cs6505

HW8: Matchings

4. Suppose you have an algorithm to find a perfect matching of a graph if one exists. Given a graph $G = (V, E)$ and a number $r(u)$ for each vertex u, find a subset of edges F s.t. for each vertex u, exactly $r(u)$ edges of F are incident to u, or declare that such a subset of edges does not exist. For example if $r(u) = 2$ for every u, then F should have two edges incident to every vertex.

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CS 6505 - Homework 8

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1. Make G' from G as follows. Take each vertex $u \in G$ and make in total $r(u)$ copies (including the original) of it. Include all these copies $\{u_1, u_2, ..., u_{r(u)}\}$ in G'.

2. Construct edges as follows. If v and w were adjacent in G, then add two vertices s_{vw} and s_{wv} in G' and add edges $v_i s_{vw}$ for all copies v_i of v, and $s_{wv} w_j$ for all copies w_j of w as well as the edge $s_{vw} s_{wv}$.

3. If G' has a perfect matching then G has a desired subset F. How do we know what the subset F is? Look at all $e \in M$ (where M is our perfect matching of G'). If e is of the form $u_i s_{uw}$ then include the edge uw in F (we do this on both ends and as we show later that if $u_i s_{vw}$ is a matched edge of G', then so is $s_{wv} w_l$ for some copy w_l). If e is of the form $s_{uv} s_{vu}$ do nothing.

4. If G does not have a perfect matching then there is no such set F.

Correctness proof: If G' has a perfect matching, then every vertex is matched. In particular, the vertices s_{vw} and s_{wv} are matched for all edges vw in the orignal graph. Since the only neighbors of s_{vw} are the copies v_i and s_{wv}, either it is matched to some v_k or it is matched to s_{wv}. If it is matched to s_{wv}, then s_{wv} is matched and therefore cannot be matched to any w_i. This case corresponds to the edge vw not being included in F. If s_{vw} is matched to some v_k, then in particular, it cannot be matched to s_{wv} which means s_{wv} is matched to something else, namely some w_l. This case corresponds to the edge vw being included in F. Now, if G' has a perfect matching and we include edges according to this procedure, we will have a set of edges F satisfying $deg_F(u) = r(u)$ for all u. Why? If G' has a perfect matching, then every vertex of G' is matched. In particular, every copy of a vertex is matched, and each matched edge of G' adjacent to some copy corresponds to exactly one edge of G. Obviously each matched edge adjacent to a copy v_i gives rise to an edge of G (because of the argument given above). We need to show that these edges never coincide. If they did, that would mean we have the matched edges in G' $v_i s_{vw}$ and $s_{wv} w_j$ and also $v_k s_{vw}$ and $s_{wv} w_l$. That can't happen as that would mean that s_{vw} would be adjacent to two matched edges which is impossible. So, this process gives us our desired set F from our perfect matching of G'. So, we see that if there is a perfect matching we get such a set F and if there is no perfect matching then no such set F exists.
Our runtime is O($\sum_{u \in V} r(u)$) (to copy all vertices) + O($\sum_{u \in V} r(u)deg(u)$) (to copy all edges) + O($T(V * max_{\{}u \in V\}\{r(u)deg(u) + deg(u)\}, E * 2max_{\{}deg(u)r(u) + 1\})$) (where T(n,m) is the time it takes to find a perfect matching on a graph with n vertices and m edges).