Math 500: Homework 9

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1. Let $p : E \to B$ be a covering map. Prove that $p^{-1}(\{b\})$ is a discrete subspace of E (i.e., that the subspace topology on $p^{-1}(\{b\})$ is the discrete topology).

   So, we want to show that every singleton $\{a\} \subseteq p^{-1}(\{b\})$ is open in the subspace topology on $p^{-1}(\{b\}) \subseteq E$, which will imply that every subset of $p^{-1}(\{b\})$ is open in the subspace topology, i.e. the subspace topology is the discrete topology on $p^{-1}(\{b\})$.

   By definition, the singleton set $\{a\} \subseteq p^{-1}(\{b\})$ open in the subspace topology if $\{a\} = V_\alpha \cap p^{-1}(\{b\})$ for some open set $V_\alpha \subseteq E$.

   We show this by double inclusion.
   First, we show that $\{a\} \subseteq V_\alpha \cap p^{-1}(\{b\})$ for some open set $V_\alpha \subseteq E$.
   That means that, since we picked $\{a\} \subseteq p^{-1}(\{b\})$, we just need to show that $\{a\} \subseteq V_\alpha$ for some open set $V_\alpha \subseteq E$. How? Note that $\{b\} \subseteq U$ implies $p^{-1}(\{b\}) \subseteq p^{-1}(U)$ (since inverse functions preserve inclusion).
   Now, $p : E \to B$ a covering map means that

   (a) $p^{-1}(U) = \bigsqcup_\alpha V_\alpha$ for disjoint open sets $V_\alpha \subseteq E$ and
   (b) $p\big|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism,

   so
   $$\{a\} \subseteq p^{-1}(\{b\}) \subseteq p^{-1}(U) = \bigsqcup_\alpha V_\alpha,$$
   That means $\{a\}$ belongs to exactly one open set $V_\alpha \in E$. So we have proven $\{a\} \subseteq V_\alpha \cap p^{-1}(\{b\})$. That's one inclusion.

   We now show that $V_\alpha \cap p^{-1}(\{b\}) \subseteq \{a\}$.
   That is, we show that $\{y\} \subseteq V_\alpha \cap p^{-1}(\{b\}) \Rightarrow \{y\} = \{a\}$.

   Well, $\{y\} \subseteq V_\alpha \cap p^{-1}(\{b\})$ if and only if $\{y\} \subseteq V_\alpha$ and $\{y\} \subseteq p^{-1}(\{b\})$ or equivalently, if and only if $\{y\} \subseteq V_\alpha$ and $p(\{y\}) \subseteq p(p^{-1}(\{b\}))$ (since both functions and inverse functions preserve inclusions). Also, $p(p^{-1}(\{b\})) = \{b\}$ because p is surjective. So yet another equivalent statement is that $\{y\} \subseteq V_\alpha$ and $p(\{y\}) = \{b\}$, which can be concisely stated as $p\big|_{V_\alpha}(\{y\}) = \{b\}$.
   So, it suffices to show that $\left(p\big|_{V_\alpha}(\{y\}) = \{b\}\right) \Rightarrow (\{y\} = \{a\})$.

   By hypothesis, $p\big|_{V_\alpha}(\{y\}) = \{b\}$, and like show above for $\{y\}$, $\{a\} \subseteq V_\alpha \cap p^{-1}(\{b\})$ implies $p\big|_{V_\alpha}(\{a\}) = \{b\}$, so we have
   $$p\big|_{V_\alpha}(\{y\}) = \{b\} = p\big|_{V_\alpha}(\{a\}),$$

but since $p\big|_{V_\alpha}$ is injective, that implies $\{y\} = \{a\}$, as desired, and we are done.

2. Let $p : E \to B$ be a covering map. Let α and β be paths in B with $\alpha(1) = \beta(0)$; let $\widetilde{\alpha}$ and $\widetilde{\beta}$ be liftings of them such that $\widetilde{\alpha}(1) = \widetilde{\beta}(0)$. Show that $\widetilde{\alpha} * \widetilde{\beta}$ is a lifting of $\alpha * \beta$.

   We want to show that $\widetilde{\alpha} * \widetilde{\beta}$ is a lifting of $\alpha * \beta$, i.e. that $p \circ (\widetilde{\alpha} * \widetilde{\beta}) = \alpha * \beta$.
   Well, what is $(\alpha * \beta)(x)$?

$$(\alpha * \beta)(x) = \begin{cases} \alpha(2x) \text{ for } x \in [0, \frac{1}{2}) \\ \beta(2x-1) \text{ for } x \in [\frac{1}{2}, 1]. \end{cases}$$

   What is $\left(p \circ (\widetilde{\alpha} * \widetilde{\beta})\right)(x)$? Well it is $p((\widetilde{\alpha} * \widetilde{\beta})(x))$.
   What is $(\widetilde{\alpha} * \widetilde{\beta})(x)$?

$$(\widetilde{\alpha} * \widetilde{\beta})(x) = \begin{cases} \widetilde{\alpha}(2x) \text{ for } x \in [0, \frac{1}{2}) \\ \widetilde{\beta}(2x-1) \text{ for } x \in [\frac{1}{2}, 1]. \end{cases}$$

   That means

$$p((\widetilde{\alpha} * \widetilde{\beta})(x)) = \begin{cases} p(\widetilde{\alpha}(2x)) \text{ for } x \in [0, \frac{1}{2}) \\ p(\widetilde{\beta}(2x-1)) \text{ for } x \in [\frac{1}{2}, 1], \end{cases}$$

   but since $\widetilde{\alpha}$ and $\widetilde{\beta}$ are liftings of α and β respectively, we know that $p \circ \widetilde{\alpha} = \alpha$ and $p \circ \widetilde{\beta} = \beta$. That implies

$$p((\widetilde{\alpha} * \widetilde{\beta})(x)) = (\alpha * \beta)(x) = \begin{cases} \alpha(2x) \text{ for } x \in [0, \frac{1}{2}) \\ \beta(2x-1) \text{ for } x \in [\frac{1}{2}, 1], \end{cases}$$

   and we are done.

3. Consider the covering map $p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ defined by

$$(p \times p)(x \times x) = (cos(2\pi x), sin(2\pi x)) \times (cos(2\pi x), sin(2\pi x)).$$

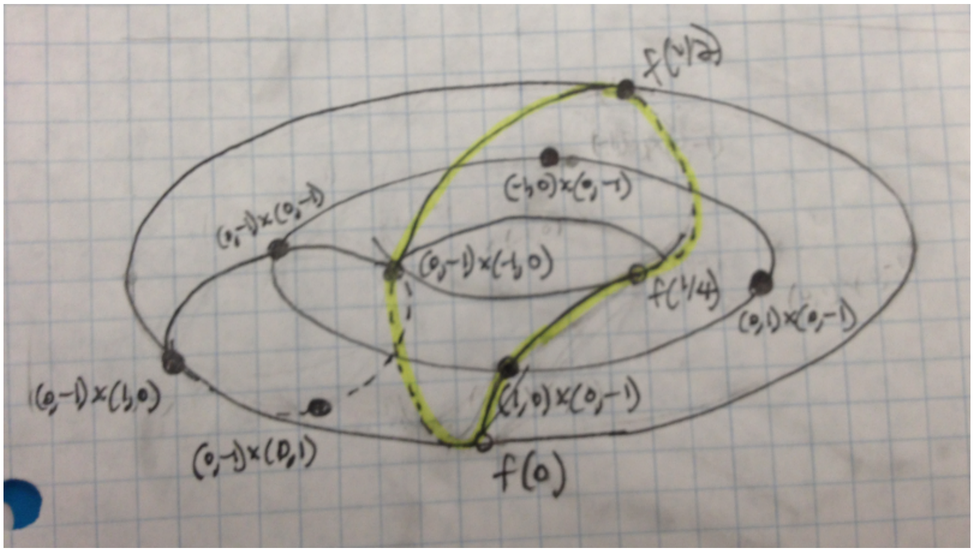
   Consider the path

$$f(t) = (cos(2\pi t), sin(2\pi t)) \times (cos(4\pi t), sin(4\pi t))$$

   in $S^1 \times S^1$. Sketch what f looks like when $S^1 \times S^1$ is identified with the doughnut surface D. Find a lifting $\widetilde{f}$ of f to $\mathbb{R} \times \mathbb{R}$ and sketch it. Well, in order to sketch f, we calculate some values of f.

$$\begin{aligned} f(0) &= (1,0) \times (1,0) \\ f\left(\frac{1}{4}\right) &= (0,1) \times (-1,0) \\ f\left(\frac{1}{2}\right) &= (-1,0) \times (1,0) \\ f\left(\frac{3}{4}\right) &= (0,-1) \times (-1,0) \\ f(1) &= (1,0) \times (1,0) \end{aligned}$$

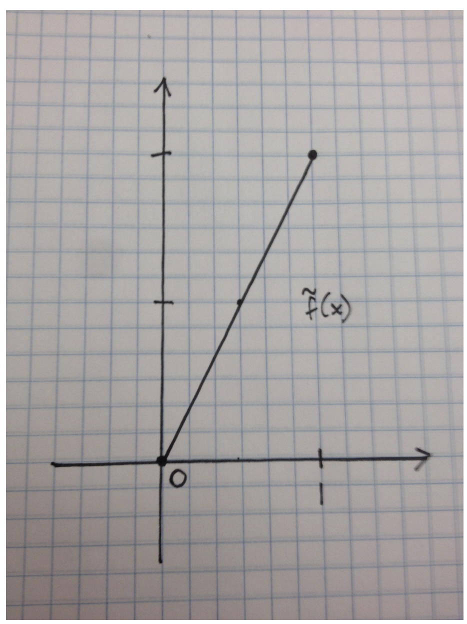
Here is my picture:



To account for my lack of drawing skills I will make clear that f goes around one copy of S^1 once and goes around the other copy twice.

Now, we need to find a function $\widetilde{f} : [0,1] \to \mathbb{R} \times \mathbb{R}$ such that $(p \times p)(\widetilde{f}(x)) = f(x)$. I claim that $\widetilde{f}(x) = x \times 2x$ is such a function. Why? Because $(p \times p)(\widetilde{f}(x)) = (p \times p)(x \times 2x) = (cos(2\pi t), sin(2\pi t)) \times (cos(4\pi t), sin(4\pi t)) = f(x)$.

Here is a picture of $\widetilde{f}(x)$ in $\mathbb{R} \times \mathbb{R}$:



4. Generalize the proof of Theorem 54.5 to show that the fundamental group of the torus is isomorphic to the group $\mathbb{Z} \times \mathbb{Z}$.

We will give an isomorphism. Namely, define $\Phi : \mathbb{Z} \times \mathbb{Z} \to \pi_1(T^2, q)$ (where $q = (1,0) \times (1,0)$) by $\Phi(n,m) = [(cos(2\pi ns), sin(2\pi ns)) \times (cos(2\pi ms), sin(2\pi ms))]$. We need to show that this a bijective homomorphism.

First, is it a homomorphism? i.e. is it true that $\Phi((n,m) + (a,b)) = \Phi((n,m)) * \Phi((a,b))$? Well,

$$\begin{aligned}\Phi((n,m) + (a,b)) &= \Phi((n+a, m+b)) \\ &= \Big[\big(cos(2\pi(n+a)s), sin(2\pi(n+a)s)\big) \times \big(cos(2\pi(m+b)s), sin(2\pi(m+b)s)\big)\Big].\end{aligned}$$

Also,

$$\begin{aligned}\Phi((n,m)) * \Phi((a,b)) &= \Big[\big(cos(2\pi ns), sin(2\pi ns)\big) \times \big(cos(2\pi ms), sin(2\pi ms)\big)\Big] \\ &\quad * \Big[\big(cos(2\pi as), sin(2\pi as)\big) \times \big(cos(2\pi bs), sin(2\pi bs)\big)\Big] \\ &= \Bigg[\Big(\big(cos(2\pi ns), sin(2\pi ns)\big) \times \big(cos(2\pi ms), sin(2\pi ms)\big)\Big) \\ &\quad * \Big(\big(cos(2\pi as), sin(2\pi as)\big) \times \big(cos(2\pi bs), sin(2\pi bs)\big)\Big)\Bigg]\end{aligned}$$

So the question is:
Are the homotopy classes $\Big[\big(cos(2\pi(n+a)s), sin(2\pi(n+a)s)\big) \times \big(cos(2\pi(m+b)s), sin(2\pi(m+b)s)\big)\Big]$ and $\Bigg[\Big(\big(cos(2\pi ns), sin(2\pi ns)\big) \times \big(cos(2\pi ms), sin(2\pi ms)\big)\Big) * \Big(\big(cos(2\pi as), sin(2\pi as)\big) \times \big(cos(2\pi bs), sin(2\pi bs)\big)\Big)\Bigg]$ equal?

i.e. are $f(s) := \big(cos(2\pi(n+a)s), sin(2\pi(n+a)s)\big) \times \big(cos(2\pi(m+b)s), sin(2\pi(m+b)s)\big)$ and $g(s) := \Big(\big(cos(2\pi ns), sin(2\pi ns)\big) \times \big(cos(2\pi ms), sin(2\pi ms)\big)\Big) * \Big(\big(cos(2\pi as), sin(2\pi as)\big) \times \big(cos(2\pi bs), sin(2\pi bs)\big)\Big)$ homotopic? I claim they are. Why is that? That is because they both wrap around one copy of S^1 $n+a$ times and wrap around the other copy $m+b$ times. So, Φ is a homomorphism.

Is it bijective? Yes, it is. Why? Well, it turns out that Φ is the lifting correspondence derived from the covering map $p \times p$ (because $p^{-1}(q) = p^{-1}((1,0) \times (1,0)) = \mathbb{Z} \times \mathbb{Z}$), and since $\mathbb{R} \times \mathbb{R}$ is simply connected (and path connected) Φ is bijective.