Cool Graph Theory Papers: From before I decided to Change My Focus To Algebra

Caitlin Beecham

August 2019-Present

1 Goals for this Semester

Read 3 papers and present in the graph theory working group (but if I get scared I don't have to, but it will make me read them, could be good because would show Robin Thomas (who occasionally attends) that I am using my research grant well, which means he might be inclined to give it to me again)

2 Graph Theory Papers that Seem Cool

- `https://arxiv.org/pdf/1709.01354.pdf`
  Graph Theory/Game Theory Paper that seems about the level of that one Yoav gave me. Cool because they decom the graph into odd cycles and trees and smaller parts. Worth looking at for that.
- `https://arxiv.org/pdf/0911.1945.pdf`
- `https://arxiv.org/pdf/1702.07179.pdf` This one's good because it seems more easy to digest than others, but less graph theoretic than others.

3 Topological Graph Polynomials in Colored Group Field Theory

`https://arxiv.org/pdf/0911.1945.pdf`

TODO:

- Read section on Tutte polynomials in this paper. `https://arxiv.org/pdf/0811.0186.pdf`
- Look at different papers about different polynomials associated with graphs because I like field theory, which relates to linear algebra because we have vector spaces over different fields which could be over extensions of a field to include certain roots. I want to see where I can go with this kind of thinking. I do not like picking a problem to solve and

then seeing what resources are useful. I much prefer reading literature, finding fun lines of thinking and then seeing which meaningful questions I can ask (conjectures I can think of) that are tackleable and might result in a paper. I much prefer to come up with my own definitions which are useful and prove things about them. In summary, I like asking my own question based on existing research, which is my aim here today. What are some cool graph polynomials? Can I come up with field extensions that encode useful graph properties from these? Today, I just want to read about different graph polynomials and take notes about what properties their roots might encode, and also think about what a meaningful vector space over such a field might be (just thinking about that on my own, not searching the literature just yet, though of course I will get there, I much more remember things when I get lost thinking about them first, then look to the literature).

4 Different Graph Polynomials

https://www.researchgate.net/profile/Johann_Makowsky/publication/221652883_From_a_Zoo_to_a_Zoology_Descriptive_Complexity_for_Graph_Polynomials/links/00b7d51aee7d4c083b000000/From-a-Zoo-to-a-Zoology-Descriptive-Complexity-for-Graph-Polynomials.pdf

4.1 Edge Difference Polynomial

Depends on a labeling of the vertices (so NOT THE SAME across isomorphism class). Say have n vertices. It's

$$P_G(X_1, \dots, X_n) = \sum_{i<j: v_i v_j \in E} (X_i - X_j).$$

Note, I could generalize to directed graphs, so if $v_i v_j \in A$ but $v_j v_i \notin A$. Use accordingly. Note that this seems useful because it reminds me of the sign function for permutations, so whether a variable has positive or negative sign could say something though of course the BIGGER point here is whether a pair is an edge or a non-edge depending on the labeling.

4.2 Chromatic Polynomial

Let $\chi(G, \lambda)$ denote the number of colorings of G using at most λ colors (so can be zero). Birkoff showed that for a fixed graph this is indeed a polynomial. This IS an invariant, so it's the same across an isomorphism class.

4.3 Tutte Polynomial

Read Tutte's account of how he arrived at idea of this polynomial. https://reader.elsevier.com/reader/sd/pii/S0196885803000411?token=312B479D435591EF121EA85D04E7E24367ACB3767BFCF46FD4953BEFF4 "Graph Polynomials" Tutte. I like this because its version in two variables has to do with the "rank generating function" of matroids. Note that this is recursively defined, and not as hard to compute as it seems. It has to do with the number of components and some recursive formula. It almost seems the meaning is harder to interpret than the formula.

4.4 Characteristic Polynomial

It's the characteristic polynomial of the adjacency matrix of the graph. Used to find eigenvalues and eigenvectors. Note that eigenvalues are the same for similar matrices (so no matter the labeling of vertices, the eigenvalues and their multiplicites will be the same). What about eigenvectors? I mean probably not, we "permuted" the coordinates, but same up to "permutation" of coordinates. I have this in quotes because we are not just permuting the rows of the adjacency matrix, but in some way we are changing the columns to keep the existing relations (edges). So, when we permute the coordinates "vertex lables", we also change more to keep the graph structure. Shall I work through an example? To make this precise? Oh okay. Our transformation by a permutation matrix $PAP^{-1} = PAP^T$ (orthogonal matrices satisfy the property that their inverse is their transpose, and permutation matrices are orthogonal). So, if $Av = \lambda v$ meaning v is an eigenvector of A. Then Pv is an eigenvector of $B := PAP^{-1}$. So, it really is JUST a permutation of the coordinates. I stand corrected. What we've done in a linear algebra sense is "changed basis" just by permuting the labels on the basis vectors.

NOTE: This polynomial is COMPLETELY determined by the eigenvalues of A.

4.5 Matching Polynomial

Don't love this, note $M(X) = P(X)$ if and only if G is a forest. ("Matching polynomial equals Characteristic polynomial" if and only if G is a forest).

4.6 There exist polynomials counting the number of induced subgraphs of a certain kind.

Let $\mathcal{H}$ be a graph property and let $\mathrm{ind}_{\mathcal{H}}(G, k)$ be the number of induced subgraphs of size k in G having property $\mathcal{H}$. Is this itself a polynomial in k? Maybe? Will determine later if need to. One can then build a polynomial in a new variable λ (Note that EVEN IF $\mathrm{ind}_{\mathcal{H}}(G, k)$ was polynomial in k, then this whole thing will NOT be polynomial in λ and k. Why? Because k is also an exponent in this expression).

$$\mathrm{gen}_{\mathcal{H}}(G, \lambda) = \sum_k \mathrm{ind}_{\mathcal{H}}(G, k)\lambda^k$$

Anyway, for $\mathcal{H}$ which is a clique, isolated point, cycle or path, these polynomials have been well-studied.

5 Other Cool Polynomials

The article said these are worth GOOGLING so do that when I have time.

- Farrell polynomials: https://www.sciencedirect.com/science/article/pii/0095895679900492
- Clique and Independent Set polynomials: http://mathworld.wolfram.com/CliquePolynomial.html, http://mathworld.wolfram.com/IndependencePolynomial.html
- Dependence polynomials https://www.sciencedirect.com/science/article/pii/0012365X9090202S
- Martin polynomials "encodes information about the families of closed paths in eulerian graphs" (note this says only in eulerian graphs, which is a small class, but say I have a graph from which I delete a forest to get an even graph, then I could use this. A line of thought here for me is decomposing a graph into a forest (a union of paths) and a remaining eulerian graph. There are polynomials describing paths as well as cycles (an eulerian graph is a disjoint union of cycles), so maybe decomposing this way could be good for isomorphism, or this result directly applies to eulerian graphs). https://www.sciencedirect.com/science/article/pii/S0095895698918536
- Go-polynomials (Didn't include URL because this seems not too useful for me).
- Rank Polynomial $R(x, y)$ (note the variables x and y are not meaningful parameters we plug in like k, λ in the chromatic polynomial, they are just used to encode rank and corank as follows). $R(x, y) = \sum_{S \subseteq E(G)} x^{r(S)} y^{s(S)}$. http://mathworld.wolfram.com/RankPolynomial.html Cool because multiplicative across connected components. Also, easy to calculate. The rank of some subgraph is just $n - c$, where n is the number of vertices and c is the number of components. It can be given in terms of the TUTTE polynomial. Also, it is related to the chromatic polynomial $\pi(x)$ in a surprisingly simple way: $\pi(x) = x^n R(-x^{-1}, -1)$. It would be interesting to think about why that is true. Though, it is not that surprising because the relationship is not actually that simple considering x appears to the -1 power and -1 is the second parameter. It actually seems we just plug in the right things to get $\pi(x)$ and it may not be as conceptually beautiful as I thought. Still worth thinking about though, since chromatic number seems like a very global concept that depends a lot on the specific structure of the graph, whereas this polynomial is a sum across all subgraphs, seems more like an "average" of local structures.
- Flow Polynomial http://mathworld.wolfram.com/FlowPolynomial.html. Weirdly, we can get this by plugging in the right things to the rank and tutte polynomials then multiplying by ± 1. It might be worth thinking about why those formulations hold. I could potentially give a talk in the graph theory working group about how all these polynomials relate to each other and why plugging in certain quantities to one gives another etc.
- Reliability Polynomial $C_G(p)$ http://mathworld.wolfram.com/ReliabilityPolynomial.html. Given G, say each edge is independently deleted with probability p. Then $C_G(p)$ is the probability that every connected component of G remains connected after randomly deleting edges this way.