirect.com/science/article/pii/S0021980069800864`

22. The contrapositive of Theorem 3 in this `https://www.sciencedirect.com/science/article/pii/S0021980069800864` says that if we have a graph G and some homormorphic image H of G and both H and G are k-colorable, then H NOT uniquely k-colorable implies that G is NOT uniquely k-colorable.

23. THIS WILL ACTUALLY BE USEFUL. Even if you look back and at first glance think it isn't, it is. Whatever problem you're currently thinking about, even if it's not this one from this file can be helped by reading and using similar techniques given here.
    `https://core.ac.uk/download/pdf/82281034.pdf`

24. `https://www.sciencedirect.com/science/article/pii/0095895680900404`

    `http://www.math.nsysu.edu.tw/~zhu/papers/circ/construction.pdf`

25. "For example, all complete graphs Kn and all odd cycles (cycle graphs of odd length) are cores. Every 3-colorable graph G that contains a triangle (that is, has the complete graph K3 as a subgraph) is homomorphically equivalent to K3. This is because, on one hand, a 3-coloring of G is the same as a homomorphism G âĘŠ K3, as explained below. On the other hand, every subgraph of G trivially admits a homomorphism into G, implying K3 âĘŠ G. This also means that K3 is the core of any such graph G. Similarly, every bipartite graph that has at least one edge is equivalent to K2.[11]"
    `https://en.wikipedia.org/wiki/Graph_homomorphism`

26. XXY noted that if we ever have that the neighborhood of one vertex is contained in the neighborhood of another vertex, then we can apply induction easily. Namely, if $N(u) \subseteq N(v)$, then one can delete u to get G', which has a 3-coloring by induction, then to extend this 3-coloring to a 3-coloring of G, one assigns $c(u) = c(v)$.

Ideas that I've realized don't seem terribly useful but am keeping around in case they do come in handy:

1. so like, hear me out: we got these cases: so, say we have a minimal counter example (namely a fork free graph with odd girth at least 5 that is 4 colorable but not 3 colorable) then we can look at the vertices of degree at least 4 and the vertices of degree ≤ 3. maybe there's some relation to the structure of the blocks of the graph. like it seems to me that the degree 4 vertices might be more likely to lie in the intersection of two blocks.

2. now, by theorem 3.1 in the dissertation, every block (namely because each block is a (maximal) 2 connected induced subgraph) of $V_{\leq 3} = \{v \in V(G) | deg(v) \leq 3\}$ is either a complete graph or an odd cycle of length at least 5. and in particular, triangle free means that it is either an odd cycle of length at least 5 or a single edge.

3. also, in the same resource, lemma 6.4.8: It G is a 4-critical graph, then either G is an odd wheel or it does not contain any wheels (a wheel is a cycle join a K_1), well duh not contain any wheel holds for us anyway since we are triangle free.

4. note: grotsch graph some minimal in some sense 4 colorable graph but not fork free so dont look at it again or use it `https://en.wikipedia.org/wiki/Gr%C3%B6tzsch_graph`

5. BIG IDEA: Are these graphs perfect? Nope. They definitely have a C_5 in fact. So, we know that $\chi(H) \neq \omega(H)$ for some induced subgraph of G. Oh, but $\omega(H) = 2$ for all subgraphs H. Oh darn. Well, we know that $\chi(H) \geq \omega(H)$, but now the fact that G is not perfect means that there definitely exists some subgraph H for which $\chi(G) \geq 3$, so that namely, G is NOT 2-colorable. Oh, clearly, since there's an odd cycle.

2 Our Results

2.1 2-Connected Graphs

To start with some intuition. I had an idea which I couldn't help but start to work out (my curiosity got the best of me). The idea was to look at a block decomposition of G. The intuition here is that if one can 3 color all blocks in G, we recall that all pairs of blocks overlap in at most one vertex so it might be easy to permute colors within one block to make them agree with others. The really useful fact here is that the set of blocks in G form a tree. So, one could successively permute colors along this tree (formally we are using induction by deleting a leaf) without affecting previous blocks considered (since there is no cycle in this block tree). I think that is the main advantage here (the fact that there are no cycles in this block tree which means that the recoloring of each block step by step cannot interact or create new conflicts with previously colored blocks).

Now, my next thought was to try to prove that every block is 3-colorable. One might wonder whether this is true (I actually have some indication that it is possible that it is). If one proceeds by induction on the number of lobes (say starting with G_0, then deleting a lobe to get G which has a 3-coloring by induction), the only possible problem arises when the lobe we delete to get a 3-coloring by induction, then add back in, consists of a single edge. However, there still might be hope in this case for a number of reasons. One has to do with unique 3-colorability and the other will be fleshed out more once I present what I have so far in the inductive proof that these blocks are 3-colorable.

I learned that there is some notion of "uniquely 3-colorable" graphs which are those in which all 3-colorings are obtained from the same 3-coloring by permuting the color classes. The fact that G is triangle-free already makes it more likely that G is not uniquely 3-colorable which is our hope. Namely, I found a paper which notes some useful things on edge density and unique colorability. First, if $\delta(G) \geq \frac{3k-5}{3k-2}n$ then G is uniquely k-colorable. So, we hope that doesn't happen. Plugging in $k = 3$ gives if $\delta(G) \geq \frac{4}{7}n$, then G is uniquely 3-colorable. We can translate this into a statement about the number of edges (to get a useful correlation). Namely, if $\delta(G) \geq \frac{4}{7}n$, then $m \geq \frac{2}{7}n^2$. So, this gives us some kind of correlation. Namely, if the minimum degree is greater than that bound, then $m \geq\geq \frac{2}{7}n^2 > \frac{1}{4}n^2$ and G is uniquely 3-colorable. Namely, we then note that G triangle-free implies that $m \leq \frac{n^2}{4}$, which means that $\delta(G) < \frac{4}{7}n$. So, there is good reason to believe that triangle-free 3-colorable graphs are not uniquely 3-colorable.

However, there are triangle-free 2-connected graphs which are uniquely 3-colorable. Here is an example (Figure 5(c) in "Constructions of Uniquely 3-Colorable Graphs" by Li and Xu), and another example ("Wolfram Alpha"), also a non-example ("Wolfram Alpha") (non-example meaning it is NOT uniquely 3-colorable). The good news is that we notice that these uniquely 3-colorable graphs have induced forks. Even better news (really exciting!) is that any uniquely 3-colorable graph MUST have a vertex of degree ≥ 4 (jot down quick proof on page 143 of "Constructions of Uniquely 3-Colorable Graphs"). (You will see in the proof I give later that we have reduced to the case in which the minimum degree of G_0 is at least 3, which means that all vertices of G have degree at least 3, except possibly u and v which might have degree 2 (just realized this... I need to adjust for this later...)) This gives us reason to hope that any fork-free, triangle-free, 2-connected graph which is 3-colorable is NOT uniquely 3-colorable, which might allow us to handle our problem case in the inductive step (equivalently put that any 2-connected, triangle-free uniquely 3-colorable graph has a fork).

Next, I note that there is an even more helpful statement on edge density and unique 3-colorability which might help in our specific case (the details of which I will get into once I present what I have of the induction proof so far). Namely, if the graphs induced by any pair of color classes are connected (define $G_{1,3}$, $G_{1,2}$, and $G_{2,3}$), then $\delta(G) > (1 - \frac{1}{k-1}n)$ is enough to imply that G is uniquely k-colorable. Now, as a finer point here, as I will explain in the proof later, our specific case we reduce to in my inductive proof will be one in which two vertices u and v (both colored 1 by induction but adjacent in G, which means we need to recolor them) lie in the same component of $G_{1,3}$ and also the same component of $G_{1,2}$. This says nothing about $G_{2,3}$ and also doesn't necessarily imply that $G_{1,3}$ or $G_{1,2}$ are connected. Only that u and v are in the same component. However, the aforementioned statement on edge density I am about to present may help us handle the case in which $G_{1,2}$, $G_{1,3}$, and $G_{2,3}$ are all connected. Now, plugging in 3 for k gives: if $G_{1,3}$, $G_{1,2}$, and $G_{2,3}$ are all connected, then $\delta(G) > (1 - \frac{1}{2})n = \frac{3}{2}n$ implies that G is uniquely 3-colorable. So, once again we hope that doesn't happen. (It not happening is not enough to guarantee that G is not-uniquely 3-colorable, but it means that is possible G is not uniquely 3-colorable). Note that $\delta(G) > \frac{3}{2}n$ implies that $m > \frac{3}{4}n^2$. So, once again, G triangle-free implies that $m \leq \frac{1}{4}n^2 < \frac{3}{4}n^2$, which implies that $\delta(G) < \frac{3}{2}n$ which means that it is possible that G is not uniquely 3-colorable, which is our hope.

I also note that G not uniquely 3-colorable is not enough to ensure we can handle our problem case, but it means that it is possible to handle it. Namely, it would mean that it is possible to change colors of the graph obtained by induction so that the vertices u and v which are adjacent in G_0 (our original graph at the inductive step) but not in the smaller graph G obtained by deleting the edge uv now receive different colors. In other words, if G is uniquely 3-colorable and the coloring of G obtained by induction assigns u and v the same color, then there is no hope to recolor them different colors. So, it is a necessary but not sufficient condition that G is not uniquely 3-colorable.

I next note that there are useful results in "Constructions of Uniquely 3-Colorable Graphs" by Li and Xu relating cuts to unique 3-colorability, which may prove useful since as you will see in my forthcoming induction proof (in progress), we are making very specific assumptions about certain cuts in certain graphs.

Finally, Li and Xu showed that uniquely 3-colorable graphs have at least $m \geq 2n - 3$ edges. In addition, since our graph G which we obtain by deleting a lobe will have at most 2 vertices of degree 2 and all the rest with degree at least 3, that says $e(G) \geq \frac{3}{2}n - 1$. So, if G is uniquely 3-colorable and triangle-free then $\max\{2n-3, \frac{3}{2}n-1\} \leq e(G) \leq \frac{1}{4}n^2$. This will help us characterize the cases in which such problematic G are uniquely 3-colorable.

Theorem 2.1. *Let $G = (V, E)$ be a triangle-free, fork-free 2-connected graph. Then G is 3-colorable. (IN PROGRESS). It has already been proved that the statement holds if G is also C_5-free so we assume that there is some C_5 in G.*

Proof. NOTE to self: if I ever need to show that some problem case can't happen because there exists a fork, I can use the discharging method to show that.

We prove the statement by induction on the number of lobes in an ear decomposition of G. First, we consider the base case in which G is a cycle. In this case, one can 2 color the cycle if it is even. Otherwise, one can 2 color all except one vertex and then assign the third color to this final vertex. Now, for the inductive step, we assume that any triangle-free, fork-free 2-connected graph with fewer lobes than G is 3-colorable and now we wish to show that G is 3-colorable. Now, if

G has a lobe consisting of at least 2 edges in which every vertex on this lobe has internal degree 2, we are done. Namely, without loss of generality, the endpoints of such a lobe have colors 1 and 2 respectively or colors 1 and 1. In either case, one deletes the internal vertices of this lobe and gets a 3-coloring on the remaining graph by induction, then replaces the lobe and colors it as follows: if the number of edges on the path is odd, one colors the internal vertices by starting at the endpoint with color 1 (if both have color 1, one picks the starting point arbitrarily) and then coloring the internal vertices with colors 2 and 3 alternatively until all internal vertices are colored. If the number of edges on the path is even, one starts at the vertex with color 1 then moves to the other endpoint coloring internal vertices with colors 3 and 2 repeatedly until all internal vertices are colored. So, it remains to consider the case in which all lobes of length greater than 1 have an internal vertex of degree 3.

In such a case, we must either delete a lobe of length at least 2 in which some internal vertices have degree at least 3, which creates difficulty in coloring these internal vertices, or delete a lobe of length 1, and try to ensure that we can recolor the colored graph we get by induction to color the endpoints of this lobe different colors. Note, that if it is ever the case that there is a vertex of degree 2 we can extend both ways until we reach a vertex of degree at least 3 (which will exist since we are assuming there is at least one lobe), then such a path will be the desired lobe. So, we have reduced to the case in which all vertices of G have degree at least 3. So every vertex has at least 3 neighbors. The triangle free property tells us that no two of these neighbors are adjacent.

For now, we choose to delete a lobe of length 1 and try to recolor the graph accordingly. Now, we delete a lobe of length 1 and obtain a 3-coloring of the remaining graph by induction. If the endpoints of this lobe receive different colors, we are done. So it remains to handle the case in which they receive the same color. Without loss of generality, say they both receive color 1. In this case, one looks at the graphs induced by vertices of colors 1 and 3, called $G_{1,3}$, and the graph induced by the vertices of colors 1 and 2, called $G_{1,2}$. Now, if in $G_{1,3}$, the endpoints of this lobe, which we call u and v, lie in different connected components, then one recolors the vertices in the component containing u as follows: recolor those with color 1 color 3 instead, and vice versa. Now, u has color 3, and we have a proper 3-coloring of G. Similarly, if u and v lie in different components of $G_{1,2}$, one permutes the color classes 1 and 2 within the component containing u and we are done. So, it remains to consider the case in which u and v lie in the same component of $G_{1,2}$ and also lie in the same component of $G_{1,3}$. We aim to recolor some vertices in $G_{1,3}$ with color 2 in order to create a cut in $G_{1,3}$ which separates u from v allowing us to permute color classes 1 and 3 within the component containing u.

Now, consider all uv-cuts in $G_{1,3}$. If there exists a uv-cut, denoted W, in $G_{1,3}$ such that all vertices $w \in W$ have no neighbors in the color class 2, then we recolor all $w \in W$ with color 2 and then permute the color classes 1 and 3 of the component containing u of the updated $G_{1,3}$. So, we now have reduced to the case in which all uv-cuts W in $G_{1,3}$ have some vertex adjacent to a vertex that receives color 2.

(In particular, this means that there is some uv-path in $G_{1,3}$ in which every vertex has a neighbor in color class 2. Though there could be many of these. Otherwise, if for all uv-paths in $G_{1,3}$ (not necessarily disjoint edge-wise or vertex-wise) there were some vertex with no neighbor in color class 2, then one could take the union over all such paths of these vertices as our desired cut). NOTE: this may actually not be important. I can probably delete this if the proof works out.

So, fix a uv-cut W. Note: I am going to present this part that I tried in which all vertices in our fixed uv-cut have color 3. It later occurred to me that some of the vertices in this cut may have color 1 while some have color 3. The paragraphs below are still restricting our view to cuts W (uv-cuts in $G_{1,3}$) in which all vertices $w \in W$ are colored 3.

We wish to permute the color classes 2 and 3 on some part of G in a smart way so that we have recolored the vertices in W with color 2 and have not created any new uv-paths in colors 1 and 3. So, for all vertices $w \in W$, we look at the component containing w in $G_{2,3}$ and permute the color classes 2 and 3. Note that in the case in which such a component is a single vertex which was colored 3 previously, this simply amounts to assigning it color 2 instead. Note that we do not update W during this process even though it may not continue to be a uv-cut in the new graph constructed at each step. Now, if the set W does in fact still separate u and v within $G_{1,3}$ we are done as these vertices have all been colored with color 2 and now u and v lie in different components of $G_{1,3}$. However, it is possible that permuting colors 2 and 3 within G created new paths in colors 1 and 3. We examine the cases in which this may happen and hope to show that there is always an induced fork or triangle in such graphs. Now, a new uv-path using colors 1 and 3 is created this way only when all vertices colored 2 in what used to be a path in colors 1 and 2 now receive color 3 or when multiple paths in $G_{1,3}$ have now been connected (by recoloring some vertices previously colored 2 with color 3) in order to form a uv-path in $G_{1,3}$. Namely, this happens when each vertex originally colored 2 in this path was in the same component of $G_{2,3}$ as some vertex $w \in W$. It is not clear whether multiple such vertices were in the same component of $G_{2,3}$ or each in the same component as different $w \in W$.

At this point I noted that if the number of vertex disjoint uv paths in colors 1 and 3 has a net loss at each step in this operation, we are done. However, I am not sure whether that can happen. I have some examples (which DO NOT meet our stated conditions of this remaining case so it is possible that this statement may be true) that I have used to think about this question. The issue is that these examples all have vertices of degree 2. In these examples, the number of disjoint uv-paths in colors 1 and 3 does not decrease. However, that does not show that it is possible that this happens, because, like I said, there are vertices of degree 2, so these are not valid examples.

TODO: have examples ready.

I then wondered: if the number of disjoint uv-paths in colors 1 and 3 increases or stays the same, is it always the case that the number of uv-paths in colors 1 and 2 decreases? I have some similar examples in which neither strictly decreases, though once again, these are not valid examples as these do not meet the conditions of the current case because they have vertices of degree 2 and might also have uv-cuts in $G_{1,3}$ that "work" without this operation. Recall that we are in the case in which none of the uv-cuts "work".

So, it remains to determine whether the above claims hold for 2-connected triangle-free, fork-free graphs, precolored by induction, in which u and v lie in the same component of $G_{1,3}$ and $G_{1,2}$ respectively, and in which for all uv-cuts W in $G_{1,3}$ (resp. $G_{1,2}$), for some $w \in W$, there exists $y \in N_G(w)$ with $c(y) = 2$ (resp. $c(y) = 3$). Also, like I said before, I am going about this operation assuming that all vertices in the cut W have color 3 (resp 2), which is a much stronger assumption than necessary. Doing so restricts us to a very small subset of the uv-cuts in $G_{1,3}$. We really should examine all uv-cuts in $G_{1,3}$ (resp. $G_{1,2}$) and assume that none of the cuts W can be recolored with color 2 (meaning we color all $w \in W$ with color 2).